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R. Rammal, Benoît Douçot. Invariant imbedding approach to localization. I. General framework and basic equations. Journal de Physique, 1987, 48 (4), pp.509-526. 10.1051/jphys:01987004804050900 . jpa-00210466

HAL Id: jpa-00210466
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Submitted on 4 Feb 2008

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J. Physique **48** (1987) 509-526 AVRIL **1987**, PAGE 509

Classification
Physics Abstracts
03.40K — 42.20 — 71.55J

Invariant imbedding approach to localization. I. General framework and basic equations

R. Rammal and B. Doucot

Centre de Recherches sur les Très Basses Températures, CNRS, B.P. 166 X, 38042 Grenoble Cedex, France

(*Reçu le 22 septembre 1986, accepté le 20 novembre 1986*)

Résumé. — On décrit un cadre général pour résoudre les problèmes de transmission d'ondes à travers un milieu désordonné. En utilisant une approche de plongement invariant, on démontre des équations différentielles vérifiées par les coefficients de réflexion et de transmission. En général, le problème de la transmission (problème aux valeurs limites) se réduit à un problème de Cauchy (à valeur initiale). Des résultats connus retrouvés et des nouvelles équations sont obtenues pour les problèmes multicanaux et les milieux dépendant du temps. L'extension de cette approche à d'autres situations est décrite brièvement, ainsi qu'une approche systématique pour étudier les équations stochastiques obtenues. Le cas d'un seul canal est utilisé comme exemple d'illustration.

Abstract. — A unified framework for solving the problems of wave transmission across a random medium is outlined. Using an invariant imbedding approach, differential equations are derived for the reflection and transmission coefficients. In general, the transmission problem, viewed as a boundary-value problem, can be reduced to an initial-value Cauchy equation, relative to the imbedding parameters. Known results are recovered and new equations pertaining to multichannel problems, time-dependent medium, etc., are obtained. The extension of this approach to other cases is outlined. A systematic method for the investigation of the stochastic differential equations so obtained is described. The case of one-dimensional linear media is used as an illustrative example.

1. Introduction.

The classical theory of transport focuses attention on the *fluxes inside* the transporting material and ends up with Boltzmann equation or linear response theory. Another point of view has been advanced by Landauer [1] where one is mainly concerned with *external quantities* such as reflected and transmitted currents. Both approaches start from first principles. The second approach has been used extensively during the recent years for the study of Anderson localization [1-6]. For instance, the electrical conductance g of a random medium can be expressed simply [1] in terms of the reflection and transmission amplitudes across the considered random medium.

For some specific cases, such as tight binding models describing linear media, it is possible to obtain explicit expressions [6] for g in terms of Green's functions, associated with the model at hand. The extension of such techniques which are mainly based on the transfer matrix method seems to be very hard. For instance, when non linearities are present in the random medium, the transfer matrix technique seems to fail completely. In this case, there is no simple way to handle the corresponding transmission problem.

The main object of this paper, the first in a series, is to present a natural framework for the implementation of Landauer's programme. The first idea is the use of the invariant imbedding equations. An interesting and recent account of this method, which has been used since the time of Stokes [7], can be found in reference [8]. The general principle of the invariant imbedding method is to address directly the emergent quantities ; the reflection and transmission coefficients. this can be done through the reduction of the initial (boundary-value) problem to a Cauchy (initial-value) problem, relative to the imbedding parameters. The reflection and transmission coefficients are then described by a closed set of non-linear differential equations.

Having the appropriate equations for the relevant quantities, one can adopt the point of view of dynamical systems where the imbedding parameters play the role of time. In such a case the known perturbation techni-

Article published online by EDP Sciences and available at http://dx.doi.org/10.1051/jphys:01987004804050900

ques of dynamical systems can be used to derive some useful results. Furthermore, when randomness is present, one can use the stochastic differential equations obtained from the invariant imbedding approach, in order to derive directly Fokker-Planck equations for the probability distributions of the emergent quantities.

The emphasis of this paper is to show the full generality of this methodology for the study of wave transmission across a random medium. To our knowledge, only the one-channel problem in one-dimensional linear medium [9] has been worked according to these lines of ideas. In the next papers of this series, new problems will be addressed following this scheme. This paper is self-contained (no prior knowledge of the invariant imbedding ideas is required) and organized as follows. In section 2, we derive the basic equations for linear problems. Two different methods are used to establish and interpret the invariant imbedding equations in the case of one channel in 1D linear medium. The multichannel transmission problems as well as the time dependent media case are then shown to give rise to matrix Riccati equations. A formal solution, obtained from the « linearization » of this set of equations is given at the end of this section. Section 3 is devoted to non-linear media, where the non-linearities are assumed to be a function of the wave field intensity. A detailed account of the results obtained for the transmission across non-linear random media will be found in reference [10]. Illustrative examples are discussed in section 4. In addition to some new results, a large number of known results are derived in a systematic way. The possible extensions of this approach are briefly discussed in the conclusion.

2. Linear problems.

In this section we shall derive the basic equations of the Invariant Imbedding approach to the transmission problems. In some particular cases, such as the one-channel problems, the same results can be obtained otherwise, using the notion of impedance for instance [11]. Here we try to be as general as possible and this in order to allow for natural extensions.

2.1 ONE-DIMENSIONAL PROBLEMS. — In order to illustrate the main steps of the Invariant Imbedding approach, let us consider the case of a plane layered medium which occupies part of the space $0 \leqslant x \leqslant L$. The complex amplitude of the field ψ is assumed to be a solution of :

$$i \frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2} - V(x)\, \psi \qquad (2.1)$$

where $V(x)$ is a given potential and $V(x) \equiv 0$ outside the layer. Here one is interested in the stationary regime, corresponding to solutions of (2.1) of the type $\psi(x, t) = e^{ik^2 t}\, \varphi(x)$. This leads to the following equation for $\varphi(x)$:

$$\frac{d^2\varphi}{dx^2} + k^2[1 + \eta(x)]\, \varphi = 0 \qquad (2.2)$$

where $\eta(x) = -V(x)/k^2$.

Let a plane wave $\varphi_0(x) = e^{ik(L-x)}$ be incident on the layer at the right. The physical quantities of main interest are the complex reflection R_L and transmission T_L coefficients across the layer. The main idea for the calculation of R_L and T_L is a follows. Instead of considering the spectrum of (2.2), it is sufficient to have a direct access to R_L and T_L. Such a task is actually possible, because R_L and T_L can be shown to be the solutions of ordinary differential equations, with respect to the « imbedding » parameter L. Indeed, to the right and left of the layer, the wave field has the form

$$\begin{aligned} &\varphi(x) = \exp[ik(L-x)] + R_L \exp[ik(x-L)] \\ &\varphi(x) = T_L \exp(-ikx)\,. \end{aligned} \qquad (2.3)$$

From the boundary conditions (continuity of $\varphi(x)$ and $\frac{d\varphi}{dx}$) on the layer boundaries, one gets from (2.2) :

$$\begin{aligned} \varphi(L) &= 1 + R_L\,, \\ \varphi'(L) &= -ik(1 - R_L) = -ik(2 - \varphi(L)) \\ \varphi(0) &= T_L\,, \\ \varphi'(0) &= -ikT_L = -ik\varphi(0)\,. \end{aligned} \qquad (2.4)$$

Equations (2.2)-(2.4) define a very well posed problem and our goal is the calculation of R_L and T_L. Before going into details, let us mention that (2.2) may describe different situations such as optical problems where $1 + \eta(x) \equiv 1 - V(x)/k^2$ plays the role of the optical index $n^2(x)$. In the present notation $\eta(x)$ can be considered as the dielectric constant.

The main first step is to transform the boundary value problem into an integral equation

$$\varphi(x\,;L) = e^{ik(L-x)} + \frac{ik}{2} \int_0^L dx'\, e^{ik|x-x'|}\, \eta(x')\, \varphi(x'\,;L) \qquad (2.5)$$

where we have used the notation $\varphi(x\,;L) \equiv \varphi(x)$. This is the first step in the invariant Imbedding method. Starting from (2.5) the main equations for R_L and T_L, considered now as functions of L, can be derived using two different methods : analytical and graphical.

2.1.1 *Analytical derivation.* — Differentiation of (2.5) with respect to L leads to

$$\frac{\partial \varphi(x\,;L)}{\partial L} = A(L)\, \varphi(x\,;L) + B(x\,;L) \qquad (2.6)$$

where

$$A(L) = ik + \frac{ik}{2}\, \eta(L)\,.\, \varphi_L\,, \quad \varphi_L \equiv \varphi(x = L\,;L) \qquad (2.7)$$

and

$$B(x\,;L)=\frac{ik}{2}\int_0^L \mathrm{d}x'\,\mathrm{e}^{ik|x-x'|}\,\eta(x')\frac{\partial\varphi(x'\,;L)}{\partial L}-$$
$$-A(L)\frac{ik}{2}\int_0^L \mathrm{d}x'\,\mathrm{e}^{ik|x-x'|}\,\eta(x')\,\varphi(x'\,;L)\,. \quad (2.8)$$

Now, it is not difficult to see that (2.8) can also be written as

$$B(x\,;L)=\frac{ik}{2}\int_0^L \mathrm{d}x'\,\mathrm{e}^{ik|x-x'|}\,\eta(x')\times$$
$$\times\left[\frac{\partial\varphi(x'\,;L)}{\partial L}-A(L)\,\varphi(x',L)\right]$$

and using (2.6)

$$B(x\,;L)=\frac{ik}{2}\int_0^L \mathrm{d}x'\,\mathrm{e}^{ik|x-x'|}\,\eta(x')\,.\,B(x',L)\,. \quad (2.9)$$

This shows in particular that $B(x,L)$ satisfies the same equation (2.5) as $\varphi(x\,;L)$ but without the source term. This implies that $B(x\,;L)=0$ and then the following equation for the field $\varphi(x\,;L)$ inside the layer :

$$\frac{\partial\varphi(x\,;L)}{\partial L}=A(L)\,\varphi(x\,;L) \quad (2.10)$$

with the boundary conditions $\varphi(x\,;L)|_{L=x}=\varphi_x$ and $\varphi(L\,;L)=\varphi_L$. Taking now the derivative of φ_L with respect to L :

$$\frac{\partial\varphi_L}{\partial L}=\frac{\partial\varphi(x\,;L)}{\partial x}\bigg|_{x=L}+\frac{\partial\varphi(x\,;L)}{\partial L}\bigg|_{x=L}=$$
$$=-ik(2-\varphi_L)+A(L)\,\varphi_L$$

and using the boundary conditions, (2.4) and (2.10) one gets $(\varphi_L=1+R_L)$

$$\frac{\mathrm{d}R_L}{\mathrm{d}L}=2\,ikR_L+\frac{ik}{2}\eta(L)\,.\,(1+R_L)^2\,,$$
$$R_L=0 \quad \text{at} \quad L=0 \quad (2.11)$$

which is an autonomous equation for the reflection coefficient R_L.

Similarly by taking (2.10) at $x=0$, one obtains the following equation for the transmission coefficient T_L :

$$\frac{\mathrm{d}T_L}{\mathrm{d}L}=ikT_L+\frac{ik}{2}\eta(L)\,.\,(1+R_L)\,T_L\,,$$
$$T_L=1 \quad \text{at} \quad L=0\,. \quad (2.12)$$

These two equations for R_L and T_L provide a complete solution of the present problem. In fact (2.11) is a Riccati equation and has a well defined solution (Cauchy problem). When R_L is known, (2.12) gives the solution for T_L. More generally, the wave field can be calculated from (2.10) :

$$\varphi(x\,;L)=\varphi_x\exp\left\{ik(L-x)+\frac{ik}{2}\int_x^L \mathrm{d}y\,\eta(y)\,.\,\varphi_y\right\}. \quad (2.13)$$

The intensity of the wave field within the medium can be expressed in terms of $|R_x|^2$. Taking $x=0$, one obtains the expression of $|T_L|^2$ as a function of the reflection coefficient. For real η, this reduces to the conservation law $|R|^2+|T|^2=1$, as its should be.

2.1.2 *Graphical derivation.* — The previous equations (2.11) and (2.12) can be derived in another way, which allows for a straighforward extension to more complicated cases. The basic idea is the interpretation of the integral equation (2.5) in terms of two elementary scattering events :

$$\varphi(x,L)\,\mathrm{d}x=\mathrm{e}^{ik(L-x)}\,\mathrm{d}x+\int_{-\infty}^{\infty}[\mathrm{e}^{ik|x-x'|}]\times$$
$$\times\left[\frac{ik}{2}\eta(x')\,\mathrm{d}x\right]\cdot[\varphi(x',L)\,\mathrm{d}x']. \quad (2.14)$$

Equation (2.14) admits the following interpretation. The amplitude (to be) at $\mathrm{d}x$ around x is the sum of two terms : the first one corresponds to the amplitude at $\mathrm{d}x$ *without scattering*, whereas the second term represents the sum over x' of the amplitudes at $\mathrm{d}x$ after scattering at x'. Each term in the second part is the product of three amplitudes : i) the amplitude at $\mathrm{d}x'$, ii) the amplitude of scattering at x' and iii) the amplitude of free propagation from x' to x. The two elementary processes are therefore : $\mathrm{e}^{ik\,\mathrm{d}x}\simeq 1+ik\,\mathrm{d}x$ (free propagation) and $\frac{ik}{2}\eta(\mathrm{d}x)\,x$ (scattering).

We compare now the reflection coefficients $R(L)$ and $R(L-\mathrm{d}L)$ corresponding respectively to the present problem with a potential $\eta(x)$ and that associated with a cut off $\eta(x)$ at $L-\mathrm{d}L$. The difference between these two coefficients is represented, to first order in $\mathrm{d}L$, in figure 1 :

$$R(L)=[1+ik\,\mathrm{d}L]\,R(L-\mathrm{d}L)[1+ik\,\mathrm{d}L]+$$
$$+\left[\frac{ik}{2}\eta(L)\,\mathrm{d}L\right]$$
$$+\left[\frac{ik}{2}\eta(L)\,\mathrm{d}L\right]R(L-\mathrm{d}L)\,.\,[1+ik\,\mathrm{d}L]$$
$$+[1+ik\,\mathrm{d}L]\,R(L-\mathrm{d}L)\left[\frac{ik}{2}\eta(L)\,\mathrm{d}L\right]$$
$$+R(L-\mathrm{d}L)\left[\frac{ik}{2}\eta(L)\,\mathrm{d}L\right]R(L-\mathrm{d}L)[1+ik\,\mathrm{d}L]\,. \quad (2.15)$$

Expanding to first order in $\mathrm{d}L$, one gets the following equation for $R(L)$

$$\frac{\mathrm{d}R}{\mathrm{d}L}=2\,ikR+\frac{ik}{2}\eta(L)\,.\,(1+R)^2$$

which is identical to (2.11).

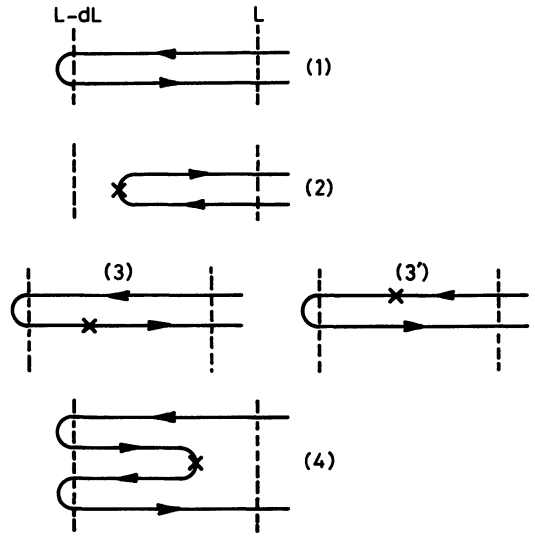

Fig. 1. — The different elementary processes leading to a non vanishing contribution to the reflection coefficient R_L. Scattering events are represented by crosses.

With the aid of figure 2, one obtains in a similar way :

$$T(L) = [1 + ik\,dL]\,T(L - dL) + \left[\frac{ik}{2}\eta(L)\,dL\right]T(L - dL) + [1 + ik\,dL]\,.\,R(L - dL) \times \left[\frac{ik}{2}\eta(L)\,dL\right]\,.\,T(L - dL)\,. \quad (2.16)$$

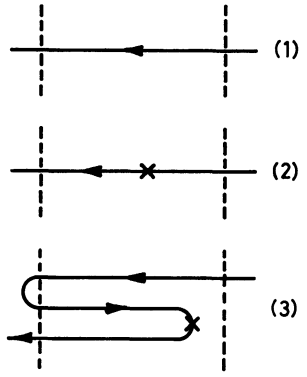

Fig. 2. — Same notation as in figure 1 for the transmission coefficient T_L.

Expanding to first order in dL, one gets similarly,

$$\frac{dT}{dL} = ik\,.\,T + \frac{ik}{2}\eta(1 + R)\,T\,. \quad (2.12)$$

This rapid derivation of (2.11) and (2.12) shows in particular the physical origin of the non-linear terms R^2 and $R.T.$: the backscattering mechanism. In addition to its relative simplicity, the present derivation has the main advantage, over the analytical approach, of its simple extension to more general cases.

Before using the Invariant Imbedding ideas in other problems, let us summarize the « logarithm » of this approach :

i) The differential equation of the wave field is first transformed into an integral form. This first step can always be performed. What we need here is just the expression of the Green functions in the regions outside the layer.

ii) Analytical approach : use the boundary conditions to relate the field at the boundary to R and T. Successive derivatives of the integral equation with respect to the Imbedding parameters lead to the desired equations.

Graphical approach : a simple interpretation of the integral equation, in terms of some elementary scattering events, is to be done first. By comparing the two problems differing just by an infinitesimal change of the Imbedding parameters, one obtains the main equations for R and T.

It is clear that the above algorithm is not limited to 1D problems and can be extended to other boundary-value problems. The main advantage is to provide a direct access to the physical quantities of main interest, through an initial value equation (Cauchy problem).

2.2 MULTICHANNEL TRANSMISSION PROBLEMS. — As a typical multichannel transmission problem we consider the following model [12]. N coupled disordered chains, of common length L, are described by the following Hamiltonian $\sum_{n,m} \mathcal{H}_{nn'}$, where

$$\mathcal{H}_{nn'} = \left[-E - \frac{1}{2m}\frac{d^2}{dx^2} + V_n(x)\right]\delta_{nn'} + t(\delta_{n-1,n'} + \delta_{n+1,n'})\,. \quad (2.17)$$

Equation (2.17) corresponds to a free propagation along the chains ($1 \leqslant n \leqslant N$) where the potential in the n-th chain is given by $V_n(x)$. The coupling constant t is the binding energy between the chains, and describe a tight binding on the perpendicular direction. There are different manners of paking the chains into a wire and this corresponds to different boundary conditions [12]. The eigenvalues problem, associated to (2.17), can be written as

$$\sum_{n'} \mathcal{H}_{nn'}\,\psi_{n'}(x) = 0 \quad (2.18)$$

where $\psi_n(x)$ is the wave field of the n-th chain and $E = k^2/2m$ is the corresponding energy. In what follows we assume a free boundary condition on the transverse direction : $\psi_0 = \psi_{N+1} = 0$. The transmission problem can be defined in an identical way as in the one channel case (see Fig. 3).

In order to derive the Invariant Imbedding equations,

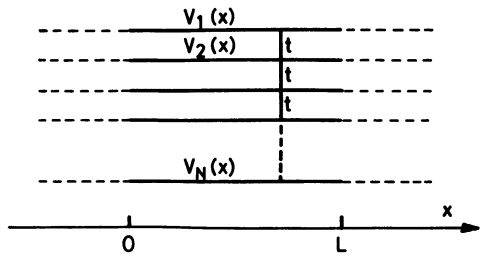

Fig. 3. — Notation for the multichannel problem.

it is convenient to diagonalize the Hamiltonian of (2.17), with respect to the transverse direction. This leads to a new basis where the channels are very well defined. The new states are given by

$$\tilde{\psi}_n(x) = \sum_{n'} \psi_{n'}(x) \cdot \left(\frac{2}{N+1} \right)^{1/2} \sin\left(\pi \frac{nn'}{N+1} \right) \tag{2.19}$$

and the Hamiltonian becomes

$$\tilde{\mathcal{H}}_{nn'} = \left(-E + \varepsilon_n - \frac{1}{2m} \frac{d^2}{dx^2} \right) \delta_{nn'} + \tilde{V}_{nn'}(x) \tag{2.20}$$

where

$$\tilde{V}_{nn'} = \sum_{n_0} V_{n_0}(x) \cdot \frac{2}{N+1} \sin\left(\pi n_0 \frac{n}{N+1} \right) \times \times \sin\left(\pi n_0 \frac{n'}{N+1} \right) \tag{2.21}$$

and

$$\varepsilon_n = 2t \cos\left[\pi n/(N+1) \right].$$

Therefore, outside the layer $[0, L]$, the eigenstates are plane waves, of wave vectors k_α, given by

$$\frac{k_\alpha^2 - k^2}{2m} = 2t \cos\left(\frac{\alpha \pi}{N+1} \right), \qquad 1 \leqslant \alpha \leqslant N . \tag{2.22}$$

Each k_α defines a channel for the incident wave to the right of the layer. Using the channel basis, the integral version of (2.18) becomes :

$$\tilde{\psi}_n(x, L) = A_n e^{ik_n(L-x)} + \frac{2m}{2ik_n} \times \times \int_0^L dx' \, e^{ik_n|x-x'|} \left[\sum_{n'} \tilde{V}_{nn'}(x') \cdot \tilde{\psi}_{n'}(x', L) \right]. \tag{2.23}$$

Defining the group velocity $v_n = k_n/m$ in channel n, (2.23) can also be written as :

$$dx \cdot \tilde{\psi}_n(x ; L) = A_n e^{ik_n(L-x)} dx + + \int_{-\infty}^{\infty} e^{ik_n|x-x'|} \left(\sum_{n'} \frac{-i\tilde{V}_{nn'}(x')}{v_n} dx \right) \times (\tilde{\psi}_{n'}(x') \, dx') . \tag{2.24}$$

Let us now consider the reflexion (R_{nm}) and transmission (T_{nm}) coefficients : n refers to the incidence channel and m to the scattering channel. Proceeding in exactly the same manner as for the one channel problem, one obtains

$$\begin{aligned} R_{nm}(L) = {} & (1 + ik_n \, dL) \cdot R_{nm}(L - dL) \cdot (1 + k_m \, dL) + \\ & + \left[\frac{-i\tilde{V}_{nm}}{v_m} dL \right] \\ & + \sum_l \left(\frac{-i\tilde{V}_{nl}}{v_l} dL \right) \cdot (1 + ik_m \, dL) \cdot R_{lm}(L - dL) \\ & + \sum_l \left(\frac{-i\tilde{V}_{lm}}{v_m} dL \right) \cdot (1 + ik_n \, dL) \cdot R_{nl}(L - dL) \\ & + (1 + ik_n \, dL) \cdot \sum_{l,l'} R_{nl}(L - dL) \\ & \times \left(\frac{-i\tilde{V}_{ll'}}{v_{l'}} dL \right) \cdot R_{l'm}(L - dL) \cdot (1 + ik_m \, dL) . \end{aligned} \tag{2.25}$$

The sums in (2.25) are taken over the intermediate channels l and l'. Expanding up to first order in dL, one obtains :

$$\frac{dR_{nm}}{dL} = i(k_n + k_m) R_{nm} + \frac{-i\tilde{V}_{nm}}{v_m} - i \sum_l \frac{\tilde{V}_{nl}}{v_l} R_{lm} - - i \sum_l R_{nl} \frac{\tilde{V}_{ln}}{v_l} - i \sum_{ll'} R_{nl} \frac{\tilde{V}_{ll'}}{v_{l'}} R_{l'm} . \tag{2.26}$$

The physical meaning of the different terms is obvious and corresponds to scattering from channel to channel. Introducing the matrices $N \times N$:

$$\chi_{\alpha\beta} = -\tilde{V}_{\alpha\beta}/v_\beta , \qquad \Delta_{\alpha\beta} = \chi_{\alpha\beta} + k_\alpha \, \delta_{\alpha\beta} ,$$

one deduces

$$\frac{d\mathcal{R}}{dL} = i\chi + i(\mathcal{R}\Delta + \Delta\mathcal{R}) + i\mathcal{R}\chi\mathcal{R} , \quad R(L=0) = 0 \tag{2.27}$$

which is a matrix differential equation for the matrix $\mathcal{R}$. Another form for this equation is

$$\frac{d\mathcal{R}}{dL} = i(\mathbb{K} \cdot \mathcal{R} + \mathcal{R} \cdot \mathbb{K}) + i(1 + \mathcal{R}) \chi (1 + \mathcal{R}) \tag{2.28}$$

where we have defined the matrix $\mathbb{K}$ as : $\mathbb{K}_{\alpha\beta} = k_\alpha \, \delta_{\alpha\beta}$ (1 is $N \times N$ identity matrix).

A similar equation can be obtained for the transmission matrix T. Indeed, following the same procedure, one has :

$$\begin{aligned} T_{nm}(L) = {} & (1 + ik_n \, dL) \, T_{nm}(L - dL) + \\ & + \sum_l \left(\frac{-i\tilde{V}_{nl}}{v_l} dL \right) \cdot T_{lm}(L - dL) \\ & + (1 + ik_n \, dL) \cdot \sum_l R_{nl}(L - dL) \\ & \times \sum_{l'} \left(\frac{-i\tilde{V}_{ll'}}{v_{l'}} dl \right) T_{l'm}(L - dL) \end{aligned} \tag{2.29}$$

and then

$$\frac{dT}{dL} = iT \cdot (\Delta + \chi \cdot \mathcal{R}) \qquad (2.30)$$

or

$$d \ln T/dL = i\mathbb{K} + i\chi(1 + \mathcal{R}) . \qquad (2.31)$$

It is not difficult to check that (2.28) and (2.30) reduce to (2.11) and (2.12) in the limit $N = 1 : \eta(L) = -V(L)/E$ and $E = k^2/2m$. Note furthermore that $\mathcal{R}$ is a symmetric matrix $R_{\alpha\beta} = R_{\beta\alpha}$ because of the time reversal invariance of the medium.

The correspondence with the generaley used notation [5] is given by :

$$\mathcal{T}_\alpha = \sum_\beta |T_{\beta\alpha}|^2 : \text{transmission in channel } \alpha$$

$$\mathcal{R}_\alpha = \sum_\beta |R_{\alpha\beta}|^2 : \text{reflection in channel } \alpha .$$

For real potentials (lossless media), the conservation law becomes : $\sum_\alpha \mathcal{R}_\alpha = \sum_\alpha (1 - \mathcal{T}_\alpha)$ and the generalized Landauer formula for the reduced conductance is written as

$$g = \left(\sum_\alpha \mathcal{T}_\alpha\right)\left(\sum_\alpha v_\alpha^{-1}\right)\Big/\sum_\beta v_\beta^{-1}(1 + \mathcal{R}_\beta - \mathcal{T}_\beta) .$$

Finally note that for a uniform medium, where $V_{n_0}(x)$ is independent of the channel index n_0, one has :

$$\tilde{V}_{nn'} = V(x)\,\delta_{nn'} , \qquad \chi_{\alpha\beta} = -\frac{1}{v_\beta} V(x)\,\delta_{\alpha\beta} .$$

This corresponds to a complete decoupling between channels and the one-channel equations are recovered.

2.3 TIME-DEPENDENT MEDIA. — Up to now, we have considered the case of time independent media. In this section we show how the invariant imbedding ideas can be used in the case of time-dependent potentials. In order to illustrate this approach, we consider the following model [13] :

$$i\frac{\partial\psi(x,t)}{\partial t} = \left[-\frac{\partial^2}{\partial x^2} + V_0(x) + V_1(x)\cos\omega t\right]\psi(x,t) \qquad (2.32)$$

where $V_1(x)$ and $V_2(x)$ are static potentials, which are zero for $x \leqslant 0$ or $x \geqslant L$ and t is time. The only time dependence is harmonic of frequency ω. A planar wave is incident to the layer at the right. The general solution of (2.32) can be written as

$$\psi(x,t) = \sum_{n=-\infty}^{\infty} \phi_n(x)\, e^{-i(E+n\omega)t} \qquad (2.33)$$

and this leads to :

$$E_n\,\phi_n = -\frac{d^2\phi_n}{dx^2} + V_0(x)\,\phi_n(x) + \frac{1}{2}V_1(x)(\phi_{n-1} + \phi_{n+1}) \qquad (2.34)$$

i.e.

$$\left(\frac{d^2}{dx^2} + k_n^2\right)\phi_n = V_0(x)\,\phi_n + \frac{1}{2}V_1(x)[\phi_{n-1} + \phi_{n+1}] .$$

Here $E_n = E + n\omega \equiv k_n^2$ define an infinite set of channels $-\infty < n < \infty$ and E is the energy of the incident wave.

The last equation can be transformed into an integral equation :

$$\phi_n(x, L) = A_n\, e^{ik_n(x-L)} + \frac{1}{2ik_n}\int_0^L dx'\, e^{ik_n|x-x'|} \times \times \left[V_0(x') + \frac{1}{2}V_1(x')(\phi_{n+1}(x') + \phi_{n-1}(x'))\right] . \qquad (2.35)$$

It is convenient to write the last term as $\sum_{n'} \tilde{V}_{nn'}(x')\,\phi_{n'}(x')$ because this allows the identification of (2.35) with (2.24). Here, $v_\alpha = 2k_\alpha$ and the matrix $\tilde{V}_{nn'}$ (of infinite order) is :

$$\tilde{V}_{nn'} = \delta_{nn'} V_0 + \delta_{n,n\pm1} V_1/2 . \qquad (2.36)$$

According to the results of the previous section, the reflection and transmission matrices $\mathcal{R}$ and T satisfy (2.27) and (2.30), with :

$$\chi_{\alpha\beta} = -\frac{1}{2k_\beta}\tilde{V}_{\alpha\beta} \quad \text{and} \quad \Delta_{\alpha\beta} = k_\alpha\,\delta_{\alpha\beta} - \frac{1}{2k_\beta}\tilde{V}_{\alpha\beta} , \quad -\infty \leqslant \alpha, \beta \leqslant \infty . \qquad (2.37)$$

The transmission coefficient across the medium is

$$\mathcal{T} = \sum_n |T_{0n}|^2 .$$

It appears that the presence of a time dependent term in the wave field equation (Eq. (2.32)) gives rise to an infinite set of coupled equations, instead of just one. Therefore, allowing for inelastic scattering « adds a new dimension » in the problem : the channel axis.

2.4 MATRIX RICCATI EQUATIONS. — The purpose of this last section is to derive a formal solution of the matrix Riccati equation, obtained above for $\mathcal{R}$:

$$\frac{d\mathcal{R}}{dL} = i\chi + i(\Delta R + R\Delta) + iR\chi R . \qquad (2.38)$$

This non-linear equation can actually be « linearized »,

because (2.38) can be shown to be equivalent to the linear system [14]

$$\begin{aligned} \frac{dU}{dL} &= i\Delta U + i\chi V\,, \quad U(0) = 0 \\ -\frac{dV}{dL} &= i\chi U + i\Delta V\,, \quad V(0) = \pi \end{aligned} \tag{2.39}$$

where $\mathcal{R}$ is related to U and V by : $\mathcal{R} = UV^{-1}$ (V non-singular).

This linearization scheme may be used to find a formal solution of (2.38). The flow defined by (2.30) can also be written as

$$\frac{dX}{dL} = -i\mathcal{D}X + \mathcal{B}X \tag{2.40}$$

where

$$X = \begin{vmatrix} U \\ V \end{vmatrix}, \quad \mathcal{D} = \begin{vmatrix} -\mathbb{K} & 0 \\ 0 & \mathbb{K} \end{vmatrix}$$

and

$$\mathcal{B} = i \begin{vmatrix} \chi & \chi \\ -\chi & -\chi \end{vmatrix}.$$

The solutions of equation (2.40) are of the form $X(L) = e^{-i\mathcal{D}L}\, Y(L)$, where $Y(L)$ is the solution of

$$\frac{dY}{dL} = C(L)\,.\,Y\,, \qquad Y(L=0) = \begin{vmatrix} 0 \\ 1 \end{vmatrix}$$

where

$$C(L) = e^{i\mathcal{D}L}\, \mathcal{B}(L)\, e^{-i\mathcal{D}L}\,.$$

Therefore,

$$Y(L) = \exp\left(\int_0^L C(L')\,dL'\right) Y(0) \equiv \begin{vmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{vmatrix} Y(0)$$

and then

$$\mathcal{R}(L) = WG_{12}\, G_{22}^{-1}\, W\,, \quad W \equiv e^{i\mathbb{K}L}\,.$$

In general the information in the potential is contained in the matrix and this allows us to follow the variations of $\mathcal{R}(L)$ with this potential. As a check, it is easy to see that for $\chi = 0 : C = 0$ and $G_{11} = G_{22} = 1$, $G_{21} = G_{22} = 0$ and then $\mathcal{R} = 0$.

3. Non-linear media.

Up to now, we have limited our approach to linear media. In this section we show how the problem of wave transmission across a non-linear medium [15] can also be worked out using the invariant-imbedding ideas. An analytical proof of the main equations is given in Appendix A. Here, the same results are obtained using the graphical method.

We consider systems described by the generalized non-linear Schrödinger equation :

$$i\frac{\partial\psi}{\partial t} = \frac{\partial^2\psi}{\partial x^2} - V(x)\,\psi + \alpha f(|\psi|^2)\,.\,\psi\,. \tag{3.1}$$

Here, $\psi(x, t)$ is the complex amplitude of the wave field, t is time and $V(x)$ is a real (or complex) regular (or random) potential. The non-linear term $f(|\psi|^2)$ is an arbitrary function of the intensity $|\psi|^2$. Examples of physical phenomena where (3.1) occurs are : non-linear optics [16] (Kerr effect), plasma physics [17], condensed matter physics [18] and astrophysics [19]. We consider (3.1) for a one-dimensional case, corresponding to a plane-layered medium, which occupies the region $0 \leqslant x \leqslant L$. Outside this region, $V(x) = 0$ and $f = 0$ identically. In the following, we consider only the stationary regime, corresponding to solutions of (3.1) of the form : $\psi(x, t) = e^{ik^2 t}\,\varphi(x)$, where $k^2 = E$ is the wave energy. This leads to the following equation for $\varphi(x)$:

$$\frac{d^2\varphi}{dx^2} + k^2[1 + \eta(x, |\varphi|^2)]\,\varphi = 0\,. \tag{3.2}$$

Here, the function $\eta(x, |\varphi|^2)$ is given by

$$\eta(x, |\varphi|^2) = [-V(x) + \alpha f(|\varphi|^2)]/k^2\,. \tag{3.3}$$

Let now a plane wave $\varphi_0(x) = A\, e^{ik(l-x)}$ be incident on the layer from the right. Then, the solution of (3.2) can be represented in the form $\varphi(x) = A\,.\,u(x)$, where $u(x)$ satisfies a similar equation, involving the intensity $w = |A|^2$ of the incident wave and $|u(x)|^2$. On the right and left of the layer, the wave field has the form :

$$\begin{aligned} u(x) &= e^{ik(L-x)} + R(L, w)\, e^{ik(x-L)}\,, && x \geqslant L \\ u(x) &= T(L, w)\, e^{ikx}\,, && x \leqslant 0\,. \end{aligned} \tag{3.4}$$

Here $R(L, w)$ and $T(L, w)$ are the complex reflection and transmission coefficients respectively. The wave number k is assumed to be the same inside and outside the layer. This restriction corresponds to an approximation where the creation of other harmonics is neglected. In what follows we show the possibility of reducing the calculation of $R(L, w)$ to an initial-value Cauchy problem, relative to the Imbedding parameters : L and w. The field at the boundary will be shown to be described by a closed non-linear partial differential equation (Eq. (3.7) below). We transform first (3.2) into an integral equation

$$u(x; L, w) = e^{ik(L-x)} + \frac{ik}{2}\int_{-\infty}^{\infty} dx'\, e^{ik|x-x'|} \times \times \eta[x, wI(x')]\,.\,u(x'; L, w)\,. \tag{3.5}$$

The interpretation of (3.5) in terms of elementary scattering events is identical to that described in section 2, for the linear problems. Using the diagrams

shown in figure 1, the same approach can be used here, to follow the variation of R when L is changed to $L - dL$. The presence of a w dependence of the scattering amplitude $\frac{ik}{2}\eta$ does not alter the diagrams (1) and (2). However, during the processes (3), (3') and (4), the incident wave intensity is modified and this leads to a modification of the reflection coefficient $R(L, w)$. For these processes, the wave field amplitude is given by

$$A \cdot \left[1 + \frac{ik}{2}\eta \cdot (1 + R)\right] \tag{3.6}$$

and the corresponding intensity is :

$$\left[1 + \frac{ik}{2}\eta \cdot (R - R^*)\right] .$$

This leads in particular to an additional contribution to $\frac{\partial R}{\partial L}$, which can be written as

$$w \frac{\partial R}{\partial w} \frac{ik}{2}\eta \cdot (R - R^*) .$$

Thus, one gets finally the following partial differential equation for $R(L, w)$

$$\frac{\partial R}{\partial L} = 2ikR + \frac{ik}{2}\eta(L, wI(L)) \cdot (1 + R)^2 + w\frac{\partial R}{\partial w} \cdot \frac{ik}{2}\eta(L, wI(L)) \cdot (R - R^*) ,$$

$$R(L = 0, w) = 0 . \tag{3.7}$$

Similarly, one obtains the following equation for the transmission coefficient :

$$\frac{\partial T}{\partial L} = ik \cdot T + \frac{ik}{2}\eta(L, wI(L)) \cdot (1 + R)\, T + w\frac{\partial T}{\partial w} \cdot \frac{ik}{2}\eta(L, wI(L)) \cdot (R - R^*) ,$$

$$T(L = 0, w) = 1 . \tag{3.8}$$

In (3.7) and (3.8), $I(L)$ is given by $I(L) = |u|^2 = (1 + R)(1 + R^*)$. To our knowledge these equations were first derived in reference [20] : a short summary of this analytical derivation is given in Appendix A. Let us conclude this section with two remarks. In the limit $w = 0$, R does not depend on w and (3.7) reduces to (2.11) as it should be. The same remark also holds for (3.8). The main difference between the linear case and that worked in this section is probably the following one. The linear problems give rise to ordinary differential equations : the only imbedding parameter is L. In contrast, non-linearities lead to partial differential equations, which can be viewed as infinite sets of ordinary differential equations. This new feature originates in the fact that both L and w are relevant imbedding parameters. In this way, the present case is actually similar to a multichannel transmission problem, where w plays the role of the channel axis. This formal distinction between linear and non-linear cases is at the origin of different physical behaviours of T as a function of L. A detailed study of the transmission across a non-linear random will be reported in the second paper of this series [10].

4. Illustrative examples.

The formulation of the transmission problems as initial-value (Cauchy) equations allows for a unified treatment of disordered media. The basic idea is the observation that the differential equation satisfied by R describes a dynamical system subjected to random perturbation coming from the potential $V(L)$. The coordinate $L \geqslant 0$ plays the role of time in (2.1). The relevant equations of motion are non-linear, in contrast with the original field equations. However, the main advantage of the dynamical system point of view is actually a direct approach to the main physical quantities : R, T, ..., etc.

In this section we shall illustrate this general approach of the simplest case : the one-channel problem. Some of the results obtained below are known and have been obtained by different authors, using somewhat different formalisms. The extension of this approach to non-linear field equations is the object of the next paper in this series [10]. While the calculations presented here are mainly performed for Gaussian white noise potentials, a straightforward extension to correlated potentials will be discussed at the end of this section.

Before the description of the present treatment, let us consider the phase portrait of the dynamical system at hand. The equation of motion for the complex reflection coefficient $R(L)$ is given by (Eq. (2.11))

$$\frac{dR}{dL} = 2ikR + \frac{ik}{2}\eta(L) \cdot (1 + R)^2 , \quad R(0) = 0 . \tag{4.1}$$

Here $\eta(L) \equiv -V(L)/k^2$ refers to the random potential. The phase space is given by $|R| \leqslant 1$. Assuming that $\eta(L) = \eta$ is independent of L, two kinds of fixed points R_∞ can appear. For $\eta \leqslant -1$, $R_\infty = -(1 + 2/\eta) \pm \frac{2}{\eta} i(-1 - \eta)^{1/2}$ are located on the boundary of the unit disc, i.e. $|R_\infty|^2 = 1$. In this case, the fixed points are attractive. In the other sector, $\eta \geqslant -1$, $R_\infty = -\left(1 + \frac{2}{\eta}\right) + \frac{2}{\eta}(1 + \eta)^{1/2}$ is the centre of a periodic orbit : the fixed points are located on the interval $[-1, +1]$ and the fixed point is reached only at $\eta = 0$.

The above phase portrait reproduces in rather simple terms the known physical picture : exponential damping at $\eta \leqslant -1$ and free propagation at $\eta \geqslant -1$. Indeed, the solution of (4.1) can be written as :

$$\frac{R(0) - R_\infty}{R(L) - R_\infty} = e^{-aL} + \frac{b}{a}(1 + e^{-aL})(R(0) - R_\infty) \tag{4.2}$$

with $a = 2ik + ik\eta(1 + R_\infty)$ and $b = -ik\eta/2$. This leads in particular to the following expression for the period of the orbits ($\eta \geqslant 1$) :

$$L^* = i\pi/a = [2k + k\eta(1 + R_\infty)]^{-1},$$

i.e. rapid revolution at $\eta \gg 1$ and slow motion at $\eta \ll 1$.

In the situations where $\eta(L)$ is random, the representative point $R(L)$ undergoes a random motion and this results in a diffusion-like motion inside the unit disc. The nature of this motion can be described by an equation for the probability density of $R(L)$.

4.1 PROBABILITY DISTRIBUTION OF THE REFLECTION COEFFICIENT. — Equation (4.1) can also be viewed as a stochastic differential equation, giving the « time » (L) evolution of the stochastic process $R(L)$ under the action of the noise term $\eta(L)$. In general, it is difficult to solve such a non-linear problem for an arbitrary $\eta(L)$. Using the Liouville Theorem, it is actually not difficult to « linearize » the present problem, as usually done for stochastic differential equations [21]. The main idea is to consider the probability density of R for which it is possible to write down a Fokker-Planck equation, giving full information about the statistics of R. For this, it is useful to use the polar coordinates : $R = r^{1/2} e^{i\theta}$, where $0 \leqslant r(L) \leqslant 1$ and $0 \leqslant \theta(L) \leqslant 2\pi$ are the new random variables. Accordingly, (4.1) becomes :

$$\frac{dr}{dL} = k\eta(L) \cdot r^{1/2}(1 - r) \sin\theta$$

$$\frac{d\theta}{dL} = 2k + \frac{k}{2}\eta(L)[2 + (r^{1/2} + r^{-1/2})\cos\theta]. \tag{4.3}$$

As usual, we introduce the density $Q(r, \theta ; L)$ of points (r, θ) in phase space. This density is a solution of the Liouville equation [21] :

$$\frac{\partial Q}{\partial L} = -\frac{\partial}{\partial r}\left[Q\frac{dr}{dL}\right] - \frac{\partial}{\partial\theta}\left[Q\frac{d\theta}{dL}\right] \tag{4.4}$$

where the derivatives of r and θ are given by (4.3). The equation for Q can be written as

$$\frac{\partial Q}{\partial L} = -k\sin\theta\left[\frac{d}{dr}\eta(L)\, Qr^{1/2}(1 - r)\right] - 2k\frac{\partial Q}{\partial\theta} - k\frac{\partial}{\partial\theta}(\eta Q) - \frac{k}{2}(r^{-1/2} + r^{1/2})\frac{\partial}{\partial\theta}(\eta Q). \tag{4.5}$$

The last equation is a very general one and holds for an arbitrary random potential $\eta(L) \equiv -V(L)/k^2$. In order to go further, it is necessary to make specific assumptions on the statistical properties of $\eta(L)$. For the sake of simplicity, η will be assumed to be a Gaussian random function with zero mean $\langle\eta(L)\rangle = 0$ and a white-noise spectrum :

$$\langle\eta(L)\,\eta(L')\rangle = g\delta(L - L').$$

Within this class of models, the average of (4.5) over the realizations of η can be performed without difficulties : this can be simply done, by using Novikov's formula (see Appendix B). What we need is just the average $\langle\eta Q\rangle$ and this can be calculated, using :

$$\frac{\delta Q}{\delta\eta} = -k\sin\theta\frac{d}{dr}[Qr^{1/2}(1 - r)] - k\frac{\partial}{\partial\theta}Q - \frac{k}{2}(r^{-1/2} + r^{1/2})\frac{\partial}{\partial\theta}[Q\cos\theta]. \tag{4.6}$$

Denoting by $W(r, \theta ; L) \equiv \langle Q(r, \theta ; L)\rangle_\eta$ the averaged probability density, one arrives at

$$\frac{\partial W}{\partial l} = \left[\sin\theta\frac{\partial}{\partial r}r^{1/2}(1 - r) + \frac{\partial}{\partial\theta} + \frac{1}{2}(r^{-1/2} + r^{1/2})\frac{\partial}{\partial\theta}\cos\theta\right]^2 W(r, \theta ; l). \tag{4.7}$$

Here, we have used the notation $l = L/\xi$, where $\xi^{-1} = \frac{1}{2}gk^2$ is the inverse of the localization length.

The above equation is an exact one, subjected to the boundary condition $W(r, \theta ; l = 0) = \delta(r)/2\pi$, because θ is uniformly distributed at $L = 0$. Further simplifications of (4.7) can occur in the limit of weak potentials or large energies. In that limit the angular variations of W can be neglected, because θ can be taken as a uniformly distributed random variable over $[0, 2\pi]$. More precisely, this condition is satisfied when : $\xi \gg 1/k$, i.e. the localization length is much larger than the wavelength $2\pi/k$ of the incident wave. Within this limit, the average over θ leads to the following evolution equation of $W(r, l) \equiv \langle W(r, \theta ; l)\rangle_\theta$:

$$\frac{\partial W}{\partial l} = 2(2r - 1)W + (1 - r)(1 - 5r)\frac{\partial W}{\partial r} + r(1 - r)^2\frac{\partial^2 W}{\partial r^2} = \frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}(1 - r)^2 W\right] \tag{4.8}$$

with $W(r, l = 0) = \delta(r)$ as an initial condition.

The so obtained equation can be cast into a more familiar form, by considering the probability distribution of the new variable $\rho = r/(1 - r)$. A straightforward algebra leads the following equation for $W(\rho, l) = (1 + \rho)^{-2} W(r, l)$:

$$\frac{\partial W(\rho)}{\partial l} = \frac{\partial}{\partial\rho}\left[\rho(1 + \rho)\frac{\partial W(\rho)}{\partial\rho}\right], \qquad W(\rho, l = 0) = \delta(\rho). \tag{4.9}$$

The above equations for $W(r)$ and $W(\rho)$ are very well known and have been derived by many authors [9, 22, 23] in the past. Furthermore, the exact solutions for

W's are known and we direct the reader to the relevant literature. Another derivation of (4.9) will be given in the next sections.

The procedure outlined here can also be used to derive the probability distribution of the impedance $Z(L)$ defined by [11]

$$R(L) = \frac{ik + Z}{ik - Z}.$$

Using the variable Z instead of R, one obtains a stochastic differential equation for Z, where the random term $\eta(L)$ enters as an additive term. This kind of Langevin-like equation can be worked as above, and leads to the same equation for $W(r\,;l)$.

It is clear that from (4.8) and (4.9) one can deduce the behaviour of the moments of r and $t = 1 - r$. In particular, if $\rho_n = \langle \rho^n \rangle$, one has [22, 23] : $\rho_1(l) = \frac{1}{2}(e^{2l} - 1)$, $\rho_2(l) = \frac{1}{12}(2\,e^{6l} - 6\,e^{2l} + 4)$, ... and $\rho_n \simeq e^{n(n+1)l}/(n+1)!$, $n \gg 1$.

4.2 MOMENTS EQUATIONS. — As a second illustration of (4.1) we shall derive the recursion equations for the moments of R and T. For this, the random potential will be assumed to be a Gaussian white noise :

$$\langle \eta(x)\,\eta(x') \rangle = g\delta(x - x'),\ \langle \eta(x) \rangle = 0\,. \quad (4.10)$$

Using the differential equations for R and R^* :

$$\begin{aligned} \frac{dR}{dL} &= 2ikR + \frac{ik}{2}\eta(L)\,.\,(1+R)^2 \\ \frac{dR^*}{dL} &= -2ikR^* - \frac{ik}{2}\eta(L)(1+R^*)^2\,. \end{aligned} \quad (4.11)$$

One deduces

$$\frac{\delta R}{\delta \eta} = \frac{ik}{2}(1+R)^2\,,\quad \frac{\delta R^*}{\delta \eta} = -\frac{ik}{2}(1+R^*)^2\,. \quad (4.12)$$

Let $f_{m,n} = \langle R^m R^{*n} \rangle$ and $K_{m,n} = \langle \eta R^m R^{*n} \rangle$. Using Novikov's formula (Appendix B), one obtains a closed set of equations for $f_{m,n} (= f^*_{n,m})$ and $K_{m,n} (= K^*_{n,m})$:

$$\begin{aligned} \frac{df_{m,n}}{dL} &= 2ik(m-n)\,f_{m,n} + \\ &+ \frac{ik}{2}m[K_{m-1,n} + 2\,K_{m,n} + K_{m+1,n}] \\ &- \frac{ik}{2}n[K^*_{n-1,m} + 2\,K^*_{n,m} + K^*_{n+1,m}] \end{aligned} \quad (4.13)$$

$$\begin{aligned} K_{m,n} = \frac{ik}{2}g[&m(f_{m-1,n} + 2\,f_{m,n} + f_{m+1,n}) - \\ &- n(f_{m,n-1} + 2\,f_{m,n} + f_{m,n+1})]\,. \end{aligned} \quad (4.14)$$

From (4.13) and (4.14), it is clear that for weak disorder or large energy, all the terms $f_{m,n}$ with $m \neq n$ can be neglected : $f_{m,n} \sim \delta_{m,n}$. This leads in particular to :

$$\frac{dr_n}{dL} = \frac{gk^2}{2}n^2(r_{n-1} - 2\,r_n + r_{n+1})\,,\quad n \geqslant 0 \quad (4.15)$$

where

$$r = RR^* \quad \text{and} \quad r_n \equiv f_{n,n} = \langle r^n \rangle\,.$$

The above result for the moments r_n of the reflection coefficient is well known and has been previously derived by different authors [23, 24]. In particular, it is not difficult to show that (4.15) describes the moment equations for the probability distribution $W(l\,;r)$ obeying the following differential equation $(0 \leqslant r \leqslant 1)$, which is identical to (4.8)

$$\frac{\partial W}{\partial l} = \frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}(1-r)^2\,W\right],\quad W(l=0, r) = \delta(r)\,. \quad (4.16)$$

Here $l = L\frac{gk^2}{2} \equiv L/\xi$, where ξ is the localization length.

Without solving (4.16) for the full probability distribution, the exponential decay of $1 - \langle r \rangle$ at large l, can be deduced from (4.15). Following [25], one can define the Laplace transform

$$\tilde{W}(\sigma, r) = \int_0^\infty e^{-\sigma l}\,W(l\,;r)\,dl$$

of $W(l, r)$ and transform (4.15) for

$$C_n(\sigma) \equiv \int_0^1 r^n\,dr\,\tilde{W}(\sigma, r)\,. \quad (4.17)$$

A straightforward algebra leads to the following recursion equation :

$$\begin{aligned} C_0(\sigma) = 1/\sigma,\ \sigma C_n(\sigma) = n^2(&C_{n-1}(\sigma) - \\ &- 2\,C_n(\sigma) + C_{n+1}(\sigma)) \end{aligned} \quad (4.18)$$

which gives a continued fraction representation for the ratio $y_n = C_{n+1}(\sigma)/C_n(\sigma)$:

$$y_{n-1} = 1/\left(\frac{\sigma + 2n^2}{n^2} - y_n\right) \equiv \frac{1}{a_n - y_n}\,. \quad (4.19)$$

The only singularities of this continued fraction are given by the roots of : $a_1^2 = 4$, i.e. $\sigma = 0$ and $\sigma = -1/4$. This implies in particular the desired result : $1 - \langle r(l) \rangle \sim e^{-l/4}$ at $l \gg 1$.

As for R, similar equations can be derived for the moments $g_{m,n} = \langle T^m T^{*n} \rangle$, where T denotes the transmission coefficient. Indeed, starting from

$$\begin{aligned} \frac{dT}{dL} &\equiv ikT + \frac{ik}{2}\eta(L)\,.\,(1+R)\,T\,, \\ \frac{\delta T}{\delta \eta} &= \frac{ik}{2}(1+R)\,T \end{aligned} \quad (4.20)$$

one deduces

$$\frac{dg_{m,n}}{dL} = ik(m-n)\,g_{m,n} + \frac{ik}{2}(m-n)\langle \eta T^m T^{*n}\rangle + \\ + \frac{ik}{2}[m\langle \eta R T^m T^{*n}\rangle - n\langle \eta T^m T^{*n} R^*\rangle]\,. \quad (4.21)$$

Keeping only the terms with $m = n$, and using the conservation relation (η real) : $RR^* + TT^* = 1$, one deduces, after a rather cumbersome algebra :

$$\frac{dg_{m,m}}{dl} = -mg_{m,m} + m^2(g_{m,n} - g_{m+1,m+1})\,. \quad (4.22)$$

Denoting $T_{m,m} \equiv \langle (TT^*)^m\rangle = \langle t^m\rangle \equiv t_m$, one finally obtains :

$$\frac{dt_m}{dl} = m(m-1)\,t_m - m^2 t_{m+1}\,. \quad (4.23)$$

Besides the above moment equations, the differential equations for T and R can be used in general to extract other information about the transmission problems. We give here two examples : transmission in the presence of dissipation and the correlation functions of the reflection coefficients.

4.2.1 *Transmission in the presence of dissipation.* — Assuming that $\eta(x)$ is the sum of two terms : $\eta(x) = \eta_1(x) + i\eta_2$, where $\eta_1(x)$ represents, as above, a Gaussian white noise potential and η_2 corresponds to a dissipation mechanism. The equations for R and T become respectively

$$\frac{dR}{dL} = 2ikR + \frac{ik}{2}(\eta_1(L) + i\eta_2)(1+R)^2\,, \qquad R(L=0) = 0 \quad (4.24)$$

$$\frac{dT}{dL} = ikT + \frac{ik}{2}(\eta_1(L) + i\eta_2)(1+R)\,.\,T\,, \qquad T(L=0) = 1\,. \quad (4.25)$$

Using the polar coordinates : $R = R_1 + iR_2 = \sqrt{r}\,e^{i\theta}$, $T = T_1 + iT_2 = \sqrt{t}\,e^{i\varphi}$, one deduces the following differential equation for $\ln t$:

$$\frac{d\ln t}{dL} = -k[\eta_1(L)\,R_2 + \eta_2(1+R_1)]\,. \quad (4.26)$$

The average over $\eta_1(L)$ leads to ($\langle R_1^2\rangle = \langle R_2^2\rangle$) :

$$\left\langle \frac{d\ln t}{dL}\right\rangle = -k\eta_2 - \frac{k^2 g}{2} \quad (4.27)$$

and then

$$\langle \ln t\rangle \simeq \exp(-L/\xi - L/\xi_d) \quad (4.28)$$

where $\xi_d = 1/k\eta_2$ and ξ has been defined above.

4.2.2 *Correlation functions of the reflection coefficients.* — In some physical problems [26] one has to consider the correlation functions of R or T at two different energies k_1 and k_2, but for the same potential $\eta(x)$. Denoting by R_1 (resp. R_2) the associated reflection coefficients, the following quantities : $\varphi_{m,n} \equiv \langle R_1^m R_2^{*n}\rangle$ may enter in different physical expressions. Other correlation functions [26] such as $\langle T_1^m T_2^{*n}\rangle$ may also be of some interest. In what follows, we shall consider $\varphi_{m,n}$ only, and we derive a recursion equation for $\varphi_{m,m}$. The two stochastic ordinary differential equations for R_1 and R_2 can be written as

$$\frac{dR_1}{dL} = 2ik_1 R_1 + \frac{ik_1}{2}\eta(L)(1+R_1)^2 \quad (4.29)$$

$$\frac{dR_2}{dL} = 2ik_2 R_2 + \frac{ik_2}{2}\eta(L)(1+R_2)^2\,. \quad (4.30)$$

Here k_1 and k_2 correspond to the two incident wave vectors. Repeating the same calculation as before, one obtains

$$\frac{d\varphi_{m,n}}{dL} = 2ik_1 m\varphi_{m,n} - 2ik_2 n\varphi_{m,n} + \\ + \frac{ik_1}{2}m[\langle \eta R_1^{m-1} R_2^{*n}\rangle \\ + 2\langle \eta R_1^m R_2^{*n}\rangle + \langle \eta R_1^{m+1} R_2^{*n}\rangle] \\ - \frac{ik_2}{2}n[\langle \eta R_1^m R_2^{*n-1}\rangle \\ + 2\langle \eta R_1^m R_2^{*n}\rangle + \langle \eta R_1^m R_2^{*n+1}\rangle\,. \quad (4.31)$$

Similarly, $K_{m,n} \equiv \langle \eta R_1^m R_2^{*n}\rangle$ can be expressed in terms of φ_{mn}'s and this leads to a closed set of differential equations for φ_{mn}'s. Neglecting terms with $m \neq n$, one deduces the following equation for $\psi_n \equiv \varphi_{n,n} = \langle R_1^n R_2^{*n}\rangle$:

$$\frac{d\psi_n}{dL} = 2in(k_1 - k_2)\,\psi_n + \frac{1}{2}gk_1 k_2 n^2(\psi_{n+1} + \psi_{n-1}) - \\ - \frac{g}{2}[k_1(3k_1 - 2k_2) + k_2(3k_2 - 2k_1)]\,n^2\psi_n\,. \quad (4.32)$$

This equation is valid at large k or small g. Assuming $k_1 = k + \Delta k$ and $k_2 = k - \Delta k$, (4.32) reduces, at $\Delta k \ll k$, to :

$$\frac{d\psi_n}{dl} = n^2(\psi_{n+1} - 2\psi_n + \psi_{n-1}) + i\beta n\psi_n\,, \qquad \psi_n(l=0) = \delta(n,0) \quad (4.33)$$

where $l = L/\xi$, and $\beta = 4\xi\,\Delta k \ll 1$. As expected (4.33) reduces to (4.15) at $\beta = 0$. In principle (4.33) allows for the calculation of ψ_n as a function of l. A perturbation calculation of $\psi_n(\beta)$ at $\beta \ll 1$ is presented in Appendix C. In this limit the exponential decay of $1 - \psi_1(l)$ is shown to be controlled by

$$\xi^* \simeq \xi\,.\,\ln(1/2\,\xi\,.\,\Delta k)$$

which is much larger than ξ at $\beta \ll 1$. The physical meaning of this new length scale is as follows. Let us denote by $\theta_{\Delta k}$ the phase difference between the two incident waves. $\theta_{\Delta k} = 0$ at $x = 0$, but increases with the travelled distance. For length scales $L \sim \xi$, $\theta_{\Delta k} \sim 0$ or a multiple of 2π, whereas for $L \sim \xi^*$, $\theta_{\Delta k} \sim \pi$. $\theta_{\Delta k}$ reaches a value $\sim \pi$ at $\xi^* \gg \xi$ because the same defects in the medium are visited at different instants. ξ^* can also be viewed as the length scale for which the overlap between two localized states is of the order of the energy difference. In fact, for a dispersion relation $\varepsilon = \varepsilon(k)$, the group velocity is given by $v_g = \partial\varepsilon/\partial k$ and to Δk is associated an energy difference $\Delta\varepsilon \sim v_g \,.\, \Delta k$. The overlap $\Delta\varepsilon \sim \exp(-\xi^*/\xi)$ between two localized states $u_1(x) \sim \mathrm{e}^{-x/\xi}$ and $u_2(x) \sim \mathrm{e}^{-(x-\xi^*)/\xi}$ leads to the expression of ξ^* obtained above.

4.3 PARAMETRIC RESONANCE : RAPID *VERSUS* SLOW VARIATIONS. — In this section we will show that the results previously obtained for the probability distribution of r can also be obtained by a different method. The basic ideas are as follows. First, not all the harmonics of the potential $\eta(x)$ play the same role. In fact, following the dynamic system analogy, we can assume that only Fourier harmonics of $\eta(x)$ at 0, $\pm 2k$ have a significant effect on the orbits. This assumption is similar to that made in the theory of parametric resonance and is valid only for weak random potentials $\eta(x)$. Therefore, we retain the following form for the potential $\eta(x) = -V(x)/k^2$:

$$V(x) = V_1(x) + V_2(x)\,\mathrm{e}^{2ikx} + V_2^*(x)\,\mathrm{e}^{-2ikx} \quad (4.34)$$

where $V_1(x)$ and $V_2(x)$ are assumed to be Gaussian white noise processes :

$$\begin{aligned} &\langle V_1(x)\rangle = 0\,, \quad \langle V_2(x)\rangle = 0\,, \\ &\langle V_1(x)\,V_1(x')\rangle = g_1\,\delta(x-x')\,, \\ &\langle V_2(x)\,V_2^*(x')\rangle = \langle V_2^*(x)\,V_2(x')\rangle = g_2\,\delta(x-x') \end{aligned} \quad (4.35)$$

The remaining correlators are assumed equal to zero.

Secondly, before performing the average over $V(x)$, rapidly varying variables can be eliminated. Such a procedure is valid to first order in g and this level of accuracy is sufficient for our purpose. In the following we shall describe this procedure of elimination of rapid variations. In order to do so, we start with the basic equations for the reflection coefficient R :

$$\frac{\mathrm{d}R}{\mathrm{d}L} = 2ikR - \frac{V(k)}{2ik}(1+R)^2\,.$$

Let us introduce the polar representation $R = \mathrm{e}^{-\rho - i\varphi}$, $\rho \geqslant 0$, $0 \leqslant \varphi \leqslant 2\pi$

$$\begin{aligned} \frac{\mathrm{d}\rho}{\mathrm{d}L} &= \frac{V(L)}{k}\sin\varphi\,.\,\sinh\rho \\ \frac{\mathrm{d}\varphi}{\mathrm{d}L} &= 2k - \frac{V(L)}{k}\,[1+\cosh\rho\,.\,\cos\varphi]\,. \end{aligned} \quad (4.36)$$

A natural change of variables is the following : $\varphi = \theta + 2kL$. The phases $\theta(L)$ and $\rho(L)$ are slowly varying quantities. Then replacing $V(x)$ in the equations of ρ and θ, and retaining only resonance terms, we obtain the following system of equations for θ and ρ :

$$-k\frac{\mathrm{d}\theta}{\mathrm{d}L} = V_1(L) + k\,.\,\cosh\rho\,.\,g(\theta) \quad (4.37a)$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}L} = -\frac{\sinh\rho}{k}\,\mathrm{Im}\,(V_2(L)\,\mathrm{e}^{-i\theta}) \equiv -\sinh\rho\,.\,g'(\theta) \quad (4.37b)$$

where $g(\theta) = \dfrac{1}{2k}\,[V_2\,\mathrm{e}^{-i\theta} + V_2^*\,\mathrm{e}^{i\theta}]$.

The next step is to use a formal solution of (4.37a) in (4.32b). Indeed, the integration of (4.37a) leads to :

$$\theta(L) - \theta_0 = -\frac{1}{k}\int_0^L \mathrm{d}x'\Big[V_1(x') + \frac{1}{2}\cosh\rho \times \times \left(V_2(x')\,\mathrm{e}^{-i\theta_0} + V_2^*(x')\,\mathrm{e}^{i\theta_0}\right)\Big]\,.$$

Here θ_0 is the initial value of θ and independent of $V(x \leqslant L)$. Using this expression in (4.37b) we deduce

$$\frac{\mathrm{d}\rho}{\mathrm{d}L} = -\frac{\sinh\rho}{k}\left\{\mathrm{Im}\,V_2\,\mathrm{e}^{-i\theta_0}\,.\left(1 + \frac{i}{k}\int_0^L \mathrm{d}x'[\ldots]\right)\right\}. \quad (4.38)$$

Here we have just retained the first terms in the expansion of $\exp(\theta - \theta_0)$. Denoting by $W(L) = V(L)\,\mathrm{e}^{-i\theta_0}$, one obtains :

$$\frac{\mathrm{d}\rho}{\mathrm{d}L} = -\frac{\sinh\rho}{k}\,\mathrm{Im}\,W_2 - \frac{\sinh\rho}{k^2}\,\mathrm{Im}\,iW_2 \times \times \int_0^L \mathrm{d}x'\Big[V_1(x) + \frac{1}{2}\cosh\rho\,(W_2 + W_2^*)\Big]. \quad (4.38\ bis)$$

The second term in (4.38 *bis*) can be replaced by its average value over $\theta(x)$ and this results in the following dynamic equation for ρ :

$$\frac{\mathrm{d}\rho}{\mathrm{d}L} = -\frac{g_2}{4k^2}\sinh\rho\,\cosh\rho + i\,\frac{\sinh\rho}{2k}\,(W_2 - W_2^*)\,. \quad (4.39)$$

This is a non-linear Langevin equation, with a real random term (multiplicative noise) :

$$\nu(x) = \frac{i}{2}\,(W_2 - W_2^*)\,,$$

of the same type, as W :

$$\langle \nu(x)\rangle = 0\,, \quad \langle \nu(x)\,\nu(x')\rangle = \frac{1}{2}\,g_2\,\delta(x-x')\,.$$

Therefore, the original dynamical system is reduced to a one random variable problem, $\rho(L)$ described by a stochastic differential equation (4.39). From (4.39) it is straightforward to obtain the following Fokker-Planck equation for $P(\rho\,;L)$:

$$\frac{\partial P}{\partial L} = \frac{g_2}{4k^2}\frac{\partial^2}{\partial \rho^2}(\sinh^2\rho\,.\,P)\,, \quad \int_0^\infty \mathrm{d}\rho P(\rho\,;L) = 1\,. \tag{4.40}$$

The localization length is identical to that previously defined: $\xi^{-1} = g_2/4k^2$. Using the correspondence $r = RR^* = \mathrm{e}^{-2\rho}$, it is very easy to check that (4.40) is identical to the Fokker-Planck equation (Eq. (4.8)), derived previously for r:

$$\frac{\partial W}{\partial l} = 2(2r-1)W + (1-r)(1-5r)\times \times\frac{\partial W}{\partial r} + r(1-r)^2\frac{\partial^2 W}{\partial r^2}\,, \quad l = L/\xi\,. \tag{4.41}$$

Let us compare the two methods leading to (4.41). In the previous section, (4.8) has been derived by performing an averaging over the rapidly varying variables (phase of R). Such a method which is appropriate for one-dimensional systems, has the disadvantage to necessitate cumbersome calculations for performing the averages. The m thod described in this section allows the derivation of the same result (Eq. (4.41)) more rapidly. Actually the new procedure is based on the fact that simplification is performed at the level of the dynamic equations, i.e. at an earlier stage of the calculation. Note that the range of validity of the above calculation is the same as previously: $\xi \gg 1/k$, i.e. $\langle V^2(x)\rangle/k^3 \ll 1$.

4.4. CORRELATED RANDOM POTENTIALS. — In this last section, we show how to generalize the previous approach to more general random potentials, in particular for spatially correlated potentials. For this we consider the case of a step-like potential $\eta(L)$, described by an Ornstein-Uhlenbeck stochastic process $\eta(L) = \pm\,\Delta$ with $\langle\eta(L)\rangle = 0$,

$$\langle\eta(L)\,\eta(L')\rangle = \Delta^2\exp(-\lambda\,|L-L'|)\,.$$

Using polar coordinates $R = r^{1/2}\mathrm{e}^{i\theta}$ and $\theta = 2kL + \varphi$, one obtains the following differential system for r and φ:

$$\frac{\mathrm{d}r}{\mathrm{d}L} = kr^{1/2}(1-r)\sin(\varphi + 2kL)\,.\,\eta(L) \equiv \equiv f_1(r,\varphi,L)\,\eta(L) \tag{4.42}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}L} = \frac{k}{2}[2 + (r^{-1/2} + r^{1/2})\cos(\varphi + 2kL)]\,.\,\eta(L) \equiv \equiv f_2(r,\varphi,L)\,.\,\eta(L)\,. \tag{4.43}$$

Let us denote by $Q(r,\varphi,L)$ the probability density of the stochastic process (r,φ). Q is a solution of the Liouville equation:

$$\frac{\partial Q}{\partial L} = -\frac{\partial}{\partial r}[Qf_1\,\eta] - \frac{\partial}{\partial\varphi}[Qf_2\,\eta]\,. \tag{4.44}$$

Taking the average over η, one deduces ($W = \langle Q\rangle$):

$$\frac{\partial W}{\partial L} = -\frac{\partial}{\partial r}f_1\langle\eta Q\rangle - \frac{\partial}{\partial\varphi}f_2\langle\eta Q\rangle\,. \tag{4.45}$$

Similarly, using the differentiation formula (Appendix B), one obtains the following equation for $W_1 = \langle\eta Q\rangle$:

$$\frac{\partial W_1}{\partial L} = -\lambda W_1 + \left\langle\eta^2\left(-\frac{\partial}{\partial r}f_1\,Q - \frac{\partial}{\partial\varphi}f_2\,Q\right)\right\rangle. \tag{4.46}$$

Taking into account the fact that $\eta(L)$ jumps between $+\Delta$ and $-\Delta$ with a rate λ^{-1}, one deduces the following closed set of coupled equations for W and W_1:

$$\frac{\partial W}{\partial L} = -\frac{\partial}{\partial r}f_1 W_1 - \frac{\partial}{\partial\varphi}f_2 W_1 \tag{4.47}$$

$$\frac{\partial W_1}{\partial L} = -\lambda W_1 - \Delta^2\frac{\partial}{\partial r}f_1 W - \Delta^2\frac{\partial}{\partial\varphi}f_2 W\,. \tag{4.48}$$

Using the previous notation, one finally gets:

$$\frac{\partial W}{\partial L} = -k\sin\theta\frac{\partial}{\partial r}[r^{1/2}(1-r)W_1] - -k\frac{\partial W_1}{\partial\theta} - \frac{k}{2}(r^{1/2} + r^{-1/2})\frac{\partial}{\partial\theta}[\cos\theta\,.\,W_1] \tag{4.49}$$

$$\frac{\partial W_1}{\partial(\lambda L)} + W_1 = -\frac{\Delta^2}{\lambda}k\frac{\partial}{\partial r}[\sin\theta r^{1/2}(1-r)W] - -\frac{k\Delta^2}{\lambda}\frac{\partial W}{\partial\theta} - \frac{k\Delta^2}{2\lambda}(r^{1/2} + r^{-1/2})\frac{\partial}{\partial\theta}[\cos\theta W]\,. \tag{4.50}$$

The case of the Gaussian-white noise potential can be recovered in the limit $\lambda\to\infty$, $\Delta^2\to\infty$, $\Delta^2/\lambda = D$. Indeed, to first order in λ^{-1}, the formal solution of (4.50) can be written as

$$W_1 \simeq -\left[1 - \frac{\partial}{\partial(\lambda L)}\right]\left[kD\frac{\partial}{\partial r}\times \times\sin\theta r^{1/2}(1-r)W + kD\frac{\partial W}{\partial\theta} + \frac{kD}{2}(r^{1/2} + r^{-1/2})\frac{\partial}{\partial\theta}(\cos\theta\,.\,W)\right] \tag{4.51}$$

Let $l = k^2 DL$ and substitute W_1 in (4.49) one obtains:

$$\frac{\partial W}{\partial l} = \left[\frac{\partial}{\partial r}\sin\theta\,.\,r^{1/2}(1-r) + \frac{\partial}{\partial\theta} + \frac{1}{2}\times \times(r^{-1/2} + r^{1/2})\frac{\partial}{\partial\theta}\cos\theta\right]^2 W \tag{4.52}$$

which is identical to the equation obtained above for

the Gaussian-white noise potential. For general values of Δ and λ no exact solution for W's has been obtained.

Besides the previous equations for the probability distributions, the moment equation can also be written down. This allows us in particular to follow the influence of a finite λ on the localization length ξ. In order to do this, one can use the differentiation formula (Appendix B) in order to derive the appropriate differential equations for $K_{m,n} = \langle \eta R^m R^{*n} \rangle$ and $f_{m,n} = \langle R^m R^{*n} \rangle$. Following the same procedure as in section 4.2, one obtains :

$$\frac{df_{m,n}}{dL} = 2ik(m-n) f_{m,n} + \frac{ik}{2} m \times \times [K_{m-1,n} + 2K_{m,n} + K_{m+1,n}] - \frac{ik}{2} n[K_{m,n-1} + 2K_{m,n} + K_{m,n+1}] \quad (4.53)$$

and

$$\frac{dK_{m,n}}{dL} + \lambda K_{m,n} = 2ik(m-n) K_{m,n} + \frac{ik\Delta^2}{2} m \times \times [f_{m-1,n} + 2f_{m,n} + f_{m+1,n}] - \frac{ik\Delta^2}{2} n[f_{m,n-1} + 2f_{m,n} + f_{m,n+1}]. \quad (4.54)$$

For weak disorder or large energy, only the diagonal terms are of order one and the non-diagonal terms can be neglected. This leads in particular to the following equation for $r_n = \langle R^n R^{*n} \rangle$:

$$\frac{dr_n}{dL} = \xi^{-1}(\lambda) n^2 (r_{n-1} + r_{n+1} - 2r_n) \quad (4.55)$$

where

$$\xi^{-1}(\lambda) = \frac{k^2 \Delta^2/\lambda}{2} \frac{\lambda^2}{\lambda^2 + 4k^2}.$$

The new length scale $\xi(\lambda)$ is the equivalent of the localization length. In particular for $\lambda \to \infty$, $\Delta^2/\lambda = D$, the Gaussian-white noise limit is recovered.

The expression for $\xi(\lambda)$ thus obtained can be also written as

$$\xi(\lambda)/\xi(\lambda = \infty) = 1 + 4k^2/\lambda^2. \quad (4.56)$$

This shows in particular that the correlations of the potential lead to an increase of the localization length in comparison with an uncorrelated potential. Our analytical expression for ξ agrees with the results of reference [27].

General correlated potentials which can be described as a Markov chain can also be worked out following the same procedure. For weak disorder, the ratio between the inverse localization lengths with and without correlations is given by the structure factor of the correlation function $\langle \eta(0)\, \eta(x) \rangle$:

$$\xi^{-1}(\text{corr})/\xi^{-1}(\text{uncorr}) = \int_{-\infty}^{\infty} dx \langle \eta(0)\, \eta(x) \rangle \cos 2kx. \quad (4.57)$$

The above example is a special case of this general relation which holds at high energies or small disorder.

5. Conclusion.

In this paper, we have shown how the Invariant Imbedding ideas can be used to reduce the wave transmission problems (boundary-values problems) to initial-value Cauchy problems. We have limited our exposition here to one-dimensional layered media. However the basic algorithm, described in section 2, is actually a very general one and can be generalized to include other cases. In addition to non-trivial linear problems (Multichannel transmission, time dependent medium, etc.) the present approach appears as the only systematic method for the study of non-linear random media. In such a case, we have derived a partial differential equation (PDE) for T and R and this contrasts with a purely linear medium. In general, a PDE can also be viewed as an infinite set of ordinary differential equations and this indicates the profound differences between linear and non-linear problems. Furthermore, an infinite set of ordinary differential equations has also been obtained for a linear time dependent medium. This common feature must be traced back to the occurrence in these two cases of a new degree of freedom in the considered system. This basic remark is at the origin of the modification of the exponential transmission decay and this either in non-linear random medium or in a time-dependent random medium. The first case will be investigated in detail in the second paper of this series [10]. A possible extension of the present approach will be the investigation of non-stationary problems, e.g. non-linear, time-dependent media, where the generation of harmonics may become an important feature. Another direction for future investigations is probably the study of the discrete versions of the Invariant Imbedding equations as well as the inclusion of new features such as external fields (magnetic, electric, etc.).

Appendix A.

In this Appendix, we outline the analytical proof of the invariant Imbedding equations, in the case of non-linear media. As was shown in the main text (Eq. (3.5)) the integral equation of the wave field can be written as follows :

$$u(x\,;L,w) = e^{ik(L-x)} + \frac{ik}{2} \times \times \int_0^L dx'\, e^{ik|x-x'|}\, \eta[x, wI(x')]\, .\, u(x', L, w). \quad (A.1)$$

Here, $\eta[x, wI(x')] \equiv [-V(x) + \alpha f(|\varphi|^2)]/k^2$, w refers to the intensity of the incident wave, and $I(x) = |u|^2 = |\varphi|^2/w$.

Differentiation of (A.1) with respect to L gives :

$$\frac{\partial u}{\partial L} = A(L, w)\, u(x, L\,; w) + B(x\,; L, w) \quad \text{(A.2)}$$

where $A(L, w) = ik + \frac{ik}{2}\,\eta(L, wI(L))\, u_L(w)$ and $u_L(w) = u(x = L\,; L, w)$. Taking the derivative of (A.1) with respect to w, it is possible to connect $B(x\,; L, w)$ to $\frac{\partial u(x\,; L, w)}{\partial w}$ as follows :

$$B(x\,; L, w) = w[A(L, w) + A^*(L, w)]\,\frac{\partial u}{\partial w}. \quad \text{(A.3)}$$

As a consequence, $u(x\,; L, w)$ satisfies the following equality :

$$\frac{\partial u(x\,; L, w)}{\partial L} = A(L, w)\, u(x\,; L, w) + \\ + [A(L, w) + A^*(L, w)]\, w\,\frac{\partial u(x\,; L, w)}{\partial w}. \quad \text{(A.4)}$$

Using now the boundary conditions :

$$u(L) = u_L = [1 + R(L, w)]\,, \quad u'(L) = -ik(2 - u_L)$$
$$u(0) = T(L, w)\,, \quad u'(0) = -iku(0)\,. \quad \text{(A.5)}$$

One deduces immediately the equation for the reflection coefficient $R(L, w)$:

$$\frac{\partial R}{\partial L} = 2\,ikR + \frac{ik}{2}\,\eta(L, wI(L)) \times \\ \times (1 + R)^2 + [A + A^*]\, w\,\frac{\partial R}{\partial w} \quad \text{(A.6)}$$

which is identical to (3.7).

Similarly, if we set $L = 0$ in (A.4), then one obtains the equation for $T(L, w)$

$$\frac{\partial T}{\partial L} = A(L, w)\, T + [A + A^*]\, w\,\frac{\partial T}{\partial w} \quad \text{(A.7)}$$

and this reproduces (3.8) as it should be.

Appendix B.

In order to perform the averages over the realizations of the random potential, it is convenient to use the well known Novikov theorem [28]. Random processes $\eta_\alpha(x)$ are assumed to be Gaussian with a mean value equal to zero. The correlation functions are arbitrary : $K_{\alpha\beta}(x, y) = \langle \eta_\alpha(x)\, \eta_\beta(y)\rangle$ where $\langle \ldots\rangle$ indicate averaging over an assembly of realizations of $\{\eta_\alpha(x)\}$, $\alpha = 1, 2, \ldots, n$.

In its simplest form, Novikov's theorem states that for a functional $\Phi[\eta]$ of $\{\eta\}$, one has the following identity

$$\langle \eta_\alpha(x)\, \Phi[\eta]\rangle = \int \mathrm{d}y\, K_{\alpha\beta}(x, y) \left\langle \frac{\delta \Phi[\eta]}{\delta \eta_\beta(y)} \right\rangle. \quad \text{(B.1)}$$

Here, $\delta\Phi/\delta\eta$ refers to the functional derivative of Φ with respect to η. When (B.1) is used twice, the following extension is obtained [29] :

$$\langle \eta^2(x)\, \Phi[\eta]\rangle = \langle \eta^2(x)\rangle \langle \Phi[\eta]\rangle + \\ + \int \mathrm{d}y \int \mathrm{d}y'\, K(x, y)\, K(x, y') \left\langle \frac{\delta^2 \Phi[\eta]}{\delta \eta(y')\, \delta \eta(y)} \right\rangle. \quad \text{(B.2)}$$

The above formulae can be generalized [30] for the mean of the product of the linear $P[\eta]$ and non-linear $R[\eta]$ functionals of the process $\eta(x)$, with a zero mean value

$$\langle P[\eta]\, R[\eta]\rangle = \langle P[\eta]\rangle \langle R[\eta]\rangle + \\ + \iint \left\langle \frac{\delta P[\eta]}{\delta \eta(y)} \right\rangle \left\langle \frac{\delta R[\eta]}{\delta \eta(y')} \right\rangle K(y, y')\, \mathrm{d}y\, \mathrm{d}y'. \quad \text{(B.3)}$$

Formulae (B.1), (B.2) and (B.3) can be proved, for example, by expanding the functional of η as a functional Taylor series, and represent a generalization of the Wick theorem for the correlation of a Gaussian random quantity with a function $f(\eta)$ of η : $\langle \eta f(\eta)\rangle = \langle \eta^2\rangle \langle f'(\eta)\rangle$, to include the case of continuous fields. Detailed proofs of (B.1), (B.2) and (B.3) will be found in the relevant references.

When the random process $\eta(x)$ is not Gaussian but a dichotomous or two step Markov process, similar formuale can be derived. For instance, if $\langle \eta(x)\rangle = 0$ and $\langle \eta(x)\, \eta(x')\rangle = \Delta^2 \exp(-\lambda\,|x - x'|)$ (Ornstein-Uhlenbeck process), then the following formula of differentiation [31]

$$\frac{\partial}{\partial x} \langle \eta(x)\, \Phi[\eta(x)]\rangle = \left\langle \eta(x)\, \frac{\partial}{\partial x}\, \Phi[\eta(x)] \right\rangle - \\ - \lambda \langle \eta(x)\, \Phi[\eta(x)]\rangle \quad \text{(B.4)}$$

can be derived for any functional $\Phi[\eta]$ of η. This formula is also valid for other types of processes and we direct the reader to [31] for further details. Note that the white noise limit is recovered at $\lambda \to \infty$, $\Delta^2 \to \infty$, $\Delta^2 \lambda^{-1} = D = \text{Const.}$, where

$$\langle \eta(x)\, \eta(x')\rangle = 2\, D\delta(x - x')\,.$$

Appendix C.

In order to calculate the correlation functions of the reflection coefficient, we follow the same procedure as in reference [22]. To first order in β, $\psi_n \equiv \langle R_1^n\, R_2^{*n}\rangle$ is the solution of the following difference-differential equation

$$\frac{\mathrm{d}\psi_n}{\mathrm{d}l} = n^2(\psi_{n+1} - 2\,\psi_n + \psi_{n+1}) + i\beta n \psi_n\,, \\ \psi_n(l = 0) = \delta_{n,0}. \quad \text{(C.1)}$$

Equation (C.1) can be transformed into a differential equation for the following generating function

$$\psi(x, l) = \sum_{n=1}^{\infty} \psi_n(l)\, x^{n-1}. \tag{C.2}$$

This leads to

$$\frac{\partial \psi(x, l)}{\partial l} = i\beta \frac{\partial}{\partial x}(x\psi) + \frac{\partial}{\partial x}\left[x \frac{\partial}{\partial x}(1-x)^2 \psi\right] + 1 \tag{C.3}$$

with $\psi(x, l = 0) = 0$ and $\psi(x, l)$ regular at $x = 0$. In terms of $\psi(x, l)$ one has

$$\langle R_1 R_2^* \rangle = \psi_1 \equiv \psi(0, l). \tag{C.4}$$

Making the change of variables : $u = x/(1-x)$ and $\psi(x, l) = (1-x)^{-2} R(u, l)$, one obtains

$$\frac{\partial R}{\partial l} = (1+u)^{-2} + \frac{\partial}{\partial u} u(u+1) \frac{\partial R}{\partial u} + i\beta \frac{\partial}{\partial u} u(u+1) R, \quad R(u, l=0) = 0. \tag{C.5}$$

The solutions of (C.5) can be chosen of the form $R(u, l) = r(u) + R_1(u, l)$, with

$$r(u) = (1+u)^{-1} + i\beta \int_0^{\infty} du_1 \frac{e^{i\beta u}}{(1+u)+u_1} \tag{C.6}$$

and $R_1(u, l)$ is the solution of :

$$\frac{\partial R_1}{\partial u} = \frac{\partial}{\partial u}\left[u(u+1)\frac{\partial R_1}{\partial u}\right] + i\beta \frac{\partial}{\partial u}[u(u+1) R_1], \quad R_1(u, l=0) = -r(u). \tag{C.7}$$

In order to solve (C.7), it is convenient to perform a Laplace transform

$$R_1(u, s) = \int_0^{\infty} e^{-sl} R_1(u, l)\, dl.$$

The function $R_1(u, s)$ is now the solution of the following equation

$$\frac{\partial}{\partial u}\left[u(u+1)\frac{\partial R_1(u, s)}{\partial s}\right] + i\beta \frac{\partial}{\partial u} \times [u(u+1) R_1(u, s)] + s R_1(u, s) = 0. \tag{C.8}$$

The solutions of (C.8) are known [22] for $\beta = 0$, and can be expressed as linear combinations of hypergeometric functions

$$R^{(\lambda)}(u) = F\left(\frac{1}{2} + i\lambda, \frac{1}{2} - i\lambda\ ; 1\ ; -u\right) \tag{C.9}$$

where $s = \frac{1}{4} + \lambda^2$.

The set $\{R^{(\lambda)}(u)\}$ is a complete family of orthogonal functions :

$$\int_0^{\infty} du\, R^{(\lambda)}(u)\, R^{(\lambda')}(u) = A(\lambda)\, \delta(\lambda - \lambda') \tag{C.10}$$

where $A(\lambda) = 1/2\ \lambda \tanh(\pi\lambda)$. Using the boundary condition :

$$\frac{-1}{1+u} = \int_0^{\infty} ds\, R_1(u, s)$$

one deduces

$$R_1(u, \lambda) = -2\lambda \tanh \pi\lambda R^{(\lambda)}(u) \times \times \int_0^{\infty} du_1 R^{(\lambda)}(u_1)/(1+u_1).$$

The last integral is equal to

$$\Gamma(1/2 + i\lambda)\ \ \Gamma(1/2 - i\lambda) = \pi/\cosh \pi\lambda$$

and then

$$R_1(u, l) = -\int_0^{\infty} d\lambda \frac{2\pi\lambda \sinh \pi\lambda}{\cosh^2 \pi\lambda} \times \times e^{-(1/4 + \lambda^2) l} F\left(\frac{1}{2} + i\lambda, \frac{1}{2} - i\lambda\ ; 1\ ; -u\right).$$

This leads to

$$R(u, l) = \frac{1}{1+u} - \int_0^{\infty} d\lambda \frac{2\pi\lambda \sinh \pi\lambda}{\cosh^2 \pi\lambda} \times \times e^{-(1/4 + \lambda^2) l} F\left(\frac{1}{2} + i\lambda, \frac{1}{2} - i\lambda\ ; 1\ ; -u\right) \tag{C.11}$$

and then

$$\psi_1(l) \equiv R(0, l) = = 1 - \int_0^{\infty} d\lambda \frac{2\pi\lambda \sinh \pi\lambda}{\cosh^2 \pi\lambda} e^{-(1/4 + \lambda^2) l} \tag{C.12}$$

which is nothing else than the average of the reflection coefficient. $\psi_1(l)$ is dominated by $\lambda \sim 0$ and then $1 - \psi_1(l) \sim \exp(-l/4)$.

The spectrum $R^{(\lambda)}(u)$ can now be used to perform a perturbation calculation at $\beta \ll 1$. For this it is sufficient to calculate the correction to $s^{(0)}(\lambda) = \frac{1}{4} + \lambda^2$ at $\lambda \sim 0$. Let us define $G(u, s)$ by $R_1(u, s) = e^{-i\beta u} G(u, s)$. Then (C.8) becomes :

$$\frac{\partial}{\partial u}\left[u(u+1)\frac{\partial G}{\partial u} e^{-i\beta u}\right] + + s\, e^{-i\beta u} G(u, x) = 0, \quad \beta \neq 0 \tag{C.13a}$$

and

$$\frac{\partial}{\partial u}\left[u(u+1)\frac{\partial R(\lambda')}{\partial u}\right] + + s^{(0)}(\lambda') R^{(\lambda')}(u) = 0, \quad \text{at} \quad \beta = 0. \tag{C.13b}$$

We neglect the phase factor $e^{-i\beta u}$ in the first term of (C.13a) and find solutions $G(u, s)$ of the form :

$$G(u,s) \equiv G^{(\lambda)}(u) = \int d\lambda'\, \varphi(\lambda')\, R^{(\lambda')}(u)\,. \quad \text{(C.14)}$$

The unknown functions $\varphi(\lambda')$ give the expansion of $G(u, s)$ on the unperturbed basis $R^{(\lambda')}(u)$. We multiply (C.13b) by $\varphi(\lambda')$ and integrate over λ' :

$$\frac{\partial}{\partial u}\left[u(u+1)\frac{\partial G^{(\lambda)}}{\partial u}\right] + \int d\lambda'\, s^{(0)}(\lambda')\, R^{(\lambda')}(u)\, \varphi(\lambda') = 0\,.$$

Comparison with (C.13a) leads to :

$$s\, e^{-i\beta u} \int d\lambda'\, \varphi(\lambda')\, R^{(\lambda')}(u) = \int d\lambda'\, s^{(0)}(\lambda')\, \varphi(\lambda')\, R^{(\lambda')}(u)\,.$$

Multiplying by $R^{(\lambda)}(u)$ and integrating over u one obtains :

$$s \int d\lambda'\, \varphi(\lambda') \int du\, R^{(\lambda)}(u)\, e^{-i\beta u R^{(\lambda')}(u)} = s^{(0)}(\lambda)\, A(\lambda)\, \varphi(\lambda)\,. \quad \text{(C.15)}$$

It is convenient to define the symmetrical kernel $\mathbb{K}(\lambda, \lambda')$ as follows :

$$\mathbb{K}(\lambda, \lambda') = \int_0^\infty du\, R^{(\lambda)}(u)\, e^{-\beta u R^{(\lambda')}(u)}\,.$$

Equation (C.15) becomes an integral equation for $\varphi(\lambda')$, s being the unknown eigenvalue

$$s \int d\lambda'\, \varphi(\lambda')\, \mathbb{K}(\lambda, \lambda') = s^{(0)}(\lambda)\, A(\lambda)\, \varphi(\lambda)\,. \quad \text{(C.16)}$$

Equation (C.16) will be solved by successive approximations, starting from the known solution at $\beta = 0$. For this, one needs the expression of the kernel $\mathbb{K}(\lambda, \lambda')$ at $\beta \ll 1$.

Note that $\mathbb{K}(\lambda, \lambda') = A(\lambda)\, \delta(\lambda - \lambda')$ at $\beta = 0$. For finite β, $\mathbb{K}(\lambda, \lambda')$ can be written as

$$\mathbb{K}(\lambda, \lambda') = \int_0^\infty du\, e^{-i\beta u}\, P_{-1/2+i\lambda}(1+2u)\, P_{-1/2+i\lambda'}(1+2u) = \frac{1}{2}\int_1^\infty dz\, e^{-i\beta(z-1)/2}\, P_{-1/2+i\lambda}(z)\, P_{-1/2+i\lambda'}(z)\,.$$

Here we have used the following identity :

$$P_{-1/2+i\lambda}(z) = F\left(\frac{1}{2}+i\lambda, \frac{1}{2}-i\lambda\,; 1\,; \frac{1-z}{2}\right).$$

The limiting behaviour of $P_{-1/2+i\lambda}(z)$ is given by :

$$P_{-1/2+i\lambda}(z \gg 1) \simeq \simeq (2\pi z)^{-1/2}\left[\frac{\Gamma(i\lambda)}{\Gamma\left(\frac{1}{2}+i\lambda\right)} e^{i\lambda \ln 2z} + \text{c.c.}\right]$$

and $$P_{-1/2+i\lambda}(z \ll 1) \simeq 1\,.$$

Therefore, the integral giving $\mathbb{K}(\lambda, \lambda')$ is dominated by the contribution at $z \gg 1$:

$$\mathbb{K}(\lambda, \lambda') \simeq \frac{1}{2}\frac{1}{2\pi}\left|\frac{\Gamma(i\lambda)}{\Gamma(i\lambda+1/2)}\right|^2 \times \times \int_1^\infty dz\, e^{-i\beta(z-1)/2}\, [e^{i(\lambda-\lambda')\ln 2z}/z + \text{c.c.}]\,.$$

Only the interval $\beta u \ll 1$ contributes to the integral and

$$\mathbb{K}(\lambda, \lambda') \simeq \frac{1}{4\pi}\left|\frac{\Gamma(i\lambda)}{\Gamma(1/2+i\lambda)}\right|^2 \times \times 2\frac{\sin[(\lambda-\lambda')\ln(1+2/\beta)]}{\lambda-\lambda'}\,.$$

Finally, one gets :

$$\mathbb{K}(\lambda, \lambda') \simeq A(\lambda)\frac{\sin(\lambda-\lambda')\alpha}{\pi(\lambda-\lambda')} \quad \text{(C.17)}$$

where $\alpha = \ln(1+2/\beta) \gg 1$. As it should be, $\mathbb{K}(\lambda, \lambda')$ reduces to $A(\lambda)\, \delta(\lambda - \lambda')$ at $\beta \to 0$. Coming back to (C.16), one has to solve :

$$s \int_0^\infty d\lambda'\, \varphi(\lambda')\frac{\sin\alpha(\lambda''-\lambda')}{\pi(\lambda''-\lambda')} = s^{(0)}(\lambda'')\, \varphi(\lambda'')\,. \quad \text{(C.18)}$$

Replacing $\varphi(\lambda')$ by the zero order solution : $\delta(\lambda - \lambda')$ in the integral, one obtains the first-order solution

$$\varphi(\lambda'') = \frac{s}{1/4+\lambda''^2}\frac{\sin\alpha(\lambda''-\lambda)}{\pi(\lambda''-\lambda)}\,.$$

Using this solution in (C.18), one deduces the new spectrum :

$$s = \pi\alpha\left[\int_{-\infty}^{\infty}\frac{d\lambda'}{1/4+\lambda'^2}\left(\frac{\sin\alpha(\lambda'-\lambda)}{\lambda'-\lambda}\right)^2\right]^{-1}$$

and this leads to

$$s = \left(\frac{1}{4}+\lambda^2\right) \times \times \left[1 + \frac{(\lambda^2-1/4)(1-e^{-\alpha}\cos 2\alpha\lambda) - 2\lambda\sin 2\alpha\lambda}{\alpha(\lambda^2+1/4)}\right]^{-1}.$$

In particular, for $\lambda \sim 0$:

$$s = \frac{1}{4}\left(1-\frac{1}{\alpha}\right)^{-1} \simeq \frac{1}{4}[1+1/\ln(2/\beta)]\,.$$

This perturbed value of the spectrum edge governs the decay of $\psi_1(l)$ at $\beta \ll 1$.

References

[1] LANDAUER, R., *Philos. Mag.* **21** (1970) 863.
LANDAUER, R., Reference [5] below, p. 38.

[2] ANDERSON, P. W., *Phys. Rev.* **B 109** (1958) 1492.

[3] MOTT, N. F. and TWOSE, W. D., *Adv. Phys.* **10** (1961) 107.

[4] Also see, *Anderson Localization,* Springer series in Solid-State Sciences **39** (1982), eds. Y. Nagaoka and H. Fukuyama (Springer, Berlin) 1982.

[5] Also see *Localization, interaction and transport phenomena,* Springer series in Solid-State Sciences **61** (1985), eds. B. Kramer, G. Bergman and Y. Bruynserade (Springer, Berlin) 1985.

[6] FISHER, D. S. and LEE, P. A., *Phys. Rev.* **B 23** (1981) 6851 ;
JOHNSON, R. and KUNZ, H., *J. Phys.* **C 16** (1983) 3895.

[7] STOKES, G., in *Mathematics and Physics Papers,* vol. 4 (Cambridge University Press, Cambridge) 1904.

[8] BELLMAN, R. and WING, G., in *An introduction to Invariant Imbedding* (John Wiley and Sons, New York) 1975.

[9] N. KUMAR, *Phys. Rev.* **B 31** (1985) 5513 ;
LANG, R. H., *J. Math. Phys.* **14** (1973) 1291 ;
HENRICH, J., *Phys. Rev.* **B 33** (1986) 5261 and references therein ;
PAPANICOLAOU, G. C., SIAM *J. Appl. Math.* **21** (1971) 13.

[10] DOUCOT, B. and RAMMAL, R., subm. to *J. Physique,* 1986.

[11] See for instance SULEM, P. L., *Physica* **70** (1973) 190 and references therein.
HALPERIN, B. I., *Phys. Rev.* **139** (1965) A 104.

[12] DOROKHOV, O. N., *Sov. Phys. JETP* **58** (1983) 606.

[13] STONE, A. D., AZBEL, M. Ya. and LEE, P. A., *Phys. Rev.* **B 31** (1985) 1707.

[14] HARNARD, J., WINTERNITZ, P. and ANDERSON, R. L., *J. Math. Phys.* **24** (1983) 1062.

[15] This problem has been considered recently in the following papers : DEVILLARD, P. and SOUILLARD, B., *J. Stat. Phys.* **43** (1986) 423 ;
FRÖHLICH, J., SPENCER, T. and WAYNE, C. E., *J. Stat. Phys.* **42** (1986) 247.

[16] KAPLAN, A. E., *Sov. Phys. JETP* **45** (1977) 896 ; *Phys. Rev. Lett.* **55** (1985) 1291.

[17] SILIN, V. P., *Sov. Phys. JETP* **26** (1968) 955.

[18] GROSS, E. P., *Nuovo Cimento* **20** (1961) 454 ;
PITAEVSKII, L. P., *Sov. Phys. JETP* **13** (1961) 451.

[19] NASSAR, A. B., *Phys. Rev.* **A 33** (1986) 3502.

[20] BABKIN, G. I. and KLYATSKIN, V. I., *Sov. Phys. JETP* **52** (1980) 416.

[21] VAN KAMPEN, N. G., *Phys. Rep.* **24C** (1976) 172.

[22] ABRIKOSOV, A. A., *Solid State Commun.* **37** (1981) 997 ;
ABRIKOSOV, A. A. and RYZHKIN, I. A., *Adv. Phys.* **27** (1978) 147.

[23] MELNIKOV, V. I., *Sov. Phys. Sol. State* **23** (1981) 444.

[24] BEREZINSKII, V. L., *Sov. Phys. JETP* **38** (1974) 620.

[25] FRISCH, U., GAUTERO, J. L., *J. Math. Phys.* **25** (1984) 1378.

[26] DOUCOT, B. and RAMMAL, R., to be published.

[27] JOHNSON, R. and KRAMER, B., in reference [5].

[28] NOVIKOV, E. A., *Sov. Phys. JETP* **20** (1965) 1290.

[29] SAN MIGUEL, M. and SANCHO, J. M., *Z. Phys.* **B 43** (1981) 361.

[30] VOLKOV, V. S. and POKROVSKY, V. N., *J. Math. Phys.* **24** (1983) 267.

[31] SHAPIRO, V. E. and LOGINOV, W. M., *Physica* **91A** (1978) 563.