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## Intermittency, self-similarity and $1/f$ spectrum in dissipative dynamical systems

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**Résumé.** — Nous étudions un système dynamique dissipatif discret qui présente une transition vers la turbulence par intermittence. Au seuil d'instabilité, ce modèle possède une structure d'homothétie interne analogue à celle des modèles de bruits de Mandelbrot. Le fait que l'on obtienne un spectre en  $1/f$  suggère que de tels spectres peuvent tirer leur origine d'une dynamique déterministe à court terme bien que non prédictible à long terme, au cœur de notre compréhension actuelle de la transition vers la turbulence.

**Abstract.** — We study a discrete dissipative dynamical system which presents a transition to turbulence *via* intermittency. At the instability threshold, this model displays a self-similar structure analogous to that of Mandelbrot's noise models. The  $1/f$  type of spectrum obtained suggests that such spectra can originate from dynamics deterministic in the short term though unpredictable in the long term, which is at the centre of the present understanding of the transition to turbulence.

**1. Introduction.** — Intermittency as a way to turbulence has recently been the subject of experimental [1] as well as theoretical [2] studies. In this article we present a simple but detailed analysis of a discrete dissipative dynamical system which displays this phenomenon and suggests a connection with the well-known widely observed  $1/f$  noise phenomenon. In the experiments either physical [1] or numerical [3, 4], intermittency presents itself as a seemingly random alternation of long regular or *laminar* phases and short irregular or *turbulent* bursts (see Fig. 1a). Present theoretical understanding rests on the theory and numerical simulation of dissipative dynamical systems [5]. Several examples are known which display the intermittency phenomenon [3, 4, 6]; one is the

celebrated Lorenz system [3]. Another one, obtained in modelling chemical reactions [6], is interesting from another point of view since it presents an  $1/f$  spectrum close to the instability threshold; one of our major points will be to show that this fact is not accidental but on the contrary typical of one of the different types of intermittency already discovered. Now a rapid glance at figure 1a is suggestive of figure 1b which is strongly reminiscent of excess noise in a transmission line as pictured by Mandelbrot [7]. Such a random noise is better understood in terms of the distribution of the duration of the *intermissions*, the time intervals between two *noisy events*. On the other hand, figure 1a was obtained from a model (to be described below) which is predictable in the short term though unpredictable in the long term. Working out its statistical properties will reveal that  $1/f$  noise can also have a deterministic origin.

In a first appendix we review briefly the essentials of the transition to turbulence *via* intermittency and we establish the connection with the model studied below. A second appendix is devoted to a discussion of the  $1/f$  spectrum described in reference [6] and to some further implications.

Even if most of the discussion is developed at a heuristic level, avoiding all delicate mathematical question, we think it may shed some light on currently still puzzling physical problems.



Fig. 1. — a) Example of intermittent signal obtained from the model defined by equation (1). b) Forgetting the underlying dynamics, one can consider this signal as a succession of noises whose distribution is of interest.

2. **The model ; some immediate properties.** — The model consists in a simple iteration :

$$x_{n+1} = f(x_n) = (1 + \varepsilon) x_n + (1 - \varepsilon) x_n^2 \pmod{1}. \quad (1)$$

Justifications for its interest are postponed to appendix I. As can be seen on figure 2,  $f$  maps the segment  $[0, 1[$  twice on itself. The origin  $x = 0$  is always a fixed point. It is stable for  $\varepsilon > 0$  : iterates of nearly all starting points will converge to it. On the contrary, when  $\varepsilon < 0$  neither  $f$  nor  $f \circ f$  nor  $f \circ f \circ f$ , etc... have stable fixed points ; let us see how the model works in this case. Define  $\tilde{x}$  by  $f(\tilde{x}) = 1$ , assume an initial condition  $x_0$  close to the origin and let the iteration proceed. The iterate  $x_n$  first increases gently but then more and more rapidly up to a point where

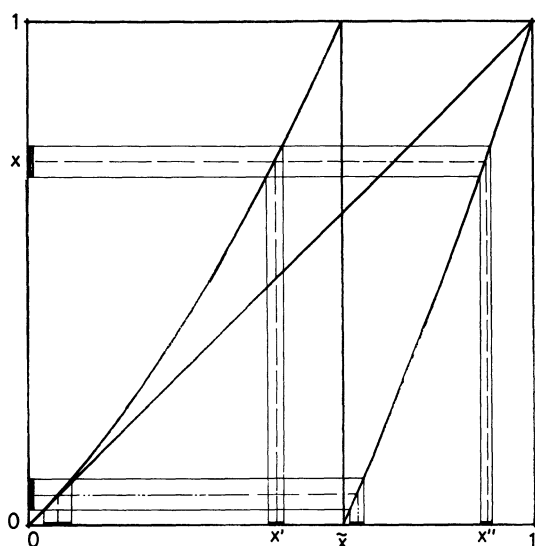


Fig. 2. — Graph of the transformation  $x \rightarrow f(x)$ . Each point has two antecedents ; for  $x$  far from the origin no singularity is expected in the invariant measure since the slope at either antecedent is much larger than 1. On the contrary for  $x$  close to the origin,  $x'$  and  $x$  are very close and one must have  $\mu(x) \sim \mu(x') \gg \mu(x'')$ .

it becomes larger than  $\tilde{x}$ . During this phase of the motion, the iterate drifts regularly away from the origin, we call it a *laminar phase*. Now due to the discontinuity in the graph of  $f$  this monotonous variation is suddenly interrupted ; a *turbulent burst* occurs which may reinject the iterate in the region  $x < \tilde{x}$ , thus beginning a new laminar phase. Hence the *intermittent* signal pictured in figure 1a.

Numerical results are obtained in starting the iteration with some initial value  $x_0$  and letting the computer run, so they appear under the form of temporal averages. Let us consider for example the characteristic exponent [5] which conveniently measures the amount of turbulence in the intermittent regime : i.e. the extent to which trajectories are sensitive to initial conditions and thus unpredictable in the long term even if they are defined by a relation

which seems perfectly deterministic (eq. (1)). Here the situation is particularly simple. An absolute error  $\delta x_0$  on  $x_0$  leads to an absolute error

$$\delta x_1 = |f'(x_0)| \delta x_0 > \delta x_0$$

on the first iterate  $x_1 = f(x_0)$  ; after  $N$  iterations one has :

$$\delta x_N = |f'(x_{N-1})| \delta x_{N-1} = \cdots = \prod_{n=0}^{N-1} |f'(x_n)| \delta x_0.$$

Let us define  $\gamma_N$  by

$$\delta x_N = \delta x_0 \exp(\gamma_N N)$$

so that

$$\gamma_N = \frac{1}{N} \ln \left\{ \prod_{n=0}^{N-1} |f'(x_n)| \right\} = \frac{1}{N} \sum_{n=0}^{N-1} \ln |f'(x_n)|,$$

the characteristic exponent  $\gamma$  is simply the limit of  $\gamma_N$  when  $N$  tends to infinity. It presents itself as a temporal average. However, as in statistical mechanics, theory more easily handles ensemble averages involving a certain probability density (measure) on phase space. The temporal average of a given function  $g(x)$

$$\bar{g} = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} g(x_n)$$

may be read as :

$$\bar{g} = \int g(x) \left[ \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \delta(x - x_n) \right] dx$$

so that

$$\mu(x) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \delta(x - x_n) \quad (2)$$

presents itself as the probability density we are looking for. It would remain to prove that it has *good* properties : independence with respect to the initial point and stability against small stochastic perturbations essentially [5b] since it is clear from its very expression that it is invariant under transformation  $f$  (a supplementary iteration amounts simply to a shift in the numbering of the iterates). A numerical estimate of this invariant density can be easily obtained by counting the number of iterates which fall in a small interval around a given point in the course of a very long run (see Fig. 3). However we would prefer to determine this probability density by analytic means. The idea is to compute the effect of a repeated application of transformation  $f$  on some initial measure (for example the usual Lebesgue measure  $\mu_0(x) \equiv 1$ ). If the  $n$ th iterate is  $\mu_n(x)$ , at step  $n + 1$  one has :

$$\mu_{n+1}(x) dx = \sum_a \mu_n(x^a) dx^a \quad (3)$$

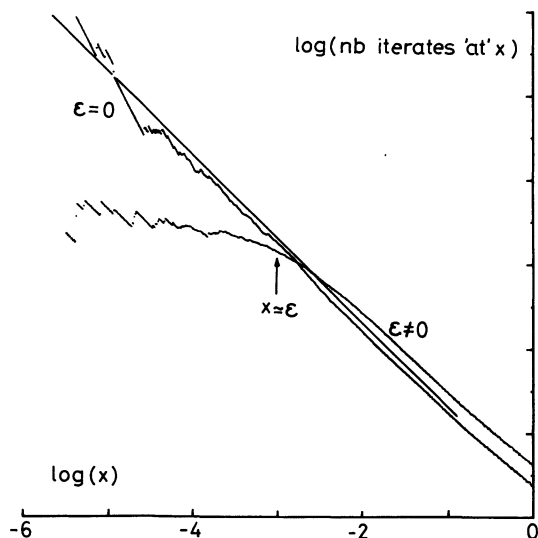


Fig. 3. — Invariant measure obtained from the numerical simulation. For  $\varepsilon = 0$ , the log of the number of iterates falling in a section around  $x$  increases as  $\log 1/x$ . For  $\varepsilon \neq 0$  here  $\varepsilon = 1/1024$ , this law is truncated at  $x \sim \varepsilon$ . The aspect of the figure at the extreme left results from the very small number of very long laminar phases starting at very small  $x_0$  which entered the statistics.

where  $\sum_a$  denotes the sum over all antecedents  $x^a$  of a given point  $x$  under  $f : x = f(x^a)$ . The measure we are looking for thus appears naturally as the fixed point of the transformation :

$$\mu(x) dx = \sum_a \mu(x^a) dx^a. \quad (3')$$

In the present (quite simple) case its existence in the intermittent regime ( $\varepsilon > 0$ ) results from a theorem [5c] due to the fact that  $|f'(x)| > 1$  everywhere. Let us compute it explicitly. As can be seen from figure 2. Each point has two antecedents one  $x' < \tilde{x}$  and the other  $x'' > \tilde{x}$ . Far from the origin, say for  $x \sim \tilde{x}$ , these two antecedents are also far from the origin.  $dx'/dx$  and  $dx''/dx$  are both of the order of  $1/2$  so that equation (3) which reads :

$$\mu(x) = \mu(x') \frac{dx'}{dx} + \mu(x'') \frac{dx''}{dx} \quad (3'')$$

can be fulfilled with  $\mu(x)$ ,  $\mu(x')$ ,  $\mu(x'')$  all of the same order of magnitude  $\tilde{\mu}$ . On the contrary when  $x$  is close to the origin,  $x' \sim x$  is also close to the origin and  $dx'/dx \sim 1$ , while  $dx''/dx$  is of the order of  $1/2$ . Then equation (3) can be fulfilled only with

$$\mu(x) \sim \mu(x') \gg \mu(x'') \sim \tilde{\mu},$$

i.e.  $\mu(x)$  tends to concentrate at small  $x$ . Assuming  $\mu$  smooth enough and expanding it, denoting  $\Delta = x - x'$ , one gets :

$$\Delta(x') \frac{d\mu}{dx'} \simeq -\mu(x') \frac{d\Delta}{dx'} + C(x)$$

where  $C(x) = \mu(x'') dx''/dx$  is slowly varying and can be considered as constant in this simple analysis. The solution reads

$$\mu(x) = \frac{x C}{\Delta} = \frac{C}{\varepsilon + (1 - \varepsilon) x}. \quad (4)$$

As long as  $\varepsilon$  is finite, the total mass  $M = \int_0^1 \mu(x) dx$  is finite but it diverges as  $\ln(1/\varepsilon)$  when  $\varepsilon \rightarrow 0_+$ . At threshold  $\varepsilon = 0$ , the condition  $|f'(x)| > 1$  is violated at the origin where  $f'(0) = 1$  and there is no non-singular invariant measure [5c]. Numerical estimates for  $\mu(x)$  as given by its temporal definition (2) are plotted in figure 3. The finite duration of the long run prevents observing *experimentally* a true divergence at  $\varepsilon = 0$ . In agreement with equation (4), one observes that  $\mu(x)$  saturates below  $x \sim 1/\varepsilon$  when  $\varepsilon > 0$ .

Now let us use the invariant measure to determine the characteristic exponent  $\gamma$ . It can be calculated through a mere integration on phase space

$$\gamma = \lim_{N \rightarrow \infty} \gamma_N = \int_0^1 \ln |f'(x)| \mu(x) dx.$$

At the limit  $\varepsilon \rightarrow 0$  one gets

$$\gamma \sim 1/\ln(1/\varepsilon). \quad (5)$$

The convergence of the corresponding numerical estimate (the temporal average) is exceptionally slow for  $\varepsilon$  small and it is very difficult to get  $\gamma$  with high precision but numerical experiments (Fig. 4) seem compatible with the theoretical estimate (5).

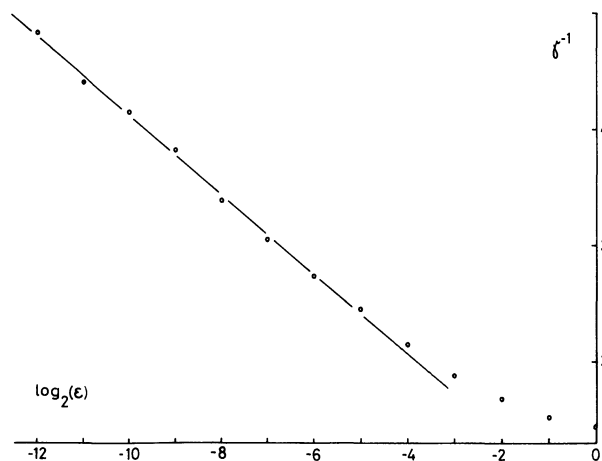


Fig. 4. — The Lyapunov number  $\gamma$  tends towards zero as  $1/\ln(1/\varepsilon)$ . Convergence was very slow and required as many as  $2 \times 10^7$  iterations for the smallest  $\varepsilon$ .

### 3. Self-similarity at the intermittency threshold. —

In this section we shall examine further the statistical properties of the process defined by equation (1)

and now understood as a random noise. In fact this is possible because a *laminar phase* of given duration is uniquely defined by its starting point. At  $\varepsilon = 0$  one has simply

$$x_{n+1} = x_n + x_n^2 \quad \text{or} \quad \Delta_n = x_{n+1} - x_n = x_n^2 \times (dn = 1).$$

Turning to a differential equation one gets

$$\frac{dx_n}{dn} = x_n^2$$

which integrates as :

$$x_n = \frac{1}{\left(\frac{1}{x_0} - n\right)}.$$

The laminar phase, that from now on we shall call an *intermission*, ends when  $x_n \geq 1$  since due to the discontinuity in the graph of  $f(x)$  the iterate can be reinjected in the vicinity of the origin starting a new intermission. The duration  $n_0$  of an intermission starting at  $x_0$  is then given by

$$n_0 = \frac{1}{x_0} - 1 \sim \frac{1}{x_0} \quad (6)$$

for  $x_0$  small enough. The number of intermissions which last longer than  $n_0$  follows immediately since they must begin at  $x < x_0$  and since they will do so if the antecedent falls in the interval  $\left[\tilde{x}, \tilde{x} + x_0 \frac{dx''}{dx}(\tilde{x})\right]$  the weight of which is

$$x_0 \frac{dx''}{dx}(\tilde{x}) \mu(\tilde{x}) \propto x_0 = 1/n_0$$

so one has

$$\mathcal{N}(n \geq n_0) \propto n_0^{-\beta} \quad \text{with } \beta = 1. \quad (7)$$

This estimate is in good agreement with numerical results presented in figure 5.

Formula (7) is the first manifestation of self-similarity [7] in this problem. Another one is given by the series of the lengths of the longest intermissions as a function of their starting time. Indeed in a numerical experiment the initial condition can always be considered as the starting point of a laminar phase. Now let  $\hat{n}$  be the length of an intermission longer than that of all other intermissions that took place before it in the course of the experiment and let  $t_i$  be the starting instant of this intermission. Figure 6 shows that long intermissions occur only a long time after the beginning of the experiment and that the starting point  $t_i$  and the length  $\hat{n}$  are roughly proportional. This is easily understood if one assumes that in an experiment involving  $N$  iterations the system can explore a region of phase space such that

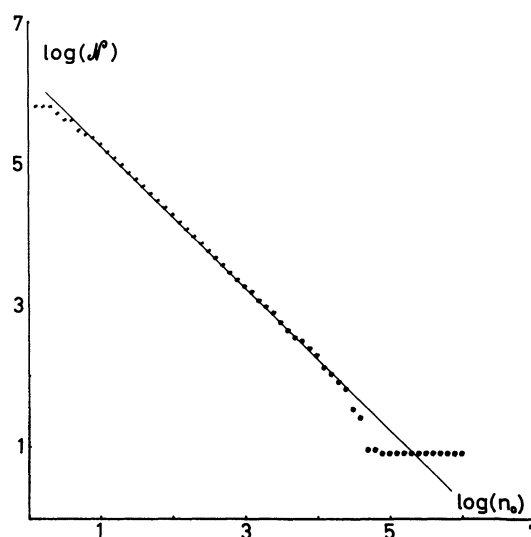


Fig. 5. — The distribution of intermissions of length  $n \geq n_0$  follows the hyperbolic Pareto law [7]  $n_0^{-1}$ . Deviations are observable at both ends, first for  $n_0$  small due to the existence of an *inner scale* the iteration step, and second for  $n_0$  large where statistics become poor.

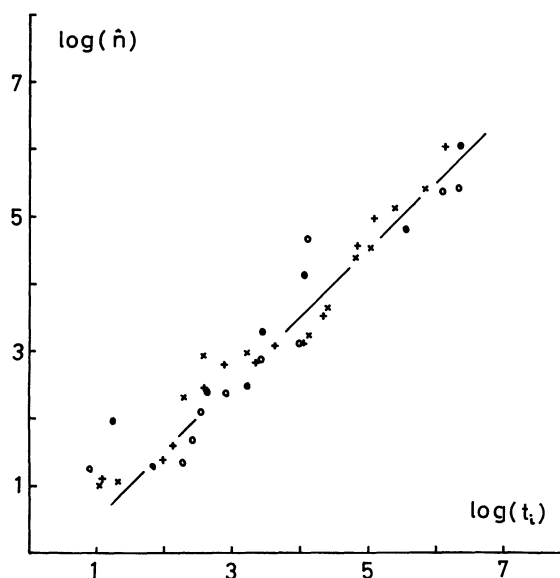


Fig. 6. — Statistical self-similarity clearly appears in this plot of the length of the longest intermission, at a given time  $\hat{n}$  as a function of its starting time  $t_i$ . Several different series starting at randomly chosen initial values  $x_0$  are given each showing a rough proportionality between  $\hat{n}$  and  $t_i$  which suggest that the domain of phase space allowed to the system at time  $N$  is restricted to  $x > x_{\min} \propto 1/N$ . Notice that a simulation which presents only one laminar phase ending at time  $N$  has to begin exactly at  $1/N$ .

$x > x_{\min} \propto 1/N$  and all the average properties of the signal are to be calculated within this restriction. In particular the average duration of intermissions will be given by

$$\bar{n}(N) = \int_1^N n_0 p(n_0) dn_0 = \int_{1/N}^1 \frac{dx_0}{x_0} \sim \ln(N) \quad (8)$$

the average number  $\nu$  of noise bursts in an experiment of length  $N$  will be

$$\nu(N) \sim N/\ln(N) \quad (9)$$

of the form  $N^\delta$  with  $\delta = 1 - \mathcal{O}(\ln)$  giving the fractal dimension of the noise according to Mandelbrot [7]. Again this is well verified for this model (see Fig. 7).

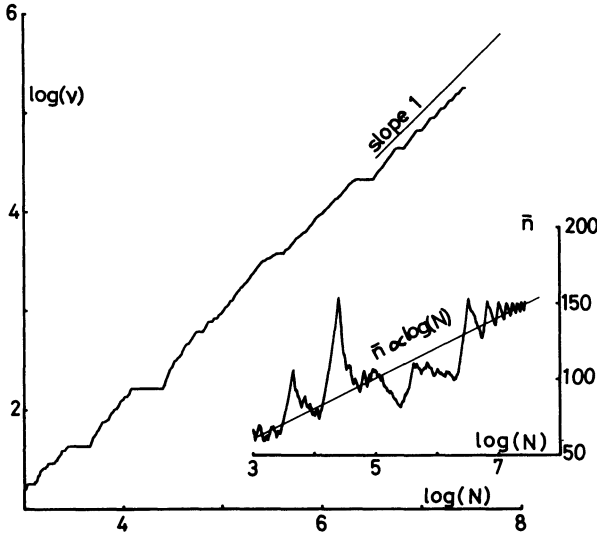


Fig. 7. — Growth of the number of *laminar phases* as a function of the number of iterations. In order to observe the behaviour predicted by our simple analysis, one has to define laminar phases by a minimal length  $n_m$  much larger than the inner scale so that they already belong to the self-similar regime depicted in figure 5. Here we have taken  $n_m = 10$ , taking  $n_m = 1$  would give  $\nu(N) \propto N$  without any logarithmic correction. Inset :  $\bar{n} = N/\nu$  grows linearly with  $\log(N)$ .

The number of events per unit time

$$\langle n(N) \rangle_0 \sim 1/\ln(N) \quad (10)$$

tends to zero logarithmically. We denote it by  $\langle \dots \rangle_0$  to recall that it presents itself as a temporal average evaluated in the course of a given experiment starting at  $(n = 0, x = x_0)$ . Since  $\langle n(N) \rangle_0$  tends to zero as the number of iterations increases, the process is not stationary. This sporadic character [7] should be related to the fact that the invariant measure is non-normalizable but this is special to the threshold  $\varepsilon = 0$ . Slightly above the threshold ( $\varepsilon \neq 0$ ), the process will appear as stationary if it is observed over a sufficiently large time, in practice a time much larger than  $1/\varepsilon$ . When  $\varepsilon \neq 0$ , the relation (6) between a starting point and the corresponding length of the intermission has to be replaced by

$$n_0 = 1/\varepsilon \ln(1 + \varepsilon/x_0)$$

which gives back (6) when  $x_0 \gg \varepsilon$  and

$$n_0 \sim 1/\varepsilon \ln(\varepsilon/x_0) \quad \text{or} \quad x_0 \sim \varepsilon \exp(-\varepsilon n_0)$$

in the opposite limit  $x_0 \ll \varepsilon$ . Then one has

$$N(n \geq n_0) \sim \varepsilon \exp(-\varepsilon n_0)/(1 - \exp(-\varepsilon n_0))$$

which gives (7) when  $n_0 \ll 1/\varepsilon$  and  $\exp(-\varepsilon n_0)$  when  $n_0 \gg 1/\varepsilon$ . Finally the average length of intermission reads

$$\bar{n}(N) = \frac{1}{\varepsilon} \left[ (1 + \varepsilon) \ln(1 + \varepsilon) - \left( \varepsilon + \frac{1}{N} \right) \times \ln \left( \varepsilon + \frac{1}{N} \right) + \frac{1}{N} \ln \frac{1}{N} \right].$$

At the limit  $N \ll 1/\varepsilon$  one recovers formula (8) whereas, when  $N$  tends to infinity,  $\bar{n}$  tends to  $\ln(1/\varepsilon)$ . As the number of iterations increases, the mean value of the signal which is just  $1/\bar{n}$  then decreases logarithmically according to (10) down to  $1/\ln(1/\varepsilon)$ .

Self-similarity is preserved at time scales smaller than  $1/\varepsilon$  which then plays the role of *outer scale* for the process. (The *outer scale* is the scale at which self-similarity is broken for an external reason, here the fact that one is not exactly at the intermittency threshold. The *inner scale* is the scale at which the process is defined, here the iteration step).

**4.  $1/f$  spectrum.** — Let us now turn to the correlations of the noise. In order to avoid difficulties about stationarity and the existence of averages we shall assume that  $\varepsilon$  is slightly larger than zero but we shall restrict to time scales much smaller than  $1/\varepsilon$  in order to remain in the self-similar regime. Then it is legitimate to assume the validity of the Wiener-Kintchine relations between the Fourier transform of the autocorrelation function and the spectral density of the process [8]. As usual the correlation function reads  $R(\tau) = \langle x(0)x(\tau) \rangle = \langle x(0) \langle x(\tau) \rangle_0 \rangle$  where, as before  $\langle x(t) \rangle_0$  is the average value of the signal when its value at  $t = 0$  is known, here when a noise occurred at  $t = 0$ . So one has  $R(\tau) \sim 1/\ln(\tau)$  in the limit of large  $\tau$ . Instead of searching directly the Fourier transform in the limit of small frequencies it may be interesting to obtain its behaviour by a qualitative *dimensional* argument. Let  $J(\omega)$  denote the Fourier transform of  $R(\tau)$ ; we have :

$$\begin{aligned} J(\omega) &= \frac{1}{\pi} \int_0^\infty R(\tau) \cos(\omega\tau) d\tau \sim \\ &\sim \int_0^\infty \langle x(\tau) \rangle_0 \cos(\omega\tau) d\tau \\ &\sim \left\langle \int_0^\infty x(\tau) \cos(\omega\tau) d\tau \right\rangle_0. \end{aligned}$$

Now consider the dominant contribution of the first period  $T = 2\pi/\omega$  and replace the cosine by a square wave, then

$$J(\omega) \sim (+1) \int_0^{T/2} x(t) dt + (-1) \int_{T/2}^T x(t) dt$$

i.e. (the number of busts during the 1st half period) minus (the number of bust during the 2nd half period). So :

$$J(\omega) \sim v(T/2) - (v(T) - v(T/2)) = 2 v(T/2) - v(T)$$

or replacing  $v$  by its value :

$$J(\omega) \sim 1/\omega(\ln \omega)^2.$$

The same result could have been derived by a more formal calculation starting with the distribution function (7) [9] but this argument is instructive in that it stresses the fact that the noise will present a non-trivial behaviour at  $\omega \rightarrow 0$  only if it possesses fractal properties. Indeed if  $v(T)$  was simply proportional to  $T$  at the limit of large  $T$ ,  $J(\omega)$  would reduce to a constant at  $\omega \rightarrow 0$ , which will be the case when  $\varepsilon \neq 0$  for  $T \gg 1/\varepsilon$  or  $\omega \ll \varepsilon$ . Now in the numerical experiment, the spectral density is obtained directly by Fourier transformation of the signal. Figure 8 displays the inverse of the power contained in a spectral line against the frequency. The slight deviation from a  $1/f$  spectrum at the lowest frequencies

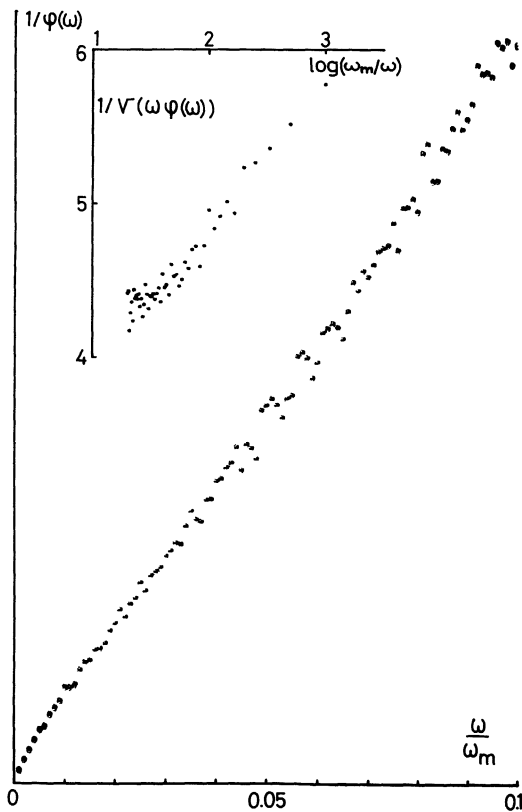


Fig. 8. — The power spectrum  $\rho(\omega)$  is obtained by Fast Fourier Transforming successive series of 2 048 iterates. About 2 000 spectra have been accumulated. Frequencies must be scaled to the maximum possible frequency  $\omega_m$  which is just the inverse of the sampling period. At threshold  $\varepsilon = 0$ , the overall  $1/f$  aspect of the spectrum is well verified; moreover the slight deviation at the lowest frequencies is compatible with the logarithmic correction predicted (see the inset).

seems compatible with the theoretical logarithmic correction. In fact this correction is important in order to make  $\int J(\omega) d\omega$  convergent at  $\omega = 0$  [9].

The process defined by (1) appears as the limit case of a more general noise with fractal dimension  $\delta$  with  $0 < \delta < 1$  and here  $\delta = 1 - \mathcal{O}(\ln)$ . Extending the argument given above one would get  $J(\omega) \sim \omega^{-\delta}$  which is also integrable at  $\omega = 0$ . Since  $\delta$  cannot be larger than 1 this naive argument cannot account for excess noise with a power spectrum of the form  $\omega^{-\zeta}$ ,  $\zeta \geq 1$  down to  $\omega = 0$ . Another limitation of the argument developed above is discussed in appendix II.

Among different interpretations of  $1/f$  noise a very old and popular one [10] involves the superposition of a continuous distribution of characteristic times each giving a Lorentzian contribution to the total spectrum :

$$J(\omega) \sim \int \frac{\tau p(\tau) d\tau}{1 + \omega^2 \tau^2}.$$

The particular form  $p(\tau) = 1/\tau$  which leads to *exact*  $1/f$  noise is obtained if one assume a kind of activation process for  $\tau$  :  $\tau = \tau_0 \exp(E/E_0)$  and a uniform distribution of *energies*  $\mathcal{D}(E) dE \equiv \mathcal{D}_0 dE$ ,  $\mathcal{D}_0$  constant. In the process defined by (1) one could also develop a similar argument :

- i) assuming that correlations over a time  $\tau$  occur when  $x(0)$  and  $x(\tau)$  happen to belong to the same intermission of length  $\tau' > \tau$ ,
- ii) noticing that the number of such intermissions is  $\sim 1/\tau$ , and
- iii) assuming a Lorentzian contribution to the spectrum, one would also get *exact*  $1/f$  noise but the derivation is clearly not satisfactory even if it leads approximately to the correct result.

**5. Conclusion.** — In this paper we have studied a quite simple model displaying intermittency, self-similarity and a  $1/f$  spectrum. Statistical properties of this model are rather easy to derive and theoretical estimates are all in nice agreement with numerical results. As explained in appendix I this model is not very natural but it derives directly from a two-dimensional model of transition to turbulence *via* intermittency. This model was not amenable to detailed analysis. The reduction to one dimension looked justified but lead to slight, significant, discrepancies between the theoretical predictions (5, 7, 9) and the corresponding numerical estimates. The origin of these discrepancies is not yet clear but the agreement observed here in the strictly 1-dimensional case definitely shows that theoretical tools can work well and that the reduction to 1-dimension is questionable.

In addition to this first answer, we have been led to a novel insight on the problem of  $1/f$ -noise : our

model works as a generator of *random* noise bursts and we have proved that there may be a hidden coherence in the distribution of random noise bursts due to the peculiar properties of dynamical systems displaying intermittency. Indeed as shown in appendix II, this is not special to the model studied above. Self-similarity at the intermittency threshold leads quite naturally to power laws for the power spectrum of the noisy signal, even if work remains to be done to predict accurately the exponent of the power law and its validity domain in specific cases. In any case, instabilities with an intermittent behaviour provide the opportunity of explaining the structure of certain random noises through short term deterministic dynamics. Theories of  $1/f$  noise based on the incoherent superposition of relaxation times of very different orders of magnitude may suffer difficulties from their *ad hoc* character [11] and it is of interest to know that other interpretations are available.

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## APPENDIX I

**Intermittency in dynamical systems.** — Recent progress in understanding the transition to turbulence rests in large part on the theory of differential dynamical systems. Such systems can be explicitly derived through modal decompositions of the macroscopic equations of motions (Navier-Stokes, Oberbeck-Boussinesq, chemical reaction diffusion equations, etc...). One is mainly interested in asymptotic states reached once transients due to initial conditions have decayed. These asymptotic states may be stationary, periodic, quasi-periodic or even chaotic. Bifurcation theory is devoted to the study of transitions between these different possible states. Turbulence has been predicted to set in after a small number of bifurcations only.

The detailed structure of the *strange attractor* which accounts for turbulence is not resolved by the general theory but left either for further study in every specific case (the Lorenz model for example) or for conjectures of broad enough generality (see Y. Pomeau p. 135 in [2a]). In the course of the analysis a theoretical tool named a Poincaré map often proves useful. It looks similar to a stroboscopic analysis of trajectories obtained in examining successive intersections of trajectories with a well chosen surface drawn in phase space. So the differential system is

reduced to a discrete time dynamical system : an iteration. In this framework, the Hopf bifurcation of a limit cycle (periodic state) towards a biperiodic state can be accounted for by an iteration on a complex variable  $z = x + iy = \rho \exp i\theta$  of the form :

$$z_{n+1} = ((1 + \varepsilon) \exp i\varphi) z_n + \lambda |z_n|^2 z_n \quad (\text{I.1})$$

with  $\lambda \in \mathbb{R}$  and  $\varphi \neq 2\pi k/n, n \leq 5$  to avoid resonance. Bifurcation theory shows that when  $\lambda < 0$  (more precisely  $\lambda \cos \varphi < 0$ ) the bifurcation is supercritical, i.e. there exists a non-trivial solution for  $\varepsilon > 0$  and the corresponding biperiodic state is stable. On the contrary, when  $\lambda > 0$  bifurcation theory predicts that the biperiodic state exists for  $\varepsilon < 0$  (subcritical) and is unstable; but it cannot say anything more, particularly about what happens for  $\varepsilon > 0$ . The conjecture which accounts for intermittency is that there is no longer any simple stable asymptotic state and that the *strange attractor* that takes place is such that the variable  $z$  of the reduced Poincaré map (I.1) has the opportunity to fall again close to the origin when it is far from it. This property can be obtained in *periodizing* the 2-dimensional complex plane by modulo conditions on components  $x$  and  $y$  of  $z$ .

Close to the origin the role of the angular variable  $\theta$  was thought unimportant and iteration (I.1) was reduced to a 1-dimensional iteration for the modulus  $\rho$  :

$$\rho_{n+1}^2 = (1 + \varepsilon)^2 \rho_n^2 + 2\lambda(1 + \varepsilon) \cos \varphi \rho_n^4 + \lambda^2 \rho_n^6 \quad (\text{I.2})$$

which leads to

$$\rho_{n+1} = (1 + \varepsilon) \rho_n + \lambda \cos \varphi \rho_n^3 + \dots \quad (\text{I.3})$$

Since the problem is 2-dimensional with rotational symmetry the natural surface element is  $2\pi\rho d\rho$  and it is equivalent to study (I.3) using  $\rho d\rho$  or to set  $\rho^2 = x$  in (I.2) and to study it using  $dx$ . The rescaling of  $x$  and the modified non-linear coefficient in (1) when compared to (I.2) are purely matters of convenience and do not affect the critical properties  $\varepsilon \rightarrow 0$  of interest in this paper.

Numerical simulation on iteration (I.1) has given  $\gamma \propto \varepsilon^\alpha$  with  $\alpha$  positive, small and (weakly) dependent on coefficient  $\lambda$ , thus unambiguously different from the logarithmic behaviour predicted by (5). In the same way, at the intermittency threshold  $\varepsilon = 0$  one gets  $N(n \geq n_0) \propto n_0^{-\beta}$  with  $\beta$  also function of  $\lambda$  and slightly different from 1 (cf. result (7)). Finally, the average number  $\nu$  of noise bursts in an experiment of length  $N$  increases as  $N^\delta$  with  $\delta$  strictly smaller than 1 in contradistinction with (9). Corrections to the 1-dimensional reduction are probably important from the conceptual viewpoint but seem to remain quite small quantitatively speaking.

## APPENDIX II

**1/f-noise in a differential dynamical system.** — In order to describe the transition to chaos in oscillating chemical reactions Yamada and Fujisaka [6] have proposed the following differential model :

$$dw_j/dt = (1 + iC_0) w_j - (1 + iC_2) |w_j|^2 w_j + (e/N) (1 + iC_1) \sum_{i=1}^N (w_i - w_j) + a \quad (\text{II.1})$$

where  $j = 1, 2, \dots, N$  and  $w_j = u_j + iv_j$  is a complex variable. For  $N = 3$ ,  $C_0 = -C_1 = C_2 = \sqrt{3}$ ,  $e/N = 0.21$ , this system presents a transition from a limit cycle behaviour for  $a \gtrsim 0.212$  to an intermittent turbulent state for  $a < 0.212$ ; a 1/f-spectrum has also been observed slightly below the intermittency threshold. Studying a Poincaré map of the flow, we have been able to show that this kind of intermittency could be understood in terms of an iteration for a certain variable  $x$  taking the form [12] :

$$x_{n+1} = (1 + \varepsilon) x_n + \beta x_n^3 \quad (\text{II.2})$$

close to the origin (+ suitable periodic conditions as before). Absence of quadratic term and symmetry under the change  $x \rightarrow -x$  are linked to the permutation properties of (II.2). An analysis similar to those of paragraph 2 and 3 leads to :

$$\gamma \propto \varepsilon^\alpha \quad \text{with} \quad \alpha = 1/2 \quad (\text{II.3a})$$

$$N(n \geq n_0) \propto n_0^{-\beta} \quad \text{with} \quad \beta = 1/2 \quad (\text{II.3b})$$

$$\nu(N) \propto N^\delta \quad \text{with} \quad \delta = 1/2. \quad (\text{II.3c})$$

Result (II.3a) has already been checked for the discrete case (II.2) (see [2b]); a similar confirmation in the differential case (II.1) will be difficult to obtain due to the *slowing down* of convergence close to the threshold and already annoying in the discrete case. Numerical simulations at threshold ( $\varepsilon = 0$ ) agree with (II.3b, c). As can be understood from the value of the exponents, the process is very sporadic, noise events become rare and rather grouped with groups well separated by comparatively long intermissions. The naive argument giving the exponent for the power law of  $J(\omega)$  leads to expect  $J(\omega) \sim \omega^{-1/2}$  and not

$\omega^{-1}$  as observed by Yamada and Fujisaka. However a close inspection of the conditions of their numerical experiments shows that their frequency window lies too close to the inner time scale and does not extend sufficiently far in the direction of low frequencies so that the exponent may be modified and one is not able to see rounding of the spectrum due to a finite distance to the threshold ( $\varepsilon = 0.09$  at  $a = 0.200$ ).

Trying to check the  $\omega^{-1/2}$  law with iteration (II.2) has led to an important restriction about the representation of the intermittent signal in terms of intermissions and noisy events which directly leads to that prediction. It should be noticed first that for  $\varepsilon = 0$  the signal is strongly non-stationary and very sporadic so that it is absolutely necessary to make  $\varepsilon > 0$  to get rid of this difficulty and obtain reproducible results. Thus the spectrum will saturate at frequencies  $\omega \sim \varepsilon$  but choosing  $\varepsilon$  small enough will allow the observation of the asymptotic power law in the intermediate frequency range far enough from the inner and outer scales. Over about two decades we have obtained

$$J(\omega) \sim \omega^{-\delta} \quad \text{with} \quad \delta \simeq 0.75 - 0.80$$

while at higher frequencies the apparent exponent increases beyond 1. The difference between the expected value 1/2 and that observed  $\delta \simeq 0.75 - 0.80$  can be understood if one realizes that in the argument of paragraph 4 we have deliberately ignored the contribution of intermissions longer than  $2\pi/\omega$  to the power spectrum at  $\omega$ . During an intermission it was implicitly assumed that the signal was zero but this is not the case. An intermission of duration  $n_0$  contributes as a regular drift to all frequencies  $\omega > 2\pi/n_0$ . A simple slope would give  $J(\omega) \sim \omega^{-2}$ . The power at frequency  $\omega$  is then a weighted average of contributions coming from intermissions shorter *and* longer than  $2\pi/\omega$ . Hence the modified exponent. A self-consistent analysis of this correction has not yet been worked out. In principle it exists also in the case of iteration (1) but the non-linear term is quadratic rather than cubic so that it repels the iterates much more vigorously away from the origin. As a consequence the number of intermissions of length larger than  $n_0$  decreases as  $1/n_0$  instead of  $1/\sqrt{n_0}$  and this can explain that their contribution is negligible at least in view of our results in figure 8.

## References

- [1] a) GOLLUB, J. P., BENSON, S. V., STEINMAN, J., in *International Conference on non-linear dynamics* J. L. Lebowitz ed., *Ann. N.Y. Acad. Sci.* (to appear);
  - b) LIBCHABER, A., MAURER, J., private communication;
  - c) BERGÉ, P., DUBOIS, M., MANNEVILLE, P., POMEAU, Y., *J. Physique Lett.* **41** (1980) L-341.
  - [2] a) POMEAU, Y., MANNEVILLE, P., in : *Intrinsic Stochasticity in Plasmas* G. Laval, D. Gresillon Eds (Ed. de Physique, Orsay) 1979, p. 239.
  - b) POMEAU, Y., MANNEVILLE, P., *Commun. Math. Phys.* **74** (1980) 189.
  - [3] MANNEVILLE, P., POMEAU, Y., *Phys. Lett.* **75A** (1979) 1; *Physica* **1D** (1980) 219.
  - [4] McLAUGHLIN, J., *J. Stat. Phys.* **15** (1976) 307.
  - [5] a) RUELLE, D., in *Mathematical Problems in Theoretical Physics* (Rome 1977), G. Dell' Antonio et al. Eds, *Lecture Notes in Physics* **80** (Springer Verlag, Berlin) 1978;
  - b) RUELLE, D., in : *International Conference on non-linear dynamics* (New York 1979), J. L. Lebowitz ed., *Ann. N.Y. Acad. Sci.* (to appear);
  - c) LASOTA, A., YORKE, J. A., *Trans. Am. Math. Soc.* **186** (1973) 481.
  - [6] YAMADA, T., FUJISAKA, H., *Supp. of the Prog. Theor. Phys.* **64** (1978) 269.
  - [7] MANDELBROT, B., *Fractals : Form, Chance and Dimension* (Freeman and Co, San Francisco) 1977.
  - [8] DAVENPORT Jr., W. B., ROOT, W. L., *An introduction to the theory of random signals and noise* (McGraw Hill, New York) 1958.
  - [9] GOLDMAN, M., DEVORET, M., Private Communication.
  - [10] VAN DER ZIEL, A., *Physica* **XVI** (1950) 359;
  - DU PRÉ, F. K., *Phys. Rev.* **78** (1950) 615.
  - [11] HOOGE, F. N., *Physica* **83B** (1976) 14.
  - [12] MANNEVILLE, P., Rapport CEA DPh-G/SPSRM/80-1734 submitted to *Phys. Lett. A*.
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