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## Anderson localization in a spatially correlated random potential under a strong magnetic field

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**Résumé.** — La densité d'états et la conductivité en présence d'un champ magnétique fort sont calculées par des développements perturbatifs pour des potentiels aléatoires Gaussiens corrélés en des points spatialement séparés, en dimension deux. Ce rapport de la longueur caractéristique des corrélations du potentiel d'impuretés et de la longueur magnétique définit un paramètre  $\eta$ . Pour  $\eta$  grand la conductivité est nulle et reste nulle dans le développement en  $1/\eta$ . Pour  $\eta$  petit on retrouve des résultats compatibles avec ceux du modèle  $\sigma$  non linéaire.

**Abstract.** — The density of state and the conductivity in the presence of a strong magnetic field are calculated by perturbation expansions for a Gaussian random potential correlated at different positions in two dimensional space. A parameter  $\eta$  is introduced which characterizes the ratio of the length defined by the impurity potential correlations to the magnetic length. For large  $\eta$ , a  $1/\eta$  expansion shows that the conductivity vanishes. For small  $\eta$  one recovers results consistent with that of the unitary non-linear  $\sigma$  model.

### 1. Introduction.

Anderson localization by impurities under the influence of a strong magnetic field in two dimensions has drawn recently a considerable interest in view of the relation with the quantized Hall effect. The existence of extended states at the centre of the Landau bands is an important problem which has been studied by theoretical calculations [1-7] and numerical simulations [8, 9].

In the usual approach of a white noise random potential the impurities at different points are statistically independent. The typical length related to the impurity potential correlations is zero. However in a strong magnetic field, in which the magnetic length  $1/\sqrt{eB}$  is small, the model of a white noise random potential is questionable. Therefore we have allowed for some spatial extent of the random potential correlation function. It is known that some materials

such as GaAs-AlGaAs heterojunctions are indeed described by such correlated random potentials [10]. In a strong field the electrons occupy the lowest Landau bands. The conductivity, as given by Kubo formulae, involves matrix elements mixing nearby Landau levels. However in the strong field limit, one can calculate the conductivity as a matrix element of an operator in the lowest Landau level  $n = 0$  and in fact obtain it through Einstein relation from a diffusion constant which may be entirely calculated within the  $n = 0$  subspace.

The calculation is simpler for the lowest Landau level since the unperturbed Green function is a simple Gaussian. For the white noise potential an exact expression for the density of states has been derived by F. Wegner [11, 12]. In addition to a direct application to the physical systems mentioned above, it is useful to consider these correlated random potentials since they provide a new expansion parameter in the theory of localization. In the white noise case the only parameter in which one can expand is  $1/N$ ,

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$N$  being the number of orbitals per site [5, 6, 13]. For the problem considered in this work we have an extra-parameter  $\eta$  which is defined from the ratio of the length, related to the impurity potential, to the magnetic length. In the limits of large and small  $\eta$  the density of states and the conductivity are exactly calculable and they can be expanded in powers of  $1/\eta$  and  $\eta$  respectively. (The white noise limit corresponds to the intermediate case  $\eta = 1$ .) The continuation to values of  $\eta$  smaller than one, though it corresponds to an unphysical situation, can be made formally term by term in the expansion. In fact though it is not apparent at this stage, the  $\eta \rightarrow 0$  limit is related to the  $1/N$  expansion about the white noise limit.

The diffusion constant has been investigated for the white noise random potential by Borel-Padé analysis [4, 7], and by an  $1/N$  expansion [5, 6]. From these new  $\eta$  and  $1/\eta$  expansions, we have obtained the diffusion constant which does not show any indication of the existence of extended states at the band centre. The possibility of expanding both in  $\eta$  and  $1/\eta$  may be useful if one wants to investigate the presence of exponentially small terms such as  $e^{-1/\eta}$  in the theory, suggested from the non-perturbative topological term of Levine *et al.* [1, 3]. Throughout this work we have restricted ourselves to zero temperature and infinite systems. The problem of edge state [14] is thus not considered in this paper.

## 2. Spatially correlated random potential.

The lowest Landau level  $n = 0$  is spanned by the orthogonal wave functions [15]

$$U_{0,m} = (2^{m+1} \pi m!)^{-1/2} z^m \exp\left(-\frac{eB}{4}|z|^2\right) (eB)^{\frac{m+1}{2}}. \quad (2.1)$$

These are the eigenfunctions of the constant magnetic field problem in the gauge in which

$$eA = (eB/2)(-y, x, 0)$$

for the eigenvalue  $\hbar eB/2m$ . In the following we have worked in units in which  $\hbar = eB = 1$ ; we have denoted  $z = x + iy$ .

The one-particle Green function restricted to the lowest Landau level, in the absence of impurities is obtained by a sum over the eigenstates (2.1) and is equal to

$$\left\langle r \left| \frac{1}{E - H} \right| r' \right\rangle = \frac{eB}{2\pi(E - \hbar\omega_c/2)} \times \exp\left\{-\frac{eB}{4}(|z|^2 + |z'|^2) + \frac{eB}{2}z^*z\right\}. \quad (2.2)$$

The random potential due to impurities is assumed

to be Gaussian, with the following characteristics

$$\begin{aligned} \langle V(r) \rangle_{av} &= 0 \\ \langle V(r) V(r') \rangle_{av} &= \frac{w}{\pi d^2} \exp(-|r - r'|^2/d^2). \end{aligned} \quad (2.3)$$

It is the effect of this new length  $d$  that we want to explore. The usual white noise problem corresponds of course to the  $d = 0$  limit.

Let us recall here some of the elements of the white noise problem. The expansion in powers of  $w$  yields a series of graphs. The propagator (2.2) being a Gaussian, any graph may be calculated from determinants of  $p \times p$  matrices, at order  $w^p$ . The generalization to the random potentials (2.3) is very similar: the propagator and the correlation functions of the random potential being Gaussian, a graph is again related to the determinant of a quadratic form. However since the correlation function of the potential is not a  $\delta$ -function, there are twice as many integrations and at order  $w^p$  we have now to consider  $2p \times 2p$  matrices. For instance for the one particle Green function averaged over the impurity potential, the expansion in powers of  $w$  at finite  $d$  leads to

$$\left\langle \left\langle r \left| \frac{1}{E - H} \right| r \right\rangle \right\rangle_{av} = \sum \gamma^{2n+1} \frac{(4\pi w)^n}{d^{2n} \det M} \quad (2.4)$$

with  $\gamma = 1/[2\pi(E - \hbar\omega/2)]$ . For example the matrix for the graph of figure 1 is

$$M = \begin{pmatrix} 1+a & 0 & -a & 0 \\ -1 & 1+a & 0 & -a \\ -a & -1 & 1+a & 0 \\ 0 & -a & -1 & 1+a \end{pmatrix} \quad (2.5)$$

with  $a = 2/d^2$ . The determinant of the above matrix is equal to

$$\det M = 1 + 4/d^2 + 8/d^4 \quad (2.6)$$

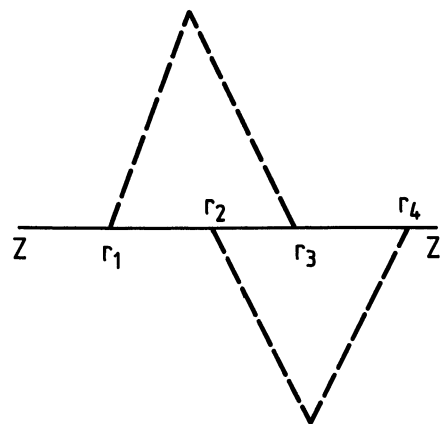


Fig. 1. — An impurity scattering graph of order  $w^2$ .

and we have (with  $\tilde{w} = 2 \pi w$ ) from this graph

$$\left\langle\left\langle r \left| \frac{1}{E - H} \right| r \right\rangle\right\rangle_{\text{av}} = \frac{\tilde{w}^2 \gamma^5}{(d^2/2 + 1)^2 + 1}. \quad (2.7)$$

In fact the combination  $d^2/2 + 1$  always appears in this expansion. Therefore we denote this quantity by  $\eta$  :

$$\eta = d^2/2 + 1. \quad (2.8)$$

In the white noise case  $\eta = 1$ , the determinant may be calculated by a direct inspection of the corresponding graph, since it is known to be equal to the number of Euler trails of the graph [11] (an Euler trail goes through each line of the graph once, in the direction of the arrows). For crossed ladder graphs the number of Euler trails is a polynomial in the number of parallel impurity lines. For instance for the graph of figure 2a,

$$v = \det M = n_1 n_2 + 1. \quad (2.9)$$

For the same graph, with an arbitrary  $\eta$ , an explicit calculation shows that

$$v = [n_1 n_2 + \eta^2] \eta^{n_1 + n_2 - 2}. \quad (2.10)$$

In fact there is a rule which allows one to find the determinant of the graph from the  $\eta = 1$  result immediately : one replaces  $n_i$  by  $n_i/\eta$  and multiply by a power of  $\eta$  related to the order of the graph.

The expansion in powers of  $w$  is an asymptotic series. We follow the approach to the white noise case and perform a Padé-Borel resummation [4] of the series. Then we parametrizes the one particle Green functions as

$$\Gamma_{\pm} = (A_1 \pm iA_2)^{-1} = C e^{\pm i\theta} \quad (2.11a)$$

$$A_1 = 2 \pi \left( E - \frac{\hbar\omega_c}{2} - \frac{1}{2\pi} \Sigma_+^R \right), \quad \Sigma_+^R = \text{Re } \Sigma_+ \quad (2.11b)$$

$$A_2 = 2 \pi \left( \frac{\varepsilon}{2} - \frac{1}{2\pi} \Sigma_+^I \right), \quad \Sigma_+^I = \text{Im } \Sigma_+ \quad (2.11c)$$

An infinitesimal  $\varepsilon$  is introduced in order to distinguish the retarded and advanced Green functions. The angle  $\theta$  is related to the value of the energy. At the band centre  $\theta$  is  $-\pi/2$ .

Calculating all the diagrams up to fourth order in  $w$  we obtain

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$$\begin{aligned} \frac{\pi\varepsilon}{A_2} = & 1 - \frac{x}{\eta} - \frac{x^2}{\eta^2 + 1} \frac{\sin 3\theta}{\sin \theta} - \frac{x^3 \sin 5\theta}{\sin \theta} \left( \frac{1}{\eta^3 + 3\eta} + \frac{3}{\eta^3 + 2\eta} \right) - \\ & - \frac{x^4 \sin 7\theta}{\sin \theta} \left( \frac{4}{\eta^4 + 3\eta^2} + \frac{1}{\eta^4 + 6\eta^2 + 1} + \frac{2}{\eta^4 + 4\eta^2} + \frac{8}{\eta^4 + 4\eta^2 + 1} \right. \\ & \left. + \frac{4}{\eta^4 + 5\eta^2} + \frac{8}{\eta^4 + 3\eta^2 + 1} \right) \end{aligned} \quad (2.12)$$


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in which

$$x = 2 \pi w c^2. \quad (2.13)$$

The two-particle Green function may also be calculated by the same technique. Defining

$$K(q) = \int d^2 r' e^{-iq(r-r')} \left\langle\left\langle r \left| (E - H + i0)^{-1} \right| r' \right\rangle\right\rangle \left\langle\left\langle r' \left| (E - H - i0)^{-1} \right| r \right\rangle\right\rangle_{\text{av}} \quad (2.14)$$

it may be expanded in powers of  $w$ , and since all the integrals are Gaussian one obtains

$$K(q) = \sum_{\text{graphs}} \frac{c}{v} \exp \left( - \left( \frac{v'}{2v} q^2 \right) \right). \quad (2.15)$$

The factor  $v'$  in the exponent of (2.15) is another determinant, again given by a simple counting problem for  $\eta = 1$ , and generalized to arbitrary  $\eta$  by a simple

replacement. For instance, for the graphs of figure 2,

$$v' = \eta^{n_1 + n_2 - 2} [n_1 n_2 + \eta(n_1 + n_2) + \eta^2]. \quad (2.16)$$

From  $K(q)$  one obtains an expansion of the diffusion constant in powers of  $w$

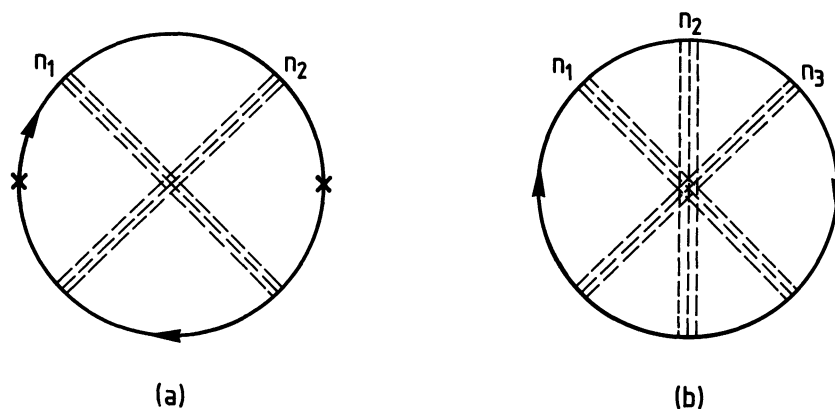
$$\frac{2\pi D}{A_2} = 1 - \sum_{\text{graphs}} \frac{2v - v'}{v^2} x^p e^{i\theta}, \quad (2.17)$$

in which  $p$  is the order of the graph and  $l$  the difference between the numbers of advanced and retarded propagators in the graph. The expansion parameter  $x$  is related to the modulus square of the propagator ; since the imaginary part is the density of states one may also write

$$x = 2 \pi^3 w(\rho^2 + \zeta^2) \quad (2.18)$$

in which

$$\zeta = \frac{1}{\pi} \text{Re } \Gamma_+.$$

Fig. 2. — Graphs of order  $1/N^2$ .

An explicit calculation yields

$$\begin{aligned} \frac{2\pi D}{A_2} = & 1 - \frac{\eta - 1}{\eta^2} x - \left( 2 \frac{\eta^2 - \eta + 1}{(\eta^2 + 1)^2} \cos 2\theta + \frac{(\eta - 1)^2}{(\eta^2 + 1)^2} \right) x^2 - \\ & - x^3 \left\{ \left( 2 \frac{\eta^3 - \eta^2 + 3\eta - 1}{(\eta^3 + 3\eta)^2} + 2 \frac{3\eta^2 - 3\eta^2 + 6\eta - 2}{(\eta^3 + 2\eta)^2} \right) \cos 4\theta \right. \\ & + \left( 2 \frac{\eta^3 - 2\eta^2 + 3\eta}{(\eta^3 + 3\eta)^2} + 2 \frac{3\eta^3 - 6\eta^2 + 6\eta - 2}{(\eta^3 + 2\eta)^2} \right) \cos 2\theta \\ & \left. + \frac{\eta^3 - 3\eta^2 + 3\eta - 1}{(\eta^3 + 3\eta)^2} + \frac{3\eta^3 - 7\eta^2 + 6\eta}{(\eta^3 + 2\eta)^2} \right\} + \dots \quad (2.19) \end{aligned}$$

In the white noise case, these series have been analysed by the Borel-Padé method. At the band centre, the sign of the successive terms of the series are alternating and the procedure leads to accurate values (compared to the exact solution) for the density of states [4, 7]. The same method was then applied to the diffusion constant, or rather the ratio  $D/A_2$  of equation (2.19); this gave a value of about 1.4 and therefore a conductivity  $\sigma$  (namely  $\sigma_{xx}$ ) at the band centre where it has a peak of  $1.4 \frac{e^2}{2\pi^2 \hbar} = 1.74 \times 10^{-5}$  mho.

From the series (2.19) for arbitrary  $\eta$ , one first sees that  $D$  roughly decreases like  $1/\eta$  at large  $\eta$  (as explained in 4). If we take numerical values of  $d \sim 300$  Å, and a cyclotron radius  $l \sim 100$  Å, i.e.  $\eta$  close to five, one finds by Borel-Padé  $\frac{2\pi D}{A_2} \simeq 0.2$  and  $\sigma_{xx} \simeq 2.8 \times 10^{-6}$  mho.

For  $\eta \sim 3$ , one has  $\sigma_{xx} \sim 5.6 \times 10^{-6}$  mho. These estimates should be useful for a comparison with the experimental values of the peak of the conductivity. In the white noise case, there is an excellent agreement between the calculation and the observation in Si-MOSFET [16] which gives  $\sigma_{xx} \simeq 1.7 \times 10^{-5}$  mho.

### 3. $1/N$ and small $\eta$ expansions.

The  $1/N$  expansion of the white noise problem, in which one introduces an internal index  $n = 1, \dots, N$

which labels the atomic orbitals at a given site, is also applicable to a random potential with arbitrary  $\eta$ . This generalization does not involve any new technical procedure and the details of the method may be found in reference [5]. The  $1/N$  expansion has a logarithmic singularity at order  $1/N^2$ . At the band centre one finds

$$\frac{2\pi D}{NA_2} = \frac{1}{\eta} \left( 1 + \frac{\eta^2}{4N^2} \ln \left( \frac{\pi \varepsilon}{NA_2} \right) + o\left(\frac{1}{N^2}\right) \right), \quad (3.1)$$

neglecting constants over  $N^2$  versus  $\ln \varepsilon/N^2$ .

This logarithmic behaviour is similar to that found in a field theoretic approach based on the non-linear  $\sigma$ -model for the unitary group [17-19]. These  $\sigma$ -models lead to a localized behaviour.

The small parameter suggested by equation (3.1) is in fact the ratio  $\eta/N$ . Therefore instead of taking a large  $N$  limit at fixed  $\eta$ , one may consider a small  $\eta$  expansion at fixed  $N$  (at  $N = 1$  for instance). The small  $\eta$  expansion of equations (2.12) and (2.19) may be performed after rescaling  $x$  (i.e.  $w$ ) as

$$x = \eta y. \quad (3.2)$$

In this limit equation (2.12) gives simply  $y = 1$ . Furthermore in the small  $\eta$  limit all the diagrams for  $\Sigma_R = \text{Re } \Sigma_+$  vanish. Hence for small  $\eta$ ,  $A_1 = 2\pi \left( E - \frac{1}{2} \hbar \omega_c \right)$  (see Eq. (2.11)). From  $y = 1$  one

obtains  $A_1^2 + A_2^2 = 2 \pi w \eta$ . The density of states which is proportional to  $A_2 C^2$  is then given by a simple semi-circle law

$$\rho = \left( \frac{\eta}{2 \pi^3 w} \right)^{1/2} (1 - v^2/4)^{1/2} \quad (3.3)$$

in which

$$v = \left( \frac{2 \pi \eta}{w} \right)^{1/2} \left( E - \frac{\hbar \omega_c}{2} \right). \quad (3.4)$$

The small  $\eta$  expansion is similar to the  $1/N$  expansion, and in fact simpler. The factors  $v$  and  $v'$  in (2.15) are simply expanded in powers of  $\eta$ . By this method keeping only logarithmic terms and neglecting constants, one finds

$$\frac{2 \pi D}{A_2} = \frac{1}{\eta} \left( 1 + \frac{\eta^2}{4} \ln \frac{\pi \varepsilon}{A_2} + O(\eta^2) \right). \quad (3.5)$$

Comparing with equation (3.1) one verifies that (3.5) agrees completely with the  $1/N$  expansion at  $N = 1$ .

This logarithmic behaviour is identical to that which was found in the unitary non-linear  $\sigma$ -model, which predicts a localized behaviour at all energies. However non-perturbative effects due to a possible additional topological term [1] would not appear in this small  $\eta$  expansion. Similarly new interaction terms in these  $\sigma$ -models such as Wess-Zumino terms [20, 21] can produce an infra-red stable fixed point and may explain the presence of extended states at the band centre.

If non-perturbative effects, such as  $(\exp - 1/\eta)$  terms, are important, they would of course not be visible in a small  $\eta$  expansion, but they would contribute to the large  $\eta$ -limit which will be considered below.

#### 4. $1/\eta$ expansion.

The large  $\eta$  limit will provide a complementary approach to the localization problem. In the large  $\eta$  limit, with  $w$  and hence  $x$ , proportional to  $\eta$ , one reads from (2.12) and (2.19) that  $\frac{\pi \varepsilon}{A_2}$  and  $\frac{2 \pi D}{A_2}$  have the same expansions in powers of  $x/\eta$  (using identities such as  $\frac{\sin 3 \theta}{\sin \theta} = 2 \cos 2 \theta + 1$ ). This is true to all orders and may be seen either from the fact that the diagrams simplify in the large  $\eta$  limit (the determinants become a simple power of  $\eta$ ) or from a path integral formulation given below. Therefore  $D$  and the conductivity vanish and in the leading large  $\eta$  limit all states are localized. Physically, in this large  $\eta$  limit, there is no more potential gradient and therefore the conductivity should vanish.

The algebra of the large  $\eta$  limit may be done simply by noticing that in the large  $d$  limit, the impurities correlation approach a constant independent of the positions.

$$\langle V(r) V(r') \rangle_{av} \rightarrow \frac{w}{\pi d^2},$$

and the series in powers of  $w$  for the Green's function becomes

$$G = \gamma \sum_0^\infty \left( \frac{2 \pi}{d^2} \gamma^2 \right)^n \frac{\Gamma(2n+1)}{\Gamma(n+1) 2^n} = \frac{\gamma}{i} \int_0^\infty dt \exp \left( it(1+i\varepsilon) - \frac{\pi w}{d^2} t^2 \gamma^2 \right) \quad (4.1)$$

with

$$\gamma^{-1} = 2 \pi \left( E - \frac{1}{2} \hbar \omega_c \right).$$

At higher orders, there are several effects but in particular  $\frac{d^2}{2}$  is replaced  $\eta = \frac{1}{2} d^2 + 1$ .

Taking the imaginary part of (4.1) one obtains a simple Gaussian for the density of states

$$\rho = \frac{1}{2 \pi} \sqrt{\frac{\eta}{w}} e^{-v^2/2} \quad (4.2)$$

with the reduced energy  $v$  of equation (3.4).

It is instructive to compare the exact result of Wegner for the white noise case  $\eta = 1$ , with the average of the small  $\eta$  and large  $\eta$  results (3.3) and (4.2). For instance at the band centre  $v = 0$ , it gives

$$\frac{1}{2} \left( \rho \sqrt{\frac{w}{\eta}} \Big|_{\eta=0} + \rho \sqrt{\frac{w}{\eta}} \Big|_{\eta=\infty} \right) = \frac{1}{2} \left( \frac{1}{\pi \sqrt{2 \pi}} + \frac{1}{2 \pi} \right) = 0.14307, \quad (4.3)$$

whereas the exact result is

$$\rho \sqrt{\frac{w}{\eta}} \Big|_{\eta=1} = \frac{\sqrt{2}}{\pi^2} = 0.14329. \quad (4.4)$$

A comparison for the whole range of energies is depicted on figure 3.

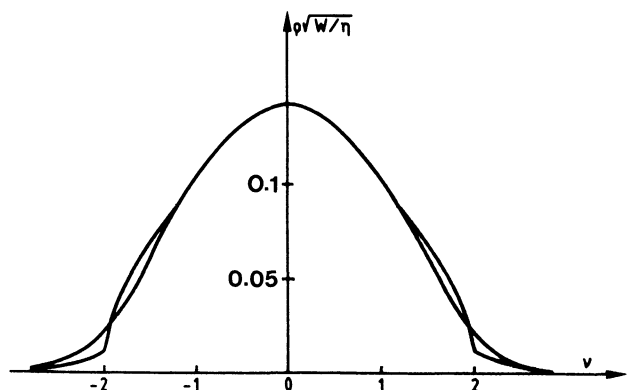


Fig. 3. — A normalized density of state  $\rho \sqrt{w/\eta}$  as a function of  $v = \sqrt{2 \pi \eta/w} (E - \hbar \omega_c/2)$ . A smooth curve represents the exact solution for  $\eta = 1$  and another curve is an average of  $\eta = 0$  and  $\eta = \infty$  solution.

Summing all the diagrams for the density of states in the large  $\eta$  limit was easy (Eq. (4.1)); an algebraic approach is much more convenient if one wants to calculate the conductivity. Let us first consider the

Green's function again and write it as a Gaussian integral over commuting variables, normalized by a Gaussian integral over Grassmannian (i.e. anti-commuting [12, 22]) variables

$$G(r, r') = -i \int \prod_r \left[ \frac{d\varphi(r) d\varphi^*(r)}{\pi} d\Psi(r) d\Psi^*(r) \right] \varphi(r) \varphi^*(r') \times \exp \left\{ i \int d^2r [\varphi^*(r) (E - H) \varphi(r) + \Psi^*(r) (E - H) \Psi(r)] \right\}. \quad (4.5)$$

The averaging over the random potential is easy since

$$\langle \exp -i \int d^2r V(r) u(r) \rangle_{av} = \exp -\frac{1}{2} \int u(r) u(r') W(r - r') d^2r d^2r' \underset{d \rightarrow \infty}{=} \exp -\frac{1}{2} \frac{w}{\pi d^2} \left( \int u(r) d^2r \right)^2 \quad (4.6)$$

One obtains therefore, in this large  $d$  limit,

$$G(r, r) = -i \int_{-\infty}^{+\infty} d\sigma \exp \left( iE\sigma - \frac{w}{2\pi d^2} \sigma^2 \right) \int \Pi \left( \frac{d\varphi d\varphi^*}{\pi} d\Psi d\Psi^* \right) \varphi(r) \varphi^*(r) \delta \left( \sigma - \int d^2r (\varphi^* \varphi + \Psi^* \Psi) \right). \quad (4.7)$$

One still has to express that  $\varphi$  and  $\Psi$  belong to the lowest Landau level, i.e. using the basis (2.1)

$$\begin{aligned} \varphi(r) &= \sum_0^{\infty} c_m U_{0,m}(r) \\ \Psi(r) &= \sum_0^{\infty} b_m U_{0,m}(r) \end{aligned} \quad (4.8)$$

in which the  $c_m$  and  $b_m$  are commuting and anticommuting variables respectively. This leads to the representation

$$\begin{aligned} \int \prod_m \left( \frac{dc_m dc_m^*}{\pi} db_m db_m^* \right) c_m c_m^* \delta \left( \sigma - \sum_n (c_n^* c_n + b_n^* b_n) \right) &= \\ = \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{it\sigma} \int \dots c_m c_m^* \exp \left[ -i(t - i\varepsilon) \sum_n (c_n^* c_n + b_n^* b_n) \right] &= \frac{1}{i} \int_{-\infty}^{\infty} \frac{dt}{2\pi} \frac{e^{it\sigma}}{t - i\varepsilon} = \theta(\sigma). \end{aligned}$$

Therefore

$$G(r, r') = -i \int_0^{\infty} d\sigma \exp \left( iE\sigma - \frac{w}{2\pi d^2} \sigma^2 \right) \sum_{m=0}^{\infty} U_{0,m}(r) U_{0,m}^*(r') \quad (4.9)$$

which coincides with the result (4.1) since

$$P_0(r, r') = \sum_m U_{0m}(r) U_{0m}(r') = \frac{1}{2\pi} \exp \left( -\frac{1}{4} |z|^2 - \frac{1}{4} |z'|^2 + \frac{1}{2} z z' \right). \quad (4.10)$$

The  $1/d^2$  expansion is generated from the representation

$$\langle V(r) V(r') \rangle = \xi + \xi \left( \exp \left[ -\frac{|r - r'|^2}{d^2} \right] - 1 \right) \equiv \xi + \xi f(r - r') \quad (4.11)$$

with

$$\xi = w/\pi d^2$$

by expanding in powers of  $f$ . At first order this leads to ( $\lambda = (2\pi)^2 \xi$ )

$$G(r, r) = \frac{-i}{2\pi} \int_0^{\infty} d\sigma \exp \left( iE\sigma - \frac{\lambda}{2} \sigma^2 \right) \left( 1 + \frac{\lambda \sigma^2}{d^2} + O\left(\frac{1}{d^4}\right) \right) \quad (4.12)$$

or equivalently to the replacement of  $\lambda$  by

$$\lambda' = \lambda \left( 1 - \frac{2}{d^2} \right) \simeq \frac{\lambda}{1 + \frac{2}{d^2}} = \frac{2 \pi w}{\eta}.$$

The two-particle Green's function, obtained by averaging the product of a retarded and of an advanced propagator, is given by the same technique

$$\begin{aligned} K(r - r') &= \left\langle r \left| \frac{1}{\varepsilon_1 - H + i0} \right| r' \right\rangle \left\langle r' \left| \frac{1}{\varepsilon_2 - H - i0} \right| r \right\rangle = \\ &= \int \varphi_1^*(r) \varphi_1(r') \varphi_2^*(r') \varphi_2(r) \exp \left\{ i(\varepsilon_1 \sigma_1 - \varepsilon_2 \sigma_2) - \frac{1}{2} \int w(r - r') (\sigma_1(r) - \sigma_2(r)) (\sigma_1(r') - \sigma_2(r')) \right\} \end{aligned} \quad (4.13)$$

in which

$$\sigma_i(r) = \varphi_i^*(r) \varphi_i(r) + \Psi_i^*(r) \Psi_i(r); \quad i = 1, 2.$$

In the large  $\eta$  limit, the integrals(4.13), after projection(4.8)into the lowest Landau, can easily be performed and yield the result

$$K(q) = e^{-q^2/2} \int_0^\infty d\sigma_1 \int_0^\infty d\sigma_2 \exp \left[ i(\varepsilon_1 \sigma_1 - \varepsilon_2 \sigma_2) - \frac{\lambda}{2} (\sigma_1 - \sigma_2)^2 \right] \quad (4.14)$$

for the Fourier transform of (4.13).

This function does not show any diffusion pole ; indeed a diffusion pole would appear in the small  $q$ , small  $\omega = \varepsilon_1 - \varepsilon_2$  limit as a contribution of the form

$$K(q) = \frac{2 \pi \rho}{Dq^2 - i\omega}. \quad (4.15)$$

A systematic  $1/\eta$  expansion may be obtained by expanding  $w(r - r')$  in powers of  $\lambda f(r - r')$ , defined in equation (4.11)

Starting from (4.13) and representing the fields by the operators  $c_m$  and  $b_m$  of (4.8), one obtains a perturbation series

$$\begin{aligned} K(q, \omega) &= \int_0^\infty d\sigma_1 \int_0^\infty d\sigma_2 \exp \left\{ i\varepsilon(\sigma_1 - \sigma_2) + \frac{i\omega}{2} (\sigma_1 + \sigma_2) - \frac{\lambda'}{2} (\sigma_1 - \sigma_2)^2 \right\} \\ &\left\{ 1 - \frac{\lambda'^2}{24 \eta^2} (\sigma_1 - \sigma_2)^4 + q^2 \left( -\frac{1}{2} - \frac{\lambda'}{2 \eta} \sigma_1 \sigma_2 + \frac{\lambda'^2}{48 \eta^2} (\sigma_1 - \sigma_2)^4 \right) \right. \\ &+ \frac{\lambda'^2}{\eta^3} \left( \frac{3}{8} \sigma_1^2 \sigma_2^2 - \frac{1}{6} \sigma_1^3 \sigma_2 - \frac{1}{6} \sigma_1 \sigma_2^3 \right) + \frac{\lambda'^3}{\eta^3} \left[ \frac{1}{48} (\sigma_1^5 \sigma_2 + \sigma_1 \sigma_2^5) \right. \\ &\left. \left. - \frac{3}{16} (\sigma_1^4 \sigma_2^2 + \sigma_1^2 \sigma_2^4) + \frac{1}{3} \sigma_1^3 \sigma_2^3 \right] + 0(1/\eta^4) \right\}, \end{aligned} \quad (4.16)$$

with

$$\varepsilon = \frac{1}{2} (\varepsilon_1 + \varepsilon_2), \omega = \varepsilon_1 - \varepsilon_2.$$

The coefficient of  $(1/\eta)^n$  is a polynomial of degree  $n$  in  $\lambda' = 2 \pi w/\eta$ . It is thus possible to obtain (4.16) directly from the series (2.19), for the diffusion constant. Indeed using the definition (4.15) of  $D$  and Einstein's relation  $\sigma_{xx} = eD\rho$ , one obtains

$$\begin{aligned} \sigma(\omega) &= \frac{e^2 \omega^2}{4 \pi \hbar} \int_{-\infty}^{+\infty} dt \int_{|t|}^\infty dy \exp \left\{ i\varepsilon t + \frac{i\omega y}{2} - \frac{\lambda'}{2} t^2 \right\} \\ &\left\{ -\frac{1}{2} - \frac{\lambda'}{8 \eta} (y^2 - t^2) + \frac{\lambda'^2}{48 \eta^2} t^4 + \frac{\lambda'^2}{24 \eta^3} \left[ -t^2(y^2 - t^2) + \frac{1}{16} (y^2 - t^2)^2 \right] \right. \\ &\left. + \frac{\lambda'^3}{48 \eta^3} \left[ \frac{1}{4} t^2(y^2 - t^2) - \frac{5}{16} t^2(y^2 - t^2)^2 \right] + 0(1/\eta^4) \right\}. \end{aligned} \quad (4.17)$$



The real part of  $\sigma(\omega)$  is then equal to

$$\begin{aligned} \text{Re } \sigma(\omega) = \frac{\omega^2 e^2}{\pi} \int_0^\infty dt \cos \varepsilon t \exp\left(-\frac{\lambda' t^2}{2}\right) & \left\{ \frac{1}{4} t - \frac{\lambda'}{24 \eta} t^3 + \right. \\ & \left. + \frac{\lambda'^2}{48 \eta^2} t^4 + \frac{1}{\eta^3} \left( -\frac{19}{1440} \lambda'^2 t^5 + \frac{\lambda'^3}{576} t^6 + \frac{\lambda'^3}{576} t^7 \right) + 0\left(\frac{1}{\eta^4}\right) \right\}. \end{aligned} \quad (4.18)$$

The function  $\text{Re } \sigma(\omega)$  is not singular at the band centre ( $\varepsilon = 0$ , and it vanishes as  $\omega^2$  in the small  $\omega$  limit; thus the  $1/\eta$  expansion does not indicate any sign of extended states. At the band centre we obtain

$$\text{Re } \sigma(\omega) = \frac{\omega^2 e^2}{4 \pi \lambda'} \left\{ 1 - \frac{1}{3 \eta} - \frac{4}{45 \eta^3} + \frac{\sqrt{2 \pi \lambda'}}{12 \eta^2} + \frac{\lambda'}{18 \eta^3} + 0(1/\eta^4) \right\}. \quad (4.19)$$

However the imaginary part of  $\sigma(\omega)$  is divergent at small  $\omega$ ; at the band centre for instance

$$\begin{aligned} \text{Im } \sigma(\omega) = -\frac{\omega e^2}{2 \sqrt{2 \pi \lambda'}} & \left\{ 1 - \frac{1}{4} \left( \frac{1}{\eta} + \frac{5}{8 \eta^3} - \frac{5}{32} \lambda'^3 \right) - \right. \\ & \left. - \frac{2 \lambda'}{\omega^2} \left( \frac{1}{\eta} + \frac{1}{4 \eta^3} + \frac{5 \lambda'}{48 \eta^3} \right) - \frac{2 \lambda'^2}{\omega^4 \eta^3} + \frac{5 \lambda'^3}{\omega^4 \eta^3} + 0\left(\frac{1}{\eta^4}\right) \right\}. \end{aligned} \quad (4.20)$$

The divergence of  $\text{Im } \sigma$ , which is already present at order  $1/\eta$ , is a bit difficult to interpret within this framework. Whether it is a true divergence, or a manifestation of the change of the behaviour of  $\text{Im } \sigma(\omega)$  from a linear law in  $\omega$  to some other power is unknown. However, from the perturbation for some particular graphs, we have certainly logarithmic singularities such as  $\ln(\omega^2 \eta)$ . Such logarithmic terms lead to the logarithmic divergence the dielectric constant which is proportional to  $\text{Im } \sigma(\omega)/\omega$  of the small  $\omega$  limit. The divergence in equation (4.20) may be interpreted as the existence of  $\ln(\omega^2 \eta)$  term in.

The density of states may be recovered from equation (4.16) in the limit of small momentum  $q \rightarrow 0$  at fixed  $\omega$ . One obtains, at the band centre,

$$\rho = \frac{1}{2 \pi} \sqrt{\frac{\eta}{w}} \left( 1 - \frac{1}{8 \eta^2} + \frac{1}{384 \eta^4} + 0\left(\frac{1}{\eta^6}\right) \right). \quad (4.21)$$

Putting  $\eta = 1$  in this  $1/\eta$  - expansion, it yields  $\rho \sqrt{w} = 0.140$ , whereas the exact value is  $\frac{1}{\pi^2} \sqrt{2} = 0.143$ .

## 5. Discussion.

We have discussed an arbitrary value of  $\eta$ . For small  $\eta$ , we recover the behaviour for the unitary non linear  $\sigma$  model. For fixed large  $\eta$ , the diffusion constant has still a  $\ln \varepsilon$  behaviour if we restrict ourselves to order  $-1/N^2$  graphs. The introduction of  $\eta$ , therefore, does not change the class of universality. Indeed, for large  $\eta$ , we can replace the summations of the number of impurity lines by integrals in the  $1/N$  expansion.

The large  $\eta$  limit has been discussed before as a classical limit in which an electron moves along an

equipotential line and these equipotentials [23, 24] become extended only at the band centre. In our treatment, the change of  $\eta$  leads to scaling of the conductivity and we have not obtained any indication of extended states within the  $1/N$  expansion for the large  $\eta$  case.

The case of infinite  $\eta$  becomes equivalent to no gradient of the random potential and it leads to vanishing conductivity. By the  $1/\eta$  - expansion, we have obtained a systematic method to calculate the density of states. For the conductivity, we find a divergence in the small  $\omega$  limit. This is interpreted as due to the  $\ln(\omega^2 \eta)$  term in the expression of  $\text{Im } \sigma(\omega)$ . From the renormalization group analysis, in analogy with the treatment of reference [25], this suggests that

$$D \exp \left[ 2 \left( D^2 - \frac{1}{\eta^2} \right) \right] = \omega. \quad (5.1)$$

However, this equation should be corrected due to the existence of a divergent term in equation (4.20). If this divergence is proportional to  $\ln(\omega^2 \eta)$ , then the dielectric constant becomes logarithmically divergent for small  $\omega$ .

In our  $1/\eta$  expansion, we have neglected possible tunnelling terms which would behave like  $e^{-\eta}$  and there is a possibility that the conductivity  $\sigma_{xx}$  behaves like  $e^{-\eta}$  at the band centre. It is interesting to perform a higher order Borel-Padé analysis for the large  $\eta$  case in order to see whether  $\sigma_{xx}$  does decrease as  $e^{-\eta}$ . However, as we have discussed, the change of  $\eta$  is just a change of scale, and we must take into account more diagrams to decide. For infinite  $\eta$ , the system has no-

potential gradient although the density of state has a Gaussian form. In this case like for a free electron gas, under a magnetic field, there is no finite conductivity, but simply a diamagnetic current at the edge of the sample. It is interesting to consider the relation of this diamagnetic current and of the Hall current in the large  $\eta$  case.

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