Linear analysis of surface deflection in Bénard-Marangoni instability

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Abstract — Linear analysis is used to study the free surface deflections of a fluid heated from below. Numerical calculations provide the regions for which the relief is convex or concave and the influence of the fluid parameters on the deflection. These theoretical results agree very well with the experimental ones obtained previously.

1. Introduction.

Bénard, in 1905, was the first to describe the regular hexagonal pattern raising in a horizontal layer of fluid heated from below [1]. A few years later, Lord Rayleigh gave the first theoretical approach to this problem, assuming that the buoyancy forces are responsible for this instability [2]. Some difficulties between theoretical predictions and experimental results appeared early: among them, the mechanism driving this instability and the form of the free surface of a cell. The latter, from the observations made by Bénard, was concave, while Jeffreys [3], using Rayleigh’s approach, showed that it would be convex. Moreover, Bénard proposed that the surface tension was responsible for the instability. Block [4], experimentally, and Pearson [5], theoretically, proved the validity of this assumption. Scriven and Sterlings [6] showed that convective motions induce concave deflections on the free surface when the instability is driven by this mechanism.

In fact, when a pool of liquid has a free surface, Rayleigh’s and Bénard’s mechanisms are both destabilizing factors, provided the liquid has a surface tension which, as usual, decreases with temperature. This is the main conclusion of a numerical approach to this problem, made by Nield [7a]. This author extended his analysis to the case of a liquid layer open to the atmosphere and overlying a layer of a porous medium saturated with the same fluid [7b]. He was able to obtain analytical solutions but, due to the unrealistic boundary conditions he assumed, his results cannot be quantitatively compared with experimental findings. In a recent paper, Davis and Homsy [8] applied the energy method to this problem. A linear analysis was also made to compare with the previous method, using a perturbative analysis, which provides the effect of the deformation on the stability. However, these authors explicitly calculated neither the characteristics of the deflection nor the influence of the different parameters on the surface relief. Therefore, they did not solve the controversy between Bénard and Jeffreys cited above.

Two theoretical works, by Zeren and Reynolds [9] and Ferm and Wollkind [10], have been devoted to the stability of a two-fluid layer under gravity and surface tension forces. Their goal was to calculate the critical parameters and the role of the surface deformations on stability. In reference [10] a comparison of the experiments made by Koschmieder [11] and Palmer and Berg [12] was made. These experiments and part of the theoretical work deal with a liquid layer under a finite air layer.

Recently, an experimental work on surface deformations in Bénard-Marangoni convection (B-M) has been published [13]. The dependence of the surface relief on the different parameters for a fluid layer open to the atmosphere was determined. The main result
was that the surface is concave above the rising fluid when the surface tension effects are dominant (i.e. for a small depth of fluid) and convex when buoyancy effect is the most important driving force (i.e. for a thicker fluid layer).

These experimental results cannot be directly compared to the theoretical ones obtained in [8-10] because in [9] the sign of the deflection is not discussed in detail and in [9] and [10] comparison is made with a double layer system, while in [13] the simple fluid is open to the atmosphere. The aim of the present work is to determine, by linear analysis, the behaviour of the surface deflection. In section 2 we discuss briefly the balance equations and boundary conditions of a fluid layer with a free and deformable surface. In section 3, the numerical results are compared with the experimental observations, and we also consider the influence of the fluid properties on the deflection.

2. Balance equations and boundary conditions (b.c.).

We consider a shallow fluid layer of depth \( d \) horizontally extended to infinity, uniformly heated from below on a rigid good conducting plate. The upper surface, open to the atmosphere, can deflect, as schematized in figure 1.

Here \( n \) and \( t \) denote normal and tangential vectors respectively, and \( h(x, y, t) \) stands for surface deflection as a function of horizontal coordinates and time. As usual, we apply the Oberbeck-Boussinesq approximation [14] to the general balance equations for the perturbations around the pure conductive stationary solution. If we denote by \( u \) and \( \theta \) the velocity and temperature perturbations respectively, the balance equations as well as the boundary conditions on the bottom conducting surface read as in the no-deflection case (see [14], p. 94). The b.c. on the free surface with small deflection can be written as [7b]

\[
\begin{align*}
\frac{\partial h}{\partial t} &= u \cdot n \\
\nabla \theta \cdot n &= - k \theta \\
- p + \rho_0 \theta \left[ h + \alpha \frac{\Delta T}{2 d} \left( h^2 + 2 k d \right) \right] + 2 \mu \frac{\partial u_z}{\partial z} &= \sigma \nabla^2 h \\
\mu \frac{\partial^2 u_z}{\partial z^2} &= - \sigma_1 \nabla^2 \left( \theta - \frac{\Delta T}{d} h \right).
\end{align*}
\]

b.c. at \( z = d + h \) can be rewritten retaining only the first terms in a Taylor expansion around \( z = d \)[6, 7b]. Second the marginal stability is assumed. This is equivalent to asserting, as in the case of no deflection, that the principle of exchanges of stability holds [15, 16]. The equations are written in a dimensionless form. The units of length, time, velocity, temperature and pressure are respectively \( d'/Kn^2, \Delta T/\nu, \nabla \) factors are introduced in coherence with the numerical method used to solve the system of equations. After some transformations the b.c. (1) at \( z = \pi \) read as

\[
\begin{align*}
u_z &= 0 \\
\frac{\partial \theta}{\partial z} &= - L_1 \theta \\
C_1 \frac{\partial}{\partial z} \left( \frac{\partial^2 u_z}{\partial z^2} + 3 \nabla^2 u_z \right) &= \nabla^2 h - B_1 \nabla^2 h \\
\frac{\partial^2 u_z}{\partial z^2} &= M_1 \nabla^2 (\theta - h).
\end{align*}
\]

In these b.c. four dimensionless parameters appear : \( C_1 = C \pi, B_1 = B/\pi^2, L_1 = L/\Pi \) and \( M_1 = M/\pi^2 \), where \( C = \mu \kappa/\sigma d, B = \rho_0 g d^3/\sigma, L = h^{\prime} d \) and \( M = \sigma_1 d \Delta T/\mu k \) are respectively the Crispation, Bond, Biot and Marangoni numbers.
Finally, the solutions in terms of normal modes are analysed:
\[
\{ u_x, \theta, h \} = \{ w(z), \theta(z), H_0 \} \exp[-i(k_x x + k_y y)]
\]
(3)
where \( k \) is the wave number. Introducing these expansions into the classical linearized equations [14], we obtain
\[
(D^2 - k^2) w - k^2 R_1 \theta = 0
\]
\[
(D^2 - k^2) \theta + w = 0.
\]
(4)
The b.c. are
\[
\begin{align*}
 w(0) &= D w(0) = \theta(0) = w(\pi) = 0 \\
 D\theta(\pi) &= -L_1 \theta(\pi) \\
 H_0 &= (k^2 M_1)^{-1} D w(\pi) + \theta(\pi) \\
 C_1 (D^2 - 3 k^2) Dw(\pi) &= (k^4 + B_1 k^2) H_0
\end{align*}
\]
(5)
where \( D = \partial/\partial z \) and \( R_1 \) is simply related to the Rayleigh number \( R = \alpha g d^3 \Delta T/\mu k \) by \( R_1 = R/\pi^4 \).
The system of equations (4) and (5) is solved numerically by means of a method proposed by Nield [7a], based on a Fourier expansion of solutions obeying the b.c. [5a].

3. Numerical results and comparison with experiments.

As usual in a linear theory, we can obtain solutions except for an undetermined amplitude \( A \). To obtain the evolution of this amplitude, a non-linear analysis must be made [8]. Nevertheless, in the experimental cases to be compared here, convective currents rise through the centre of the cells (1-hexagons) [19], implying this amplitude is always positive. Our goal here is to discuss the sign of the deflection and, therefore, we will assume \( A = +1 \) in the following.

The dependence of \( H_0 \) on \( R/R_0 \) is plotted in figure 2 for three values of \( C \) and for \( L = 0.1 \) and \( B = 4 \), values corresponding to the oil used in the experiments [12] (\( R_c \) is the critical value of \( R \) and \( R_0 \) is \( R \) when \( \sigma_1 = 0 \)). The main feature in this figure is a change in the deflection sign at \( R/R_0 = 0.174 \).

Some data obtained in the experiments quoted in [12] are gathered in figure 3. In it, two well-separated regions can be distinguished. For small \( R/R_0 \) the free surface is concave, i.e. the deflection is negative, while for higher \( R/R_0 \) it is convex, i.e. the deflection is positive. For \( 0.171 \leq R/R_0 \leq 0.175 \), a third region can be distinguished in which the deflection is «hybrid» (concave-convex or convex-concave) [13], corresponding broadly to a flat surface. Therefore a good agreement between the theoretical value \( R_c/R_0 = 0.174 \) and the experimental results is obtained for the convex-concave transition. Figures 2 and 3 also agree in the sign of the deflection, confirming that both Bénard [1] and Jeffreys [3] were right because for very shallow pools \( (R_c/R_0 < 0.174) \) the free surface
is concave, while for greater $R_e/R_0$, buoyancy forces dominate and then the surface is convex [13].

We thereby determined the influence of the dimensionless numbers $C$, $L$ and $B$ on the transition threshold. Variations on $B$ and $C$ do not change the reference $R_0$, but when $L$ varies from 0.1 to 1 000 (from an insulating to a conducting free surface), $R_0$ changes from 685.5 to 1 106.04. We performed the numerical analysis using different $C$, $L$ and $B$

<table>
<thead>
<tr>
<th>$L$</th>
<th>$R_e/R_0$</th>
</tr>
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<tbody>
<tr>
<td>0.1</td>
<td>0.174</td>
</tr>
<tr>
<td>1</td>
<td>0.176</td>
</tr>
<tr>
<td>10</td>
<td>0.130</td>
</tr>
<tr>
<td>1 000</td>
<td>0.122</td>
</tr>
</tbody>
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Now the influence of the pool depth on the deflection $h$ is examined. It appears, on the one hand, in the dimensionless parameter $h$, and, on the other hand, through the dimensionless parameters $R_e$, $B$, $C$ and $L$. Taking an arbitrary reference depth $d^*$, we numerically determined this influence through a parameter $r = d/d^*$. The results are given in figure 4. For small $r$, the deflection $h$ is concave. At some value (in the experimental case compared here, for $R_e/R_0 = 0.174$) $h$ vanishes, and as $r$ increases, $h$ also increases. The experimental results in references [13] and [20] agree with these conclusions. When the deflection is positive, it increases with the layer depth until it reaches a maximum. For greater values of $r$ it decreases. This is due to the antagonistic behaviour of the numbers $R_e$, $L$, $B$, $C$ in which $d$ appears. In order to check this conclusion, it would be interesting to measure the deflection in very thick layers. Besides, a weakly non-linear analysis could specify the influence of each physical parameter and in particular the distance to the threshold $\varepsilon = (R - R_e)/R_e$, which is experimentally known [13].

Finally, to determine the influence of the surface deflection on the stability, we have assumed a linear development of the critical Marangoni number $M_e$ on $C$. (In fact, it is more coherent from the physical point of view to take $M_e$ as a perturbed parameter,

![Fig. 4. Surface deflection $h$ as a function of the layer depth in arbitrary units. Here $r = d/d^*$ is the relation between the depth and an arbitrary reference $d^*$.

![Fig. 5. The parameter $M^{(1)}$ as a function of $R_e/R_0$.](image-url)
in view of the small region in which the Marangoni effects dominate (Fig. 2)).

\[ M_e = M_e^{(0)} + M_e^{(1)} \alpha + \cdots \]  

(6)

where \( M_e^{(0)} \) is the corresponding value without deflection. The parameter \( M_e^{(1)} \) as a function of \( R_e/R_0 \) is plotted in figure 5. These results are in agreement with those obtained by Davis and Homsy [7] for similar corrections on \( R_e \). This correction acts as a destabilizing factor when negative, i.e., for concave surfaces, and as a stabilizing factor when positive, i.e., when the surface is convex. But it is noteworthy that it is very small (< 0.01%) and negligible for practical purposes.

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References