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Résumé. — Nous construisons un système dynamique simple qui présente une transition vers la turbulence par intermittence de type I. Nous démontrons que la stochasticité apparaît dès le franchissement du seuil de bifurcation. De plus, bien que la transition entre régime laminaire et régime turbulent soit abrupte, certaines propriétés statistiques du système varient continûment.

Abstract. — We study a discrete dynamical system which presents a transition to turbulence *via* type I intermittency. We demonstrate that chaos is already present just beyond the threshold of intermittency. Moreover, although the transition from laminar to turbulent regime is abrupt, certain statistical properties of the system vary continuously.

Intermittency as a way to turbulence has been recently the subjet of theoretical [1] as well as experimental [2, 3, 4] studies. In this paper, we analyse the statistical properties of certain intermittent transitions to turbulence. For this purpose we introduce a discrete dynamical system which accounts for the so-called type I intermittency [1].

Such an intermittent behaviour occurs in the well-known Belousov-Zhahotinski chemical reaction [3]. In Rayleigh Benard thermogravitational instability, turbulence may also take place *via* an intermittent transition for high values of the Prandtl number, as shown by P. Bergé *et al.* [4] and by Maurer *et al.* [2]. The former authors have even constructed a simple model derived from the Baker transformation which enables them to explain the observed transition in terms of type I Intermittency.

Intermittent behaviour also appears in the study of ordinary differential equations. Let us quote the example of the saturation of plasma instability through the non-linear decay of a linearly unstable H.F. wave into linearly damped L.F. modes [8]. At low values of the growth rate, the wave amplitudes exhibit regular oscillations — as shown in figure 1*a*. At higher values of the growth rate (Fig. 1*b*) seemingly stable oscillations of the wave amplitudes are randomly interrupted by turbulent bursts. Increasing further the growth rate, the duration of the laminar periods decreases more and more until the system reaches a fully chaotic regime (Fig. 1*c*). Such a transition from a stable periodic behaviour to a seemingly random alternation of long laminar periods and short turbulent bursts denotes an intermittent transition to turbulence. Moreover, if the intermittent signal is recorded long enough, it appears that the number of oscillations between two bursts has an upper bound. This feature is typical of type I intermittency.

For such a transition to turbulence, a question arises : is chaos already present just beyond the onset of intermittency, even though the behaviour appears regular most of the time ? And, if so, what are the statistical properties of the system near the transition ?

Usually, for a real system, it is impossible to conclude about the stochastic or non-stochastic nature of a process. But, fortunately, most features of the intermittent transition are model independent. Indeed, we can associate to such a system a one dimensional map which keeps every important feature of intermittency. We may, for instance, consider successive energy peaks of one wave in the previous example. This gives a sequence of numbers, $X_1, X_2 \dots X_N \dots$, connected by a finite difference equation of the form,

$$X_{N+1} = f(X_N, \varepsilon).$$

The control parameter ε is here a smooth function of the growth and damping rates. When increasing ε, the mapping $X_{N+1} = f(X_N, \varepsilon)$ displays an intermittent transition from the laminar to chaotic regime which reproduces that of the real system. Such mappings

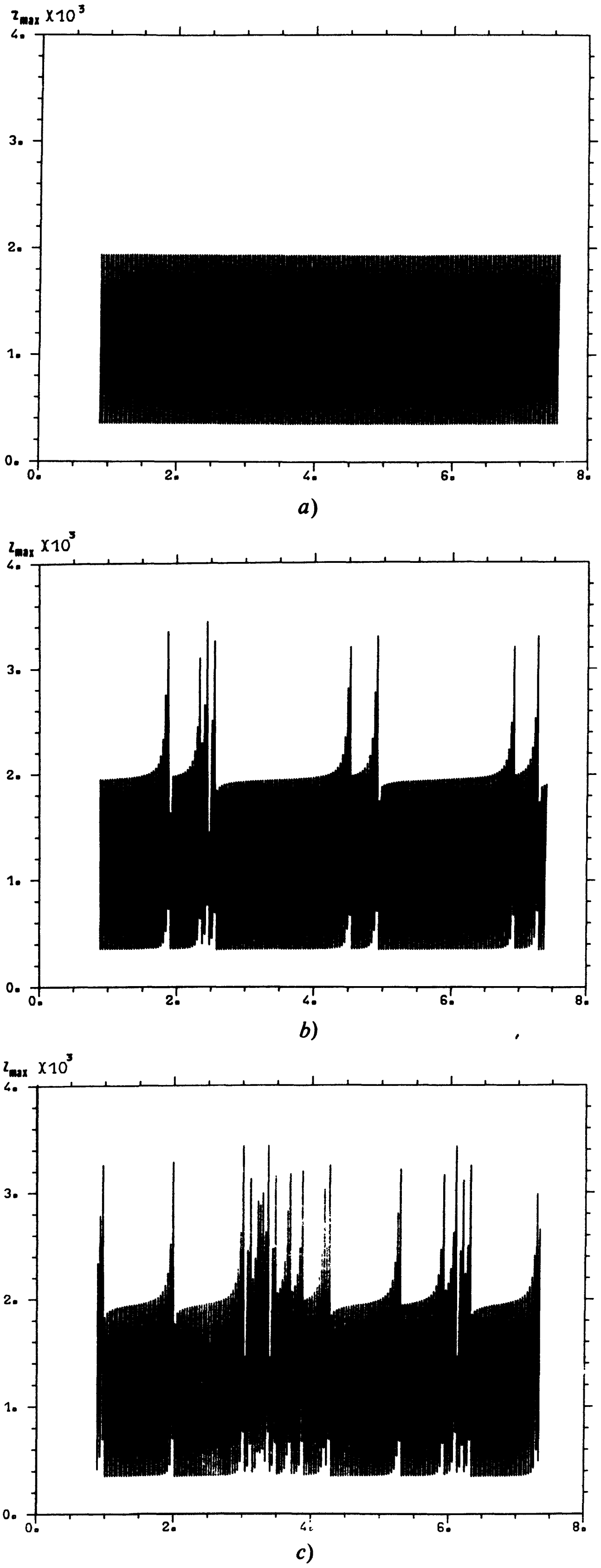

Fig. 1. — Successive peak values of the low-frequency wave energy in the decay problem : *a*) in the « laminar regime » ; *b*) in the « turbulent regime » near the threshold ; *c*) in the « turbulent regime » far from the threshold.

may be also deduced from experimental data [4]. The explicit form of f is of course linked to the specific problem one considers. However, every family of 1.D maps displaying type I intermittency presents generic features to be described hereafter.

Here we construct such a mapping. For this model, it is possible to demonstrate that even at the intermittency threshold, neighbouring trajectories separate exponentially in the long term. Hence the system bifurcates from a stable periodic regime to a stochastic one, endowed with strong statistical properties (ergodicity, strong-mixing...). Moreover, the probability of finding asymptotically the system in a given state of phase space, is well described in both regimes by an invariant (under iteration) measure. In the laminar regime, it reduces to a δ-function (which pictures the constant value of the peak energy for the oscillations in the decay problem). We show that this invariant measure continuously varies through the intermittent transition to turbulence, and we give its approximate expression near the threshold.

Let us recall the characteristic features of a map which accounts for type I Intermittency. As shown in [1], the « regular » oscillations between two bursts can be described by the following generic local form of the difference equation : $x_{n+1} = f(x_n, \varepsilon)$

$$x_{n+1} = x_n + \varepsilon + x_n^2 . \tag{1}$$

Indeed for negative values of the control parameter ε, the map has two fixed points ; $\pm \sqrt{-\varepsilon}$. The stable value $(-\sqrt{-\varepsilon})$ accounts for the stable periodic regime before the intermittent transition. If ε is positive, the fixed points of (1) vanish. Starting from a slightly negative value of x, the successive iterates x_n's, as given by (1), drift slowly towards positive values. This slow drift corresponds to the laminar period (Fig. 1*b*). Far away from 0, the local form (1) is no longer valid. The long distance behaviour, which affects the burst structure has only to insure the reinjection of the iterate in the region $X \sim 0$, thus beginning a new laminar phase.

Here we introduce such a map $x_{N+1} = f(x_N, \varepsilon)$ which locally has the generic form (1), and which moreover is tractable for analytical study of type I intermittency. Namely, we consider the family of mappings of the interval [0, 1] onto itself defined as follows (Fig. 2). Let x_0 be a point of [0, 1/2], b a positive constant and ε a control parameter. Then we define f by,

$$x_{n+1} = x_0 + \frac{x_n - x_0 + \varepsilon}{1 - b(x_n - x_0)} \quad \text{if} \quad x_A < x_n < x_B \tag{2}$$

$$x_{n+1} = 2\, x_n \bmod 1 \ \text{ if } 0 \leqslant x_n \leqslant x_A \text{ or } x_B \leqslant x_n \leqslant 1$$

where

$$x_{\substack{A\\B}} = \frac{1}{4\,b}\left(1 + 3\,bx_0 \mp \sqrt{(1 - bx_0)^2 + 8\,b(x_0 - \varepsilon)}\right).$$

In order to insure $0 < x_A < x_0 < x_B < 1/2$, the

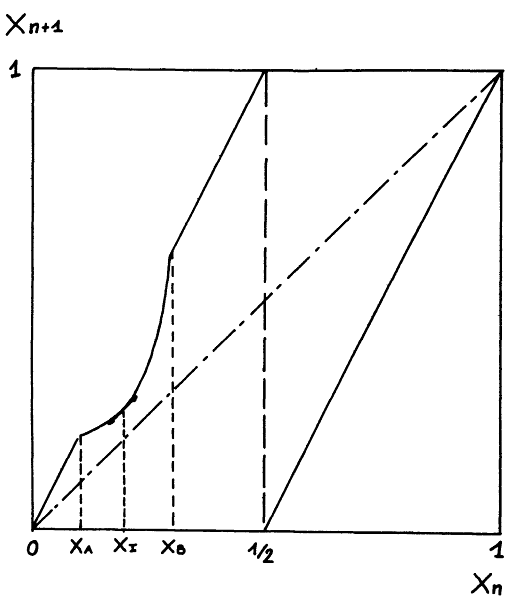

Fig. 2. — The map of the interval [0, 1] onto itself described by equations (2).

constants b and x_0 satisfy $\frac{1}{1 - 3x_B} < b < \frac{1}{x}$, and $|\varepsilon| < x_0 < \frac{1}{4}$.

We can make this map C^N, $N \geqslant 2$, by slightly modifying f in neighbourhoods of x_A, x_B as small as we like.

The origin is always an unstable fixed point. For $\varepsilon = 0$, x_0 is a stable fixed point. On the contrary, for $\varepsilon > 0$, f has no stable fixed point. For $\varepsilon \to 0_+$, a small channel is created near x_0 between the representative curve $x_{n+1}(x_n)$ and the bissectrix $x_{n+1} = x_n$ (Fig. 2). In the vicinity of $x_I \sim x_0 - \varepsilon/2$, where the slope of f is 1, the curve $x_{n+1}(x_n)$ may be approximated by a parabola of the generic form (1). Hence if we assume an initial condition close to x_A, the iterate x_n first gently increases in the vicinity of x_0, and then more and more quickly up to a point where it becomes larger than 1/2 (« laminar phase »). Now due to the discontinuity in the graph of f, this monotonic variation is suddenly interrupted. A « turbulent » burst occurs which returns the iterate into the region $x_A \leqslant x \leqslant x_B$.

It is worthwhile to look at this mapping, since we know an exact expression of the iterate $f^n(x)$, provided that x and $f^k(x)$, $0 \leqslant k \leqslant n$, lie in the interval $[x_A, x_B]$:

$$f^n(x) = x_0 + \sqrt{\frac{\varepsilon}{b}}\,\mathrm{tg}\left[n\,\mathrm{Arc\,tg}(\sqrt{b\varepsilon}) + \mathrm{Arc\,tg}\left(\sqrt{\frac{b}{\varepsilon}}[x - x_0]\right)\right].$$

Hence the number L of iterations required to drift through the whole channel $[x_A, x_B]$ writes:

$$L = E\left\{\frac{1}{\sqrt{\varepsilon b}}\left[\mathrm{Arc\,tg}\left([x_B - x_0]\sqrt{\frac{b}{\varepsilon}}\right) - \mathrm{Arc\,tg}\left([x_A - x_0]\sqrt{\frac{b}{\varepsilon}}\right)\right]\right\}. \quad (3)$$

L is also an upper bound for the duration of a laminar phase. In the limit $\varepsilon \to 0_+$, L diverges as $\pi/\sqrt{\varepsilon b}$.

If we increase ε from 0 to a finite value $\varepsilon_0 \ll 1$, the bursts are not significantly affected whereas the « laminar phases » become shorter and shorter. Thus the recorded signal looks more and more chaotic as pictured in figure 1. In order to determine if the system is already stochastic near the threshold, and to make sure it presents no stable periodic orbit (however complicated it may be), we test the dependence on the initial conditions.

Indeed a small error δx_1 on x_1 leads to an error $\delta x_2 \sim f'(x_1)\,\delta x_1$ on its first iterate $x_2 = f(x_1)$. After N iterations one gets in the limit where $\delta x_1 \ll 1$

$$\delta x_N \sim f'(x_{N-1})\,\delta x_{N-1}$$

or

$$\delta x_N \sim \prod_{i=1}^{N-1} f'(x_i)\,\delta x_1$$

where $\quad x_i = f^i(x_1)$.

Here, the derivative $f'(x)$ can be either smaller than 1 (in the interval $[x_A, x_I]$) or greater than 1 (anywhere else). Then δx_n may be either greater or smaller than δx_1 depending on the initial condition x_1 and on the number of iterations performed. However for such a one-dimensional map the sensitive dependence to initial conditions is related to its so-called expanding character. A mapping is expanding if there exists some integer M and some real numbers $\lambda > 1$ and $C > 0$ such that, for any integer N greater or equal to M, the following inequality holds:

$$\operatorname*{Inf}_{x \in [0,1]} |(f^N)'(x)| \geqslant C\lambda^N.$$

Therefore if the mapping is expanding, trajectories exponentially separate in the long term since for δx_1 small enough.

$$|\delta x_n| \geqslant C\lambda^N |\delta x_1|$$

with $\lambda > 1$.

Let us define

$$\gamma_x = \lim_{N\to\infty} \frac{1}{N} \mathrm{Log}\,|f^{N'}(x)| \quad (4)$$

the exponent γ_x is related to the amount of stochasticity and to the expanding character of the map. For an expanding map, γ_x is positive for almost every x in [0, 1].

Hence we first show that for $\varepsilon > 0$, the mapping defined by equations (2) is expanding. In order to demonstrate this property for the mapping (2), we calculate a lower bound for the derivative $(f^N)'(x)$ for a given ε, the number N of iterations being fixed. Let us first consider L successive iterates of a point x. As previously defined by (3), L presents the number of iterations required to drift from x_A to x_B. A lower bound of $(f^L)'(x)$, is obtained for a trajectory which spends as much time as possible in the interval $[x_A, x_I]$ where $f'(x)$ is smaller than 1. Hence,

$$\operatorname*{Inf}_{x \in [0,1]} (f^L)'(x) \geqslant (f^L)'(x_A).$$

Moreover at the end of a laminar period, at least two iterations are required to send back the image of x into the interval $[x_A, x_B]$ and

$$\operatorname*{Inf}_{x \in [0,1]} (f^{L+2})'(x) \geqslant 4(f^L)'(x_A).$$

Besides, if p is any integer smaller than $L + 2$ and if I is the number of iterations necessary to drift from x_A to x_I, the following inequality holds since $f'(x) \leqslant 1$ in $[x_A, x_I]$.

$$\operatorname*{Inf}_{x \in [0,1]} (f^p)'(x) \geqslant (f^I)'(x_A).$$

Now let N be any integer. We can write

$$N = E\left(\frac{N}{L+2}\right) + q \quad \text{with} \quad q < L + 2$$

and the derivative $(f^N)'(x)$ has for a lower bound :

$$[4(f^L)'(x_A)]^{E\left(\frac{N}{L+2}\right)} (f^I)'(x_A).$$

Is $4(f^L)'(x_A)$ greater than 1 ? Its exact expression is easily deduced from (2) :

$$(f^L)'(x_A) = \frac{\varepsilon + b[f^L(x_A) - x_0]^2}{\varepsilon + b[x_A - x_0]^2}.$$

Let ξ_B be the preimage of x_B that belongs to $[0, 1/2]$

$$\xi_B = x_0 + \frac{x_B - x_0 - \varepsilon}{1 + b(x_B - x_0)}.$$

As $\xi_B < f^L(x_A) < x_B$, we get

$$(f^L)'(x_A) > \frac{\varepsilon + b(\xi_B - x_0)^2}{\varepsilon + b(x_A - x_0)^2}$$

and the quantity $4(f^L)'(x_A)$ is larger than unity if $0 < bx_0 < 2/3$. Therefore the mapping defined by (2) is expanding for $0 < \varepsilon < x_0$ if we choose b and x_0 such that

$$x_0 < 2/9$$

$$\frac{1}{1 - 3x_0} < b < \frac{2}{3x_0}.$$

Indeed setting

$$c = \frac{(f^L)'(x_A)}{4(f^L)'(x_A)} > 0$$

and

$$\lambda = [4(f^L)'(x_A)]^{1/L+2} > 1,$$

as soon as $N > L + 2$ the derivative $(f^N)'(x)$ satisfies the inequality :

$$\operatorname{Inf} (f^N)'(x) > c\lambda^N.$$

Here, the constant $c < 1$ denotes the existence of the domain $[x_A, x_I]$ where $f'(x)$ is less than 1. The existence of such a region could at first view jeopardize the expanding character of the map. In fact it turns out that the domain where $f'(x)$ is greater than 1 prevails for this model, as is shown by the value of $\lambda > 1$. For N large enough the derivative $(f^N)'(x)$ is everywhere larger than 1. As a consequence every periodic orbit of the mapping is unstable. Thus the behaviour of the associated dynamical system is unpredictible in the long term as soon as the threshold $\varepsilon = 0$ is crossed.

An estimation of the amount of turbulence in the asymptotic regime, is given by the exponents γ_x. From the definition (4), then

$$\gamma_x > \operatorname{Log} \lambda = \frac{1}{L+2} \operatorname{Log} 4[f^{L'}(x_A)].$$

In the limit where $\varepsilon \to 0_+$, the maximum duration of a laminar phase, L, diverges as $L = \frac{\pi}{\sqrt{\varepsilon b}}$, and $\frac{1}{L+2} > \frac{\sqrt{\varepsilon b}}{\pi}$. On the other hand, the derivative $f^{L'}(X_A)$ remains finite if $\varepsilon \to 0_+$. In a first approximation $f^{L'}(X_A)$ does not depend on $\varepsilon \to 0_+$, and

$$\operatorname{Log}[4 f^{L'}(X_A)] = k(b, x_0).$$

Therefore we get a lower bound for γ_x which is of the form

$$\gamma_x > k(b, x_0)\sqrt{\varepsilon} \quad \text{for any } x.$$

In order to describe from a statistical point of view, the asymptotic behaviour of the system, for $\varepsilon > 0$, we are now looking for an invariant probability density, $\mu(x)$, on phase space $x \in [0, 1]$. Due to its expanding character the mapping (2) possesses such an invariant density, which varies continuously from point to point [5]. Moreover, as shown in reference [5], the measure, $\mu(x)\,dx$, endows the system with strong statistical properties such as ergodicity and mixing. Ergodicity enables us to calculate phase space averages instead of time averages. The invariance of the measure $\mu(x)\,dx$, under transformation f implies that $\mu(x)\,dx = \mu(x')\,dx' + \mu(x'')\,dx''$, where x' and x'' are the preimage of x in $[0, 1/2]$ and $[1/2, 1]$ respectively.

Hence $\mu(x)$ satisfies

$$\mu(x) = \frac{\mu(x')}{f'(x')} + \frac{\mu(x'')}{f'(x'')}. \quad (5)$$

Here we do not solve exactly equation (5). But we are going to look for an approximate expression of $\mu(x)$ in the limit where $\varepsilon \to 0_+$. As the trajectory spends most of the time near $x = x_0$, whereas it only briefly visits the region outside the channel $[x_A, x_B]$ we expect the density $\mu(x)$ to present a sharp peak near x_0 and a small value far from this point. Hence, for x near x_0, we drop the second term on the R.H.S. of (5), which writes

$$\mu(x) = \frac{(1 + \varepsilon b)}{1 + b(x - x_0)^2} \mu\left[x_0 + \frac{x - x_0 - \varepsilon}{1 + b(x - x_0)}\right]. \quad (6)$$

An exact solution of (6) is a Lorentzian centred in $x = x_0$. Hence, we obtain

$$\mu(x) = \frac{A\sqrt{\varepsilon}}{\varepsilon + b(x - x_0)^2} + 0(\sqrt{\varepsilon})$$

for x in the vicinity of x_0.

On the other hand, as $\mu(x)$ is a continuous function of x, we get

$$\mu(x) \sim B\sqrt{\varepsilon} \text{ outside } [x_A, x_B].$$

In fact, it can be demonstrated that the ratio of the density values far from x_0 and near x_0 is of order ε [6]. Then, if we normalize $\mu(x)$ in such a way that the total weight of the interval is 1, the constants A and B are of order of unity. Hence near the intermittent threshold, the probability density in phase space is concentrated in a narrow region of width $\sqrt{\varepsilon/b}$ around x_0. The peak value being of order $1/\sqrt{\varepsilon}$, the total weight of this region is almost equal to 1. If ε is decreased from positive to negative values, the invariant measure continuously evolves from a Lorentzian peak ($\varepsilon > 0$) to a δ-function ($\varepsilon \leqslant 0$). Of course, this invariant probability density may be numerically computed, by examining the distribution of the successive iterates of a point in the course of a very long run (Fig. 3), or by subtler means [7].

A consequence of the existence of the invariant measure $\mu(x)\,dx$, is the ergodicity of the trajectories as soon as the intermittent threshold $\varepsilon = 0$ is crossed. Hence the exponents γ_x, previously introduced, do not depend on x, $\gamma_x = \gamma$. The exponent γ is nothing else than the so-called characteristic exponent which measures the average rate of divergence for neighbouring trajectories. In agreement with previous conjectures (4), we can check that γ continuously depends on ε and is of order $\sqrt{\varepsilon}$ if $\varepsilon \to 0_+$,

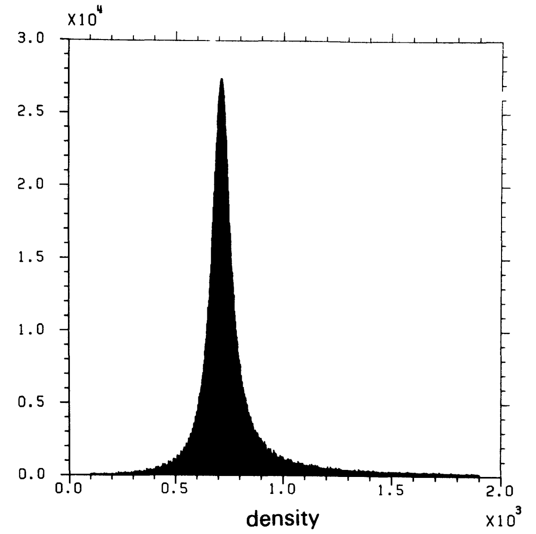

Fig. 3. — The invariant probability density in the neighbourhood of x_0 near the stochasticity threshold.

$$\gamma = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \text{Log} \mid f'(x_i) \mid = \int_0^1 \text{Log} \mid f'(x) \mid \mu(x)\,dx \sim \sqrt{\varepsilon}.$$

Hence type I intermittency appears as an abrupt transition between two very different asymptotic regimes, namely periodic and turbulent. However the quantities which measure the amount of stochasticity smoothly vary with the control parameter ε, thus yielding a certain continuous character for such a transition. In principle from the knowledge of the invariant measure, we can determine the properties of the correlation function.

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