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► **To cite this version:**

F. Benamira, J.P. Provost, G. Vallée. Separable and non-separable spin glass models. Journal de Physique, 1985, 46 (8), pp.1269-1275. 10.1051/jphys:019850046080126900 . jpa-00210071

**HAL Id: jpa-00210071**

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Submitted on 4 Feb 2008

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# LE JOURNAL DE PHYSIQUE

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*J. Physique* 46 (1985) 1269-1275

AOÛT 1985, PAGE 1269

Classification  
*Physics Abstracts*  
05.50 — 75.50K

## Separable and non-separable spin glass models

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(Reçu le 4 février 1985, accepté le 2 avril 1985)

**Résumé.** — On introduit une famille de modèles qui interpole entre les modèles séparables et le modèle de Sherrington-Kirkpatrick. Ceci permet une meilleure compréhension des différences entre les modèles séparables et non séparables en particulier en ce qui concerne l'extensivité du logarithme de la fonction caractéristique des couplages aléatoires, la brisure de la symétrie des répliques et la nature des paramètres d'ordre. Cette famille contient des modèles « réalistes » comportant des paramètres ajustables susceptibles de mieux rendre compte des résultats expérimentaux que le modèle S.K.

**Abstract.** — A family of models which interpolates between the separable models and the Sherrington-Kirkpatrick (S.K.) model is introduced. This allows a better understanding of the differences between separable and non-separable models, in particular as concerns the extensivity of the logarithm of the characteristic function of the random couplings, the breaking of the replica symmetry and the nature of the order parameters. This family contains true spin glass models with adjustable parameters which might account for the experimental situation better than the S.K. model.

### 1. Introduction.

If one focus attention on the statistical properties of the random coupling constants, the available solvable mean field spin glass models can be divided in two classes. The first class, made of the so-called separable models [1-5] is characterized by coupling constants  $J_{ij}$  which are a finite sum of products over  $i$  and  $j$  of random variables associated with each site  $i$ . These models are exactly solvable without calling for the replica method and possess « natural » order parameters; they retain the experimental fact that the  $N^2$  true random couplings  $V(x_i - x_j)$  ( $V$  being the interaction potential) depend on  $N$  random variables (the positions  $x_i$  of the magnetic impurities) and are

therefore correlated. However they lack a major feature of spin glasses, i.e. the existence of an infinite number of free energy valleys [6]. From a technical point of view, this is reflected by the fact that their solution can be recovered by the symmetric replica method [5, 6]. The models of the second class, on the contrary, exhibit a rich structure of the set of equilibrium states which is thought to be essential to account for the experimental results (failure of ergodicity, large spectrum of relaxation times, etc...). The most famous model in this class is the Sherrington-Kirkpatrick (S.K.) model [7] and the « simplest » [8] one is the random energy model of Derrida [9]; both deal with independent Gaussian random coupling constants. Their solution relies on a special hierarchical scheme of breaking of the replica's symmetry [10] which induces an exotic but physically interesting ultrametric topology in the space of pure states [11].

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It leads to a new type of order parameter whose values are the correlations between replicas and which is interpreted in terms of the overlaps of the pure states [12].

From an experimental and a theoretical point of view it is desirable to dispose of intermediate models which retain the favourable features of both classes. The aim of this paper is precisely to introduce a family of models which interpolates between the separable models and the S.K. model. These models are obtained from separable models by allowing the number of random variables associated with each site  $i$  to go to infinity. One may find a motivation for this extension by looking at the expression of the random couplings in terms of the Fourier transform of the interaction potential

$$V(x_i - x_j) \sim N^{-1} \sum_k \hat{V}(k) e^{ikx_i} e^{-ikx_j}.$$

In the separable models the coupling constants mimic an approximation of  $V$  by a finite sum over  $k$ . The models considered in the following amount to take a number of terms proportional to  $N$  ( $N$  goes to infinity in the thermodynamical limit) in the sum and to replace  $\hat{V}(k)$  by a staircase function with a finite number of values. (Let us recall that, for the R.K.K.Y. potential,  $\hat{V}(k)$  is almost constant for  $k < 2k_F$ .)

In this paper we concentrate on those general features which allow a better understanding of true spin glass models and enlighten the comparison between the S.K. model and the separable ones. The detailed study of specific new models and the discussion of their ability to describe the experimental situation will be made elsewhere. Our main results are the following. First the S.K. model can be recovered as a limit case in a family which also contains other true spin glass models. For such models the logarithm of the characteristic function of the random coupling constants is an extensive quantity (in contrast with the separable models). The Parisi ansatz is applicable to any model of the family. In the case of separable models, this scheme of breaking of the replica symmetry leads to the symmetric solution. For all these models two types of coupled order parameters appear, one of them being the natural order parameter of separable models. These two order parameters coincide only for the S.K. model. Finally we argue that all the models likely to describe true spin glasses have a common critical behaviour but their de Almeida-Thouless line is different from that of the S.K. model and may lie below it in the  $H.T.$  plane.

The paper is organized as follows. In section 2 we introduce the family of models and discuss some examples. In section 3 we derive the coupled equations for the order parameters within the replica method and examine their general structure. Section 4 is devoted to the comparison of the separable and non-separable models. We conclude in section 5 by a discussion on the physical interpretation of the two

types of order parameters. All along the paper the S.K. model and the Van Hemmen (V.H.) model [4] serve as illustrative examples.

## 2. Description of the models.

We consider a family of models with infinite range, 2-spins random interactions of the form

$$H(J, S) = - N^{-1} \sum_{(i,j)} (\xi_i, \mathbb{J} \xi_j) S_i S_j \quad (1)$$

where  $\mathbb{J}$  is a  $p \times p$  symmetric matrix and  $\xi_i$  are  $N$  independent identically distributed Gaussian  $p$ -vectors with zero mean. Through a redefinition of the matrix  $\mathbb{J}$ , the components  $\xi_{i\mu}$  of the vectors  $\xi_i$  may always be made independent with variance one. (Although some calculations could be done without restricting ourselves to a Gaussian probability law, this choice is essential for the obtention of the main results of this paper.)

Hamiltonians of the form (1) have already been considered for various kinds of random variables but only for  $p$  finite ( $p = 1, 2, 3, \dots$ ) [1-5]. The simplest one is the V.H. Hamiltonian whose  $\mathbb{J}$  matrix is

$$\mathbb{J}_{\text{V.H.}} = \begin{bmatrix} 0 & J \\ J & 0 \end{bmatrix}.$$

As already mentioned in the introduction, such models are not able to describe true spin glasses. However, this is no longer the case if one allows  $p$  to go to infinity.

As an illustrative example let us recover the S.K. model in a limit  $p \rightarrow \infty$ . This is best discussed in terms of the characteristic function  $\phi$  of the random coupling constants. Let  $J_{ij} = (\xi_i, \mathbb{J} \xi_j)$  for any  $i$  and  $j$ ; then  $\phi$  is a function of  $2^{-1} N(N+1)$  independent variables  $u_{ij}$  (the elements of a symmetric matrix  $\mathbb{U}$ ):

$$\phi(u) = E_{\xi_i} \left( \exp \left\{ \frac{1}{2} \sum_{ij} u_{ij} J_{ij} \right\} \right). \quad (2)$$

For independent Gaussian variables  $\xi_{i\mu}$ ,  $\phi(u)$  reads [13]:

$$\phi(u) = [\det(\mathbb{1} - \mathbb{U} \otimes \mathbb{J})]^{-\frac{1}{2}} = \prod_{\mu\rho} (1 - u_\rho J_\mu)^{-\frac{1}{2}} \quad (3)$$

where  $u_\rho$  and  $J_\mu$  are the eigenvalues of  $\mathbb{U}$  and  $\mathbb{J}$ .

$\phi(u)$  has to be compared with the characteristic function  $\phi_{\text{S.K.}}(u)$  of the S.K. coupling constants which are independent Gaussian variables with zero mean and variance  $NJ^2$ :

$$\phi_{\text{S.K.}}(u) = \exp \left\{ \frac{NJ^2}{2} \sum_{ij} u_{ij}^2 \right\} = \exp \left\{ \frac{NJ^2}{2} \sum_\rho u_\rho^2 \right\}. \quad (4)$$

It is easy to verify that the equality

$$\prod_\mu (1 - u_\rho J_\mu)^{-\frac{1}{2}} = \exp \left\{ \frac{NJ^2}{2} \sum_\rho u_\rho^2 \right\} \quad (5)$$

is satisfied in the limit  $N \rightarrow \infty$  for  $\mathbb{J}$  matrices of dimension  $N^{2+\varepsilon}$  whose eigenvalues are alternately  $\pm JN^{-\frac{1}{2}(1+\varepsilon)}$ . In full rigor,  $\varepsilon$  must be strictly positive in order to obtain formula (5). However it will become clear further (formula (8) and section 3) that the condition  $\varepsilon > -1$  is sufficient to recover the S.K. model in the thermodynamical limit.

More generally the above calculation suggests that the logarithm of the characteristic function  $\phi$  of the random coupling constants is an extensive quantity for true spin glass models. For future convenience let us introduce the functions

$$f(z) = - (2N)^{-1} \sum_{\mu} \ln(1 - J_{\mu} z) \quad (6)$$

and

$$\varphi(u) = N^{-1} \ln \phi(u) = \text{Tr } f(\mathbb{U}). \quad (7)$$

For the S.K. model  $f$  (or  $\varphi$ ) is a well defined function in the thermodynamical limit :

$$f_{\text{S.K.}}(z) = \lim_{N \rightarrow \infty} - (2N)^{-1} \frac{N^{2+\varepsilon}}{2} \ln \left( 1 - \frac{J^2 z^2}{N^{1+\varepsilon}} \right) = \frac{1}{4} J^2 z^2. \quad (8)$$

But this is also true for a large number of models. A sufficient condition is that all distinct eigenvalues  $J_{\sigma}$  of  $\mathbb{J}$  have a multiplicity proportional to  $N$  and that the sum  $\sum_{\sigma} \ln(1 - J_{\sigma} z)$  be convergent. In particular, the S.K. model appears as an extremal case ( $a = b$  goes to zero) in a two-parameter family of models whose  $\mathbb{J}$  matrices have dimension  $b^{-1} N$  and alternate finite eigenvalues  $\pm a^{1/2} J$  :

$$f_{a,b}(z) = - (4b)^{-1} \ln(1 - aJ^2 z^2). \quad (9)$$

On the contrary, for  $p$  finite,  $f$  (or  $\varphi$ ) tends to zero for  $N$  large. This is the case for the V.H. model :

$$f_{\text{V.H.}}(z) = - (2N)^{-1} \ln(1 - J^2 z^2) \quad (10)$$

which also appears as a limiting case ( $a = 1$ ,  $b = 2^{-1} N$  goes to infinity) in the above two parameters family.

### 3. Determination of the gap equations.

The free energy  $F$  of a spin glass is a non-symmetric double expectation value  $E_{S_i}$  over the spins (sum on the spin configurations) and  $E_{\xi_i}$  over the random coupling constants (quenched average over the disorder also denoted by  $\overline{\quad}$ ) :

$$- \beta F = E_{\xi_i} [\ln E_{S_i} [\exp \{ - \beta H(J, S) \}]] = \overline{\ln Z}. \quad (11)$$

In the replica method, based on the identity  $\overline{\ln Z} = \lim_{n \rightarrow 0} n^{-1} (\overline{Z^n} - 1)$ , one introduces  $n$  copies  $\alpha$  (replicas) of the system and calculates :

$$\overline{Z^n} = E_{\xi_i} E_{S_i^{\alpha}} \left[ \exp \left\{ - \beta \sum_{\alpha=1}^n H(J, S^{\alpha}) \right\} \right] \quad (12)$$

with the hope that the limit  $n \rightarrow 0$  can be properly defined. The replica Hamiltonian  $H_R = \sum_{\alpha} H(J, S^{\alpha})$

being quadratic

$$H_R = - \sum_{\alpha, \mu} (2N)^{-1} J_{\mu} \left( \sum_i \xi_{i\mu} S_i^{\alpha} \right)^2 \quad (13)$$

and the  $\xi_i$ 's Gaussian,  $\overline{Z^n}$  can be obtained by applying twice the wellknown Gaussian transform :

$$\exp ax^2 = E_v [\exp \{ (2a)^{\frac{1}{2}} vx \}]. \quad (14)$$

(In this expression  $E_v$  denotes an expectation value with respect to the Gaussian random variable  $v$  with mean zero and variance one.) Introducing  $n$  Gaussian  $p$ -vectors  $\mathbf{v}_{\alpha}$  one gets successively :

$$\overline{Z^n} = E_{\xi_i} E_{S_i^{\alpha}} E_{\mathbf{v}_{\alpha}} \left[ \exp \left\{ \sum_{\mu i \alpha} \left( \frac{\beta J_{\mu}}{N} \right)^{\frac{1}{2}} \xi_{i\mu} S_i^{\alpha} v_{\alpha\mu} \right\} \right] \quad (15)$$

$$= E_{S_i^{\alpha}} E_{\mathbf{v}_{\alpha}} \left[ \exp \left\{ \frac{\beta}{2N} \sum_{\mu i} J_{\mu} \left( \sum_{\alpha} S_i^{\alpha} v_{\alpha\mu} \right)^2 \right\} \right] \quad (16)$$

$$= E_{\mathbf{v}_{\alpha}} \left[ \left[ E_{S^{\alpha}} \left[ \exp \left\{ \frac{\beta}{2N} \sum_{\alpha\beta} (\mathbf{v}_{\alpha}, \mathbb{J} \mathbf{v}_{\beta}) S^{\alpha} S^{\beta} \right\} \right] \right]^n \right]. \quad (17)$$

At this stage of the calculation it is important to note that  $\overline{Z^n}$  only depends on the replica variables and that the vectors  $\mathbf{v}_{\alpha}$  only appear in (17) through the quantities :

$$J_{\alpha\alpha} = 2^{-1} (\mathbf{v}_{\alpha}, \mathbb{J} \mathbf{v}_{\alpha}); \quad J_{\alpha\beta} = (\mathbf{v}_{\alpha}, \mathbb{J} \mathbf{v}_{\beta}) \quad \alpha \neq \beta. \quad (18)$$

Since the random vectors  $\mathbf{v}_{\alpha}$  and  $\xi_i$  have the same probability law, the  $J_{\alpha\beta}$ 's obey the same statistics as

the original coupling constants  $J_{ij}$  ( $J_{ii}$  included) and contain all the information on the disorder. Therefore it will not be surprising that the characteristic function  $\phi$  already introduced in section 2 emerges from (17).

The expectation values over  $S^{\alpha}$  and  $\mathbf{v}_{\alpha}$  can be disentangled by constraining  $N^{-1} \beta J_{\alpha\beta}$  to equal  $\lambda_{\alpha\beta}$ , with the aid of  $2^{-1} n(n+1)$  Lagrange multi-

pliers  $q_{\alpha\beta}$ . The average over the spins yields the generating function of the correlations between replicas :

$$E_{S^\alpha} \left[ \exp \left\{ \sum_{\alpha \geq \beta} \lambda_{\alpha\beta} S^\alpha S^\beta \right\} \right] = \exp \{ g(\lambda) \} \quad (19)$$

whereas the average over the random vectors  $v_\alpha$ , as expected, leads to :

$$E_{v_\alpha} \left[ \exp \left\{ \frac{\beta}{2} \sum_{\alpha, \beta} q_{\alpha\beta} J_{\alpha\beta} \right\} \right] = \phi(\beta q). \quad (20)$$

Finally one gets

$$\overline{Z}^n = \left( \frac{N}{2i\pi} \right)^{\frac{n(n+1)}{2}} \int_{-i\infty}^{i\infty} \int_{-\infty}^{\infty} \prod_{\alpha \geq \beta} dq_{\alpha\beta} d\lambda_{\alpha\beta} \times \exp \{ -NG(\lambda, q) \} \quad (21)$$

with

$$G(\lambda, q) = \sum_{\alpha \geq \beta} \lambda_{\alpha\beta} q_{\alpha\beta} - \phi(\beta q) - g(\lambda). \quad (22)$$

For  $N$  large, the integral (21) is calculated by the steepest descent method. At the saddle point  $\lambda_{\alpha\beta}$  and  $q_{\alpha\beta}$  ( $\alpha \geq \beta$ ) satisfy the gap equations :

$$\lambda_{\alpha\beta} = \frac{\partial \phi(\beta q)}{\partial q_{\alpha\beta}}; \quad q_{\alpha\beta} = \frac{\partial g(\lambda)}{\partial \lambda_{\alpha\beta}}. \quad (23)$$

In the particular case of Ising spins ( $(S^\alpha)^2 = 1$ ) one should not introduce the variables  $\lambda_{\alpha\alpha}$  and  $q_{\alpha\alpha}$ ; however the correct result is still given by (23), its solution  $q_{\alpha\alpha} = 1$  ensuring that  $\lambda_{\alpha\alpha}$  disappears from (22).

In order to see how formulae (22) and (23) work, let us verify that expression (8) effectively corresponds to the original S.K. model. The function  $\phi(\beta q)$  then reads :

$$\phi(\beta q) = \frac{1}{4} \beta^2 J^2 \text{Tr} \mathbb{Q}^2 = \frac{1}{4} n \beta^2 J^2 + \frac{1}{2} \beta^2 J^2 \sum_{\alpha > \beta} q_{\alpha\beta}^2 \quad (24)$$

and the first equation (23) yields :

$$\lambda_{\alpha\beta} = \beta^2 J^2 q_{\alpha\beta} \quad (\alpha \neq \beta). \quad (25)$$

Putting this expression for  $\lambda_{\alpha\beta}$  in (22) one recovers the familiar expression :

$$G_{\text{S.K.}}(q) = -\frac{1}{4} n \beta^2 J^2 + \frac{1}{4} \beta^2 J^2 \sum_{\alpha \neq \beta} q_{\alpha\beta}^2 - \ln \left[ E_{S^\alpha} \left[ \exp \left\{ \frac{1}{2} \beta^2 J^2 \sum_{\alpha \neq \beta} q_{\alpha\beta} S^\alpha S^\beta \right\} \right] \right] \quad (26)$$

where only the quantities  $q_{\alpha\beta}$  appear.

For the S.K. model, Parisi [10] has proposed to solve the gap equations, in the limit  $n$  going to zero, by an artful parametrization of the matrix  $\mathbb{Q}$  (whose elements are  $q_{\alpha\beta}$ ). Recently the Parisi's ansatz has

been shown to lead to the exact results for the random energy model where both variables  $q_{\alpha\beta}$  and  $\lambda_{\alpha\beta}$  appear [8]. It is a noticeable feature of equations (23) to be also compatible with this ansatz. Indeed, it is known that the  $q_{\alpha\beta}$  given by the second equation (23) are the elements of a Parisi matrix  $\mathbb{Q}$  if the  $\lambda_{\alpha\beta}$ 's are the elements of a Parisi matrix  $\mathbb{A}$ ; in turn, if  $\mathbb{Q}$  is of Parisi's type, the first equation (23) tells us that this is also true for  $\mathbb{A}$  because the matrix  $\mathbb{A} = \beta \phi'(\beta \mathbb{Q})$  is a function of the  $\mathbb{Q}$  matrix and the Parisi matrices form an algebra. This ansatz ensures that the value of  $G(\lambda, q)$  at the saddle point is proportional to  $n$ , which allows us to obtain the free energy per spin  $N^{-1} F$  through the limit  $\beta^{-1} \lim_{n \rightarrow 0} n^{-1} G$ .

#### 4. Comparison of the models.

In order to compare the models with  $p$  finite and those with  $p$  infinite, it is useful to have at one's disposal the expressions of the free energy, of the order parameters, and of the entropy at zero temperature for symmetric replicas. (We discuss further the validity of the replica symmetry in the different cases.) With this hypothesis, equations (22) and (23) read :

$$\lim_{n \rightarrow 0} n^{-1} G = -\frac{1}{2} \lambda(q-1) - [f(\beta(1-q)) + \beta q f'(\beta(1-q))] - E_z [\ln [2 \cosh (z \lambda^{\frac{1}{2}})]] \quad (27)$$

and

$$\lambda = 2 \beta^2 q f''(\beta(1-q)); \quad q = 1 - E_z [\cosh^{-2}(z \lambda^{\frac{1}{2}})]. \quad (28)$$

The critical temperature  $\beta_c^{-1}$  which corresponds to the departure of  $q$  and  $\lambda$  from zero is given by :

$$2 \beta_c^2 f''(\beta_c) = 1. \quad (29)$$

Finally, the value of the entropy at zero temperature is :

$$S_0 = f(z_0) - z_0 f'(z_0) \quad \text{with} \quad \pi z_0^2 f''(z_0) = 1. \quad (30)$$

When the dimension of  $\mathbb{J}$  is finite, one might be tempted to set  $f = 0$  (or  $\phi = 0$ ) in the thermodynamical limit. However, one must realize that the function  $f$  is a finite sum of logarithmic functions which may be singular. Therefore, although  $f''$  is proportional to  $N^{-1}$ , a finite value of  $\lambda$ , in the low temperature phase, can be obtained from (28) by fixing the quantity  $\beta(1-q)$  to a value which corresponds to a singularity of  $f$ ; this fixed value is nothing but the inverse critical temperature  $\beta_c$ .

The fact that, in the low temperature phase,  $\beta(1-q)$  is frozen and  $f$  is singular has several consequences. The freezing of  $\beta(1-q)$  implies that the magnetic susceptibility remains constant. More important is the remark that,  $f$  and  $f'$  being less singular than  $f''$ , their singular behaviour is not sufficient to compensate (in contradistinction to  $f''$ ) the factor  $N^{-1}$  to which they are proportional. It follows that the

function  $f$ , which carries the information on the random coupling constants and is responsible for the existence of a phase transition, does not contribute to the free energy in the thermodynamical limit. The same remark ensures that the entropy at zero temperature  $S_0$  given by (30) is zero. It also explains the validity of the hypothesis of symmetric replicas, as shown in the appendix.

For the special case of the V.H. model the singularity of  $f$  occurs when  $\beta J(1 - q) = 1$ . Below the critical temperature ( $\beta_c J = 1$ ) the actual solution of (28) is such that

$$q = 1 - (\beta J)^{-1}; \quad 1 = E_z[\beta J \cosh^{-2}(z\lambda^{1/2})] \quad (31)$$

and the free energy reads

$$N^{-1} F = (2\beta J)^{-1} \lambda - E_z[\ln [2 \cosh(z\lambda^{1/2})]] \quad (32)$$

(up to terms of order  $N^{-1/2}$ ). Equations (31) and (32) are those obtained in references [4] and [6] for the special case of Gaussian variables  $\xi_{i\mu}$ . The constancy of the magnetic susceptibility in this case has already been noticed [14].

When the dimension of  $\mathbb{J}$  is infinite, one expects the hypothesis of symmetric replicas to be invalid since, in the «replica philosophy», the breakdown of this symmetry is considered to be the signature of true spin glasses. For the S.K. model, de Almeida and Thouless [15] have shown that the symmetric solution is indeed unstable. Such an instability is difficult to prove directly for a general function  $f$  but is strongly suggested by the fact that the entropy at zero temperature  $S_0$  given by (30) can take an unphysical negative value. One can show that it is indeed the case for the two parameter family of models specified by the expression (9) of  $f_{a,b}$ . (For  $a = b$  going to zero one recovers the S.K. value  $S_0 = - (2\pi)^{-1}$ .) The same is true for models whose  $\mathbb{J}$  matrix has a dimension proportional to  $N$  and constant (non alternate) eigenvalues  $J_\mu = J$ . Therefore, the class of models considered in section 2 contains several candidates for the description of true spin glasses.

The detailed study of the relevance of such models to account for the experimental results lies beyond the scope of this paper. Let us simply mention some of their expected properties. Near the critical temperature ( $q_{\alpha\beta}$  small) the expansion of  $\phi(\beta q)$  begins with a term proportional to  $\text{Tr } \mathbb{Q}^2$  for all these models. Their critical behaviour is, therefore, similar to that of the S.K. model. However, the presence of cubic and quartic terms in this expansion will change the de Almeida-Thouless line which for the S.K. model disagrees with the experimental curve [16]. Qualitatively one expects that, for models with a singular function  $f$  and an inverse critical temperature  $\beta_c$  located near a singularity, this line lies below the S.K. one in the  $(H - T)$  plane; the reason is that, loosely speaking, such models lie between the S.K.

model ( $f_{s.k.}$  not singular) and the V.H. model ( $f_{v.h.}$  singular with  $\beta_c$  on the singularity) for which there is no replica symmetry breaking.

### 5. Interpretation of the order parameters $q_{\alpha\beta}$ and $\lambda_{\alpha\beta}$ .

For the models with  $p$  finite,  $q$  and  $\lambda$  play different roles. In the (symmetrical) replica approach  $q$  is determined first, through the singularity of  $f$ , and the ensuing equation for  $\lambda$  is the one that would be obtained by conventional mean field theory. So,  $\lambda$  may appear as the «natural» order parameter. (For example the original V.H. order parameter is  $(\lambda/2 \beta^2 J^2)^{1/2}$ ). On the contrary, for the S.K. model the parameters  $q_{\alpha\beta}$  and  $\lambda_{\alpha\beta}$  are physically equivalent since they are proportional. Let us now look at the general case.

At a formal level, one is struck by the symmetrical role played by the functions  $\phi(\beta q)$  (formulae (7) and (20)) and  $g(\lambda)$  (formula (19)) in the expression of the free energy and by the duality relation between the parameters  $q_{\alpha\beta}$  and  $\lambda_{\alpha\beta}$ . As a function of the  $q$ 's,  $\phi$  contains all the information on the randomness of the coupling constants while, as a function of the  $\lambda$ 's,  $g$  contains all the information on the statistics of the spins. The gap equations (23) couple the  $q$ 's and the  $\lambda$ 's; they show that  $q_{\alpha\beta}$  is the mean value of  $S^\alpha S^\beta$  with respect to the Boltzmann factor  $\exp \left\{ \sum_{\alpha \geq \beta} \lambda_{\alpha\beta} S^\alpha S^\beta \right\}$  and that  $\beta^{-1} \lambda_{\alpha\beta}$  is the mean value of  $N^{-1} J_{\alpha\beta}$  with respect to the Boltzmann factor  $\exp \left\{ \beta \sum_{\alpha \geq \beta} q_{\alpha\beta} J_{\alpha\beta} \right\}$ . This duality also appears in the expression  $\sum_{\alpha \geq \beta} \lambda_{\alpha\beta} q_{\alpha\beta}$  of the internal energy.

At the level of the replica Hamiltonian  $H_R$ , one can interpret  $q_{\alpha\beta}$  and  $\lambda_{\alpha\beta}$  in terms of quenched thermodynamical averages. The interpretation of  $q_{\alpha\beta}$  is well known : by adding to  $H_R$  the quantity  $N^{-1} y \sum_i S_i^\alpha S_i^\beta$  one easily verifies that  $q_{\alpha\beta}$  describes the correlation between replicas :

$$q_{\alpha\beta} = \overline{\left\langle N^{-1} \sum_i S_i^\alpha S_i^\beta \right\rangle_{H_R}} \quad (33)$$

One can obtain the equivalent expression for  $\lambda_{\alpha\beta}$  by adding to  $H_R$  the quantity  $X(\mathbf{u}) = N^{-1} \sum_{i\alpha} S_i^\alpha(\xi_i, \mathbb{J}\mathbf{u}_\alpha)$

which depends on arbitrary vectors  $\mathbf{u}_\alpha$ . It is clear from formula (15) that, for the calculation of  $Z^n(\mathbf{u})$ ,  $v_{\alpha\mu}$  must now be replaced by  $v'_{\alpha\mu} = v_{\alpha\mu} + (N^{-1} \beta^{-1} J_\mu)^{1/2} \mathbf{u}_{\alpha\mu}$  in the expression (20) of  $\phi(\beta q)$ . The measure on the  $v'_{\alpha\mu}$  being of the form

$$\exp \left\{ - \frac{\mathbf{v}'_\alpha{}^2}{2} \right\} \exp \left\{ (N\beta)^{-1/2} (\mathbf{v}'_\alpha, \mathbb{J}^{1/2} \mathbf{u}_\alpha) + \frac{1}{2} (N\beta)^{-1} (\mathbf{u}_\alpha, \mathbb{J}\mathbf{u}_\alpha) \right\}$$

$\phi(\beta q)$  can be rewritten as an expectation value on centred Gaussian vectors (which we again note  $\mathbf{v}_\alpha$ ) :

$$\phi(\beta q) = E_{\mathbf{v}_\alpha} \left[ \exp \left\{ \frac{\beta}{2} \sum_{\alpha\beta} q_{\alpha\beta} (\mathbf{v}_\alpha, \mathbb{J} \mathbf{v}_\beta) + (N\beta)^{-1/2} (\mathbf{v}_\alpha, \mathbb{J}^{1/2} \mathbf{u}_\alpha) + \frac{1}{2} (N\beta)^{-1} (\mathbf{u}_\alpha, \mathbb{J} \mathbf{u}_\alpha) \right\} \right]. \quad (34)$$

Applying the differential operator  $\sum \partial u_{\alpha\mu} \partial u_{\beta\mu}$  to the above expression (34) and to  $\overline{Z^n(\mathbf{u})}^\mu$  and setting  $\mathbf{u} = 0$  one gets :

$$\lambda_{\alpha\beta} = (N^{-1} \beta)^2 \left\langle \sum_{ij} (\xi_i, \mathbb{J}^2 \xi_j) S_i^\alpha S_j^\beta \right\rangle_{H_R} - \delta_{\alpha\beta} N^{-1} \beta \text{Tr } \mathbb{J}. \quad (35)$$

For the models with  $p$  finite the vector  $\mathbf{W} = N^{-1} \sum_i S_i(\mathbb{J} \xi_i)$  is self-averaging and is the natural order parameter ;  $\lambda$  is simply the norm of this vector and the fact that only the norm of  $\mathbf{W}$  appears in the gap equation is due to the Gaussian character of the  $\xi_i$ 's. For the S.K. model one remarks that the quenched average factorizes since, according to formula (25) :

$$\begin{aligned} N^{-1} \overline{\langle (\xi_i, \mathbb{J}^2 \xi_j) S_i^\alpha S_j^\beta \rangle_{H_R}} &= \delta_{ij} J^2 \overline{\langle S_i^\alpha S_j^\beta \rangle} \\ &= N^{-1} \overline{\langle (\xi_i, \mathbb{J}^2 \xi_j) \rangle} \overline{\langle S_i^\alpha S_j^\beta \rangle}. \end{aligned} \quad (36)$$

In the limit  $n \rightarrow 0$ ,  $q_{\alpha\beta}$  and  $\lambda_{\alpha\beta}$  become functions  $q(x)$  and  $\lambda(x)$  on the interval  $[0, 1]$ . In the same way as for the S.K. model, and under the same assumptions (clustering property of the pure states), the derivative  $dx/dq$  can be identified with the probability distribution  $P(q) = \overline{P_j(q)}$  of the overlaps  $q$  between the pure equilibrium states of the system. We do not know whether the function  $\lambda(x)$  has a similar interpretation, for instance in terms of the mean probability distribution of the scalar products  $\mathbf{W}_\alpha \cdot \mathbf{W}_\beta$  of the vectors  $\mathbf{W}_\alpha$  associated with pure states  $\alpha$ .

### Appendix.

Let us show that the replica symmetry breaking scheme of Parisi applied to models with  $p$  finite leads to the symmetrical solution. Due to the hierarchical structure of this scheme it is sufficient to establish this result for the first step (one breaking) ; at this stage, with conventional notations, expression (22) reads :

$$\begin{aligned} \lim_{n \rightarrow 0} n^{-1} G &= \frac{1}{2} [-m\lambda_0 q_0 + (m-1)\lambda_1 q_1] - \frac{\lambda_1}{2} - \frac{1}{m} E_{z_0} [\ln [E_{z_1} [\cosh^m(z_0 \lambda_0^{1/2} + z_1(\lambda_1 - \lambda_0)^{1/2})]]] - \\ &- \frac{1}{m} [f(\beta(1 - mq_0 + (m-1)q_1)) + (m-1)f(\beta(1 - q_1)) + m\beta q_0 f'(\beta(1 - mq_0 + (m-1)q_1))]. \end{aligned} \quad (A.1)$$

Its extremalization with respect to  $q_0$  and  $q_1$  yields :

$$m\lambda_0 = 2m\beta^2 q_0 f''(\beta(1 - mq_0 + (m-1)q_1)) \quad (A.2)$$

$$(m-1)\lambda_1 = 2 \frac{m-1}{m} [\beta f'(\beta(1 - mq_0 + (m-1)q_1)) - \beta f'(\beta(1 - q_1)) + m\beta^2 q_0 f''(\beta(1 - mq_0 + (m-1)q_1))]. \quad (A.3)$$

Setting apart the possibilities  $m = 0$  and  $m = 1$  which correspond to the symmetrical situation as a direct inspection of (A.1) shows, it is clear from (A.2, 3) that the  $\lambda$ 's can remain finite in the thermodynamical limit only if  $f''$  is singular. However since  $f''$  is more singular than  $f'$ , the leading contribution in (A.3) comes from  $f''$  and one gets the symmetrical solution  $\lambda_1 = \lambda_0$  in the limit  $N$  going to infinity.

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