



**HAL**  
open science

## Reflection of light by a random layered system

E. Bouchaud, M. Daoud

► **To cite this version:**

E. Bouchaud, M. Daoud. Reflection of light by a random layered system. Journal de Physique, 1986, 47 (9), pp.1467-1475. 10.1051/jphys:019860047090146700 . jpa-00210342

**HAL Id: jpa-00210342**

**<https://hal.science/jpa-00210342>**

Submitted on 4 Feb 2008

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Classification  
*Physics Abstracts*  
 41.10H — 71.55J

## Reflection of light by a random layered system

E. Bouchaud and M. Daoud

Laboratoire Léon Brillouin (\*), CEN-Saclay, 91191 Gif-sur-Yvette Cedex, France

(Reçu le 30 mai 1985, révisé le 12 mai 1986, accepté le 15 mai 1986)

**Résumé.** — Nous envisageons la transmission et la réflexion de la lumière par un milieu lamellaire. L'indice optique de chaque lamelle est une constante qui fluctue d'une lamelle à une autre. Dans la limite d'un grand nombre  $N$  de lamelles, le coefficient de transmission s'annule exponentiellement et le milieu devient un réflecteur parfait. Une méthode nouvelle et simple nous permet d'évaluer la longueur de pénétration  $\xi$  à l'incidence normale et pour l'angle critique de réflexion totale du milieu homogène équivalent. Dans le cas d'une distribution gaussienne de largeur  $\zeta$  de la partie fluctuante de l'indice nous trouvons respectivement  $\xi \sim \zeta^{-2}$  et  $\xi_c \sim \zeta^{-2/3}$  dans ces deux cas. Nous discutons également les propriétés du coefficient de réflexion et donnons des résultats de simulation numérique. Ces résultats sont également valables pour la transmission des neutrons.

**Abstract.** — We consider the transmission and reflection of light by a layered medium where the optical index of every layer is a constant which fluctuates from one layer to another. In the limit of a very large number  $N$  of layers, the transmission coefficient vanishes exponentially, and the medium becomes an ideal reflector. A new simple method is used to evaluate the penetration depth  $\xi$  for normal incidence and at the critical angle for total reflection of the homogeneous equivalent medium. For a Gaussian distribution with width  $\zeta$  of the fluctuating part of the index, we find  $\xi \sim \zeta^{-2}$  and  $\xi_c \sim \zeta^{-2/3}$ , respectively. We also discuss the properties of the reflection coefficient and show computer simulations. We expect our results to be also valid for neutron transmission.

### 1. Introduction.

One-dimensional localization of electrons has been extensively studied since the early work of Anderson [1]. Both numerical and analytical results were obtained [2-6]. However this concept was used only recently to study the propagation of waves through random media [5]. Extensive work is currently being done in such specific problems as chains of random impedances [7], absorption properties of three-dimensional random media [8, 9] and sound propagation through layered random media [10]. In this paper we consider the transmission and reflection of light by a random layered medium. The system we consider is a succession of  $N$  infinite layers with equal width  $l$  (see Fig. 1). The refractive index of each layer is fixed, but fluctuates around an average value  $n$  for successive layers. We neglect absorption completely ( $n$  is real). Possible realizations of the system we have in mind are as follows.

a) Every layer is a mixture of two components. The composition fluctuates for different layers.

b) One might also think of a liquid polarized in a flow. If this liquid flows through a succession of cavities with random width, the velocity distribution is random, and so will be the index.

Propagation of light across such layered media was considered some time ago by Sulem and Frisch [18] who showed that the transmission coefficient should be an exponentially decreasing function of the number  $N$  of layers. However they did not relate the penetration depth of the incident beam to randomness. This is part of what is done in the present paper. Moreover, to our knowledge, this is the first time that one considers the difference between the reflectivity of such layered system for different incidence angles, and more precisely between normal and critical angles of incidence. The latter is related to the band edge problem in Anderson localization which was very recently considered by Derrida and Gardner [11, 12], whose work followed the work of Kappus and Wegner [13] on the band centre anomaly. The approach we use in section 4 for critical incidence (total reflection for the equivalent homogeneous system) is however quite different from that in reference [11] : here we use a simple renormalization argument to derive the scaling

(\*) Laboratoire commun CEA-CNRS.

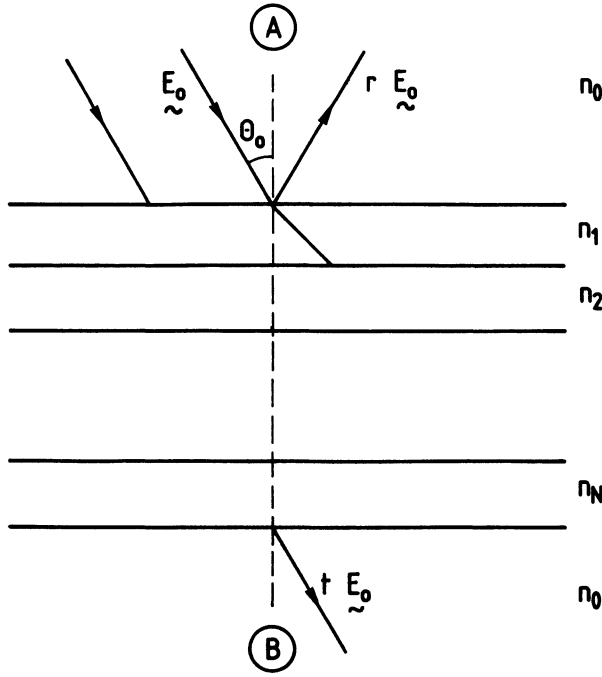


Fig. 1. — Sketch of the layered system.

properties of the penetration length and the reflection coefficient.

Another motivation for the present work is that measurements of the reflection coefficient by such layered systems should not be very hard to perform by light or neutron scattering.

In the following, we will assume that the refraction index of layer  $k$  is :

$$n_k = n(1 + \eta_k) \tag{1}$$

where the  $\eta_k$ 's are random independent variables, with a probability distribution  $P(\eta)$  which we will assume to be a Gaussian for convenience.

$$P(\eta) = \left( \frac{1}{2\pi\zeta^2} \right)^{1/2} \exp\left( -\frac{\eta^2}{2\zeta^2} \right). \tag{2}$$

Suppose we have  $N$  such layers embedded in a medium with index  $n_0$ , and an incident plane wave comes with an incidence angle  $\theta_0$ , as shown in figure 1. We wish to know the transmission and reflection coefficients by this layered system. The basic conservation equations are given in section 2. The discrete nature of the problem is kept and leads to a product of random matrices that was not considered so far. Section 3 gives some general results about the transmission coefficient. Section 4 deals with critical incidence. We show that the penetration length  $\xi$  varies with the width  $\zeta$  of the distribution of  $\eta_k$  as  $\xi \sim \zeta^{-2/3}$ ; we also discuss the properties of the distribution of reflection coefficients. In section 5, the scaling laws for the same properties are derived for normal incidence. Some numerical results are also discussed.

## 2. Conservation law in the layered system.

Consider a set of  $N$  infinite layers with equal width  $l$  along the  $z$  direction, embedded in two semi-infinite media with optical index  $n_0$ . An electromagnetic plane wave is sent with an incidence angle  $\theta_0$  with the normal  $z$  to the layers, as shown in figure 1. We assume that the incident electric field is monochromatic and linearly polarized along a direction perpendicular to the plane of incidence

$$E_0 = E_0 \exp i(\mathbf{q}_0 \cdot \mathbf{r} - \omega t) \mathbf{y} \tag{3}$$

where  $E_0$  is the complex amplitude, and  $\mathbf{y}$  a unit vector normal to the plane of incidence. We also assume that there is no current distribution at the surface of the layers, so that there is conservation of the tangential components of the electric and magnetic fields at every interface. Let  $E_t^{(k)}$  and  $E_r^{(k)}$  be the (complex) amplitudes of the transmitted and reflected electric fields in the  $k$ th layer, at  $z_k = (k - 1)l$ . The conservation law may be written [14], using transfer matrices

$$E^k = M_k E^{k-1} \quad (1 \leq k \leq N + 1) \tag{4}$$

where

$$E^k = \begin{pmatrix} E_t^{(k)} \\ E_r^{(k)} \end{pmatrix}$$

and

$$M_k = \frac{1}{2} \begin{bmatrix} \left( 1 + \frac{K_{k-1}}{K_k} \right) e^{iK_{k-1}l} \left( 1 - \frac{K_{k-1}}{K_k} \right) e^{-iK_{k-1}l} \\ \left( 1 - \frac{K_{k-1}}{K_k} \right) e^{iK_{k-1}l} \left( 1 + \frac{K_{k-1}}{K_k} \right) e^{-iK_{k-1}l} \end{bmatrix} \tag{5}$$

with

$$K_k = \frac{n_k \omega}{c} \cos \theta_k = q_k \cos \theta_k \tag{6}$$

with  $\theta_k$  the refraction angle in layer  $k$ , with boundary conditions

$$E_t^0 = E_0; \quad E_r^0 = E_r; \quad E_t^{N+1} = E_t; \quad E_r^{N+1} = 0,$$

$E_0, E_t$  and  $E_r$  being the incident, transmitted in medium B, reflected in medium A fields, and  $K_0 = 0$  in the argument of the exponentials of relation (5). Introducing

$$\varepsilon = 1 - \left( \frac{n_0}{n} \right)^2 \sin^2 \theta_0 \tag{7a}$$

$$\tilde{\kappa} = n \frac{\omega}{c} \tag{7b}$$

we may relate  $K_k$  to the random variable  $\eta_k$  using

relation (6)

$$K_k^2 = \bar{k}^2(\varepsilon + 2 \eta_k + \eta_k^2). \quad (8)$$

It is however more convenient to use another set of variables which are the total electric and magnetic fields :

$$X_k = E_t^{(k)} + E_r^{(k)} \quad (9a)$$

$$Y_k = iK_k(E_t^{(k)} - E_r^{(k)}). \quad (9b)$$

We then have

$$\begin{pmatrix} X_k \\ Y_k \end{pmatrix} = Q_k \begin{pmatrix} X_{k-1} \\ Y_{k-1} \end{pmatrix} \quad (10)$$

where  $Q_1 = 1$  is the identity matrix and

$$Q_{k+1} = \begin{pmatrix} \cos K_k l & \frac{1}{K_k} \sin K_k l \\ -K_k \sin K_k l & \cos K_k l \end{pmatrix} \quad (11)$$

$(2 \leq k \leq N + 1)$

With the usual convention that in case  $K$  is a complex number,  $K = i\bar{K}$ ,

$$\cos Kl = \cosh \bar{K}l, \quad \text{and} \quad \sin Kl = i \sinh \bar{K}l.$$

Note that the elements of  $Q_k$  are real.

Iteration of relation (9) leads to

$$\begin{pmatrix} X_{N+1} \\ Y_{N+1} \end{pmatrix} = \tau_N \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} \quad (12)$$

with

$$\tau_N = \prod_{i=N+1}^1 Q_i. \quad (13)$$

To our knowledge, this is the first time that one looks at the properties of matrices such as  $\tau_N$ , with the  $Q_k$  s given by equation (10) where the  $K_k$ 's are random variables.

Defining the transmission and reflection coefficients for the amplitudes

$$t = E_t/E_0 \quad (14a)$$

$$r = E_r/E_0 \quad (14b)$$

equation (12) is equivalent to

$$\begin{pmatrix} t \\ iK_0 t \end{pmatrix} = \tau_N \begin{pmatrix} 1 + r \\ iK_0(1 - r) \end{pmatrix}. \quad (15)$$

We can easily check from equation (11) that all the  $Q_i$ 's are unimodular matrices, and so is  $\tau_N$ . Using (15) it may be easily shown that this property is equivalent to the

conservation of energy

$$|r|^2 + |t|^2 = 1. \quad (16)$$

This is done in appendix 1.

### 3. Transmission coefficient and exponential growth of $\tau_N$ .

It was shown, under some restrictive conditions which are fulfilled by  $\tau_N$ , that when  $N$  goes to infinity the norm of the product of  $N$  random matrices diverges exponentially [15] :

$$\|\tau_N\| \sim e^{\gamma N} \quad (\gamma N \gg 1) \quad (17)$$

where the so-called Lyapunov exponent  $\gamma$  is positive. Note that  $\gamma$  has to be related to the randomness, and thus to the width  $\zeta$  of the distribution function. We will see some special cases below. Note also that, although the norm of the matrix diverges, the determinant of  $\tau_N$  is unity. Finally, it is shown in appendix 1 that the transmission coefficient  $t = E_t/E_0$  is related to a norm of  $\tau_N$

$$|t|^2 = \frac{4}{\|\tau_N\|^2 + 2} \quad (18)$$

so that  $|t|$  goes exponentially to zero

$$|t| \sim e^{-\gamma N} \quad (\gamma N \gg 1). \quad (19)$$

Relation (19) implies that the intensity transmitted by the layered system is exponentially small whatever the extent of randomness in the successive layers, and whatever the angle of incidence. Even a very simple system consisting in successive layers where the average index  $n$  is the same as in the semi-infinite homogeneous medium  $n_0$  will not transmit the incident wave, and thus acts as a perfect reflector as long as the number of random successive layers is larger than a characteristic value

$$N^* \sim \gamma^{-1} \quad (20)$$

in what follows, we relate the Lyapunov exponent  $\gamma$  to the width  $\zeta$  of the distribution of the fluctuation in  $n_k$  for two simple cases, namely at the critical angle  $\theta_c$  for total reflection, and for normal incidence.

A final property of  $\tau_N$  is shown in appendix 2 : when  $N$  goes to infinity the asymptotic form of  $\tau_N$  is

$$\tau_N \sim e^{\gamma N} P(\zeta) \quad (21)$$

where  $P(\zeta)$  tends to a projection operator :

$$[P(\zeta)]^2 = P(\zeta)$$

and thus

$$\text{Tr} \tau_N \sim e^{\gamma N} \quad (22)$$

a relation which will be useful in next section.

#### 4. Critical incidence.

Critical incidence angle corresponds to total reflection for the homogeneous system of index  $n$

$$n = n_0 \sin \theta_c.$$

Equivalently it may be defined by  $\varepsilon = 0$  in equation (8).

If there were no randomness in the refraction index of the layers, all the  $Q_k$  s would be equal to

$$Q = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}. \quad (23)$$

For *small* randomness, we consider a perturbation of  $Q$  to first order in  $\eta$ . Using (8) with  $\varepsilon = 0$ , we expand (11) :

$$Q_{k+1} \simeq \begin{pmatrix} 1 - \bar{k}^2 l^2 \eta_k & l \left( 1 - \frac{1}{3} \bar{k}^2 l^2 \eta_k \right) \\ - 2\bar{k}^2 l \eta_k & 1 - \bar{k}^2 l^2 \eta_k \end{pmatrix} \quad (24)$$

which is unimodular to first order in  $\eta$ . We stress that we are not interested here in the exact value of the Lyapunov exponent but rather in its scaling with  $\zeta$  (see Eq. (27) below).

The renormalization transformation performed in appendix 3 leads to

$$\tau(N, \zeta) = A \tau \left( \frac{N}{2}, 2\sqrt{2} \zeta \right) A^{-1}. \quad (25)$$

Using (22) and (25) we get

$$N \gamma(\zeta) = \frac{N}{2} \gamma(2\sqrt{2} \zeta) \quad (26)$$

and thus

$$\gamma \sim \zeta^{2/3} \quad (27)$$

and, from relation (19) the transmission coefficient is

$$|t| \sim e^{-N \zeta^{2/3}} \quad (28)$$

and thus a penetration length, from (28) and (20)

$$\xi \sim N^* l \sim \zeta^{-2/3} l. \quad (29)$$

Equation (27) was derived recently by Gardner and Derrida [11] for the one-dimensional Anderson localization near the band edge by a different method.

Let us now discuss the reflection properties of the system. We define the vanishing quantity

$$\varphi \equiv 1 - r. \quad (30)$$

Setting

$$\begin{aligned} \varphi(N, \zeta) &\equiv \varphi & t(N, \zeta) &\equiv t \\ \varphi \left( \frac{N}{2}, 2\sqrt{2} \zeta \right) &\equiv \varphi' & t \left( \frac{N}{2}, 2\sqrt{2} \zeta \right) &\equiv t'. \end{aligned}$$

Equation (25) together with (15) lead to

$$A^{-1} \begin{pmatrix} t \\ iK_0 t \end{pmatrix} = \tau \left( \frac{N}{2}, 2\sqrt{2} \zeta \right) A^{-1} \begin{pmatrix} 2 - \varphi \\ iK_0 \varphi \end{pmatrix} \quad (31a)$$

$$\begin{pmatrix} t' \\ iK_0 t' \end{pmatrix} = \tau \left( \frac{N}{2}, 2\sqrt{2} \zeta \right) \begin{pmatrix} 2 - \varphi' \\ iK_0 \varphi' \end{pmatrix} \quad (31b)$$

Thus we can relate  $t$ ,  $t'$ ,  $\varphi$  and  $\varphi'$  to the elements of  $\tau \left( \frac{N}{2}, 2\sqrt{2} \zeta \right)$ . Because the latter is unimodular, we get the following condition :

$$\varphi \varphi' - 4 \varphi' + 2 \varphi = t t'. \quad (32)$$

Since  $t$  and  $t'$  are exponentially small, we assume that  $t t'$  as well as  $\varphi \varphi'$  may be neglected in (32). Then (32) becomes

$$\varphi' = \frac{\varphi}{2}.$$

Thus

$$\varphi \left( \frac{N}{2}, 2\sqrt{2} \zeta \right) = \frac{1}{2} \varphi(N, \zeta). \quad (33)$$

Since equation (16) is valid for both sets of coefficients, we find that to the leading order in  $\zeta$ ,  $\varphi$  is pure imaginary, and from (33) we get

$$\text{Im } \varphi \sim \zeta^{2/3} \quad (34)$$

and

$$\text{Re } \varphi \simeq (\text{Im } \varphi)^2 \sim \zeta^{4/3}. \quad (35)$$

Figure 2 shows the results of computer simulations :  $\text{Im } \varphi$  converges well when  $N$  increases. It also shows that  $\varphi$  is sample dependent : the limit of  $\text{Im } \varphi$  depends on the chosen sample. This result has a simple physical meaning : after a limited number of layers, the transmitted intensity is already very small. Then the partial wave which actually probes a large number of layers contributes very poorly to the reflection coefficient. Thus the actual values of the indices of the first layers are very important for the reflection properties. This means that the same experiment on different samples with a constant value of  $\zeta$  leads to different values for  $\text{Im } \varphi$  : there is a distribution  $P(\text{Im } \varphi)$  for  $\text{Im } \varphi$ . The meaning of equation (34) is that this distribution is a universal function of  $(\text{Im } \varphi)/\zeta^{2/3}$  rather than of  $(\text{Im } \varphi)$  itself. This was done numerically. Figure 3 shows the result of computer simulations for different values of  $\zeta$ . Because of the superposition of the different curves, it shows that the distribution  $P(\text{Im } \varphi)$  is a universal function of  $(\text{Im } \varphi)/\zeta^{2/3}$ , in agreement with our predictions.

#### 5. Normal incidence.

In the previous section we showed that, at the critical angle, the incident beam is reflected. We stress that the reflection is the effect of randomness in the optical index. What is characteristic of the critical angle is the scaling variable  $N \zeta^{2/3}$  which includes the disorder.

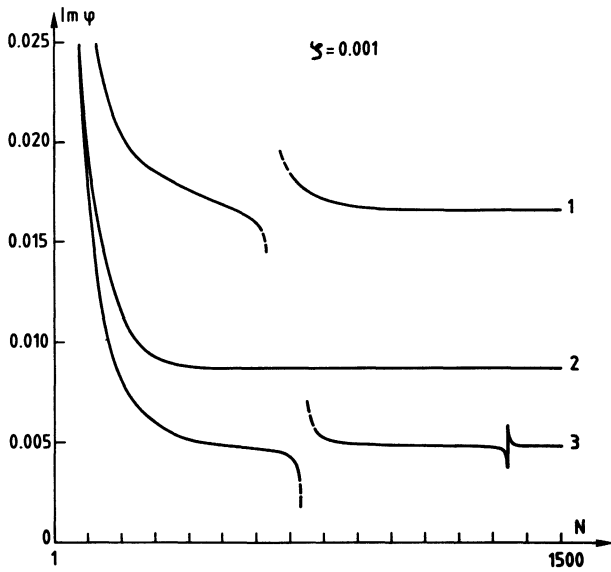


Fig. 2. — The imaginary part of  $\varphi$  as a function of the number  $N$  of layers for  $\zeta = 0.001$ . The limit reached by  $\text{Im } \varphi$  when  $N$  is large enough depends on the sample (curves 1, 2 and 3). Possible bursts such as those on (1) and (3) are probably related to local interference conditions and depend on the actual distribution of indices in the sample. They become smaller as  $N$  increases and do not alter the asymptotic behaviour.

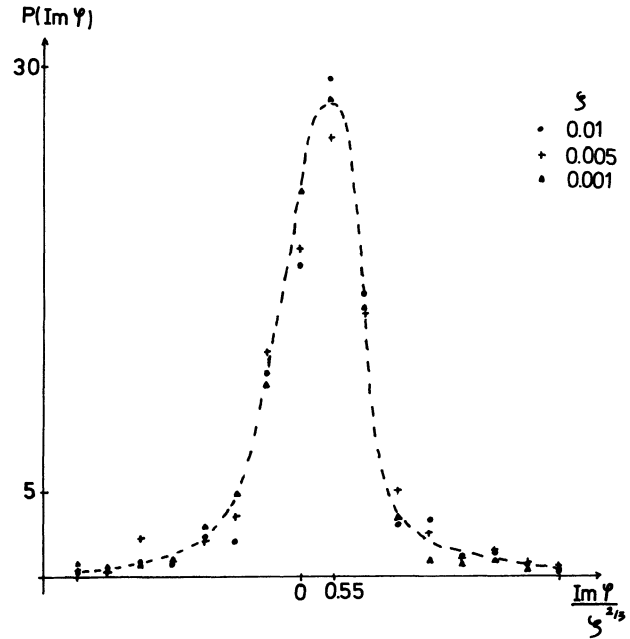


Fig. 3. — The distribution  $P(\text{Im } \varphi)$  as a function of the scaled variable  $\text{Im } \varphi / \zeta^{2/3}$  for three different values of  $\zeta$ . For each value of  $\zeta$ , 400 samples were studied. Every plotted point corresponds to an average over 400/16 results. The dotted curve is a guide to the eye.

In this section, we consider the case of normal incidence to show that there is still total reflection (for an infinite system), but that the scaling is different. When the incidence angle  $\theta_0$  is zero, we get, from equation (7a)

$$K_k = \bar{k}(1 + \eta_k). \tag{36}$$

The  $Q_k$  matrices are particularly simple when  $\bar{k}l = 2p\pi$  with  $p$  being an integer. For convenience, we will only consider this case in what follows. Then equation (11) becomes

$$Q_{k+1} = \begin{pmatrix} \cos \bar{k}l\eta_k & \frac{\sin \bar{k}l\eta_k}{\bar{k}(1 + \eta_k)} \\ -\bar{k}(1 + \eta_k) \sin \bar{k}l\eta_k & \cos \bar{k}l\eta_k \end{pmatrix} \tag{37}$$

If there were no randomness,  $Q_k$  would reduce to the identity matrix. As in section 4, for small randomness we consider a perturbation to the first order in  $\eta$ : Equation (37) becomes:

$$Q_{k+1} \simeq \begin{pmatrix} 1 & l\eta_k \\ -\bar{k}^2 l\eta_k & 1 \end{pmatrix} \tag{38}$$

Note that  $Q_k$  is still unimodular to the first order in  $\eta_k$  which we consider here. The renormalization procedure is simpler than for critical incidence: grouping

the matrices by sets of two, we get

$$\tau(N, \zeta) = \tau\left(\frac{N}{2}, \sqrt{2}\zeta\right) \tag{39}$$

leading to a different scaling for  $\gamma$ . In the same way as in last section, we get

$$\gamma \sim \zeta^2 \tag{40}$$

and thus, for the transmission coefficient, the penetration length, and the reflection coefficient:

$$|t| \sim e^{-N\zeta^2} \quad (N\zeta^2 \gg 1), \tag{41}$$

$$\xi \simeq N^* l \sim \zeta^{-2} l, \tag{42}$$

$$\text{Im } \varphi \sim \zeta^2, \tag{43}$$

and

$$\text{Re } \varphi \simeq \frac{1}{2} (\text{Im } \varphi)^2 \sim \zeta^4. \tag{44}$$

As discussed in section 4,  $\text{Im } \varphi$  is sample dependent. In the present case of normal incidence, the distribution of  $\varphi$  was determined by Sulem [17], using a continuous wave equation.

It may be shown that the exponent 2 holds for any incidence angle as long as the condition  $\bar{k}le = 2\pi p$  is fulfilled. We conjecture that this value of the exponent holds for any value of  $\theta_0$  different from  $\theta_c$ . As a matter of fact, this exponent may be derived from a mean-field calculation where the multiple reflections

are not taken into account. This is somewhat similar to the random phase approximation of Anderson *et al.* [3]. Note however that it cannot lead to the right scaling at critical incidence.

## 6. Conclusion.

We have shown that a set of layers with random indices is a perfect reflector of an incident wave when the number of layers is large enough. The reflection coefficient  $r$  is sample dependent but follows a simple scaling law with disorder  $\zeta$ . The imaginary part of  $\varphi \equiv 1 - r$  might be observed by an interference experiment between the incident and reflected beams. The exponential decrease in the transmitted amplitude is closely related to the Anderson localization of electrons in a disordered potential. This analogy is more direct if one considers the transmission of neutrons by a layered system. It is possible then to relate the index  $n$  to the scattering potential by a single element and to the density of scatterers. Thus we also expect an incident neutron beam to be reflected by a layered medium exactly in the same way as the electromagnetic waves we considered above. There is however a difference in the order of magnitudes for both cases: one might reasonably assume a width  $\zeta \sim 10^{-2}$  for the distribution of indices in the case of light, whereas  $\zeta$  is rather of the order of  $10^{-6}$  for neutrons, layers with 1 micron width are also possible. This leads to penetration lengths of the order of some microns for light, and 1 cm for neutrons, at  $\theta_0 = \theta_c$ . For normal incidence the typical length for light is of the order of 1 cm and the corresponding conclusions might still be checked experimentally. One limitation for our calculation is absorption. Our results are valid only when the typical length for absorption is larger than the penetration depth discussed above.

Finally, we note that interesting experiments by neutron scattering might be performed with magnetic materials. Depending on the polarization of neutrons, the magnetic part of the neutron refraction index changes sign. Thus the critical angles are very different for neutrons polarized up or down. The penetration lengths are thus also very different because of the different power law behaviour of  $\zeta$  as a function of  $\zeta$  for  $\theta_0 = \theta_c$  and  $\theta_0 \neq \theta_c$  discussed above. Thus if a beam of nonpolarized neutrons is sent onto a layered system, the neutrons polarized up are totally reflected while a large fraction of those polarized down should be transmitted, so that the layered medium acts as a polarizer.

## Acknowledgments.

The authors are much indebted to J. L. Pichard for very stimulating discussions on Anderson localization. E. Bouchaud wishes to thank J. P. Bouchaud and P. Le Doussal for interesting discussions.

## Appendix I.

We calculate using equation (14),

$$\begin{bmatrix} t \\ iK_0 t \end{bmatrix} = \tau_N \begin{bmatrix} 1 + r \\ iK_0(1 - r) \end{bmatrix} \quad (\text{A.1})$$

Let

$$\tau_N = \begin{bmatrix} a_N & b_N \\ c_N & d_N \end{bmatrix}. \quad (\text{A.2})$$

Defining :

$$\varphi = 1 - r \quad (\text{A.3})$$

equations (A.1) and (A.2) lead to :

$$\begin{cases} t = a_N(2 - \varphi) + iK_0 b_N \varphi \\ iK_0 t = c_N(2 - \varphi) + iK_0 d_N \varphi \end{cases} \quad (\text{A.4})$$

which is readily solvable

$$\varphi = \frac{2(a_N + ic_N/K_0)}{a_N + d_N + i\left(\frac{c_N}{K_0} - K_0 b_N\right)} \quad (\text{A.5})$$

$$t = \frac{2}{a_N + d_N + i\left(\frac{c_N}{K_0} - K_0 b_N\right)} \quad (\text{A.6})$$

We may deduce the reflection and transmission coefficients for the intensities :

$$|r|^2 = \frac{(a_N - d_N)^2 + (K_0 b_N + c_N/K_0)^2}{(a_N + d_N)^2 + (b_N K_0 - c_N/K_0)^2} \quad (\text{A.7})$$

$$|t|^2 = \frac{4}{(a_N + d_N)^2 + (b_N K_0 - c_N/K_0)^2} \quad (\text{A.8})$$

Using (A.7) and (A.8) it is straightforward to show that  $|r|^2 + |t|^2 = 1$  is equivalent to

$$a_N d_N - b_N c_N = 1. \quad (\text{A.9})$$

On the other hand, one can see on relation (A.8) that  $|t|^2$  may be expressed in the following way :

$$|t|^2 = \frac{4}{\text{Tr}[(\sigma\tau_N)(\sigma\tau_N)^+] + 2} \quad (\text{A.10})$$

with

$$\sigma = \begin{pmatrix} 0 & 1/K_0 \\ K_0 & 0 \end{pmatrix}$$

and

$$\text{Tr}[(\sigma\tau_N)(\sigma\tau_N)^+] = \|\tau_N\| \quad (\text{A.11})$$

is a norm in the space of real  $2 \times 2$  matrices.

## Appendix II

Let us consider the first representation we used in section 2, equation (5). It has been proved [16] through time reversal invariance considerations that for a non-absorbing medium the product  $\prod_{k=N+1}^1 M_k$  has

the simple form :

$$\prod_{k=N+1}^1 M_k = \begin{bmatrix} 1/t^* & -r^*/t^* \\ -r/t & 1/t \end{bmatrix} \quad (\text{A.18})$$

where  $t^*$  and  $r^*$  are the complex conjugates of  $t$  and  $r$  respectively.

This assertion holds, whatever the angle of incidence. Let  $P_k$  be the passage matrix from the first to the second representation

$$Q_k = P_k^{-1} M_k P_{k-1}. \quad (\text{A.12})$$

Then

$$\begin{aligned} \tau_N &= Q_{N+1} Q_N \dots Q_1 \\ &= P_{N+1}^{-1} \left( \prod_{k=N+1}^1 M_k \right) P_0 \end{aligned} \quad (\text{A.13})$$

with

$$P_0 = P_{N+1} = \begin{pmatrix} 1 & -i/K_0 \\ 1 & i/K_0 \end{pmatrix}. \quad (\text{A.14})$$

Using equation (21) we get

$$\tau_N = \frac{1}{2} \begin{pmatrix} \frac{t+t^*-(r^*t+rt^*)}{|t|^2} & \frac{i}{K_0} \frac{t^*-t+rt^*-tr^*}{|t|^2} \\ iK_0 \frac{(t-t^*+rt^*-tr^*)}{|t|^2} & \frac{t+t^*+r^*t+rt^*}{|t|^2} \end{pmatrix}. \quad (\text{A.15})$$

the most general form of  $2 \times 2$  projector is

$$P = \begin{pmatrix} \frac{1-a}{2} & \frac{1-a^2}{2c} \\ \frac{c}{2} & \frac{1+a}{2} \end{pmatrix} \quad (\text{A.16})$$

in order to identify  $\tau_N$  to  $e^{\gamma N} P$ , we must have

$$e^{\gamma N} = \frac{t+t^*}{|t|^2} \quad (\text{A.17})$$

then we can calculate  $a$  and  $c$  in (A.16) as functions of  $r, t, r^*$  and  $t^*$ . The form given in (A.16) may be obtained only if

$$|r|^2 = 1.$$

This is realized in the limit when  $N$  is much larger than  $N^* \sim \gamma^{-1}$  because we know that in such limit

$$|t|^2 \sim e^{-2\gamma N} \ll |r|^2 \approx 1.$$

### Appendix III.

When the medium is homogeneous, all the  $Q_k$ s are equal, and at criticality they have the following expression :

$$Q = \begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix}. \quad (\text{A.18})$$

We note that  $Q$  is the fixed point of the following transformation

$$A^{-1} Q^2 A = Q \quad \text{with} \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}. \quad (\text{A.19})$$

In order to perform a perturbation of  $Q$  in the random case up to the lowest order in  $\eta_k$ , we write :

$$Q_{k+1} = \begin{bmatrix} 1 - \kappa^2 l^2 \eta_k & l \left( 1 - \frac{\kappa^2 l^2 \eta_k}{3} \right) \\ -2\kappa^2 l \eta_k & 1 - \kappa^2 l^2 \eta_k \end{bmatrix} \quad (\text{A.20})$$

Note that  $Q_{k+1}$  remains unimodular, up to the order of approximation retained.

We then have

$$\begin{aligned} A^{-1} Q_{k+1} Q_k A &= \begin{bmatrix} 1 - 2(\eta_k + \eta_{k-1}) \kappa^2 l^2 & l \left( 1 - \frac{2\kappa^2 l^2 (\eta_k + \eta_{k-1})}{3} \right) \\ -2(2\kappa^2 l (\eta_k + \eta_{k-1})) & 1 - 2\kappa^2 l^2 (\eta_k + \eta_{k-1}) \end{bmatrix} + \kappa^2 l^2 (\eta_k - \eta_{k-1}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ A^{-1} Q_{k+1} Q_k A &= Q_{k'} + \kappa^2 l^2 (\eta_k - \eta_{k-1}) \sigma_z \quad \text{with} \quad \sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \quad (\text{A.21})$$

$Q_{k'}$  has the same form as the initial matrices  $Q_{k-1}$  and  $Q_k$ , with the random variable  $\eta_k$  being replaced by  $2(\eta_k + \eta_{k-1})$ .

However the transformation we have performed has generated the term  $\kappa^2 l^2 (\eta_k - \eta_{k-1}) \sigma_z$  which was absent from the initial matrices  $Q_k$ .

Let us consider a set of more general random matrices :

$$\tilde{Q}(x, y, z) = \begin{bmatrix} 1 - \alpha x + y & l \left( 1 - \frac{\alpha}{3} x + z \right) \\ -\frac{2\alpha}{l} x & 1 - \alpha x - y \end{bmatrix} \quad (\text{A.22})$$



where  $\alpha = k^2 l^2$  and  $x, y$  and  $z$  are three non-correlated centred Gaussian small random variables :

$$\langle xy \rangle = \langle xz \rangle = \langle yz \rangle = 0.$$

Thus  $\tilde{Q}$  depends on three parameters  $\sigma_1^2 = \langle x^2 \rangle$ ,  $\sigma_2^2 = \langle y^2 \rangle$ ,  $\sigma_3^2 = \langle z^2 \rangle$ .

The matrices  $Q_k$  are of the form  $\tilde{Q}$ , with :

$$\begin{cases} x = \eta_k \\ y = z = 0. \end{cases}$$

Let us now make the following product

$$A^{-1} \tilde{Q}(x, y, z) \tilde{Q}(x', y', z') A = \begin{bmatrix} 1 - \alpha[2(x + x')] + [\alpha(x - x') + (y + y')] & l \left[ 1 - \frac{\alpha}{3}[2(x + x')] + \frac{1}{2}[(y - y') + (z + z')] \right] \\ -\frac{2\alpha}{l}[2(x + x')] & 1 - \alpha[2(x + x')] - [\alpha(x - x') + (y + y')] \end{bmatrix} \quad (\text{A. 23})$$

The matrix obtained has exactly the same structure as the initial matrix  $\tilde{Q}(x, y, z)$ , with three new random variables :

$$\begin{cases} X = 2(x + x') \\ Y = \alpha(x - x') + (y + y') \\ Z = \frac{1}{2}(y - y') + \frac{1}{2}(z + z'). \end{cases} \quad (\text{A. 24})$$

One can check that :  $\langle XY \rangle = \langle XZ \rangle = \langle YZ \rangle = 0$ .

The parameters :  $\sigma_1'^2 = \langle X^2 \rangle$ ,  $\sigma_2'^2 = \langle Y^2 \rangle$ ,  $\sigma_3'^2 = \langle Z^2 \rangle$  which characterize  $\tilde{Q}(X, Y, Z)$  are related to  $\sigma_1^2, \sigma_2^2, \sigma_3^2$  by the following relations :

$$\begin{cases} \sigma_1'^2 = 8 \sigma_1^2 \\ \sigma_2'^2 = 2 \alpha^2 \sigma_1^2 + 2 \sigma_2^2 \\ \sigma_3'^2 = \frac{1}{2} \sigma_2^2 + \frac{1}{2} \sigma_3^2. \end{cases} \quad (\text{A. 25})$$

Thus we have a linear relation between the vectors  $(\sigma_1'^2, \sigma_2'^2, \sigma_3'^2)$  and  $(\sigma_1^2, \sigma_2^2, \sigma_3^2)$ . The eigenvalues of this linear transformation are equal to 8, 2 and 1/2.

The matrices  $Q_k$  are characterized by a one component vector  $(\sigma_1^2 = \zeta^2, \sigma_2^2 = 0, \sigma_3^2 = 0)$ . This vector can be expressed in function of the eigenvectors  $V_1, V_2$  and  $V_3$  associated respectively to the eigenvalues 8, 2 and 1/2 ;  $(\sigma_1^2 = \zeta^2, \sigma_2^2 = 0, \sigma_3^2 = 0) = a\zeta^2 V_1 + b\zeta^2 V_2 + c\zeta^2 V_3$ ,  $a, b$  and  $c$  are three constants which depend on  $\alpha$  and could be calculated. The important fact to note for the following discussion is that  $a \neq 0$ . We can write :

$$A^{-1} Q_{k+1}(a\zeta^2 V_1, b\zeta^2 V_2, c\zeta^2 V_3) Q_k(a\zeta^2 V_1, b\zeta^2 V_2, c\zeta^2 V_3) A = Q_k \left( 8 a\zeta^2 V_1, 2 a\zeta^2 V_2, \frac{a}{2} \zeta^2 V_3 \right). \quad (\text{A. 26})$$

In fact, we shall be interested only in the scaling behaviour of the Lyapounov exponent with  $\zeta^2$ . Thus we shall retain only the most relevant perturbation, which is associated with the largest eigenvalue (equal to 8). The second eigenvalue, equal to 2, should correspond to a correction to the scaling laws. The third one is smaller than 1 and corresponds to a non relevant perturbation of the homogeneous case.

Thus, neglecting the less relevant terms, we can write equation (A. 26) in the following way :

$$A^{-1} Q_{k+1}(\zeta^2) Q_k(\zeta^2) A \simeq Q_k(8 \zeta^2). \quad (\text{A. 27})$$

And thus :

$$\begin{aligned} \tau(N, \zeta^2) &= A(A^{-1} Q_{N+1}(\zeta^2) Q_N(\zeta^2) A) \dots (A^{-1} Q_2(\zeta^2) Q_1(\zeta^2) A) A^{-1} \\ &= A(Q_{N/2}(\zeta^2) \dots Q_1(\zeta^2)) A^{-1} \\ \tau(N, \zeta^2) &= A\tau\left(\frac{N}{2}, 8 \zeta^2\right) A^{-1}. \end{aligned} \quad (\text{A. 28})$$

## References

- [1] ANDERSON, P. W., *Phys. Rev.* **109** (1958) 1492.
  - [2] ISHII, K., *Suppl. Prog. Th. Phys.* **53** (1973) 77.
  - [3] ANDERSON, P. W., THOULESS, D. J., ABRAHAMS, E., FISHER, D. S., *Phys. Rev. B* **22** (1980) 3519.
  - [4] ANDERECK, B., ABRAHAMS, E., *J. Phys. C* **13** (1980) L383.
  - [5] ANDERSON, P. W., *Solid State Sci.* **61** (1984) 12 (Springer Verlag).
  - [6] PAPANICOLAOU, G. C., preprint.
  - [7] AKKERMANS, E., MAYNARD, R., *J. Physique* **45** (1984) 1549.
  - [8] JOHN, S., *Phys. Rev. Lett.* **53** (1984) 2169.
  - [9] JOHN, S., *Phys. Rev. B* **31** (1985) 304.
  - [10] BALUNI, V., WILLEMSEN, J., *Phys. Rev. A* **31** (1985) 3358.
  - [11] DERRIDA, B., GARDNER, E., *J. Physique* **45** (1984) 1283.
  - [12] DERRIDA, B., *Non linear equations in field theory.* Lecture Notes in Physics (Springer Verlag) to be published.
  - [13] KAPPUS, M. and WEGNER, F., *Z. Phys. B* **45** (1981) 15.
  - [14] BORN, M., WOLF, E., *Principles of Optics* (Pergamon Press) 1980.
  - [15] FURSTENBERG, F., *Trans. Am. Math. Soc.* **108** (1963) 420.
  - [16] PICHARD, J. L., Thesis, Orsay (1984). Available at SPSRM, CEN-Saclay, 91191 Gif-sur-Yvette Cedex.
  - [17] SULEM, P. L., *Physica* **70** (1973) 190.
  - [18] SULEM, P. L. and FRISCH, U., *J. Plasma Phys.* **8** (1972) 217.
-