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## Irregular wavy flow due to viscous stratification

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**Résumé.** — En utilisant l'approximation de tension superficielle forte, nous déterminons une équation asymptotique décrivant l'évolution non linéaire d'une interface perturbée dans un écoulement de Poiseuille stratifié en viscosité  $\Phi_\tau + \Phi\Phi_\xi + \Phi_{\xi\xi} + \Phi_{\xi\xi\xi} = 0$ . Bien que pleinement déterministe, cette équation possède des solutions en forme d'ondes quasi-périodiques fluctuant de façon irrégulière.

**Abstract.** — By use of the strong surface-tension approximation, an asymptotic equation is derived to describe the nonlinear evolution of the disturbed interface in viscosity stratified-Poiseuille flow :  $\Phi_\tau + \Phi\Phi_\xi + \Phi_{\xi\xi} + \Phi_{\xi\xi\xi} = 0$ . While fully deterministic, this equation is capable of generating solutions in the form of irregularly self-fluctuating quasi-periodic waves.

### 1. Introduction.

As shown by Yih [1], the plane interface of viscosity-stratified Poiseuille flow is unstable to long-wave perturbation (see also [2-4]). A similar effect of long-wave instability occurs when a film of viscous liquid flows down a vertical (or inclined) wall [5]. As a result of this instability, irregularly self-fluctuating waves develop on the surface of the film [6, 7].

In the strong surface-tension approximation, the equation of the film surface is

$$\Phi_\tau + \Phi\Phi_\xi + \Phi_{\xi\xi} + \Phi_{\xi\xi\xi} = 0 \quad (1)$$

where  $\Phi$  is a suitably scaled perturbation. Equation (1) though fully deterministic, turns out to be capable of generating solutions in the form of irregularly fluctuating quasi-periodic waves [8-14]. This behaviour of the solution is in good agreement with well-known experimental observations of film flow [15-17].

The aim of this communication is to show that a similar phenomenon of interface turbulence should also take place in viscosity-stratified flows.

### 2. Fundamental equations and basic solutions.

Consider two layers of liquid flowing between two parallel plates (Fig. 1). In order to capture the effect of instability in its simplest and purest form, we shall assume that the two layers are of equal thickness  $d$  and equal density  $\rho$ , differing only in their viscosities  $\mu_1$  and  $\mu_2$ . An example of a system of these specifications is motor oil and water, which have similar densities ( $\rho_{1,2} = 0.9; 1 \text{ g/cm}^3$ ), significantly different viscosities ( $\mu_{1,2} = 0.4; 0.01 \text{ poise}$ ) and fairly strong surface tension ( $\sigma \cong 40 \text{ dyne/cm}$ ).

In terms of suitably chosen nondimensional variables, the flow equations can be written as follows :

$$\begin{aligned} (u_1)_x + (v_1)_y &= 0 \\ (u_1)_t + u_1(u_1)_x + v_1(u_1)_y + (p_1)_x &= R^{-1}[(u_1)_{xx} + (u_1)_{yy}] \\ (v_1)_t + u_1(v_1)_x + v_1(v_1)_y + (p_1)_y &= R^{-1}[(v_1)_{xx} + (v_1)_{yy}] \\ \text{for } -1 < y < h(x, t) \end{aligned} \quad (2)$$

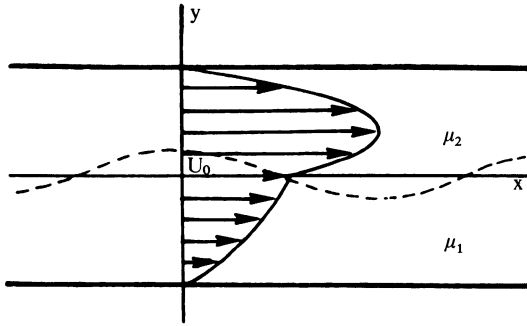


Fig. 1. — Diagram of two-layer Poiseuille flow. The dashed line represents the disturbed interface.

$$\begin{aligned}
 (u_2)_x + (v_2)_y &= 0 \\
 (u_2)_t + u_2(u_2)_x + v_2(u_2)_y + (p_2)_x &= mR^{-1}[(v_2)_{xx} + (u_2)_{yy}] \\
 (v_2)_t + u_2(v_2)_x + v_2(v_2)_y + (p_2)_x &= mR^{-1}[(v_2)_{xx} + (v_2)_{yy}] \\
 \text{for } h(x, t) < y < 1.
 \end{aligned} \tag{3}$$

Here  $u_i, v_i$  are the nondimensional components of the liquid velocity in units of  $U_0$  — the velocity of the undisturbed flow interface (Fig. 1);  $x, y$  are nondimensional space coordinates in units of  $d$ ;  $t$  is non-dimensional time, in units of  $d/U_0$ ;  $p_i$  — is the nondimensional pressure, in units of  $\rho U_0^2$ ;  $h(x, t)$  is the non-dimensional perturbation of the interface, in units of  $d$ ;  $m = \mu_2/\mu_1$ ;  $R = \rho U_0 d/\mu_1$  is the Reynolds number.

At the plates ( $y = \pm 1$ ) we have the non-slip conditions :

$$v_1(x, -1, t) = u_1(x, -1, t) = u_2(x, 1, t) = v_2(x, 1, t) = 0. \tag{4}$$

At the interface ( $y = h(x, t)$ ) the following conditions hold :

Continuity of velocity :

$$u_1 = u_2, \quad v_1 = v_2. \tag{5}$$

Impermeability of the interface :

$$h_t + h_x u_2 - v_2 = 0. \tag{6}$$

Continuity of shearing stress :

$$2 h_x [(v_1)_y - (u_1)_x] + (1 - h_x^2) [(u_1)_y + (v_1)_x] = 2 m h_x [(v_2)_y - (u_2)_x] + m(1 - h_x^2) [(u_2)_y + (v_2)_x]. \tag{7}$$

Discontinuity of normal stress due to surface tension  $\gamma$  :

$$\begin{aligned}
 h_x^2 [-p_1 + 2 R^{-1}(u_1)_x] - 2 R^{-1} h_x [(u_1)_y + (v_1)_x] - p_1 + 2 R^{-1}(v_1)_y &= \\
 = h_x^2 [-p_2 + 2 m R^{-1}(u_2)_x] - 2 m R^{-1} h_x [(u_2)_y + (v_2)_x] - p_2 + 2 m R^{-1}(v_2)_y + \gamma h_{xx} (1 + h_x^2)^{-3/2}
 \end{aligned} \tag{8}$$

where  $\gamma = \sigma/\rho d U_0^2$  is the nondimensional surface tension.

The solution corresponding to undisturbed unidirectional flow is

$$\begin{aligned}
 h &= h^{(0)} = 0, \quad v_i = v_i^{(0)} = 0, \\
 p_i^{(0)} &= -R^{-1}(m+1)x + \text{const.} \\
 u_i^{(0)} &= A_i y^2 + B_i y + 1, \quad i = 1, 2
 \end{aligned} \tag{9}$$

where

$$A_1 = mA_2 = -1/2(m+1), \quad B_1 = mB_2 = -1/2(m-1).$$

We shall assume below that the flow rate  $Q$  is constant, i.e.,

$$\begin{aligned}
 \int_{-1}^h u_1 dy + \int_h^1 u_2 dy &= \int_{-1}^0 u_1^{(0)} dy + \int_0^1 u_2^{(0)} dy = \frac{1 + 14m + m^2}{12m}, \quad \text{i.e.,} \\
 Q &= \left( \frac{1 + 14m + m^2}{12m} \right) U_0 d.
 \end{aligned} \tag{10}$$

**3. Strong surface-tension approximation.**

If the parameter  $\gamma$  is large,

$$\gamma = \varepsilon^{-2} \gg 1 \tag{11}$$

then the structure of the perturbation  $h$ , by analogy with the situation in thin film flow [7], is expected to be as follows :

$$h = h[\varepsilon(x + \beta t), \varepsilon^2 t, \varepsilon]. \tag{12}$$

The phase velocity  $\beta$  is determined in the course of the solution process.

On the basis of (12) we introduce new scaled variables :

$$\zeta = \varepsilon(x + \beta t), \quad \eta = y, \quad s = \varepsilon^2 t. \tag{13}$$

The solution of problem (2-8) is sought in the form of an asymptotic expansion :

$$\begin{aligned} u_i &= u_i^{(0)}(\eta) + \sum_{n=1} \varepsilon^n U_i^{(n)}(\zeta, \eta, s) \\ v_i &= \sum_{n=1} \varepsilon^n V_i^{(n)}(\zeta, \eta, s) \\ p_i &= p_i^{(0)}(\zeta \varepsilon^{-1} - \beta s \varepsilon^{-2}) + \sum_{n=0} \varepsilon^n P_i^{(n)}(\zeta, \eta, s) \\ h &= \sum_{n=1} \varepsilon^n H^{(n)}(\zeta, s). \end{aligned} \tag{14}$$

In the course of the calculations, which are largely similar to those done in [7], it is found that

$$\beta = - \frac{3 m^2 + 10 m + 3}{m^2 + 14 m + 1} \tag{15}$$

$$\begin{aligned} \frac{\partial H^{(1)}}{\partial s} - \frac{16(m+1)(m-1)^3}{(m^2+14m+1)^2} H^{(1)} \frac{\partial H^{(1)}}{\partial \zeta} + \\ + \frac{R(m+1)(m-1)^2(7m^6 - 226m^5 + 4169m^4 + 6436m^3 + 4169m^2 - 226m + 7)}{1680m^2(m^2+14m+1)^3} \frac{\partial^2 H^{(1)}}{\partial \zeta^2} + \\ + \frac{R(m+1)}{3(m^2+14m+1)} \frac{\partial^4 H^{(1)}}{\partial \zeta^4} = 0. \end{aligned} \tag{16}$$

If  $m \simeq 1$ , the relationship (15) and equation (16) simplify to

$$\beta = -1 \tag{17}$$

$$\frac{\partial H^{(1)}}{\partial s} - \frac{(m-1)^3}{8} H^{(1)} \frac{\partial H^{(1)}}{\partial \zeta} + \frac{R(m-1)^2}{240} \frac{\partial^2 H^{(1)}}{\partial \zeta^2} + \frac{R}{24} \frac{\partial^4 H^{(1)}}{\partial \zeta^4} = 0. \tag{18}$$

Hence, the dispersion relation corresponding to linear analysis of the stability of the trivial solution  $H^{(1)} = 0$  is

$$\omega = \frac{R(m-1)^2}{240} k^2 - \frac{R}{24} k^4 \quad (H^{(1)} \sim \exp(\omega s + ik\zeta)). \tag{19}$$

The fact that the coefficient of the third term in equation (19) is positive implies long-wave instability of the interface due to viscous stratification ( $m \neq 1$ ). The positive coefficient of the fourth term ensures relaxation of the short-wave perturbations due to surface tension. Elementary rescaling brings equations (16), (18) to the parameter-free form (1).

Thus, it is indeed true that viscous stratification can generate an effect of interface turbulence similar to that observed in thin-film flow. It would be interesting

to set up an experiment demonstrating this phenomenon under laboratory conditions.

In conclusion, we would like to mention the recent paper of Michelson [18] on the time-independent solutions of equation (1). The analytical and numerical investigation undertaken in that paper shows that the set of time-independent solutions of this equation is remarkably rich. Along with periodic solutions, one has an infinite set of quasi-periodic and chaotic solutions corresponding to a disturbed interface. It was

previously thought that the turbulence in equation (1) is non-steady in nature and is generated by the interaction of several space modes. In the light of Michelson's results, it is quite possible that this turbulence is related to the set of steady chaotic solutions of equation (1) which is an attractor for the time-dependent problem.

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