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Nonlinear reversible hydrodynamics of liquid crystals and crystals

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Résumé. — La théorie hydrodynamique unifiée, non linéaire réversible des cristaux liquides nématiques, des cristaux, des smectiques A, des cholestériques et des smectiques C est présentée.

A côté des contributions convectives qui sont attendues mais néanmoins nouvelles, des termes non linéaires sont obtenus dans le tenseur des contraintes et dans l'équation pour les variables hydrodynamiques qui caractérisent les symétries dérangées.

Un bref résumé est donné pour les nématiques. Les résultats sont conformes avec les théories données plus tôt. Après cela nous discutons le cas des cristaux. Après les smectiques A, nous recherchons les termes qui décrivent les rotations du directeur. L'influence d'un champ magnétique extérieur est discutée aussi. Le traitement hydrodynamique des cholestériques est décrit en insistant sur les différences entre les cristaux liquides smectiques A et les cholestériques respectivement.

Finalement la théorie hydrodynamique non linéaire des smectiques C est donnée. Les équations montrent des contributions non linéaires nombreuses reflétant les symétries brisées différentes des smectiques C et l'interaction entre elles. Comme un fait remarquable la dépendance de ces coefficients non linéaires sur toutes les quantités scalaires qui sont présentes dans les smectiques C est établie. Ceci a été fait également pour les autres systèmes présentés, mais ceux-ci possèdent moins de quantités scalaires ce qui montre encore une fois le rôle spécial joué par les smectiques C comme un système biaxial.

Pour une application de la théorie présentée sur des problèmes spécifiques il est, naturellement, nécessaire d'ajouter des contributions irréversibles, calculées par exemple par Martin, Parodi, Pershan, de Gennes, Moritz et Franklin aux équations hydrodynamiques non linéaires réversibles.

Abstract. — The unified nonlinear reversible hydrodynamics of nematics, crystals, smectics A, cholesterics and smectics C is presented.

Besides the expected but nonetheless new convective terms, additional nonlinear contributions are obtained in the stress tensor and in the equation for the hydrodynamic variables characterizing the broken symmetries.

A brief section treats nematic liquid crystals. The results found are in agreement with previously derived theories. Then we discuss crystals. Next smectics A are considered including the examination of terms which describe rotations of the director. The influence of an external magnetic field is discussed as well. The hydrodynamic treatment of cholesterics is described, where special emphasis is laid on the investigation of the differences between smectics A and cholesterics.

Finally the nonlinear hydrodynamics of smectics C is given. The equations show numerous nonlinear contributions reflecting the different broken symmetries of smectics C and the interplay between them. As a quite remarkable fact the dependence of these nonlinear coefficients on all scalar quantities present in smectics C is established. This has been done for the other presented systems as well but those have less scalars illustrating once more the special role played by smectics C as a biaxial system.

In order to apply the presented theory to specific problems it is, of course, necessary to supplement the nonlinear reversible hydrodynamic equations by irreversible contributions derived e.g. by Martin, Parodi and Pershan; de Gennes; Moritz and Franklin.

1. Introduction. — During the last years there has been an increasing interest from the experimental [1-6] as well as from the theoretical groups [7-14] in the study of hydrodynamic instabilities in liquid crystals,

especially in nematics but more recently also in smectics A and cholesterics.

Up to now the theoretical treatments of the several instabilities in liquid crystals (convective instabilities

under the influence of electric fields, magnetic fields, shear torques and thermal gradients) have mainly dealt with the discussion of the linear regime far below the critical point. Therefore it was sufficient to use the linearized hydrodynamic equations for the description of the dynamic behaviour of the systems under investigation [8, 11, 15, 16]. The linearized phenomenological hydrodynamics of nematics [17, 18], smectics A [19, 20], smectics C [19], cholesterics [21, 22], crystals and solids [19, 23, 24-26] has been given previously. (For a discussion of linearized hydrodynamics in terms of correlation functions applied to simple fluids and nematics compare refs. [27-29].)

In addition to these rigorous hydrodynamic approaches to the study of liquid crystals and other system there also exist continuum type theories for liquid crystals [15, 30-35] which deal with more variables thus containing nonhydrodynamic contributions, and it were mainly these theories which have been applied to the study of hydrodynamic instabilities in liquid crystals up to now.

If one is interested in the nonlinear dynamics of hydrodynamic instabilities one has to use the fully nonlinear hydrodynamic equations as is well known from the study of the Bénard and Taylor instabilities [36-43] occurring in simple fluids. In this case the relevant hydrodynamic equations are the Navier Stokes equations supplemented by an equation for the temperature. It is familiar from the study of these systems that the nonlinear reversible terms in the equations are crucial for the understanding of the dynamical properties near the threshold of the instability. The nonlinear irreversible terms are less important and can be neglected without harm (for a discussion of this point for the Bénard instability cf. e.g. ref. [44]).

In the case of liquid crystals the situation is more complicated. Up to now it seems to be an open question which nonlinear terms are the most important ones near the threshold of hydrodynamic instabilities. This question has to be discussed for each instability, separately.

Because the condition of vanishing entropy production (for the reversible terms) is much more stringent than the requirement of positivity of the entropy production (for the irreversible terms) it is less difficult to set up a systematic approach for the reversible nonlinearities.

Concerning the nonlinear irreversible terms in liquid crystals it should be noted that there occur e.g. for nematics, besides the director dependent viscosities and thermal conductivities [15], numerous additional terms (compare ref. [45]).

Therefore we focus our attention in the present paper on a study of the nonlinear reversible terms in the hydrodynamic equations of liquid crystals.

A second motivation for the present work is a genuine interest in the formulation of the nonlinear hydrodynamics for ordered systems (systems with

broken symmetries). The present study is to our knowledge the first attempt to give a nonlinear formulation of the complete reversible hydrodynamics of liquid crystals (except for the case of nematics where theories have already been given).

The paper is organized as follows :

In paragraph 2 we describe the derivation of the nonlinear reversible hydrodynamics of ordered systems. The hydrodynamic method reviewed there is based on refs. [46, 47, 27, 48]. In the next paragraph we present the application of the general theory to the nonlinear reversible hydrodynamics of nematics (3.1), solids (3.2), smectics A (3.3), cholesterics (3.4) and smectics C (3.5).

2. Phenomenological hydrodynamics. — 2.1 HYDRODYNAMIC REGIME.

It is the aim of hydrodynamics to give the time evolution of variables which vary on time scales where many collisions take place. If τ denotes a characteristic time for microscopic interactions, the hydrodynamic regime is confined to frequencies ω satisfying the condition $\omega\tau \ll 1$. Or to phrase it differently : any disturbance which maintains local thermodynamic equilibrium must have temporal variations which are slowly on a scale of τ .

Similar arguments apply to the consideration of spatial variations. If l is a characteristic microscopic length the wave number k of the hydrodynamic excitations must satisfy $kl \ll 1$. This condition may yield a rather stringent restriction on the region of validity of hydrodynamics. For cholesteric liquid crystals, e.g. the pitch $2\pi q_0^{-1}$ is usually of the order of 5 000 Å, and has to be considered as a microscopic length. Thus the wavelength of any hydrodynamic excitation must be much greater, which implies for example that *light scattering* experiments have to be carried out in cholesteric liquid crystals in the IR if one wishes to deal with the hydrodynamic regime. For the other hydrodynamic systems to be discussed here the restrictions on the hydrodynamic regime are less stringent.

In the case of smectics A, e.g., we find a spacing of the smectic layers of the order of 10 Å and therefore it is possible to carry out light scattering experiments in smectics in the hydrodynamic regime in the visible region of the spectrum.

2.2 HYDRODYNAMIC VARIABLES.

— The first problem we have to face when we formulate the hydrodynamic equations for a given system is the determination of the hydrodynamic variables, i.e. we have to find out those variables having variations that are slow in time and space. First of all there are local densities of conserved quantities. The temporal evolution of these densities is given by local conservation laws which follow from the continuous symmetries of any nonrelativistic system. These are the most

familiar hydrodynamic variables : density ρ , energy ε , momentum density $\vec{g}$.

Following Martin, Parodi, Pershan [19] $\vec{g}$ is the total momentum density and not only the momentum density associated with the centre of mass. Theories of liquid crystals distinguishing between momentum due to rotation about the centre of mass and momentum due to centre of mass motion would require the introduction of additional variables and an additional equation of motion for an angular-momentum type quantity. But such a treatment inevitably yields modes which decay in microscopic times and therefore it is not possible to justify the inclusion of these additional variables in a purely hydrodynamic theory.

In addition to the variables discussed up to now there is another class of hydrodynamic variables occurring in systems with spontaneously broken continuous symmetries.

In such systems there exist variables $\{x^\alpha\}$ describing the broken symmetries, for example in nematics $\{x^\alpha\} = \vec{n}_\alpha$ the director, or in antiferromagnets $\{x^\alpha\} = \vec{N}$, the staggered magnetization. Since homogeneous changes of these additional variables cause no

restoring forces, gradients $\{x^\alpha\}$ will enter the free energy F . Their restoring forces $\partial F / \partial \nabla_i x^\alpha$ will tend to zero with wave number k , leading to hydrodynamic, long wavelength and low frequency excitations [21, 29]. For a free energy of the form

$$F = K_{ij}^{\alpha\beta}(\{x^\gamma\}) (\nabla_i x^\alpha) (\nabla_j x^\beta),$$

which depends explicitly on $\{x^\alpha\}$, e.g. the Frank free energy for nematics, not only $d\nabla_i x^\alpha$, but also dx^α changes the free energy. The restoring force on the latter, however, tends to zero with k^2 and is nonlinear in $\{\nabla_i x^\alpha\}$. In the following we will take into account both types of energy changes. There is even the possibility that higher order derivatives of $\{x^\alpha\}$ ($\nabla_i \nabla_j x^\alpha$) have an influence on the dynamics.

2.3 FUNDAMENTAL RELATION. — The hydrodynamic description is based on the assumption of local thermodynamic equilibrium. Therefore, we can apply conventional thermodynamics, locally. Choosing the energy density ε as a thermodynamic potential we have

$$\varepsilon V = E = E(V, \rho V, g_i V, \sigma V, \rho V \nabla_i x^\alpha, \rho V x^\alpha, \rho V \nabla_i \nabla_j x^\alpha) \quad (2.1)$$

where V is the volume and $\nabla_i x^\alpha$ is assumed to be an intensive variable (i.e. a mass density not a volume density). Latin indices characterize vector and tensor components, while greek indices denote the various spontaneously broken symmetries of the system. Summation over repeated indices is always implied if not states otherwise. With the help of Euler's relation we obtain for bulk hydrodynamics

$$d\varepsilon = \mu d\rho + T d\sigma + V_i dg_i + \Phi_i^\alpha d\nabla_i x^\alpha + h^\alpha dx^\alpha + \psi_{ij}^\alpha d\nabla_i \nabla_j x^\alpha \quad (2.2)$$

and

$$p = -\varepsilon + \mu\rho + T\sigma + V_i g_i. \quad (2.3)$$

It should be noted that the observables in eq. (2.2) vary in space and time. The changes $d\dots$ refer to distinct points in space and time ($x(t), t$). If specialized to temporal changes, $d\dots$ is, therefore, the material derivative $d/dt = \partial/\partial t + V_k \nabla_k$.

The thermodynamic conjugates or generalized forces are given by

$$\begin{aligned} \mu &\equiv \left. \frac{\partial \varepsilon}{\partial \rho} \right|_{\sigma, g_i, x^\alpha, \nabla_i x^\alpha, \nabla_i \nabla_j x^\alpha}; & T &\equiv \left. \frac{\partial \varepsilon}{\partial \sigma} \right|_{\rho, g_i, x^\alpha, \nabla_i x^\alpha, \nabla_i \nabla_j x^\alpha} \\ V_i &\equiv \left. \frac{\partial \varepsilon}{\partial g_i} \right|_{\rho, \sigma, x^\alpha, \nabla_i x^\alpha, \nabla_i \nabla_j x^\alpha}; & \Phi_i^\alpha &\equiv \left. \frac{\partial \varepsilon}{\partial \nabla_i x^\alpha} \right|_{\rho, \sigma, g_i, x^\alpha, \nabla_i \nabla_j x^\alpha} \\ h^\alpha &\equiv \left. \frac{\partial \varepsilon}{\partial x^\alpha} \right|_{\rho, \sigma, g_i, \nabla_i x^\alpha, \nabla_i \nabla_j x^\alpha}; & p &\equiv - \left. \frac{\partial E}{\partial V} \right|_{\rho V, \sigma V, g_i V, x^\alpha V, \nabla_i x^\alpha V, \nabla_i \nabla_j x^\alpha} \\ \psi_{ij}^\alpha &\equiv \left. \frac{\partial \varepsilon}{\partial \nabla_i \nabla_j x^\alpha} \right|_{\rho, \sigma, x^\alpha, \nabla_i x^\alpha, g_i} \end{aligned} \quad (2.4)$$

where Φ_i^α, h^α tend to zero in the homogeneous limit; they are derived from the same gradient energy (cf. the last section).

If $\{y^\alpha\}$ is an extensive variable, the chemical potential μ is replaced by

$$\tilde{\mu} \equiv \mu - \rho^{-1} \Phi_i^\alpha \nabla_i y^\alpha - \rho^{-1} h^\alpha y^\alpha - \rho^{-1} \psi_{ij}^\alpha \nabla_i \nabla_j y^\alpha$$

in (2.2) and the pressure is given by

$$p = -\varepsilon + \tilde{\mu}\rho + V_i g_i + T\sigma + \Phi_i^\alpha \nabla_i y^\alpha + h^\alpha y^\alpha + \psi_{ij}^\alpha \nabla_i \nabla_j y^\alpha. \quad (2.5)$$

The Gibbs relation (2.2) is the fundamental starting point of all hydrodynamic theories.

2.4 EQUATIONS OF MOTION. — The equations of motion for the conserved quantities can be written in the form

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla_i g_i &= 0 \\ \frac{\partial \sigma}{\partial t} + \nabla_i \left(\sigma V_i + \frac{q_i}{T} \right) &= \frac{R}{T} \\ \frac{\partial g_i}{\partial t} + \nabla_j (g_j V_i + p \delta_{ij}) + \nabla_j \sigma_{ij} &= 0 \\ \frac{\partial \varepsilon}{\partial t} + \nabla_i [(\varepsilon + p) V_i] + \nabla_j (\sigma_{ij} V_i) + \nabla_i j_i^\varepsilon &= 0 \end{aligned} \quad (2.6)$$

or

$$\begin{aligned} \frac{d\rho}{dt} &= -\rho \nabla_i V_i \\ \frac{d\sigma}{dt} &= -\sigma \nabla_i V_i - \nabla_i \left(\frac{q_i}{T} \right) + \frac{R}{T} \\ \frac{dg_i}{dt} &= -g_i \nabla_j V_j - \nabla_j \sigma_{ij} - \nabla_i p \\ \frac{d\varepsilon}{dt} &= -(\varepsilon + p) \nabla_i V_i - \nabla_j (V_i \sigma_{ij}) - \nabla_i j_i^\varepsilon - V_i \nabla_i p. \end{aligned} \quad (2.7)$$

As it is obvious from the structure of these equations the terms which are already present in simple fluids have been split off and Galilei invariance has been taken into account explicitly ($g_i = \rho V_i$ for liquid crystals and crystals).

For the variables characterizing the broken symmetries the dynamic equations read ⁽¹⁾, ⁽²⁾

$$\frac{\partial}{\partial t} x^\alpha + V_j \nabla_j x^\alpha + Z^\alpha = 0 \quad (2.8a)$$

or

$$\frac{d}{dt} x^\alpha = -Z^\alpha. \quad (2.8b)$$

Eqs. (2.6) or eqs. (2.7), respectively, are not independent of each other, but they are connected by the Gibbs relation (2.2). The currents q_i , j_i^ε , σ_{ij} and Z^α and the dissipation function R (which acts as a source term) consist in the most general case of reversible as well as irreversible and linear as well as non-linear contributions.

Because the quantities q_i , j_i^ε , σ_{ij} and Z^α contain the terms which account for the $\{x^\alpha\}$ and thus reflect the symmetries of the specific hydrodynamic system under consideration they cannot be considered for all materials simultaneously. Therefore we must discuss the explicit structure of q_i , j_i^ε , σ_{ij} and Z^α for each system separately. Nevertheless the currents have to satisfy a number of constraints which will be discussed in the next section.

2.5 CONSTRAINTS ON THE CURRENTS. — In general the additional terms appearing in eqs. (2.6), (2.8) will be connected with phenomenological parameters (reversible and irreversible transport parameters), which are subject to the following constraints on the possible structure of the currents :

i) Entering the Gibbs relation (2.2) with eqs. (2.6) and (2.8) we obtain

$$R = \nabla_i (q_i - j_i^\varepsilon) - \frac{q_i}{T} \nabla_i T + h^\alpha Z^\alpha + \Phi_i^\alpha \nabla_i Z^\alpha + \psi_{ij}^\alpha \nabla_i \nabla_j Z^\alpha - \Sigma_{ij} \nabla_j V_i \quad (2.9)$$

where

$$\Sigma_{ij} = \sigma_{ij} - \Phi_j^\alpha \nabla_i x^\alpha - \psi_{kj}^\alpha \nabla_k \nabla_i x^\alpha - \psi_{jk}^\alpha \nabla_i \nabla_k x^\alpha.$$

In the reversible case eq. (2.9) simplifies further

$$\nabla_i j_i^{R\varepsilon} = h^\alpha Z^{R\alpha} + \Phi_i^\alpha \nabla_i Z^{R\alpha} + \psi_{ij}^\alpha \nabla_i \nabla_j Z^{R\alpha} - \Sigma_{ij}^R \nabla_j V_i \quad (2.10)$$

where

$$\Sigma_{ij}^R = \sigma_{ij}^R - \Phi_j^\alpha \nabla_i x^\alpha - \psi_{kj}^\alpha \nabla_k \nabla_i x^\alpha - \psi_{jk}^\alpha \nabla_i \nabla_k x^\alpha$$

the superscript R denotes the reversible part of the corresponding currents.

ii) In the equations for the conserved variables we have already taken into account Galilei invariance. Thus the additional currents σ_{ij} and j_i^ε are not allowed to contain terms proportional to $\vec{v}$ whereas terms proportional to $\nabla \vec{v}$ are still possible. For the equations of the nonconserved variables Galilei invariance will be discussed for each hydrodynamic system separately.

iii) In order to guarantee conservation of angular

⁽¹⁾ One could also obtain $\frac{d}{dt} \nabla_i x^\alpha = -\nabla_i Z^\alpha$. However, this convective term $V_k \nabla_k \nabla_i x^\alpha$ implies that there are more than α independent dynamical equations and that $\nabla_i x^\alpha$ is not always a gradient field ($\partial/\partial t (\vec{\nabla} \times \vec{\nabla} x^\alpha) \neq 0$). For both reasons $\frac{d}{dt} \nabla_i x^\alpha = -\nabla_i Z^\alpha$ has to be rejected.

⁽²⁾ When dealing with extensive variables $\{y^\alpha\}$ instead of the intensive ones $\{x^\alpha\}$, eq. (2.8a) must be replaced by

$$\dot{y}^\alpha + \nabla_i (V_i y^\alpha) = -\rho Z^\alpha.$$

momentum locally, the total stress tensor and, thus, σ_{ij} must satisfy [49, 19, 50]

$$\sigma_{ij} = \frac{1}{2}(\sigma_{ij} + \sigma_{ji}) + \nabla_k \Pi_{ijk} \quad (2.11)$$

with

$$\Pi_{ijk} = -\Pi_{jik}.$$

3. Nonlinear reversible hydrodynamic equations.

— 3.1 NEMATIC LIQUID CRYSTALS. — In this chapter we rederive concisely the nonlinear reversible hydrodynamics of nematic liquid crystals. Although the results are already well known [15, 16], we present them here for tutorial reasons and since we can transfer some of the considerations to our treatment of smectic liquid crystals in the following chapters.

From a macroscopic point of view nematic liquid crystals are liquids with spontaneously broken rotational invariance. This broken symmetry shows up in the existence of a preferred axis, whose direction is described by the unit vector $\vec{n}$. According to the discussion in paragraph 2.3, the Gibbs relation (2.2) takes the form

$$d\varepsilon = \mu d\rho + T d\sigma + V_i dg_i + \Phi_{ij} d\nabla_j n_i + h_i dn_i \quad (3.1)$$

where Φ_{ij} and h_i have to vanish in the homogeneous limit $k \rightarrow 0$ like k and k^2 , respectively; higher gradient terms $\sim d\nabla_j \nabla_k n_i$ can be neglected in nematics. However, since $\vec{n}^2 = 1$, only those δn_i for which

$$n_i \delta n_i = 0 \quad (3.2)$$

enter the hydrodynamic description. The constraint (3.2) projects out the components of $\vec{n}$, which are not connected with a broken symmetry. Indeed, the rotational symmetry is not completely broken, since rotations about $\vec{n}$, which are described by $\delta\vec{n} \propto \vec{n}$, are still possible.

By choosing the vector $\vec{n}$ for the description of the nematic state, we lost the fact, that for the preferred axis head and tail are undistinguishable [15]. Thus,

we have to postulate, that all hydrodynamic equations are invariant under the transformation

$$\vec{n} \leftrightarrow -\vec{n}. \quad (3.3)$$

Now we have to specify the yet unknown current Z_i in the hydrodynamic equation for n_i

$$\frac{\partial}{\partial t} n_i + V_k \nabla_k n_i + Z_i = 0 \quad (3.4)$$

$\vec{n}$ is even under time reversal as well as under spatial inversion (parity). With the help of these symmetry arguments and taking into account the constraints (3.2) and (3.3) we find the following reversible contributions to Z_i :

$$Z_i^{(1)} = \alpha \delta_{ij}^{\text{tr}} n_k (\nabla_l V_k + \nabla_k V_l) \quad (3.5)$$

with

$$\delta_{ij}^{\text{tr}} = \delta_{ij} - n_i n_j$$

which is a straightforward generalization of the linearized expression [29]. Note that the phenomenological parameter α may yet be a function of the scalar variables of the system (e.g. ρ, σ or p, T).

As a second term in the expression for Z_i we get

$$Z_i^{(2)} = \frac{\beta}{2} \delta_{ij}^{\text{tr}} n_k (\nabla_l V_k - \nabla_k V_l) = -\beta \delta_{ij}^{\text{tr}} (\vec{\omega} \times \vec{n})_l \quad (3.6)$$

with $\vec{\omega} = 1/2 \text{curl } \vec{v}$. In linearized theories it is argued, that β must be equal to 1, since a rigid rotation of a nematic sample as a whole is not allowed to contribute to the hydrodynamics of $\vec{n}$ [15]. The subtraction of this global rotation is just achieved by $\beta = 1$ (or by the replacement $\frac{d}{dt} \rightarrow \frac{d}{dt} - \vec{\omega} \times$ in eq. (3.4)). In nonlinear theories the result $\beta = 1$ can strictly be derived from angular momentum conservation. We proceed, therefore, in deriving the stress tensor with the parameter β yet unspecified.

Vanishing entropy production requires counter terms to $Z_i^{(1)}$ and $Z_i^{(2)}$ in σ_{ij} and j_i^e , which are easily found by eq. (2.10)

$$\begin{aligned} \sigma_{ij} &= \alpha [n_i (h_j - \nabla_l \Phi_{jl}) + (i \leftrightarrow j)] + \frac{\beta}{2} [n_i (h_j - \nabla_l \Phi_{jl}) - (i \leftrightarrow j)] + \Phi_{kj} \nabla_i n_k \\ j_i^e &= \Phi_{ki} (Z_k^{(1)} + Z_k^{(2)}). \end{aligned} \quad (3.7)$$

The antisymmetric part of σ_{ij} must be zero or a total divergence in order to preserve angular momentum (cf. eq. (2.11)). However

$$\begin{aligned} \sigma_{ij}^{\text{anti}} \varepsilon_{ijk} &= [-\beta \nabla_k n_i \Phi_{jk} + \beta n_i h_j + \beta \Phi_{jk} \nabla_k n_i + \Phi_{kj} \nabla_i n_k] \varepsilon_{ijk} \\ &= [-\beta \nabla_k n_i \Phi_{jk} + (\beta - 1) (h_j n_i + \Phi_{jk} \nabla_k n_i)] \varepsilon_{ijk}. \end{aligned}$$

This equality holds since the energy ε is rotational invariant, i.e.

$$d\varepsilon = 0 = h_i \Omega_{ij} n_j + \Phi_{ij} \Omega_{jk} \nabla_k n_i + \Phi_{ij} \Omega_{ik} \nabla_j n_k \quad (3.8)$$

for any antisymmetric matrix Ω_{ij} [15, 16].

Thus, only $\beta = 1$ fulfills condition (2.11). In conclusion, the reversible currents read

$$\begin{aligned} Z_i &= (\alpha + \frac{1}{2}) n_i (h_j - \nabla_l \Phi_{jl}) + (\alpha - \frac{1}{2}) n_j (h_i - \nabla_l \Phi_{il}) \\ \sigma_{ij} &= \alpha [n_i (h_j - \nabla_l \Phi_{jl}) + n_j (h_i - \nabla_l \Phi_{il})] + \frac{1}{2} [\Phi_{kj} \nabla_i n_k + \Phi_{ki} \nabla_j n_k] - \frac{1}{2} \nabla_k (n_i \Phi_{jk} - n_j \Phi_{ik}) \\ j_i^e &= n_i \Phi_{ki} [(\alpha + \frac{1}{2}) \nabla_k V_l + (\alpha - \frac{1}{2}) \nabla_l V_k] . \end{aligned} \quad (3.9)$$

For the irreversible contributions to Z_i , σ_{ij} , j_i^e and q_i we refer to existing linear and nonlinear theories [15, 17, 28, 45].

The set of hydrodynamic eqs. (2.6), (3.4) and (3.8) has to be closed by equations of state linking the generalized forces with the variables :

$$\begin{aligned} \vec{V} &= \frac{1}{\rho} \vec{g} \\ \delta\mu &= \lambda \delta\rho + \gamma \delta\sigma \\ \delta T &= T C_v^{-1} \delta\sigma + \gamma \delta\rho \\ \Phi_{ij} &= K_{ijlm} \nabla_m n_l \end{aligned} \quad (3.10)$$

and

$$h_i = \left. \frac{\partial K_{ijlm}}{\partial n_i} \right|_{\nabla_n} (\nabla_j n_k) (\nabla_m n_l)$$

with

$$\lambda \equiv \frac{\partial^2 \varepsilon}{\partial \rho^2}; \quad \gamma \equiv \frac{\partial^2 \varepsilon}{\partial \sigma \partial \rho}; \quad C_v \equiv \frac{\partial^2 \varepsilon}{\partial \sigma^2}$$

$$\begin{aligned} K_{ijlm} &\equiv \frac{\partial^2 \varepsilon}{\partial (\nabla_m n_l) \partial (\nabla_j n_i)} = K_1 \delta_{jl}^{\text{tr}} \delta_{im}^{\text{tr}} + \\ &+ K_2 \varepsilon_{ijp} n_p \varepsilon_{lmq} n_q + K_3 \delta_{il}^{\text{tr}} n_j n_m \end{aligned}$$

where the partial derivatives are performed while keeping the appropriate variables of $\vec{g}$, ρ , σ , n_i and $\nabla_j n_l$ constant. The expression $\varepsilon = \frac{1}{2} K_{ijlm} (\nabla_j n_i) (\nabla_m n_l)$ corresponds to the well-known Frank free energy and K_1 , K_2 and K_3 are the Frank elastic constants [15, 29].

It should be stressed, that nowhere in the above derivation of the hydrodynamic equations the director $\vec{n}$ was restricted to be constant in equilibrium. Thus, we may apply equations (3.4) to (3.10) to hydrodynamic motions in inhomogeneous textures, as well.

3.2 CRYSTALS. — In this section we will derive the nonlinear reversible hydrodynamics for a crystal, which is the simplest example of a system with spontaneously broken translational symmetry. We only specify the occurring tensors (susceptibilities, transport parameters) for the case of isotropic solids; a generalization to the various lattice structures is straightforward. We deal with imperfect solids in the sense, that in equilibrium it is not required that each lattice site is occupied by one atom [19].

Translations are described by the Galilei invariant vector $\vec{R}$, which characterizes displacements of the lattice sites [19, 23, 29]. In solids the translational

invariance in all three space directions is spontaneously broken.

Then, the Gibbs relation takes the form

$$d\varepsilon = T d\sigma + \rho d\mu + \vec{V} \cdot d\vec{g} + \Phi_{ij} d\nabla_j R_i \quad (3.11)$$

where $\Phi_{ij} = \Phi_{ji}$ as can easily be seen from the structure of the elastic energy.

A translation of the solid as a whole does not contribute to the hydrodynamic motion. Therefore, unlike R , $\dot{R}$ is not a Galilei invariant quantity and one arrives at the following dynamical equation for R

$$\dot{R}_i + V_k \nabla_k R_i - V_i + Z_i = 0 . \quad (3.12)$$

The existence of the term $-V_i$ in the equation of motion for R_i has already been discussed by Martin, Parodi and Pershan. It can also be justified in a microscopic framework. Using Mori's projector formalism one obtains for the elements of the frequency matrix coupling R_i and g_j

$$\omega_{R_i g_j} = \alpha_{ij} .$$

In the isotropic case one can choose $\alpha_{ij} = \delta_{ij}$ without loss of generality (cf. ref. [29] for a detailed discussion of this point). The phenomenological current Z_i which is determined by symmetry considerations (Z_i transforms like a velocity) reads

$$Z_i = \alpha_{ijkl} R_j \nabla_k V_l . \quad (3.13)$$

This term can contain R_i (and not $\nabla_j R_i$) since it is not derived from a free energy. Of course, for a homogeneous translation this term vanishes ($\nabla_i V_j = 0$).

Vanishing entropy production requires then (cf. eq. (2.10))

$$\sigma_{ij} = -\Phi_{ij} - \alpha_{kmji} R_m \nabla_l \Phi_{kl} + \Phi_{ij} \nabla_l R_j \quad (3.14)$$

and

$$j_i^e = \alpha_{jmk} R_m \Phi_{ji} \nabla_k V_l$$

while angular momentum conservation is guaranteed by (cf. eq. (2.11))

$$\alpha_{ijkl} = \alpha_{ijlk} . \quad (3.15)$$

The α -tensor is less symmetric than the elasticity tensor in Hooke's law, since $\alpha_{ijkl} \neq \alpha_{jikl}$ in general. Hence, α_{ijkl} will contain, as a rule, more phenomenological parameters, generally.

For isotropic solids, however, the tensor α_{ijkl} contains two independent phenomenological parameters $\alpha_{ijkl} = \alpha_1 \delta_{ij} \delta_{kl} + \alpha_2 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$, where α_1 and α_2 are still allowed to be functions of the scalar variables (ρ , σ or p , T ; and $\bar{R}^2$).

The unusual and new nonlinear terms provide a

(reversible) coupling of the linear modes e.g. transverse and longitudinal sound. They allow for instabilities and they will play an important role in the melting process as well.

Finally, the equations of state close the set of equations

$$\begin{aligned}\Phi_{ij} &= \hat{\chi}_{ijkl} \nabla_l R_k + \hat{\gamma}_{ij}^{(1)} \delta \rho + \hat{\gamma}_{ij}^{(2)} \delta \sigma \\ \delta T &= T C_v^{-1} \delta \sigma + \gamma \delta \rho + \hat{\gamma}_{ij}^{(2)} \nabla_j R_i \\ \delta \mu &= \lambda \delta \rho + \gamma \delta \sigma + \hat{\gamma}_{ij}^{(1)} \nabla_j R_i\end{aligned}\quad (3.16)$$

where the susceptibilities γ , λ , C_v , $\hat{\chi}_{ijkl}$ and $\hat{\gamma}_{ij}^{(\nu)}$ ($\nu = 1, 2$) are, as usual, defined as second partial derivatives of the energy density. Especially we have

$$\begin{aligned}\hat{\chi}_{ijkl} &= \chi_{ijkl} + \frac{1}{2} \{ \chi_{ijpl} \nabla_p R_k + \chi_{pjkl} \nabla_p R_i + \chi_{pikl} \nabla_p R_j \} \\ \hat{\gamma}_{ij}^{(\nu)} &= \gamma_{ij}^{(\nu)} + \frac{1}{2} \{ \gamma_{pj}^{(\nu)} \nabla_p R_i + \gamma_{pi}^{(\nu)} \nabla_p R_j \}\end{aligned}\quad (3.17)$$

where χ_{ijkl} and $\gamma_{ij}^{(\nu)}$ are the corresponding susceptibilities of linear elasticity ⁽³⁾, ⁽⁴⁾. The fact that there occur no additional phenomenological parameters with the nonlinear contributions is due to the structure of the nonlinear elastic energy.

For isotropic solids $\gamma_{ij}^{(1,2)} = \gamma^{(1,2)} \delta_{ij}$ and

$$\chi_{ijkl} = \chi_1 \delta_{ij} \delta_{kl} + \chi_2 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}).$$

In general the susceptibility tensor has the same structure and contains the same number of independent phenomenological parameters as the elasticity tensor, since

$$\chi_{ijkl} = \chi_{klij} = \chi_{jikl} = \chi_{jilk}.$$

Eqs. (3.12)-(3.15) and eq. (2.6) constitute the complete nonlinear reversible hydrodynamics of imperfect crystals to order k^2 ⁽⁵⁾.

3.3 SMECTICS A LIQUID CRYSTALS. — Smectic A liquid crystals are characterized macroscopically by a spontaneously broken translational symmetry in one direction [15]. This preferred direction is denoted by the unit vector $\vec{n}$. Consequently, smectics A liquid

crystals are similar to crystals in the $\vec{n}$ -direction, and to ordinary liquids in the plane perpendicular to $\vec{n}$.

Like in nematics the state of the system is equally well described by $-\vec{n}$ instead of $\vec{n}$; thus the hydrodynamic equations have to show a $\vec{n} \leftrightarrow -\vec{n}$ invariance.

The translation is described as in the last chapter by the displacement of the layers $\bar{R}$.

Since the translational invariance is broken only in the $\vec{n}$ -direction we have to impose the constraint

$$\vec{n} \times \vec{R} = 0. \quad (3.18)$$

Clearly, for a linearized theory the quantity $n_{\bar{k}} \nabla_j R_i$ reduces to the usual variable $\nabla_j x$, with $x = \vec{n} \cdot \vec{R}$ [19].

i) *Smectics A without external fields.* — When orientating effects on $\vec{n}$ (external fields, walls, etc.) are absent, the dynamics of smectics A consists of several contributions. First there are fluctuations of the layers parallel $\vec{n}$ which would even be present if $\vec{n}$ could be completely fixed, thus leading to $\Phi_{ij} d\nabla_j R_i$ in the Gibbs relation. Second, the direction of $\vec{n}$ can fluctuate as well, i.e. there occur rotations of $\vec{n}$. Like in nematics, homogeneous rotations of $\vec{n}$ do not change the free energy. Thus the Gibbs relation contains terms $\psi_{ij} d\nabla_j n_i$, where ψ_{ij} is proportional to $\nabla_i n_k$. However, there is also a term proportional to $\nabla_i n_k$ present in the Gibbs relation, since the gradient energy ε_g (cf. ref. [51]) of the R 's depends on $\vec{n}$ leading to a contribution to $d\varepsilon = h_i dn_i$ with h_i depending on $(\nabla_i R_j)$ $(\nabla_i R_m)$ and vanishing with k^2 . From simple geometric considerations one finds

$$\delta n_i = -n_k \nabla_i^{\text{tr}} R_k \quad (n_i \delta n_i = 0)$$

for infinitesimal changes δn_i [15].

⁽³⁾ The necessity to take into account nonquadratic contributions in the elastic energy has been pointed out to the authors by one of the referees.

⁽⁴⁾ If one considers in the free energy terms which are cubic in the strain ($\varepsilon^{(3)} = \psi_{ijklmn} \varepsilon_{ij} \varepsilon_{kl} \varepsilon_{mn}$), one gets further contributions to Φ_{ij}

$$\Phi_{ij}^{(3)} = \frac{\partial \varepsilon^{(3)}}{\partial \nabla_i R_j}$$

which are quadratic in ∇R : Usually the ψ_{ijklmn} are very small and therefore we refrain from including them.

⁽⁵⁾ When expanding the currents and the dynamic equations in k , the multiplication of the highest order terms (in k) in these expansions yields contributions which are not systematic. Therefore the terms proportional k^3 in (3.14) must be discarded.

Replacing δn_i by gradients of $\vec{R}$ via this linearized relation is sufficient to give the terms cubic in $\nabla_i R_j$ and quadratic in $(\nabla_i \nabla_j R_k)$ in the free energy. In the following $\vec{n}$ means always its equilibrium value, which we assume to be constant. Thus the Gibbs relation can be written

$$d\varepsilon = T d\sigma + \mu d\rho + \vec{V} \cdot d\vec{g} + \Phi_{ij} d\nabla_j R_i + \psi_{ijk} d\nabla_k \nabla_j R_i. \quad (3.19)$$

There is no term proportional to dR_i in the Gibbs relation since the gradient energy depends not on R_i . Furthermore it should be noticed that in the linearized theory only gradients $\nabla \vec{n}$ enter the description because of the condition

$$\nabla_i R_j = \nabla_j R_i$$

which excludes rotations.

In the nonlinear theory both, rotations and translations, enter the Gibbs relation.

The thermodynamic force Φ_{ij} splits up into a longitudinal and a transverse (with respect to $\vec{n}$) part

$$\Phi_{ij} = n_i(n_j \Phi + h_j). \quad (3.20)$$

For the term $d\nabla_j \nabla_k R_i$ it is sufficient to keep only transverse gradients because longitudinal gradients are already present to lower order in k ; nonlinearities would be of even higher order in k .

Thus ψ_{ijk} can be written

$$\psi_{ijk} = n_i \delta_{jk}^{\text{tr}} \psi$$

with

$$\psi = K_1 n_i \delta_{jk}^{\text{tr}} \nabla_j \nabla_k R_i. \quad (3.21)$$

The equations of state which express the thermodynamic forces in terms of the variables read

$$\begin{aligned} V_i &= \frac{1}{\rho} g_i \\ \delta T &= T C_v^{-1} \delta \sigma + \gamma \delta \rho + \gamma_2 (n_i n_j \nabla_i R_j - \frac{1}{2} n_m n_j (\nabla_i^{\text{tr}} R_m) (\nabla_i^{\text{tr}} R_j)) \\ \delta \mu &= \lambda \delta \rho + \gamma \delta \sigma + \gamma_1 (n_i n_j \nabla_i R_j - \frac{1}{2} n_m n_j (\nabla_i^{\text{tr}} R_m) (\nabla_i^{\text{tr}} R_j)) \\ \Phi &= \chi_{\parallel} (n_i n_j \nabla_j R_i - \frac{1}{2} n_k n_i (\nabla_m^{\text{tr}} R_k) (\nabla_m^{\text{tr}} R_i)) + \gamma_1 \delta \rho + \gamma_2 \delta \sigma \\ h_j &= - \delta_{jr}^{\text{tr}} (\nabla_r R_i) n_i [\chi_{\parallel} n_i n_m \nabla_m R_i + \gamma_1 \delta \rho + \gamma_2 \delta \sigma]. \end{aligned} \quad (3.22)$$

The δ 's denote deviations from the equilibrium values. A static linear coupling between Φ and $\delta \rho$ has already been introduced by de Gennes [15].

The equation of motion for $\vec{R}$ reads $\left(\frac{d}{dt} R_i + Z_i = 0 \right)$

$$\frac{\partial R_i}{\partial t} + V_k \nabla_k R_i - n_i n_j V_j + f_{kl} R_i \nabla_k V_l = 0 \quad (3.23)$$

with $f_{kl} = f_1 \delta_{kl}^{\text{tr}} + f_2 n_k n_l$ where the phenomenological parameters f_1, f_2 may still depend on the scalar quantities ρ, σ or p, T and R^2 . Reversibility requires *via* eq. (2.10) for the stress tensor

$$\begin{aligned} \sigma_{ij} &= - n_i n_k \Phi_{kj} + f_{ji} R_i (- \nabla_k \Phi_{lk} + \nabla_k \nabla_m \psi_{lkm}) + n_i n_l \nabla_k \psi_{lkj} + \\ &\quad + \Phi_{kj} \nabla_i R_k + \psi_{ijk} \nabla_k \nabla_i R_l + \psi_{lkj} \nabla_k \nabla_l R_i \end{aligned} \quad (3.24)$$

and

$$j_i^{\text{e}} = - n_j n_l \psi_{jik} \nabla_k V_l + (\psi_{jim} \nabla_m - [\nabla_m \psi_{jmi}] + \Phi_{ji}) f_{kl} R_j \nabla_k V_l.$$

The symmetry condition (2.11) for the total stress tensor and thus for σ_{ij} remains to be demonstrated.

There are terms symmetric by construction (e.g. f_{ij} -terms). Furthermore all ψ_{ijk} -terms together can be brought into the required form. The remaining part

$$- n_i n_k \Phi_{kj} + \Phi_{kj} \nabla_i R_k$$

is symmetric up to order k^2 , thus guaranteeing angular momentum conservation for the lowest three orders in k : k^0, k^1 and k^2 . Higher order terms in k in σ_{ij} would be symmetrized by higher order terms in k in the current Z_i , which we have discarded, since we are discussing a hydrodynamic theory.

ii) *Influence of a magnetic field.* — We will now discuss the influence of an external, low frequency and homogeneous magnetic field. As usual [15] the magnetic field is taken into account perturbatively. The following considerations are equally valid for an electric field. The magnetic field couples to the director *via* an energy $\varepsilon_H = - \frac{1}{2} \chi_a (\vec{H} \cdot \vec{n})^2$ [15, 16, 29].

There are two possible equilibrium states :

$$\vec{H} \parallel \vec{n} \quad \text{and} \quad \vec{H} \perp \vec{n} \quad \text{for} \quad \chi_a > 0 \quad \text{and} \quad \chi_a < 0$$

respectively. In the former case fluctuations δn_i (with $n_i \delta n_i = 0$) give rise to energy fluctuations

$$d\varepsilon_H = \chi_a H^2 \delta n_i dn_i.$$

This additional contribution to the Gibbs relation is incorporated into our hydrodynamic equations by the replacement $(\delta n_i = -n_k \nabla_i R_k)$

$$\Phi_{ij} \rightarrow \Phi_{ij} + n_i h_j^H \quad (3.25)$$

with

$$h_j^H = \chi_a H^2 n_k \delta_{jm}^{\text{tr}} \nabla_m R_k \quad (\chi_a > 0).$$

In the latter case $\vec{H} \perp \vec{n}$, there are fluctuations δn_i normal to $\vec{n}$ and normal to $\vec{H}$. They are taken into account by

$$h_j^H = |\chi_a| H_j H_k n_l \nabla_k R_l \quad (\chi_a < 0). \quad (3.26)$$

The transverse gradients of R_i occur in the same k -order as the longitudinal gradients (cf. eqs. (3.22), (3.20)), but are of lower k -order than the transverse gradients considered in the last section. However, h_j^H is proportional to H^2 .

If H is sufficiently strong and k small enough, h_j of eq. (3.22) ($\chi_a > 0$) can be neglected in comparison with h_j^H of eq. (3.25); in that case the ψ_{ijk} -contributions are superfluous as well. Then the equations of motion are identical to those of section 3.3i) with the interpretation $\Phi_{ij} = n_i n_j \Phi + n_i h_j^H$.

In the case $\vec{H} \perp \vec{n}$, however, h_j^H of equation (3.26) ($\chi_a < 0$) contains only transverse gradients with respect to $\vec{n}$, which are parallel to $\vec{H}$, while h_j of eq. (3.22) contains as well gradients of $\vec{R}$, which are transverse to both, $\vec{n}$ and $\vec{H}$. Hence, h_j (and ψ_{ijk}) cannot be neglected completely even in strong magnetic fields. In addition the replacement (3.25) is not the only effect of magnetic fields. The biaxiality leads to a more complicated form of the tensors involved, e.g.

$$f_{ij} = f_1(\delta_{ij} - n_i n_j - H_i H_j H^{-2}) + f_2 n_i n_j + f_3 H_i H_j H^{-2}.$$

This biaxiality will show up in scattering experiments. In addition, for both cases there is the possibility, that all transport parameters, which are not explicitly denoted as constants, may depend on H^2 and $(\vec{H} \cdot \vec{n})^2$.

3.4 CHOLESTERIC LIQUID CRYSTALS. — From a macroscopic point of view smectics A and cholesterics are quite similar to each other, since the broken symmetry is the same in both cases. Indeed, the linearized hydrodynamics of smectics A and cholesterics turned out to be isomorph to the lowest order in k [19]. The rather different range of the hydrodynamic regime in both systems, however, was already mentioned above. In cholesteric liquids the director $\vec{n}$ is arranged in a helical structure in such a way, that $\vec{n}$ is parallel in each layer, but changes from layer to layer. The helical axis (pitch axis) $\vec{p}$, normal to the layers, denotes the direction, in which the translational symmetry is spontaneously broken. Thus, $\vec{p}$ is the analogue to $\vec{n}$ in smectics A.

Because of the close analogy between smectics A and cholesterics we can take over the Gibbs relation (3.18) and the equations of state (3.22) and (3.19) [51] of smectics A to cholesterics by the simple replacement $\vec{n} \rightarrow \vec{p}$. However, there is a difference between smectics A and cholesterics with respect to the symmetries. The mirror symmetry with respect to the layers in smectics A is not present in cholesterics, since left and righthanded helices are distinguishable from each other. Therefore, the hydrodynamic equations of cholesterics have not to show a $\vec{p} \leftrightarrow -\vec{p}$ invariance. For the same reason there exists a pseudoscalar q_0 , connected with the equilibrium pitch p_0 of the helix by $q_0 = 2\pi p_0^{-1}$ and defined by

$$q_0 \equiv -(\vec{n} \cdot \text{rot } \vec{n})_{\text{eq.}} \quad \text{or} \quad q_0 = (p_i \nabla_i \varphi)_{\text{eq.}}$$

[15, 21, 22], where φ is the twist angle ($\varphi \equiv q_0 \vec{p} \cdot \vec{r}$; $\vec{r}$ = position vector). The quantity q_0 is constant, since the equilibrium pitch is constant. The temperature and pressure dependence of the non-equilibrium pitch is contained in our description by the static coupling between $\nabla_j R_i$ and δT and $\delta \mu$ (cf. eq. (3.22) with $\vec{n} \rightarrow \vec{p}$) via the susceptibilities γ_2 and γ_1 respectively.

The lack of this invariance and the existence of q_0 allows for certain terms in the hydrodynamic equations having no analogues in smectics A. In detail we find

$$\begin{aligned} Z_i &= Z_i^A - \frac{p_i p_k}{2 q_0} (\text{curl } \vec{V})_k + g_{kj} q_0 p_i \nabla_k V_j \\ \sigma_{ij} &= \sigma_{ij}^A - \left(\frac{1}{2 q_0} \varepsilon_{ijk} + p_k g_{ji} q_0 \right) (\nabla_l \Phi_{kl} - \nabla_l \nabla_p \psi_{kpl}) + \beta \varepsilon_{jlk} p_s R_m (\nabla_k R_m) \nabla_l \Phi_{si} \\ j_i^E &= j_i^{EA} + [\Phi_{ji} - (\nabla_k \psi_{jki}) + \psi_{ijk} \nabla_k] \left[-\frac{1}{2 q_0} \varepsilon_{qji} + p_k q_0 g_{ji} \right] \end{aligned} \quad (3.27)$$

where Z_i^A , σ_{ij}^A (and j_i^{EA}) are the appropriate expressions for smectics A (cf. eqs. (3.23), (3.24) with $\vec{n}$ replaced by $\vec{p}$). The tensors f_{ij} and g_{ij} are of the axial form $A_{ij} = A_1(\delta_{ij} - p_i p_j) + A_2 p_i p_j$ thus containing four phenomenological transport coefficients f_1 , f_2 , g_1 ,

g_2 , which may still depend on all scalar variables (σ , ρ or p , T and R^2). In the structure of the tensor f_{ij} we disregard the term $f_3 \varepsilon_{ijk} p_k$, which is allowed by symmetry, but violates angular momentum conservation. The analogous term $g_3 \varepsilon_{ijk} p_k$ is contained in

equation (3.27) with $g_3 = -1/2 q_0^2$; this equality is obtained by the observation, that

$$\dot{\phi} = \frac{\partial}{\partial t} q_0 p_i R_i$$

describes a hydrodynamic motion only, if it exceeds the homogeneous rotation $\frac{1}{2} p_j p_i (\text{curl } \vec{V})_i$, which has therefore, to be subtracted [21].

The phenomenological transport parameter β is a constant; this term can be rewritten as

$$\beta \varepsilon_{ijk} \nabla_l (p_s R_m \Phi_{si} \nabla_k R_m),$$

thus, conserving angular momentum. In this term we have omitted a term of even higher order in k ($\Phi_{ij} \rightarrow \Phi_{ij} - \nabla_k \psi_{ijk}$).

It is the only term, which discriminates between right and lefthanded helices.

If linearized, eq. (3.27) agrees only in lowest order in k with Martin, Parodi and Pershan [15] and in addition with Lubensky [21] with respect to the curl $\vec{V}$ term. However, there is a further term, not vanishing by linearization, namely $g_{kl} q_0 p_i \nabla_k V_l$ which is of higher order in k than $-p_i p_j V_j$, but of the same order as the curl $\vec{V}$ -term and the additional, purely nonlinear terms. While the curl $\vec{V}$ -term couples R to the shear diffusion [21], the g_{kl} -terms contribute to the first sound spectrum, although in higher order in k than the $p_i p_j V_j$ -term. In the nonlinear case (q_{kl} depending on T or p) these g_{kl} -terms cause a coupling among all modes of the system (including heat conduction). In the terms containing q_0 and in the β -term of σ_{ij} the difference between the hydrodynamics of smectics A and cholesterics is shown up.

For the influence of an external, static and homogeneous magnetic field $\vec{H}$ we can discriminate between two cases:

$$\varepsilon_H = -\frac{1}{2} \chi_a (\vec{H} \cdot \vec{n}) \quad \text{with} \quad \chi_a > 0 \quad \text{or} \quad \chi_a < 0.$$

In the former case in equilibrium $\vec{H} \parallel \vec{n}$ is favoured, which destroys the cholesteric phase, since $\vec{n}$ is not uniform in equilibrium [15]. In the latter case $\vec{H} \perp \vec{n}$ which implies $\vec{H} \parallel \vec{p}$. This corresponds to the case $\chi_a > 0$ or $\vec{H} \parallel \vec{n}$ in smectics A. As was discussed there, the influence of the magnetic field can be described by the replacement

$$\Phi_{ij} \rightarrow \Phi_{ij} + |\chi_a| p_i p_k H^2 \delta_{jm}^{\text{tr}} \nabla_m R_k. \quad (3.28)$$

The Gibbs relation for smectics C can be written (cf. eq. (3.1) and (3.18))

$$d\varepsilon = T d\sigma + \mu dp + \vec{V} \cdot d\vec{g} + \Phi_{ij} d\nabla_j R_i + h_i dn_i + \chi_{ijk} d\nabla_k \nabla_j R_i + \psi_{ij} d\nabla_j n_i \quad (3.32)$$

with

$$\begin{aligned} \Phi_{ij} &= p_i p_k \frac{\partial \varepsilon}{\partial (\nabla_j R_k)} \bigg|_{\vec{n}, \nabla_j n_i, \sigma, \rho, g_i, \nabla_i \nabla_j R_k} \\ \psi_{ij} &= \delta_{ik}^3 \frac{\partial \varepsilon}{\partial (\nabla_j n_k)} \bigg|_{\vec{n}, \nabla_j R_i, \sigma, \rho, g_i, \nabla_i \nabla_j R_k} \\ h_i &= \delta_{ik}^3 \frac{\partial \varepsilon}{\partial n_k} \bigg|_{\nabla_j n_i, \nabla_j R_i, \sigma, \rho, g_i, \nabla_i \nabla_j R_k} \\ \chi_{ijk} &= \delta_{jm}^3 \delta_{kl}^3 p_i p_p \frac{\partial \varepsilon}{\partial (\nabla_l \nabla_m R_p)} \bigg|_{\vec{n}, \nabla_j n_i, \sigma, \rho, g_i, \nabla_i \nabla_j R_k} \end{aligned}$$

3.5 SMECTIC C LIQUID CRYSTALS. — Smectic C liquid crystals are, like smectics A, characterized by the existence of a layer structure. In the direction normal to the layers, denoted by the unit vector $\vec{p}$, the translational symmetry is spontaneously broken [19, 15]. The associated hydrodynamic variable is the displacement vector $\vec{R}$, with the constraint

$$\vec{p} \times \vec{R} = 0. \quad (3.29)$$

Like in smectics A rotations of $\vec{p}$ (which is analogue to $\vec{n}$ in smectics A) will be expressed by gradients of $\vec{R}$. In the following $\vec{p}$ always denotes its equilibrium value, which we assume to be constant.

Smectic C liquid crystals differ, however, from smectics A, because there the averaged direction of the axes of the molecules is parallel to the layer normal $\vec{p}$, while in smectics C the axes of the molecules are arranged parallel to a direction $\vec{n}$, which is tilted from $\vec{p}$ by a polar angle $\theta_0 = \cos^{-1}(\vec{n} \cdot \vec{p})$.

On account of the existence of the second preferred axis $\vec{n}$, rotational symmetry with respect to axes normal to $\vec{n}$ is spontaneously broken. But only those rotations δn_i enter hydrodynamics, which are normal to $\vec{n}$ (like in nematics) and normal to $\vec{p}$, since a homogeneous rotation of $\vec{n}$ about $\vec{p}$ does not change the energy of the system (there is a degeneracy with respect to the azimuth angle). Therefore, inhomogeneous rotations $\nabla_j n_i$ (with $0 = p_i \nabla_j n_i$) experience a restoring force vanishing in the homogeneous limit $k \rightarrow 0$. Thus, the additional hydrodynamic variable connected with the broken rotational symmetry in smectics C is n_i under the constraints [19]

$$p_i \delta n_i = 0 = n_i \delta n_i. \quad (3.30)$$

The constraints (3.30) guarantee, that the polar angle θ_0 is always constant, since fluctuations of θ_0 are associated with an elastic energy not vanishing in the homogeneous limit and thus, leading to non-hydrodynamic modes.

By the equilibrium structure of smectic C liquid crystals it is obvious, that only a simultaneous inversion of $\vec{n}$ and $\vec{p}$ does not change the configuration. Thus, all hydrodynamic equations must be invariant under the simultaneous replacements

$$n_i \leftrightarrow -n_i \quad \text{and} \quad p_i \leftrightarrow -p_i. \quad (3.31)$$

where

$$\delta_{ij}^3 = \delta_{ij} - n_i n_j - p_i p_j.$$

The gradient energy up to cubic nonlinearities and up to fourth order gradients in the quadratic terms reads

$$\begin{aligned} \varepsilon_g = & \frac{\chi_{||}}{2} (p_k p_l \nabla_k R_l)^2 - \frac{\chi_{||}}{2} p_k p_l (\nabla_k R_l) (\nabla_i R_m) (\nabla_j R_m) \delta_{ij}^3 + \\ & + [p_k p_l \nabla_k R_l - \frac{1}{2} (\nabla_i R_m) (\nabla_j R_m) \delta_{ij}^3] [\gamma_1 \delta\rho + \gamma_2 \delta\sigma] \\ & + B_{jl} \delta_{ik}^3 (\nabla_l n_k) (\nabla_j n_i) + C_{jl} \delta_{ik}^3 p_m p_q (\nabla_l n_k) (\nabla_j n_i) (\nabla_m R_q) \\ & + D_{jm} \delta_{il}^3 n_k (\nabla_i \nabla_j R_k) (\nabla_m n_l) + A (p_k \delta_{il}^3 \nabla_i \nabla_l R_k)^2 \end{aligned} \quad (3.33)$$

with

$$\begin{aligned} C_{ij} &= C_1 n_i n_j + C_3 \delta_{ij}^3 + C_2 p_i p_j + C_{12} (n_i p_j + n_j p_i) \\ B_{jl} &= B_1 n_j n_l + B_2 p_j p_l + B_3 \delta_{jl}^3 + B_{12} (n_j p_l + p_j n_l) \\ D_{jm} &= D_1 \tilde{n}_j \tilde{n}_m + D_3 \delta_{jm}^3 \\ \tilde{n}_k &= n_k - p_k \cos \theta_0. \end{aligned}$$

For its linear part (3.33) agrees with de Gennes [15]. Since θ_0 is held constant, $n_i \nabla_i R_j = 0$.

In order to set up the equations of motion we can take over the results obtained for smectics A and for nematics (under the stronger constraint (3.30)).

In addition, however, we must allow for the biaxiality of smectics C and we must look for terms coupling R_i and n_i directly. Finally we get

$$\frac{\partial}{\partial t} n_k + V_j \nabla_j n_k - \delta_{ik}^3 (\vec{\omega} \times \vec{n})_i + \delta_{km}^3 (\alpha_1 n_l + \alpha_2 p_l) (\nabla_m V_l + \nabla_l V_m) + f_5 p_j \delta_{km}^3 V_m \nabla_l (\Phi_{jl} - \nabla_i \chi_{jli}) = 0 \quad (3.34)$$

$$\frac{\partial}{\partial t} R_i + V_l \nabla_l R_i - p_i p_l V_l + f_{kl} R_i \nabla_k V_l + f_5 p_i \delta_{km}^3 V_k (h_m - \nabla_j \psi_{mj}) = 0$$

with

$$\begin{aligned} \omega_i &= \frac{1}{2} \varepsilon_{ilk} \nabla_l V_k \\ f_{kl} &= f_1 \delta_{kl}^3 + f_2 p_k p_l + f_3 n_k n_l + f_4 (n_k p_l + n_l p_k). \end{aligned}$$

Comparing eq. (3.34) with the analogous equations for nematic and smectics A we notice the following differences. Because of the constraint $p_i \delta n_i = 0$ not present in nematics, there occur δ_{km}^3 (instead of $\delta_{km} - n_k n_m$) in various terms of eq. (3.34). For the solid body rotation term $(\vec{\omega} \times \vec{n})$, this has the effect, that only $\vec{\omega}$ parallel to $\vec{p}$, occurs in the equation of motion. This is obvious, since in smectics C n_k describes rotations only about $\vec{p}$ and, therefore, the solid body rotation only about $\vec{p}$ has to be subtracted from $\dot{n}_i$. The biaxiality of smectics C shows up in the form of the tensor f_{kl} ($f_1 = f_3$ and $f_4 = 0$, for smectics A) and in the existence of the phenomenological parameter α_2 , which is zero in nematics ($\alpha_1 = \alpha$, there). The purely nonlinear terms with the phenomenological parameter f_5 describe a new, direct dynamical coupling of R_i and $\dot{n}_i$, i.e. a inhomogeneous coupling of translational fluctuations along $\vec{p}$ and rotational

fluctuations about $\vec{p}$ in the presence of a velocity $\vec{v}$ perpendicular to $\vec{p}$ and to $\vec{n}$. It should be noted, that all seven phenomenological reversible transport parameters $f_1, f_2, f_3, f_4, f_5, \alpha_1, \alpha_2$ may still depend on the scalar variables of the system, i.e. p, T or ρ, σ and R^2 , and especially on the scalar product $\vec{n} \cdot \vec{p}$.

The yet unspecified part of the stress tensor σ_{ij} is obtained with the help of eq. (2.10).

By means of the relation

$$\begin{aligned} p_l \varepsilon_{ijk} (\psi_{kj} \nabla_i n_k - n_i \nabla_k \psi_{jk} + n_i h_j) = \\ = p_l \varepsilon_{ijk} \nabla_k (n_i \psi_{jk}) \end{aligned} \quad (3.35)$$

which reflects the fact, that homogeneous rotations about $\vec{p}$ do not change the energy, and which is the analogue to eq. (3.8) in nematics, we can cast σ_{ij} into a form, required by eq. (2.11).

Finally, σ_{ij} reads

$$\begin{aligned} \sigma_{ij} = & - \Phi_{kj} p_i p_k + p_i p_l \nabla_k \psi_{lkj} - f_{ji} R_k (\nabla_l \Phi_{kl} - \nabla_l \nabla_m \chi_{klm}) + \frac{1}{2} (\psi_{kj} \nabla_i n_k + \psi_{ki} \nabla_j n_k) + \\ & + (\alpha_1 n_i + \alpha_2 p_i) (h_j - \nabla_k \psi_{jk}) + (\alpha_1 n_j + \alpha_2 p_j) (h_i - \nabla_k \psi_{ik}) + \frac{1}{2} \nabla_m (n_j \psi_{im} - n_i \psi_{jm}) \\ & + \Phi_{kj} \nabla_i R_k + \chi_{kjl} \nabla_l \nabla_i R_k + \chi_{lkj} \nabla_k \nabla_i R_l \end{aligned} \quad (3.36)$$

and

$$j_i^e = -p_j p_l \chi_{jik} \nabla_k V_l + (\chi_{jim} \nabla_m - [\nabla_m \chi_{jmi}] + \Phi_{ji}) f_{kl} R_j \nabla_k V_l + \\ + n_l \psi_{ki} [(\alpha_1 + \frac{1}{2}) \nabla_k V_l + (\alpha_1 - \frac{1}{2}) \nabla_l V_k] + p_l \psi_{ki} \alpha_2 (\nabla_k V_l + \nabla_l V_k) \\ + (\Phi_{ki} - [\nabla_j \chi_{kji}] + \chi_{kij} \nabla_j) f_5 p_k \delta_{mq}^3 V_q (h_m - \nabla_s \psi_{ms}) + \psi_{ki} f_5 p_s \delta_{km}^3 V_m \nabla_l (\Phi_{sl} - \nabla_j \chi_{slj}) .$$

An external magnetic field leads to the occurrence of the nonhydrodynamic variable n_i with $p_i \delta n_i \neq 0$ and to modes with a gap. In that case other nonhydrodynamic variables (e.g. $\delta\theta_0$) would have to be included. Such a treatment is beyond the aim of the present work.

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