



**HAL**  
open science

# Mean-field and spin-rotation phenomena in fermi systems : the relation between the leggett-rice and lhuillier-laloë effects

K. Miyake, W.J. Mullin, P.C.E. Stamp

► **To cite this version:**

K. Miyake, W.J. Mullin, P.C.E. Stamp. Mean-field and spin-rotation phenomena in fermi systems : the relation between the leggett-rice and lhuillier-laloë effects. *Journal de Physique*, 1985, 46 (5), pp.663-671. 10.1051/jphys:01985004605066300 . jpa-00210006

**HAL Id: jpa-00210006**

**<https://hal.science/jpa-00210006>**

Submitted on 4 Feb 2008

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Classification

Physics Abstracts

67.50 — 67.20 — 51.60 — 05.60

## Mean-field and spin-rotation phenomena in Fermi systems : the relation between the Leggett-Rice and Lhuillier-Laloë effects

K. Miyake(\*), W. J. Mullin and P. C. E. Stamp(†)

Laboratory for Low Temperature Physics, Hasbrouck Laboratory, University of Massachusetts, Amherst, MA 01003, U.S.A.

(Reçu le 5 novembre 1984, accepté le 24 janvier 1985)

**Résumé.** — On montre que le terme responsable des effets de rotation des spins identiques dans l'intégrale de collision de la théorie de Lhuillier et Laloë (LL) d'un gaz quantique non dégénéré est équivalent, dans la limite des systèmes dilués, au terme de précession dans le « champ moléculaire » qui donne lieu à l'effet Leggett-Rice (LR) dans un système de Fermi dégénéré. Cette équivalence est établie (a) en considérant le terme de rotation des spins à basses températures et (b) en obtenant des équations hydrodynamiques valides à toute température à partir de l'équation de Landau-Silin dans l'approximation de l'onde 1. Le facteur  $\mu$  de rotation des spins obtenu par (b) est en accord avec la valeur de LR à basse température et de LL à hautes températures. La constante de diffusion  $D_0$  ainsi obtenue possède le comportement correct à basse température, y compris le facteur standard de correction de champ moyen ; à hautes températures,  $D_0$  a la forme donnée par LL, multipliée par un facteur de correction de champ moyen. L'importance de ce facteur est mise en lumière par le fait qu'il donne le deuxième coefficient du viriel pour la pression.

**Abstract.** — The term in the Boltzmann equation collision integral causing identical-particle spin rotation in the Lhuillier-Laloë (LL) theory of a non-degenerate quantum gas is shown to be equivalent in the dilute limit to the molecular field precessional term giving rise to the Leggett-Rice (LR) effect in a degenerate Fermi system. This equivalence is shown (a) by considering the spin-rotation term at low temperatures and (b) by deriving hydrodynamic equations valid for all temperatures from the Landau-Silin equation in the *s*-wave approximation. The spin-rotation factor  $\mu$  resulting from (b) is found to agree with the LR value at low temperatures and with the LL value at high temperatures. The diffusion constant  $D_0$  that results has the proper low temperature behaviour, including the standard mean-field correction factor ; at high temperatures  $D_0$  has the LL form times *a* the mean-field correction factor. The importance of the missing mean-field term is illustrated by showing that it gives rise to second virial corrections to the pressure.

### 1. Introduction.

Considerable study of polarized hydrogen and helium systems has been stimulated by the description by Lhuillier and Laloë [1] (LL) of spin-rotation phenomena in Boltzmann gases. Such effects, which are the result of two-particle exchange scattering events, provide a theoretical background for recent observations of spin waves [2] in H<sub>2</sub>, relaxation times [3] in gaseous <sup>3</sup>He↑, and spin diffusion [4] in <sup>3</sup>He↑-liquid <sup>4</sup>He mixtures.

Around 1970, Leggett and Rice [5, 6] (LR) predicted that spin-echo measurements of diffusion would show molecular-field effects. The hydrodynamic equations derived by LL for the magnetization and spin currents of Boltzmann gases are identical in form to those derived from Landau theory by LR for strongly interacting degenerate Fermi fluids.

The relationship between spin-rotation and molecular-field effects has remained somewhat nebulous although the analogies between them have led to a widespread feeling that they must be closely related. Here we attempt to make explicit the relationship. We find that in dilute systems the two effects do indeed have the same origin. For dense systems such a clear statement cannot be made. Before discussing

(\*) Present address : Department of Physics, Nagoya University, Nagoya 464, Japan.

(†) Present address : Department of Physics, Univ. of Lancaster, Lancaster LA1 4YB, U.K.

how we approach this problem, we review some of the details of the two phenomena.

Lhuillier and Laloë [1] investigated the scattering of two quantum particles and the implications that their results have on the collision integral in the Boltzmann equation. After a particle of polarization  $S$  has been scattered by a target  $S'$ , the polarization of the transmitted wave is found to have precessed about the constant direction of  $S + S'$ . This phenomenon occurs even when the interaction potential is spin independent; it is a purely quantum effect arising mainly from the interference of the transmitted wave and the exchange portion of the scattered wave in the forward direction.

If the  $s$ -wave scattering length is  $a$  and the de Broglie wavelength of the particles is  $\lambda$ , then the cross section for this interference scattering is  $\tau \sim \lambda a$ . For  $\tau$  to be important relative to the other scattering processes of cross section  $\sigma \sim a^2$ , one must have  $\tau/\sigma \sim \lambda/a > 1$ . This condition implies having low temperature. However, since these results are used in conjunction with an analysis of transport *via* the kinetic equation for a Boltzmann gas, the average density  $n$  must be small enough for the validity of that description. It turns out that there is a fairly wide experimental window such that these conditions  $a < \lambda < n^{-1/3}$  are satisfied.

The transport properties of a non-degenerate gas depend crucially on two-body scattering which enters into the collision integral of the Boltzmann kinetic equation. One can see intuitively that, in a spin-echo or spin-wave experiment, spin-rotation scattering is important because spins will approach each other while polarized along different axes. In the spin-echo experiment magnetization in the  $z$ -direction is tipped by an angle  $\phi$ . A field gradient along  $z$  causes a helical spin pattern. Spins from different regions diffuse together and collide while canted with respect to one another. Spin rotation then affects the rate at which magnetization is transported through the gas. Spin waves can be set up by the spin-echo procedure with a small tipping angle. They propagate *via* the spin-rotation process.

Hydrodynamic equations describing the evolution of the spin current  $\mathbf{J}$  and the magnetization density  $\mathbf{M}$  of the gas have been derived by LL from the kinetic equation. Spin rotation introduces a term  $\mu \mathbf{J} \times \mathbf{M}$  into the equation for  $\partial \mathbf{J} / \partial t$ . In a spin-echo experiment this term results in a precession of the spin current about the magnetization leading to a diminution of the effective diffusion constant and a phase shift of the echo. These effects have been observed [4] in experiments on dilute solutions of  $^3\text{He}$  in superfluid  $^4\text{He}$  above the degeneracy temperature of the  $^3\text{He}$  system.

The term  $\mu \mathbf{J} \times \mathbf{M}$  also leads to the coherent spin oscillations originally predicted by Bashkin [7] and then by LL and further examined theoretically by Levy and Ruckenstein [8]. These spin waves have been detected [2] in polarized atomic hydrogen by observing the free induction decay. In another experiment

a coupling of the spin waves to the transverse magnetization leads to a frequency shift of the spin-spin relaxation time. This latter phenomena has been observed [3] in  $^3\text{He}$  gas polarized by optical pumping techniques.

Note that LL have predicted that spin-rotation can occur in either Fermi or Bose systems; the effect has now been observed in both of these.

The coupled hydrodynamic equations for  $\mathbf{J}$  and  $\mathbf{M}$  derived by LL have exactly the same form as those derived by LR for a degenerate Fermi fluid. This latter derivation was based on the Landau-Silin (LS) equation [9-11] for Fermi liquids. The implications of this theory for a spin-echo experiment included, of course, the impeded diffusion and the spin-echo phase shift mentioned above. Spin waves had been predicted earlier by Silin [9]. The basic physics of these processes in degenerate fluids involves the precession of a spin about the « molecular field » created by the background fluid through which the spin moves. The molecular field effects appear in the drift term of the LS equation in contrast to the non-degenerate case discussed above in which spin rotation terms originate from the collision integral.

The change in diffusion constant predicted by LR was observed [12] in experiments on liquid  $^3\text{He}$  well below the Fermi temperature. The molecular field parameter to which the  $\mu$  of LL is analogous depends on the Landau parameters  $F_0^a$  and  $F_1^a$ . That experiment provided the first determination of  $F_1^a$ . Spin waves in the degenerate regime have been observed in electron systems [13] and more recently in liquid  $^3\text{He}$  [14] and in  $^3\text{He}$ - $^4\text{He}$  mixtures [15].

The problem of sorting out the relationship between spin-rotation and molecular-field effects is approached here in two ways. First, in section 2, we consider the low-temperature limit of the spin-rotation term in the LL collision integral and compare it with the appropriate term in the Landau-Silin drift term. These are found to be identical in the dilute limit.

The second approach (Sect. 3) is to consider the applicability of the Landau-Silin kinetic equation, in the dilute or  $s$ -wave limit, at arbitrary temperatures. By use of variational and relaxation-time approximations we are able to derive hydrodynamic equations for  $\mathbf{J}$  and  $\mathbf{M}$  valid for all temperature ranges. The equations and the expressions derived from them for  $\mu$  and for the longitudinal diffusion constant  $D_0$  are found to agree with those of LR in the degenerate regime and with those of LL in the Boltzmann statistics limit. This result verifies the correspondence, in the  $s$ -wave limit, of the physical origins of the two limits. We find, however, that there is a numerically small correction to  $D_0$  that occurs as expected in the degenerate limit, but that persists unexpectedly into the Boltzmann limit. This effect arises from a mean-field term in the Landau-Silin equation *other* than the spin-precession terms. An analogous term is missing in the LL formalism.

For temperatures below the Boltzmann limit we find statistical corrections to  $D_0$  that appear in the form of a virial expansion. This result suggests that we study the pressure as it can be derived from the Landau-Silin kinetic equation. This investigation allows us to see the importance of the « other » molecular-field term — without it the second virial coefficient would be incorrect. This work is reported in the Appendix.

Some of the points we make in this paper seem somewhat implicit in the work of Levy and Ruckenstein [8]. Meyerovich [16], on the other hand, explicitly applies the Landau-Silin equation to the dilute Boltzmann limit. However, our work goes a bit beyond either of those applications in its point of view and in deriving results valid for all temperatures.

Before launching into the details of our discussion it is useful to make some clarifications relative to three of the length scales occurring in the discussion. These are the average interparticle spacing  $r_0 = n^{-1/3}$ ; the thermal wavelength  $\lambda$ ; and the potential range,  $a$ .

The non-degenerate state is described by Boltzmann statistics and occurs when  $\lambda \ll r_0$ . This state can be further subdivided depending on whether  $\lambda$  is larger or smaller than  $a$ . In the first case pair scatterings of identical particles must be described quantum mechanically and exchange effects (leading to spin-rotation, for example) are important. This is the situation studied by LL. If  $\lambda \ll a$  then the gas is completely classical.

In the degenerate regime  $\lambda \gg r_0$  and the de Broglie wavelength becomes, for Fermions, the inverse Fermi wave vector  $k_F^{-1} \sim r_0$ . Within this state we can distinguish dilute and dense Fermi systems depending on the size of  $k_F a$ .

It is easy to become confused by these length scales when considering partial wave expansions. Such expansions are generally series in  $ka$ . For the Boltzmann gas  $k \sim \lambda^{-1}$  and many partial waves may be needed to describe a simple two-body interaction. In this limit the importance of many-body effects depends on other parameters, namely  $\lambda/r_0$  or  $a/r_0$ . By contrast, in a degenerate Fermi system the partial wave expansion is a series in  $k_F a \sim a/r_0$ . If a dilute system is under consideration so that many-body effects are negligible then it makes no sense to go beyond  $s$ -wave scattering terms because many-body effects would be as important as the higher-order partial waves.

## 2. Low-temperature limit of the spin-rotation term.

As an introduction to the subject of spin-rotation we consider here both the LL and LR formalisms and show how the precessional terms of each correspond at the kinetic equation level. We do this by considering the low-temperature limit of the LL term.

The Boltzmann equation satisfied by the operator  $\rho(\mathbf{r}, \mathbf{p})$  (the Wigner transform of the one-atom density operator — a  $2 \times 2$  matrix in the spin — 1/2 case) has been shown by LL [1] to be

$$\begin{aligned} \frac{\partial \rho}{\partial t}(\mathbf{r}, \mathbf{p}) + \frac{1}{m} \mathbf{p} \cdot \nabla_r \rho(\mathbf{r}, \mathbf{p}) = & \int d\mathbf{q}' v_r \int d^2 \hat{\mathbf{q}} \left\{ \sigma_k(\theta) [n(\mathbf{p}'_2) \rho(\mathbf{p}'_1) - n(\mathbf{p}_2) \rho(\mathbf{p}_1)] + \right. \\ & + \frac{\varepsilon}{2} \sigma_k^{\text{ex}}(\theta) [[\rho(\mathbf{p}'_1), \rho(\mathbf{p}'_2)]_+ - [\rho(\mathbf{p}_1), \rho(\mathbf{p}_2)]_+] \\ & \left. + i \frac{\varepsilon}{2} \tau_k^{\text{ex}}[\rho(\mathbf{p}'_1), \rho(\mathbf{p}'_2)] \right\} - \frac{i\varepsilon}{2} \int d\mathbf{q} \frac{q}{m} \tau_{\text{fwd}}^{\text{ex}}(k) [\rho(\mathbf{p}), \rho(\mathbf{p} - \mathbf{q})]. \quad (1) \end{aligned}$$

The parameters  $\mathbf{r}, \mathbf{p}$  are position and momentum variables, respectively, throughout the gas;  $n(\mathbf{r}, \mathbf{p}) = \text{Tr}_{\text{spin}} \rho(\mathbf{r}, \mathbf{p})$ ;  $\varepsilon = \pm 1$  for Bosons and Fermions, respectively. We refer the reader to reference [1] for the definitions of the various momenta and the cross sections.

If we consider the case where  $\rho$  can be diagonalized in the same spin basis for all  $\mathbf{r}$ , then the only terms surviving in (1) are those containing the standard cross sections  $\sigma_k$  and  $\sigma_k^{\text{ex}}$ . However, the term we are most concerned with here is the last, containing  $\tau_{\text{fwd}}^{\text{ex}}(k)$ . This cross section, in which

$$k = q/2\hbar, \quad (2)$$

can be expressed in terms of phase shifts  $\delta_l$  as shown by LL :

$$\tau_{\text{fwd}}^{\text{ex}} = \text{Re} \frac{4\pi}{k^2} \sum_l (-1)^l (2l+1) e^{i\delta_l} \sin \delta_l. \quad (3)$$

In the low-energy (low-temperature) limit

$$\tau_{\text{fwd}}^{\text{ex}} \cong \frac{4\pi}{k^2} \delta_0 = -\frac{4\pi a}{k} \quad (4)$$

where  $a$  is the  $s$ -wave scattering length.

To see the relationship between (1) and the equations of Leggett [6], we multiply (1) by a Pauli matrix  $\sigma$  and take the spin trace to get an equation for  $\sigma_p = \text{Tr}_s(\sigma \rho(r, p))$ . This calculation has been carried out by LL who find

$$\frac{\partial \sigma_p}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla_r \sigma_p - \frac{\varepsilon}{2} \int d\mathbf{q} \frac{q}{m} \tau_{\text{fwd}}^{\text{ex}}(k) \sigma_p \times \sigma_{p-q} = \mathbf{I}_p \quad (5)$$

where  $\mathbf{I}_p$  is the trace of  $\sigma$  times all the terms in the collision integral other than the one in  $\tau_{\text{fwd}}^{\text{ex}}$  which has now been transferred to the left side of equation (5).

The LS equation for the same quantity, but in the degenerate limit, and as written by Leggett [6], is

$$\frac{\partial \boldsymbol{\sigma}_p}{\partial t} + \sum_{i=xyz} \left\{ v_{pi} \frac{\partial \boldsymbol{\sigma}_p}{\partial r_i} - \frac{\partial \mathbf{h}_p}{\partial r_i} \frac{\partial n_p}{\partial p_i} \right\} + \frac{2}{\hbar} \boldsymbol{\sigma}_p \times \mathbf{h}_p = \mathbf{I}_p \quad (6)$$

where the effective magnetic field is

$$\mathbf{h}_p = -\frac{1}{2} \hbar \gamma \mathbf{H} + \int d\mathbf{p}' f_{pp'}^a \boldsymbol{\sigma}_{p'} \quad (7)$$

with  $\gamma$  the gyromagnetic ratio and  $\mathbf{H}$  the external magnetic field. The second term involving  $f_{pp'}^a$ , which is the coefficient of  $\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}'$  in Landau's interaction function, is the « molecular field ». If we take  $\mathbf{H} = 0$  as in LL, then the last term on the left side of equation (6) is

$$\frac{2}{\hbar} \int d\mathbf{p}' f_{pp'}^a \boldsymbol{\sigma}_p \times \boldsymbol{\sigma}_{p'} = \frac{2}{\hbar} \int d\mathbf{q} f_{p,p-q}^a \boldsymbol{\sigma}_p \times \boldsymbol{\sigma}_{p-q} \quad (8)$$

This is seen to have the same form as the spin-rotation term of equation (5) and will be identical to it if

$$f_{p,p'}^a = \frac{\hbar |\mathbf{p} - \mathbf{p}'|}{4m} \tau_{fwd}^{\text{ex}}(\mathbf{p} - \mathbf{p}') \quad (9)$$

where we have taken  $\varepsilon = -1$  for Fermions. In the dilute low-temperature limit, with equations (2) and (4), equation (9) becomes

$$f_{p,p'}^a = -\frac{2\pi\hbar^2 a}{m} \quad (10)$$

Expression (10) is indeed the appropriate relation for the antisymmetric part of the Landau interaction function in the dilute limit [17]. Note that, even if the polarization is not small, the function  $f_{pp'}^a$ , defined by equation (7) will remain the coefficient of  $\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}$  in the Landau Fermi liquid interaction function [18].

These results make it reasonable to suppose that, in the dilute limit, the spin-rotation effect of LL and the molecular-field precession term in the Leggett-Rice effect arise from precisely the same physical principles.

### 3. The Landau-Silin equation at arbitrary temperature.

In order to further investigate the relationship between the LR formalism for degenerate systems and the LL relations for Boltzmann gases, we develop analogous relations valid, in the  $s$ -wave limit, for all temperatures. Our approach is based on the Landau-Silin kinetic equation. In the dilute case there is no reason to believe that this kinetic equation should be restricted in validity to the degenerate regime [16]. Using this fact we are able to see how the degenerate treatment goes continuously over into the non-degenerate limit

and how completely or not the results agree with those of the LL Boltzmann equation.

Three points are relevant here :

(a) If we go beyond low  $T$ , Fermi liquid theory in general and the Landau-Silin equation in particular are not valid for an arbitrarily dense fluid. For the dilute system, however, particles and quasiparticles are identical and finite lifetime effects are not a problem. Dilution in the degenerate regime limits us to considering only  $s$ -wave interactions as mentioned in the introduction.

(b) If a transverse (spin-echo) experiment is carried out on a highly polarized system, even at very low temperature, quasiparticles will find themselves far from the Fermi surface and Fermi liquid theory will not provide a valid description except, again, in the dilute case [19]. (As an example, think of the situation in a system having a large up-spin Fermi sphere and a small down-spin sphere when a few of the up spins are tipped over.)

(c) It follows that a continuous connection between the spin-rotation phenomena of LL and the mean-field effects of LR cannot be made at any level beyond the  $s$ -wave interaction approximation. To go beyond the  $s$ -wave approximation to the interaction function  $f_{pp'}$  in the degenerate state is to include many-body effects [17] as discussed in section 1. But the Boltzmann equation approach of LL is by nature a strictly two-body theory. Although at high temperatures two-body scatterings may be described properly by many partial waves, keeping any beyond the  $s$ -wave in making a continuous transition of the theory to very low  $T$  would be inconsistent.

To proceed we take equation (6) to be valid at all temperatures in the dilute limit. The basic assumption involved in deriving equation (6) is that third or higher order terms in  $\delta n$ ,  $\nabla_p$ , or  $\boldsymbol{\sigma}_p$  are negligible [6, 11]. The discussion is limited by this assumption to low polarization. To go beyond such a restriction coupled equations for the distribution function  $n_p$  and  $\boldsymbol{\sigma}_p$  would have to be solved [6, 9]. As equation (6) stands, the distribution function  $n_p$  is the equilibrium Fermi function.

Upon assumption of the  $s$ -wave relationship, equation (10), for  $f_{pp'}^a$  we find for the effective field equation (7)

$$\mathbf{h}_p = -1/2 \hbar \gamma \mathbf{H} - fn\mathbf{M} \quad (11)$$

where  $n$  is the density,  $f$  is the interaction constant given by

$$f \equiv \frac{2\pi\hbar^2 a}{m}, \quad (12)$$

and

$$\mathbf{M}(\mathbf{r}, t) = \frac{1}{n} \int d\mathbf{p}' \boldsymbol{\sigma}_{p'}(\mathbf{r}, t) \quad (13)$$

is the polarization.  $\mathbf{M}$  is defined to range from  $-1$  to  $+1$ . In general, we have

$$\frac{\partial n_p}{\partial p_i} = \frac{\partial n_p}{\partial \varepsilon_p} v_{pi} \quad (14)$$

with  $v_{pi} = \partial \varepsilon_p / \partial p_i$ , the particle velocity. Putting these results in equation (6), and neglecting the small effect of the field gradient [6], we find

$$\frac{\partial \sigma_p}{\partial t} + \sum_i v_{pi} \frac{\partial}{\partial r_i} \left[ \sigma_p + fn\mathbf{M} \frac{\partial n_p}{\partial \varepsilon_p} \right] - \gamma \sigma_p \times \mathbf{H} - \frac{2}{\hbar} f \sigma_p \times \mathbf{M} = \mathbf{I}_p. \quad (15)$$

The last term on the left side of (15) is the molecular field precessional term which, as we have seen in section 2, is the same as the spin-rotation term of LL. The remaining drift terms in equation (15) do not exactly correspond to those of LL. The difference is the other molecular field effect (the second term in square brackets of Eq. (15)). We will follow this term through the derivation of the spin-hydrodynamic equations.

First integrate equation (15) over momentum to find

$$\frac{\partial \mathbf{M}}{\partial t} + \sum_i \frac{\partial}{\partial r_i} J_i(\mathbf{M}) - \gamma \mathbf{M} \times \mathbf{H} = 0 \quad (16)$$

where

$$J_i(\mathbf{M}) = \frac{1}{n} \int d\mathbf{p} v_{pi} \sigma_p \quad (17)$$

is the  $i$ th component of the spin current carrying polarization  $\mathbf{M}$  (the notation is such that, for example,  $J_x(M_x)$  is that part of the current of  $x$ -polarization, travelling in the  $x$ -direction). Equation (16) is just the equation of continuity for a spin current and is identical with the LL and LR results.

To find an equation for the time dependence of  $J$  we make a variational approximation. We take

$$\sigma_p = -\alpha \frac{\partial n_p}{\partial \varepsilon_p} \left( \mathbf{M} + \sum_l \hat{\mathbf{x}}_l C_l \cdot \mathbf{p} \right) \quad (18)$$

where  $\hat{\mathbf{x}}_l$  are the Cartesian unit vectors  $\hat{\mathbf{x}}$ ,  $\hat{\mathbf{y}}$ , and  $\hat{\mathbf{z}}$  and the  $C_l$  are variational parameters. If  $V$  is the volume, the constant  $\alpha$  is given by

$$\alpha^{-1} = \frac{1}{nV} \int d\mathbf{p} \left( -\frac{\partial n_p}{\partial \varepsilon_p} \right). \quad (19)$$

For the degenerate limit  $\alpha = (2/3) \varepsilon_F$ ; in the Boltzmann case  $\alpha = kT$ . With this form the magnetization is properly located at the Fermi surface for the degenerate system and the form reduces to that of LL [1] in the Boltzmann case because then  $\partial n_p / \partial \varepsilon_p \rightarrow \beta n_p$

Using equation (18) it is easily verified that equation (13) is satisfied and that, by equation (17),

$$J_i(\mathbf{M}) = 3 \alpha m \sum_l \hat{\mathbf{x}}_l C_{li} \quad (20)$$

where  $C_{li}$  is the  $i$ th component of the vector  $\mathbf{C}_l$ .

Multiplication of equation (15) by  $v_{pi}$  and integration over  $\mathbf{p}$ , together with use of equations (17)-(20), leads to the result

$$\frac{\partial J_i(\mathbf{M})}{\partial t} + \frac{\alpha}{m} \left( 1 - \frac{nf}{\alpha} \right) \frac{\partial \mathbf{M}}{\partial r_i} - \frac{2}{\hbar} nf J_i(\mathbf{M}) \times \mathbf{M} - \gamma J_i(\mathbf{M}) \times \mathbf{H} = \mathcal{J}_i(\mathbf{M}) \quad (21)$$

in which

$$\mathcal{J}_i(\mathbf{M}) = \frac{1}{n} \int d\mathbf{p} v_{pi} \mathbf{I}_p. \quad (22)$$

Since the form of  $\mathbf{I}_p$  and hence  $\mathcal{J}_i(\mathbf{M})$ , are not actually known, we follow Leggett's procedure [6] and make the assumption

$$\mathcal{J}_i(\mathbf{M}) = -\frac{\mathcal{J}_i(\mathbf{M})}{\tau_D} \quad (23)$$

where  $\tau_D$  is a diffusive relaxation time. The quantity  $\mathbf{I}_p$  which appears in equations (6) and (15) and which determined  $\mathcal{J}_i(\mathbf{M})$  arises from the density matrix form of the collision integral which would be the appropriate generalization of the LL collision integral. Such a generalization does not yet exist<sup>(1)</sup> and so we are unable to justify the form of equation (23) or to give a rigorous expression for  $\tau_D(T)$ . If we assume  $\tau_D$  is the same relaxation time that occurs in longitudinal spin diffusion then we can identify it in special cases from the general form of the diffusion constant. In the classical limit we know that

$$D_0 = \frac{kT}{m} \tau_D. \quad (24)$$

Use of this form and the results for  $D_0$  of reference [1] give  $\tau_D$  in the  $s$ -wave limit as

$$\tau_D = \frac{3}{32} \frac{1}{a^2 n} \left( \frac{m}{\pi kT} \right)^{1/2}. \quad (25)$$

Similarly, in the degenerate case, an expression for  $\tau_D$

<sup>(1)</sup> A form for a collision integral [20], applicable to the degenerate case, and based on results of Wölfle, has been used by Pal and Bhattacharyya to study the Leggett-Rice effect. It is not obvious that all effects due to the non-diagonal character of the distribution function have been included, and it would be interesting to compare this form with one arrived at by a derivation analogous to that of reference [1]. Unfortunately such a derivation is not at all straightforward [21].

could be found from the work of Hone [22]. We do not need this form explicitly and do not quote it.

Equation (17) has the steady state solution [6]

$$J_i(\mathbf{M}) = -\frac{D_0}{1 + (\mu M)^2} \times \left\{ \frac{\partial \mathbf{M}}{\partial r_i} + \mu \left( \mathbf{M} \times \frac{\partial \mathbf{M}}{\partial r_i} \right) + \mu^2 \left( \mathbf{M} \cdot \frac{\partial \mathbf{M}}{\partial r_i} \right) \mathbf{M} \right\} \quad (26)$$

where

$$\mu = -\frac{2}{\hbar} n f \tau_D(T), \quad (27)$$

$$D_0 = \frac{\alpha}{m} \left( 1 - \frac{n}{\alpha} f \right) \tau_D(T). \quad (28)$$

Equation (26) is the form for the spin current that arises in both the degenerate [6] and Boltzmann cases. Our result spans both those regimes. We can now compare our results for  $\mu$  and  $D_0$  with those of references [1] and [6]. In the degenerate limit Leggett shows

$$\mu = \frac{2}{\hbar} \frac{n \left( f_0^a - \frac{1}{3} f_1^a \right)}{1 + \frac{1}{3} F_1^a} \tau_D \quad (29)$$

and

$$D_0 = 1/3 v_F^2 (1 + F_0^a) \tau_D. \quad (30)$$

In the *s*-wave case  $f_0^a = -f$ ,  $f_1^a = 0$  and  $F_1^a = N(0) f_1^a$  where the density of states at the Fermi surface is given by

$$N(0) = \frac{3}{2} \frac{n}{\varepsilon_F}. \quad (31)$$

With these equalities and  $\alpha = 2/3 \varepsilon_F$  it is easily seen that the results of equations (27) and (28) are identical with those of equations (29) and (30), respectively, for the dilute degenerate case. This is no surprise, of course, because our result is just a special case of Leggett's derived in a somewhat different way. Note that the mean-field correction factor  $(1 + F_0^a)$  of equation (30) appears as  $(1 - 3/2 n f / \varepsilon_F)$  in equation (28).

In the Boltzmann gas limit substitution of equation (25) into equation (27) yields

$$\mu = -\frac{3}{8} \left( \frac{\pi \hbar^2}{m a^2 k T} \right)^{1/2} \quad (32)$$

This expression agrees with the *s*-wave approximation for  $\mu$  of LL [1]. Using  $\alpha = kT$  we also find

$$D_0 = \frac{kT}{m} (1 - \beta n f) \tau_D. \quad (33)$$

This expression differs from the usual high temperature

result, equation (24), by the factor  $(1 - \beta n f)$ . This correction, which is numerically very close to one, arises from the other mean-field term mentioned in the paragraph after equation (15). It is this term which also gives rise to the  $(1 + F_0^a)$  factor in the degenerate case. While numerically close to one this mean-field correction ought, in principle, to be present in a complete two-body analysis such as that given by LL. Laloë has suggested [21] that the LL analysis might indeed give rise to such a term if collisional retardation was taken into account in an LL-type analysis.

A small digression concerning the presence of such molecular-field terms in a microscopic derivation of the Boltzmann equation is perhaps appropriate here. Kadanoff and Baym [23] have shown that usual forms of the Boltzmann equation result if self-energy terms on the left side of their Green's function equation of motion are treated differently than those on the right side. While both of these sets of self-energy terms arose from a group of collisional integrals they were segregated onto separate sides of the equation because the authors saw their roles as falling into two classes. Terms on the left side were said to describe « kinetic effects of the potential, i.e., how the potential changes the energy-momentum relation from that of free particles... », while those on the right describe « the dynamical effects of collisions, i.e., how the collisions transfer particles from one energy-momentum configuration to another ». The kinetic terms give rise to the mean-field terms of the Landau equation.

It is interesting then that the LL formalism similarly results in two types of collision terms : those giving rise to entropy production and those that do not [1]. All of the latter are introduced by particle indistinguishability. The analogy with the findings of reference [22] are quite striking and suggest strongly that further investigation may show that the « missing » mean-field term can result from the LL formalism.

In this regard it is also noteworthy that the LL treatment gives rise to spin-rotation terms [1, 24] which do *not* appear in the high temperature limit of the Landau-Silin equation. Among these are contributions to  $\mu$  from lateral scattering. These extra terms are of order  $(a/\lambda)^3$  at low temperatures.

Returning to our discussion of equation (33), we note that the term  $\beta n f$  is of the order of magnitude of a virial coefficient correction. Of the same order or larger are statistical corrections to the high temperature value of  $\alpha$  of equation (19). The next order correction in  $n\lambda^3$  can be shown to give the result

$$D_0 = \frac{kT}{m} (1 + 2^{-5/2} n\lambda^3 - \beta n f) \tau_D. \quad (34)$$

Obviously we could provide exact results for  $\alpha$  valid through all temperatures including those intermediate between degenerate and Boltzmann states. However, this is not very useful without having a comparable

expression for  $\tau_D(T)$  valid for all  $T$ . If such a general  $\tau_D$  were available it might provide a theoretical basis on which to evaluate the results of the above-mentioned experimental measurements [4] of  $\mu$  and  $D_0$  in dilute solutions of  $^3\text{He}$  in  $^4\text{He}$ . In those experiments deviations from the LL predictions were observed as the temperature was lowered from the Boltzmann region to just above the Fermi temperature.

We note that in the  $s$ -wave case there are no statistical corrections contained in the expression for  $\mu$ , equation (27). The only temperature dependence is that contained in  $\tau_D(T)$ .

The virial expansion form of equation (34) suggests that we attempt to derive virial corrections to the pressure from the kinetic equation. This is done in the Appendix where we find the mean-field term to play an important role.

We could derive more elaborate expressions containing many partial waves for  $\mu$  and  $D_0$  in the Boltzmann limit by substituting a more complete expression (See Appendix, Eq. (A16)) into the Landau-Silin equation. This procedure would not be inconsistent with the claims of paragraph (c) at the beginning of this section. The resulting forms for  $\mu$  and  $D_0$  would be valid only for the Boltzmann limit and no connection could be made for the partial wave expression with degenerate results beyond just the  $s$ -wave term. We have not derived such expressions for  $\mu$  and  $D_0$ ; however the derivation for the pressure in the Appendix illustrates this procedure.

#### 4. Summary and discussion.

By comparing the low-temperature limit of the spin-rotation term of the LL kinetic equation to the mean-field spin precession term of the Landau-Silin equation we have shown that these two terms are identical in form for a dilute system. This result indicates that they arise from the same physical effect. From the detailed derivation of the effect given by LL it is known that spin-rotation arises from the interference of the exchange part of the scattered wave with the transmitted beam, or with the scattered wave itself, in the interaction of two spins polarized along different axes. The mean-field spin precession term of the LS equation arises in the drift term while the corresponding spin-rotation term of LL arises from the collision integral terms which contain commutators of the distribution matrix. These terms have time-reversal behaviour opposite to the usual collision terms, and as LL have shown [1], have nothing to do with entropy production. Thus it is not surprising that they can be transformed into drift terms as they appear in the LS equation.

By considering the extension of the LS equation for the dilute Fermi system to arbitrary temperatures we have been able to derive hydrodynamic equations for  $\mathbf{J}$  and  $\mathbf{M}$  and expressions for  $\mu$  and  $D_0$  that encompass both the LL and LR results, at least in  $s$ -wave approxi-

mation. That the results agree with those of both formalism proves that, in this limited situation at least, the physics involved in each is identical.

The Boltzmann limit for the diffusion constant found from the LS equation shows the effects of mean-field terms not contained in the LL expressions. We have shown that such a term is important in giving rise to the correct second virial term in the pressure.

Our work has shown how it might be possible to construct a theory of spin-rotation valid for arbitrary temperature if certain advances were made. The first of these involves finding  $\tau_D$  at all temperatures including those intermediate between degenerate and Boltzmann limits. An analytic expression for  $\tau_D$  seems unlikely but a numerical one might be feasible. The second improvement necessary would be the extension of the LS equation to arbitrarily large polarizations. This involves not only solving coupled equations for  $n_p$  and  $\sigma_p$  as explained in section 3, but also generalizing the interactions as discussed by Meyerovich [16].

#### Appendix.

Here we show how an expression for the gas pressure may be derived from the Landau kinetic equation applied at arbitrary temperature. We find many-body statistical corrections to the pressure but a second virial potential correction. The latter correction is seen to be a result of the mean-field term present in the Landau equation and illustrates that such terms can be important in the Boltzmann limit as well as in the degenerate case. Many-body potential corrections are, of course, not found because we must make dilute case assumptions for the Landau equation to be valid out of the degenerate region.

The Landau kinetic equation, for the case of diagonal density matrix  $\rho$  (with elements  $n_{p\sigma}$ ) is

$$\frac{\partial n_{p\sigma}}{\partial t} + \mathbf{v}_{p\sigma} \cdot \nabla n_{p\sigma} + \nabla_r \epsilon_{p\sigma} \cdot \nabla_p n_{p\sigma} = I_{p\sigma}. \quad (\text{A.1})$$

Within Landau theory we write

$$\nabla_r \epsilon_{p\sigma} = \sum_{\sigma'} \int d\mathbf{p}' f_{p\mathbf{p}'}^{\sigma\sigma'} \nabla_r n_{p'\sigma'}. \quad (\text{A.2})$$

The momentum conservation hydrodynamic equation can be derived [11] from (A.1) by multiplying by  $\mathbf{p}$ , integrating over  $\mathbf{p}$ , and summing on  $\sigma$ . The result is

$$\frac{\partial g_\alpha}{\partial t} + \sum_\beta \frac{\partial}{\partial r_\beta} \Pi_{\alpha\beta} = 0 \quad (\text{A.3})$$

where the momentum density is

$$g_\alpha = \sum_\sigma \int d\mathbf{p} p_\alpha n_{p\sigma} \quad (\text{A.4})$$

and the stress tensor is

$$\Pi_{\alpha\beta} = T_{\alpha\beta} + S_{\alpha\beta}. \quad (\text{A.5})$$

In (A.5), we have

$$T_{\alpha\beta} = \sum_{\sigma} \int \mathbf{d}\mathbf{p} m v_{p\alpha} v_{p\beta} n_{p\sigma} \quad (\text{A.6})$$

$$S_{\alpha\beta} = \frac{1}{2} \delta_{\alpha\beta} \sum_{\sigma\sigma'} \int \mathbf{d}\mathbf{p} \int \mathbf{d}\mathbf{p}' f_{pp'}^{\sigma\sigma'} n_{p\sigma} n_{p'\sigma'} \quad (\text{A.7})$$

$S_{\alpha\beta}$  arises directly from the molecular-field effect, namely, the last term on the left side of equation (A.1).

Equation (A.6) is easily seen to give

$$T_{\alpha\beta} = \frac{2}{3} n \bar{\varepsilon} \delta_{\alpha\beta} \quad (\text{A.8})$$

where  $\bar{\varepsilon}$  is the average single-particle kinetic energy.

At high temperatures if we keep the first two terms in the expansion of  $\bar{\varepsilon}$  in  $n\lambda^3$ , we find

$$T_{\alpha\beta} \cong \delta_{\alpha\beta} nkT(1 + 2^{-7/2} n\lambda^3). \quad (\text{A.9})$$

In the same limit we can replace  $n_{p\sigma}$  in equation (A.7) by the Boltzmann distribution

$$n_{p\sigma} \cong \frac{1}{Z} e^{-\beta\varepsilon_{p\sigma}} \quad (\text{A.10})$$

where

$$Z = \sum_{\sigma} \int \mathbf{d}\mathbf{p} e^{-\beta\varepsilon_{p\sigma}} \quad (\text{A.11})$$

Then we find

$$S_{\alpha\beta} = \delta_{\alpha\beta} n^2 kTb \quad (\text{A.12})$$

with

$$b = \frac{1}{2kTZ^2} \sum_{\sigma\sigma'} \int \mathbf{d}\mathbf{p} \int \mathbf{d}\mathbf{p}' f_{pp'}^{\sigma\sigma'} e^{-\beta(\varepsilon_{p\sigma} + \varepsilon_{p'\sigma'})}. \quad (\text{A.13})$$

Since  $\Pi_{\alpha\beta}$  is diagonal the hydrostatic pressure at high temperatures is just

$$P = nkT(1 + nB) \quad (\text{A.14})$$

where

$$B = 2^{-7/2} n\lambda^3 + b. \quad (\text{A.15})$$

Vetrovec and Carneiro [24] have shown that, in the dilute limit  $f_{pp'}$  can be written in terms of scattering phase shifts  $\delta_l$  as

$$f_{pp'}^{\sigma\sigma'} = -\frac{4\pi\hbar^2}{mk} \left[ \delta_{\sigma,\sigma'} \sum_{l(\text{even})} (2l+1) \delta_l + 2(\delta_{\sigma,\sigma'} + \delta_{\sigma,-\sigma'}) \sum_{l(\text{odd})} (2l+1) \delta_l \right] \quad (\text{A.16})$$

from which, as they indicate, it follows that (A.15) agrees with the standard phase shift expression [25] for the second virial coefficient.

If we are able to make an  $s$ -wave approximation we can develop an expression for  $P$  valid for all temperatures. Write

$$f_{pp'}^{\sigma\sigma'} = f(1 - \sigma\sigma'). \quad (\text{A.17})$$

Then

$$S_{\alpha\beta} = 2\delta_{\alpha\beta} f n_+ n_- \cong \frac{1}{2} \delta_{\alpha\beta} n^2 f \quad (\text{A.18})$$

for low polarization. Thus

$$P = \frac{2}{3} n \bar{\varepsilon} + \frac{1}{2} n^2 f. \quad (\text{A.19})$$

For high  $T$  this result is an  $s$ -wave version of equation (A.14). On the other hand for low  $T$ , using  $\bar{\varepsilon} = 3/5 \varepsilon_F$ , we get

$$P = \frac{2}{5} n \varepsilon_F + \frac{1}{2} n^2 f. \quad (\text{A.20})$$

The first term is the Pauli pressure. The last is the second virial potential correction. We could get many-body potential corrections valid at low  $T$  by considering more general forms of  $f_{pp'}^{\sigma\sigma'}$ .

#### Acknowledgments.

We would like to thank F. Laloë and C. Lhuillier for many useful discussions.

#### References

- [1] LHUILLIER, C. and LALOË, F., *J. Physique* **43** (1982) 197; 225;  
LHUILLIER, C., *J. Physique* **44** (1983) 1.
- [2] JOHNSON, B. R., DENKER, J. S., BIGELOW, N., LEVY, L. P., FREED, J. H. and LEE, D. M., *Phys. Rev. Lett.* **52** (1984) 1508.
- [3] NACHER, P. C., TASLEVIN, G., LEDUC, M., CRAMPTON, S. B. and LALOË, F., *J. Physique Lett.* **45** (1984) L-441.
- [4] GULLY, W. J. and MULLIN, W. J., *Phys. Rev. Lett.* **52** (1984) 1810.
- [5] LEGGETT, A. J. and RICE, M. J., *Phys. Rev. Lett.* **20** (1968) 586; *Phys. Rev.* **21** (1968) 506.
- [6] LEGGETT, A. J., *J. Phys. C* **12** (1970) 448.
- [7] BASHKIN, E. P., *JETP Lett.* **33** (1981) 8 (*Pis'ma Zh. Eksp. Teor. Fiz.* **33** (1981) 11).
- [8] LEVY, L. P. and RUCKENSTEIN, A. E., *Phys. Rev. Lett.* **52** (1984) 1512.

- [9] SILIN, V. P., *JETP* **6** (1958) 945 (*Zh. Eksp. Teor. Fiz.* **33** (1957) 1227).
- [10] LANDAU, L. D., *JETP* **3** (1956) 1058 ; **32** (1957) 59.
- [11] BAYM, G. and PETHICK, C., *Physics of Liquid and Solid Helium*, Part II, ed. K. H. Benneman and J. B. Ketterson (John Wiley and Sons, New York) 1978, Ch. 1.
- [12] CORRUCINI, L. R., OSCHEROFF, D. D., LEE, D. M. and RICHARDSON, R. C., *Phys. Rev. Lett.* **27** (1971) 650 ; *J. Low Temp. Phys.* **8** (1972) 229.
- [13] SCHULTZ, S. and DUNIFER, G., *Phys. Rev. Lett.* **18** (1967) 283.
- [14] MASUHARA, N., CANDELA, D., EDWARDS, D. O., HOYT, R. F., SCHOLZ, H. N., SHERRILL, D. S. and COMBESCOT, R., *Phys. Rev. Lett.* **53** (1984) 1168.
- [15] OWERS-BRADLEY, J. R., CHOCHOLACS, H., MUELLER, R. M., BUCHAL, Ch., KUBOTA, M. and POBELL, F., *Phys. Rev. Lett.* **51** (1983) 2120.
- [16] MEYEROVICH, A. E., *J. Low Temp. Phys.* **53** (1983) 485 ; Also see,
- BASHKIN, E. P. and MEYEROVICH, A. E., *Adv. Phys.* **30** (1981) 1.
- [17] ABRIKOSOV, A. A. and KHALATNIKOV, I. M., *JETP* **6** (1957) 888 (*Zh. Eksp. Teor. Fiz.* **33** (1957) 1154) ; BASHKIN, E. P. and MEYEROVICH, A. E., *JETP* **47** (1978) 992 (*Zh. Eksp. Teor. Fiz.* **74** (1978) 1904).
- [18] ABRIKOSOV, A. A. and DZVALOSHINSKII, I. E., *JETP* **8** (1958) 535 (*Zh. Eksp. Teor. Fiz.* **35** (1958) 771).
- [19] STAMP, P. C. E., Thesis, University of Sussex, 1983 (unpublished).
- [20] PAL, A. and BHATTACHARYYA, P., *J. Low Temp. Phys.* **51** (1983) 265. WOLFLE, P., *Z. Phys.* **232** (1970) 38.
- [21] LALOË, F., private communication.
- [22] HONE, D., *Phys. Rev.* **121** (1961) 669.
- [23] KADANOFF, L. P. and BAYM, G., *Quantum Statistical Mechanics* (Benjamin) 1962, Chs. 9, 11.
- [24] LHUILLIER, C. and LALOË, F., *Phys. Rev. Lett.* **54** (1985) 1207.
- [25] VETROVEC, M. B. and CARNEIRO, G. M., *Phys. Rev. B* **22** (1980) 1250.
- [26] LANDAU, D. and LIFSHITZ, E. M., *Statistical Physics* (Addison-Wesley, Reading, MA) 1969, Ch. VII.