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APPLICATIONS OF TOPOLOGY TO THE STUDY OF ORDERED SYSTEMS (*)

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Résumé. — Cet article est une version détaillée d'un preprint récent dans lequel le problème était analysé d'une manière très concise. On donne ici d'abord une introduction pédagogique à la théorie de l'homotopie puis ses applications à l'étude de configurations non singulières dans les systèmes ordonnés tels que $^3\text{He-A}$, les cristaux liquides nématiques, etc. On examine aussi brièvement le rôle de l'hamiltonien pour décider de l'observabilité et de la stabilité de ces configurations.

Abstract. — This paper is a detailed version of a recent preprint wherein the above topic was analysed rather concisely. A pedagogical introduction to homotopy theory is followed by its applications to the study of non-singular configurations in ordered systems such as $^3\text{He-A}$, nematic liquid crystals etc. The role of the hamiltonian in deciding the observability and stability of these configurations is briefly examined.

1. **Introduction.** — Several years ago Finkelstein [1] pointed out and illustrated with several examples, how a branch of topology, called homotopy theory could be profitably used by physicists. Only recently, following the invention of monopoles, instantons etc. have particle theorists come to appreciate this fact. Even more recent is the work of Blaha [2] and Thoulouse and Kléman [3] wherein homotopy theory was used to classify singularities or defects in ordered systems such as ferromagnets, nematics etc. In a recent and somewhat concise preprint [4] I had reverted to the approach pioneered by Finkelstein, namely to the study of topology stable but non-singular configurations in ordered systems and discussed the conditions on the energy functional \mathcal{H} for their observability. The present work is an expanded version of reference [4] and contains several details omitted therein, in addition to an introduction to homotopy theory provided for nonexperts.

2. **Introduction to homotopy theory.** — The ordered systems in question will be described by a field $y(x)$ defined over the points x of some spatial domain X . The symbol y can stand for a unit spin vector \mathbf{s} , if the system is a ferromagnet, the director \mathbf{n} , if it is a nematic etc. Let us denote by Y , the manifold of

possible values for y . For example $Y = S^1$, a circle, if y is a two component vector of fixed length ; $Y = S^2$ a 2-sphere, if y is a three component vector of fixed length ; $Y =$ solid sphere of radius 2π if y is a 3-vector of length $\leq 2\pi$ and so on.

Let us begin with X , a one dimensional interval, parametrised by $0 \leq x \leq 2\pi$. Consider some field distribution $y(x)$. So each point $x_i \in X$, there is an image point $y(x_i) \in Y$. As x varies from 0 to 2π , the image point traces out a curve in Y starting at $y(0)$ and ending at $y(2\pi)$. Let us restrict ourselves to fields obeying $y(0) = y(2\pi) = y_0$, where y_0 is called the base point. A compact way to write this is $y(\partial X) = y_0$, where ∂X denotes the boundary points of X . In this case the image point traces out a closed curve anchored at y_0 . In every configuration of the system there exists such an image loop and *vice versa*. The study of the system thus reduces to the study of the corresponding closed loop in Y .

Suspending for a while the question why, let consider the classification of the configurations of the loop into classes such that :

- i) any two loops belonging to the same class can be continuously distorted into each other, and
- ii) loops belonging to different classes cannot be continuously distorted to each other. The classes are called homotopy classes and members of a given class are said to be homotopic to each other.

Consider for example the case where $Y = E^2$ the two dimensional Euclidean space which we may

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take to be the x - y plane. Let y_0 be any point, say $(1, 1)$. It is evident that there is only one homotopy class : any loop can be deformed into any other and in particular to a point loop at y_0 . In terms of the field $y(x)$, it means that any configuration $y(x)$ can be continuously deformed to the constant map $y(x) = y_0$.

Consider next $Y = E^2 - (0, 0)$. (The exclusion of the null vector from the allowed values can occur for example if y is the magnetic field in a plasma, see Finkelstein and Weil [5].) Let y_0 be any point in Y . It is clear that while any loop not enclosing the origin can be collapsed to y_0 , those which do enclose the origin cannot be. The latter can be further classified by means of an integer m , whose magnitude and sign specify the number of times and the sense in which the origin is encircled. It is also clear that maps belonging to distinct values of m are non-homotopic. Consider the set $\{y^0, y^1, y^2, \dots\}$, where y^m stands for all maps in which the loop encircles $(0, 0)$ m times. It turns out that this set forms a group, that is to say, there exists a law of combination of the elements which obeys the group axioms. The group is called the first homotopy group of Y , and denoted by $\Pi_1(Y)$. The subscript 1 tells us that X acquires the form of a closed one dimensional surface, i.e., a loop, which topologically is equivalent to a circle of 1-sphere S^1 . Our interest here will be limited to the set of elements in $\Pi_1(Y)$ and not involve their group structure. For an introduction to the latter, see for example reference [6] and additional references contained therein.

Now, we have seen that the elements of $\Pi_1[E^2 - (0, 0)]$ are in one to one correspondence with the integers m . A study of the group also reveals that their law of combination is identical to the addition of the corresponding integers. One denotes by Z_∞ the set of all integers forming a group under addition. These results may be summarized as

$$\Pi_1(E^2 - (0, 0)) = Z_\infty. \quad (2.1)$$

Notice that the choice of base point y_0 plays a very minor role. It will not be explicitly discussed in the future. If our region X is two dimensional and all points on the perimeter ∂X map onto a single point y_0 , X will assume the form of a closed two-dimensional surface, topologically equivalent to a 2-sphere S^2 . The group in question is $\Pi_2(Y)$. Likewise if X were a three dimensional region of space, the group to study is $\Pi_3(Y)$.

A word about the condition $y(\partial X) = y_0$. For infinite systems, the requirement that $y \rightarrow a$ constant y_0 at infinity ensures the finiteness of the energy, assuming the existence of gradient terms. In finite systems this boundary condition may be forced by an external field or the walls of a container. In any event it is only with this condition that X becomes a closed surface in Y and the groups become the well studied $\Pi_n(Y)$.

We now turn to the question that has no doubt

tested the patience of the reader : why the homotopy classification ? There are two reasons as far as we are concerned :

i) As long as the time evolution is a smooth and continuous change, it follows that the homotopy index, such as the integer m , is a constant of the motion. We thus have a conservation law without discussing the hamiltonian in any detail, except to assume it produces smooth changes in the state of the system.

ii) We can anticipate a family of metastable configurations one from each class. Consider the energy functional, which we shall refer to as the hamiltonian \mathcal{H} . In each class, there must be some configuration that minimizes it. This is a metastable state : it's stability within the class is a result of its being a minimum, while passage to other classes is topologically forbidden. We are reminded of the following situation in one dimensional quantum mechanics : while finding the absolute minimum of the energy yields the ground state, we can get the first excited state by minimising in the odd parity sector (assuming \mathcal{H} is parity invariant).

With this motivation we proceed to study some of the groups that will interest us.

3. The groups $\Pi_m(S^n)$ and $\Pi_m(P^n)$. — We will be interested here only in the groups $\Pi_m(S^n)$ and $\Pi_m(P^n)$, where P^n is the projective n -sphere, that is the n -sphere S^n with diametrically opposite points identified. Both m and n will be less than or equal to three. Although the elements of these groups are listed in text books, let us consider one concrete example that is amenable to intuitive analysis : the group $\Pi_1(S^1)$.

The aim is to map a closed loop X onto a circle $Y = S^1$ which is also parametrised by $0 \leq y \leq 2\pi$. There are first of all maps in which X never completes a turn around Y . An example is given in figure 1. All such loops may be deformed to a point, say at $y = 0$. We label these maps by an integer $m = 0$ which counts the number of times X covers or wraps around Y . There is another equivalent definition of m . Pick any point y of Y and consider all the

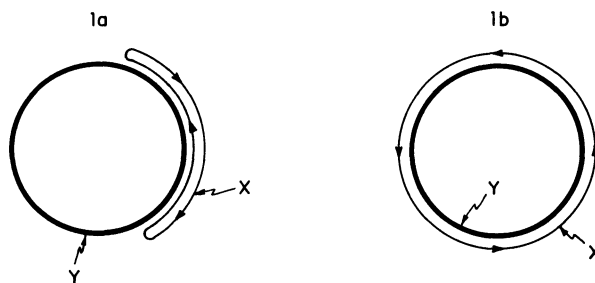


FIG. 1. — a) This map is not a constant, but is homotopic to the constant map $y(x) = y_0$. It has $m = 0$; b) This map has $m = 1$. The identity map $y = x$ is a special case.

points $x \in X$ that map onto it. To each one assign a number ± 1 , this sign being that of dy/dx at that point. Then m is the algebraic sum of these numbers. Yet another definition is

$$m = \frac{1}{2\pi} \int_0^{2\pi} (dy/dx) dx .$$

It is a very practical way of finding m when our intuition breaks down. (These recipes are easily generalised to the case where X and Y are n -spheres, with dy/dx being replaced by the Jacobian $J(y/x)$. If Y is P^n ; the rule can be applied after disregarding the identification of antipodal points). In figure 1a, if we choose a point y untouched by the loop X , m is clearly 0, while if we choose a point to which two points of X are mapped, the opposite signs of dy/dx cancel to give $m = 0$.

Consider next the case where X goes around Y once clockwise (Fig. 1b). A special case called the identity map, is $y(x) = x$. It is clear $m = 1$, following any of the three definitions. (Note that the *identity map* $y = x$ is a nontrivial map and must not be confused with the *identity element* of the group which corresponds to trivial $m = 0$ map.) Likewise $m = -1$ if X goes around Y counterclockwise. Clearly m can be any integer and maps labelled by different m are nonhomotopic. We may summarize our results as :

$$\Pi_1(S^1) = Z_\infty . \tag{3.1}$$

Strictly speaking what we have established is not the equality of the two groups, but only the corresponding sets.

It can be similarly shown that $\Pi_2(S^2) = Z_\infty$. For example in the identity ($m = 1$) map, X covers Y as the skin of an orange covers it. If $m > 0$, X wraps around Y m times.

Having illustrated the general principles by means of the preceding discussion, let us accept without proof the following results true for $n \geq 2$:

$$\begin{aligned} \Pi_n(S^m) = \Pi_n(P^m) = Z_\infty , \quad n = m \\ = I , \quad n < m \end{aligned} \tag{3.2}$$

where I is the trivial group with just the identity element. The exception to the rule, $n = 1$, will not be discussed in any detail here.

Among the maps with $n > m$; the only nontrivial one with $n \leq 3$ is $\Pi_3(S^2)$. Hopf [7] showed that

$$\Pi_3(S^2) = \Pi_3(P^2) = Z_\infty \tag{3.3}$$

and provided representative maps from each class called Hopf maps. Since X and Y are spheres of different dimensionality, the integer m has a somewhat complicated interpretation, which we will discuss in due course.

Our mathematical preparation is complete. We now turn to the applications.

4. Applications to configuration with singularities or defects. — A singular point $x \in X$ is characterized by the fact that as we approach x from many directions we get many limiting values for y . Although singular maps are not our subject, we review briefly the interesting work of Thoulouse and Kléman [3] because our machinery makes it easy to do so and because the relationship between this work and theirs is worth understanding.

Let X be a two dimensional region. Let us surround it by a closed loop or circle S^1 and study the variation of y on it. If the map on S^1 is a nontrivial element of $\Pi_1(Y)$, (that is, not homotopic to the constant map) a defect or singular point lies within. To see this, imagine slowly shrinking the loop S^1 in size. The map on it must also change smoothly, and hence remain in the same class. Ultimately the loop must shrink to an infinitesimal one surrounding a point. The distribution on this infinitesimal loop is non constant by assumption. As a result, y approaches different values as we approach this point from different directions. If X is a three dimensional region a similar analysis in terms of $\Pi_2(Y)$ is applicable.

Notice the analogy between the way in which a defect in the interior manifests itself via the map on the boundary and

- i) Gauss' law, which relates the charge in a volume to the flux leaving a surrounding surface and
- ii) Cauchy's theorem, relating the integral of an analytic function on a closed contour to the residues to the singularities within.

Now, just as opposite charges or poles with opposite residues may neutralize each other at the boundary, so may two (or more) defects annul each other to produce a trivial map on the boundary. To see how this happens however, we must know how to combine the maps, that is, the group structure, which is unfortunately behind the scope of this paper.

Although defect analysis also involves homotopy theory, there ends the similarity with our present analysis. A defect theorist is interested in maps with nontrivial behaviour at the boundary and a defect inside, while we deal with maps trivial at the boundary $y(\partial X) = y_0$, with no defect or singularity inside. We now turn to our subject proper.

5. The nematic liquid crystal in two space dimensions. — Let us first take as the two dimensional region X the entire x - y plane. The order parameter is the director \mathbf{n} which we parametrize as $\mathbf{n} = (n_x, n_y, n_z) = (\sin \beta \cos \alpha, \sin \beta \sin \alpha, \cos \beta)$, where α and β are the azimuthal and polar angles on S^2 . Since $\mathbf{n} = -\mathbf{n}$ physically, $Y = P^2$ and not S^2 . This distinction is however irrelevant for the topological group we will consider. We impose the condition that at spatial infinity $\mathbf{n} = (0, 0, 1)$, that is $\beta = 0$. This makes the relevant group $\Pi_2(P^2) = Z_\infty$.

Given an energy functional or hamiltonian \mathcal{H} , one metastable configuration from each class may be found by minimising \mathcal{H} within that class. This non-trivial task has been done by Belavin and Polyakov [8] (here-after referred to as BP) for the case

$$\mathcal{H} = \int a(\nabla\mathbf{n})^2 d^2x \quad (5.1)$$

which describes a nematic with all three elastic constants equal. It was shown by BP that :

- a) in each class $\mathcal{H} \geq 8\pi a |m|$
- b) the minimum is attained by the configurations

$$w = e^{i\alpha} \cot(\beta/2) = \prod_{i,j} \left(\frac{z - z_i}{\lambda} \right)^{m_i} \left(\frac{\lambda}{z - z_j} \right)^{n_j} \quad (5.2)$$

where $z = x + iy$; and λ is an arbitrary scale parameter. The restriction $\sum m_i > \sum n_j$ ensures that as $|z| \rightarrow \infty$; $\beta \rightarrow 0$. The homotopy index is $m = \sum m_i$ and it measures the number of points in the z plane that yields the same w . Woo [9] has shown that these minima are the only metastable configurations.

Consider next the more realistic case of the nematic in a cylinder of radius R , axis perpendicular to the x - y plane, and walls such that \mathbf{n} likes to be parallel to them, that is $\mathbf{n} = (0, 0, 1)$ at the walls. The topological classification is then the same as in the BP case. One hopes that there will exist a one to one correspondence between the BP configurations and the metastable configurations in the finite cylinder. The reasoning is that as one adiabatically shrinks the infinite cylinder (the x - y plane) the configurations within will evolve adiabatically. But upon examining the equations I find that all configurations are point-like, unlike in the BP case. Such configurations are of no physical interest. We are thus faced with an important fact : even if the configuration space of a system admits interesting homotopy classes, the configurations that minimise \mathcal{H} in each class may be physically unobservable or even non-existent. We would now like to trace the origin of this disaster in the case of the \mathcal{H} in eq. (5.1).

Our analysis follows closely that of Derrick [10] and is dimensional in nature. We start with the fact that \mathcal{H} in eq. (5.1) is scale invariant, that is to say, two configurations $\mathbf{n}(x)$ and $\mathbf{n}(x/\lambda)$, related by a scale transformation have the same energy. That is

$$\begin{aligned} \mathcal{H}[\mathbf{n}(x/\lambda)] &\equiv \int_{\text{all space}} d^2x [\nabla_x \mathbf{n}(x/\lambda)]^2 = \\ &= \int_{\text{all space}} d^2(x/\lambda) [\nabla_{x/\lambda} \mathbf{n}(x/\lambda)]^2 = \mathcal{H}[\mathbf{n}(x)]. \end{aligned}$$

Consequently, the minimum in each class is attained, not by one configuration, but a family related by scale parameter λ (see eq. (5.2)). (The family is actually even bigger, but that does not concern us here.) The BP solutions thus represent a system in neutral

equilibrium with respect to scale changes. This is why the adiabatic arguments are invalid : no matter how smoothly the walls of the cylinder are brought in (from infinity to R) the BP configurations collapse under the push to zero size, with no restraining forces.

What we need is an \mathcal{H} with terms that grow as the configuration shrinks. One such example is

$$\mathcal{H} = \int [a(\nabla\mathbf{n})^2 + b(\nabla\mathbf{n})^4] d^2x. \quad (5.3)$$

The $(\text{grad})^4$ term averts the complete collapse, since it grows as $1/d^2$, d being the size of the configuration, where by size I mean the spatial distance over which the field differs appreciably from y_0 . With the walls pushing in and the b term pushing out, there will be some size d_0 which minimises the energy. Several remarks are in order :

i) Since the system is in reality discrete, there is a fundamental length l_0 , which is of order molecular dimensions; the size d_0 of the continuum solution should be much greater than l_0 for it to have a sensible counterpart in the discrete case.

ii) The $(\text{grad})^4$ term is only an example and any term of that dimensionality, such as $(\nabla_x \mathbf{n})^4$ is equally good. These will be referred to as b -terms.

iii) The domain X must be a finite cylinder. But for the walls, the $1/d^2$ nature of the b -term will cause the configuration to expand without limit in the process of minimising its energy.

iv) If X were a 3-dimensional region, the a and b terms, which contribute as d and $1/d$ respectively, will lead to an equilibrium size d_0 , even if X is all of space.

v) Configurations larger than the equilibrium one may also be observed under some conditions. These would of course shrink towards the size d_0 . If however the shrinking is slow, they may possibly be observed in the process. This is not a far fetched possibility. Certain closed loop structures in nematics have been photographed in the process of collapse, which takes a few seconds [11].

vi) There are no general principles, as in quantum field theory, that forbid b -terms or even higher gradients, though we cannot vouch for their size. That there must be some b -term can be seen as follows. Although we treat the system as a continuum, we know there is an underlying length l_0 . Fields that vary appreciably over the scale must be excluded, that is, we must impose a cut off. The b -terms, whose contributions to \mathcal{H} grow rapidly for such configurations effect such a cut-off. One cannot say more about the size of the b -terms in general.

vii) The above points i)-vi), dimensional in nature, apply to other systems as well and not just nematics.

There is one BP configuration that can be observed in part in a finite cylinder, quite independent of the b -terms. It is the $m = 1$ map

$$w = e^{i\alpha} \cot(\beta/2) = \left(\frac{z}{\lambda}\right) = \left(\frac{x + iy}{\lambda}\right) = \left(\frac{\rho}{\lambda}\right) e^{i\varphi}$$

ρ and φ being cylindrical coordinates on the plane. Notice that at $\rho = \lambda$, $\beta = \pi/2$ and $\varphi = \alpha$, that is, the director lies in the x - y plane and is everywhere radial. It is clear that the introduction of a cylinder, whose axis is along the z -direction, and whose walls lie on $\rho = \lambda$, will not disturb the configuration within, provided \mathbf{n} likes to be *normal* to the walls. Such a configuration has in fact been observed and studied by Cladis and Kléman [12], William, Pieranski and Cladis [13] and Meyer [14]. To the best of my knowledge none of the other BP configurations can be observed in this manner.

6. The nematic liquid in three space dimensions. —

Let X be all of three space, parametrised by the spherical coordinates r , θ and φ . Given that $\mathbf{n} \rightarrow (0, 0, 1)$ as $r \rightarrow \infty$, X becomes equivalent to S^3 and the relevant group is $\Pi_3(P^2) = Z_\infty$. Let us discuss briefly how representative maps from each class, called Hopf maps, are obtained and the significance of the index m .

Let us first trade the coordinates r , θ and φ for those describing S^3 . Rather than use the angular coordinates that span the surface of S^3 , we prefer four cartesian coordinates x_1, x_2, x_3 and x_4 constrained by

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1. \tag{6.1}$$

We have chosen the radius of the sphere S^3 representing X to be unity. The relation between the two sets of coordinates is

$$\begin{aligned} x_1 &= \frac{r^2 - 1}{r^2 + 1} \\ x_2 &= \frac{2r \cos \theta}{1 + r^2} \\ x_3 &= \frac{2r \sin \theta \cos \varphi}{1 + r^2} \\ x_4 &= \frac{2r \sin \theta \sin \varphi}{1 + r^2} \end{aligned} \tag{6.2}$$

Note that the north and south poles of S^3 , $x_1 = \pm 1$ respectively, go to $r = \infty$ and 0 respectively. Our map is the three dimensional version of the familiar stereographic projections of S^2 on to a plane. The maps $w(x_1, x_2, x_3, x_4)$, where $w = e^{i\alpha} \cot(\beta/2)$ are more easily described in terms of complex coordinates $z_1 = x_1 + ix_2$ and $z_2 = x_3 + ix_4$ constrained by $|z_1|^2 + |z_2|^2 = 1$. The Hopf maps have the feature

$$w(z_1, z_2) = w(z_1/z_2 = z). \tag{6.3}$$

It can easily be verified that the allowed range for the variable z is the entire complex plane. Now, we already know that if X is the entire x - y plane and $Y = P^2$ (or S^2), there exist maps which fall into classed labelled by an integer m , the BP configurations being representative elements from each class. In fact the Hopf map of degree m ,

$$w_m(z) = e^{i\alpha_m} \cot\left(\frac{\beta_m}{2}\right) = z^m \tag{6.4}$$

is just the BP function of index m with no poles and all m zeros at the origin. Retracing our steps back to r , θ , and φ , we get

$$\begin{aligned} \alpha_m &= m \left(\tan^{-1} \frac{2r \cos \theta}{1 - r^2} - \varphi \right) \\ \beta_m &= 2 \cot^{-1} \left\{ \frac{(1 - r^2)^2 + 4r^2 \cos^2 \theta}{4r^2 \sin^2 \theta} \right\}^{1/2} \end{aligned} \tag{6.5}$$

The broad features of w_m are as follows. Centered at the origin is a circle of unit radius lying in the x - y plane called the core on which $\beta = \pi$ and around which the field varies most and the energy is most concentrated. The core is surrounded by tori of increasing size and decreasing β . Their cross sections in the half plane $\varphi = \varphi_0$ are circles, whose centers move away from the core as they grow in size (Fig. 2). The cross sections are like the closed lines of magnetic field that would surround the core if it were a current carrying loop. The largest of these tori with $\beta = 0$ is bounded by the x -axis and the sphere at infinity. If we move once around the circular cross section of the torus $\beta = \beta_0$ in the $\varphi = 0$ plane, α_m will range between 0 and $2\pi m$ (see Fig. 2). In other words, as one describes this circle in space, the image point completes m turns around the sphere $Y = P^2$ starting at the point $\alpha = 0, \beta = \beta_0$ and staying on the latitude $\beta = \beta_0$. In any other plane $\varphi = \varphi_0$, the only

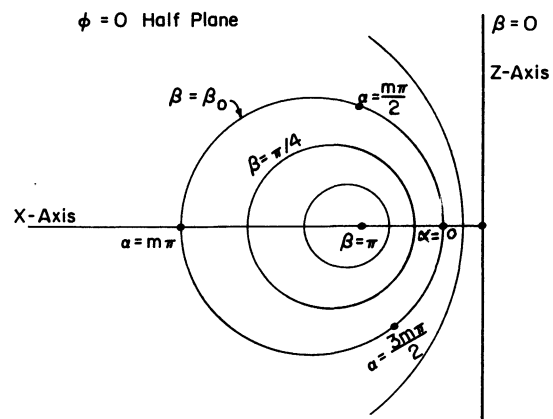


FIG. 2. — A cross section in the $\varphi = 0$ half plane of the Hopf map. The core ($\beta = \pi$) comes out of the paper as a point. Each circle corresponds to a given β . On any one circle $\beta = \beta_0$; α varies from 0 to $2\pi m$.

difference is that the image point starts out at $\alpha = m\varphi_0$, $\beta = \beta_0$ and completes m terms around the latitude $\beta = \beta_0$. The natural coordinates for these maps are the toriodal ones, μ , η and φ [15], in terms of which

$$\begin{aligned} \alpha_m &= -m(\varphi + \eta) \\ \beta_m &= 2 \tan^{-1} (\sin h\mu)^{|m|}. \end{aligned} \quad (6.6)$$

The Hopf maps are merely representatives from each class. To find metastable configurations one must find the minima of \mathcal{H} within each class. If a and b terms are present and b is large enough observable metastable configurations can be expected. The theory also predicts that Hopf maps larger than the stable one may be observed collapsing if the process is slow enough. Now such configurations with a core, have been seen in nematic liquids [11]. Although they start out with arbitrary shapes they become circular as they shrink. It is tempting to identify these with Hopf maps or more generally with maps homotopic to them. A more detailed experimental study is needed, especially near the core, to confirm this hypothesis. On the theoretical side remains the question of minimising \mathcal{H} in the presence of a and b terms. Enz [16] has solved the problem approximately and obtains maps homotopic to the Hopf map, following a route very different from ours.

7. ³He-A in three space dimensions. — Let X be once again all of 3-space. The order parameter is a triad of orthonormal vectors \mathbf{l} , \mathbf{x}_1 and \mathbf{x}_2 , where \mathbf{l} is the orbital angular momentum, while \mathbf{x}_1 and \mathbf{x}_2 determine the complex phase in the plane perpendicular to \mathbf{l} . Our boundary condition is that at spatial infinity the triad assumes a standard form with \mathbf{x}_1 , \mathbf{x}_2 and \mathbf{l} pointing along the x , y and z axes respectively. The triad at any interior point with position vector \mathbf{r} may be obtained by performing an $SO(3)$ rotation on the standard triad at infinity. Let the vector angle $\mathbf{\Omega}(\mathbf{r})$ specify by its magnitude and direction, the angle and axis of the rotation. Since $\mathbf{\Omega}$ is a vector of arbitrary orientation and magnitude $\leq 2\pi$, $Y =$ solid sphere of radius 2π . Since all points at the boundary refer to the same (identity) element, the topology of Y is S^3 . The careful reader would have noticed that our Y actually double counts the elements of $SO(3)$, in the following sense. Although the $SO(3)$ rotation angle can range from 0 - 2π about any axis, a rotation by an amount θ about any axis is physically equivalent to a rotation by $2\pi - \theta$ with the axis reversed. Thus we need only a solid sphere of radius π . Our sphere S^3 thus represents $SU(2)$, for which $0 \leq |\mathbf{\Omega}| \leq 4\pi$. It can be readily shown that Y for $SO(3)$ is equivalent to P^3 (see appendix). We shall continue to use $Y = S^3$ since the distinction is inconsequential : $\Pi_3(P^3) = \Pi_3(S^3)$.

Notice that the manifolds X and Y are both solid spheres with points on the surface identified, and

are thus equivalent to S^3 . At least one nontrivial map suggests itself readily : we rescale X to a sphere of radius 2π , and then do the identity mapping on to Y . The general solution to this problem is

$$\begin{aligned} \mathbf{\Omega}(\mathbf{r}) &= 2\pi f(r) \frac{\mathbf{r}}{r}, & f(0) &= 0 \\ & & f(\infty) &= 1. \end{aligned} \quad (7.1)$$

One specific example of the smooth function $f(r)$ is :

$$\mathbf{\Omega}(\mathbf{r}) = \frac{2\pi r}{\sqrt{r^2 + \lambda^2}} \left(\frac{\mathbf{r}}{r} \right) \quad (7.2)$$

where λ is an arbitrary scale. The homotopy index is $m = 1$ for this identity map, since to each point of Y ; there corresponds just one point of X . For the same reason, we can get a map of index m by taking a sphere Y of radius $2\pi m$ (wherein each distinct element of $SU(2)$ appears m times) and mapping X on to it :

$$\mathbf{\Omega}(\mathbf{r}) = \frac{2\pi m r}{\sqrt{\lambda^2 + r^2}} \left(\frac{\mathbf{r}}{r} \right). \quad (7.3)$$

The configuration may be visualised as follows. At the origin since $\mathbf{\Omega} = \mathbf{0}$, the triad is in standard form. As we move away in some direction ; the triad rotates about an axis parallel to that direction, and by an angle that goes from 0 to $2\pi m$ as r goes from 0 to ∞ . Figure 3 shows the situation along some special directions for the case $m = 1$.

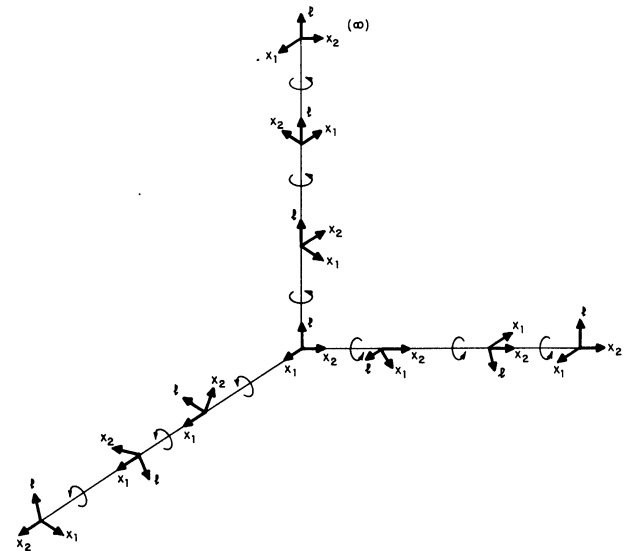


FIG. 3. — The $m = 1$ identity map in ³He-A. As we leave the origin along any ray, the triad twists along that axis and completes a 2π revolution as we reach infinity.

As always we have only presented examples from each class. The determination of metastable states involves minimising the hamiltonian in each class.

8. **A word of caution.** — We conclude our discussion with the following words of caution : the entire preceding analysis is an approximation. This is a reflection of the fact that although we represent our media as continuum, they are in reality discrete, characterised by some length l_0 , which may be a lattice spacing, molecular dimension etc. Recall that the continuum field $y(x)$ is usually defined as the average of a microscopic variable over a small but macroscopic volume surrounding the point x . This volume is much larger than l_0 . If the field $y(x)$ so defined varies very smoothly, that is, its scale for variations is much larger than l_0 , the medium may be reliably described as a continuum, and our topological analysis applied to it. If however the system wanders off into configurations where it cannot be represented meaningfully by a continuum function, we must abandon our analysis, and in particular the topological conservation laws.

Consider the following example. Imagine a configuration that is very large and smooth and is described by a continuum function $y(x)$ that belongs to a class distinct from the vacuum. Imagine rescaling this configuration to smaller and smaller sizes. The continuum approximation will break down at one point. If we keep going till only a few lattice sites or molecules are involved we have essentially reached the vacuum. (This would not happen in a real continuum, for a configuration, however much reduced in scale, will contain all the details of the original map and hence the homotopy index.) The fact that the homotopy index may thus be destroyed is a fatal blow to our analysis in principle, but not necessarily so in practice if one of two things happens :

i) In the course of performing the above rescaling, we encounter states of very high energy. This is what would happen if we had a sufficiently large b -term : once we scale down to a size d_0 , further reduction in size would cost a lot of energy, thereby presenting an effective barrier. It is of course necessary that d_0 be much larger than l_0 for our arguments, based on the continuum estimate of energy, to be valid.

ii) Even though there is no barrier, the time taken for the system to go through the above mentioned stages is large. In such a case we have an approximate conservation law and a homotopy classification valid over sizeable periods.

The shrinking configuration is a special case of a general phenomenon of the system going into configurations that vary so rapidly that the continuum description breaks down. Our topological analysis is useful if such excursions are either slow or suppressed by energy barriers due to the gradient terms in \mathcal{H} . Even if one picks on a specific system with a well defined hamiltonian, estimating the time scales for these processes is a very difficult task. The purpose of this digression is not to attack this

problem but only to point out its existence and the assumptions implicit in our analysis.

9. **Summary and discussions.** — We employed homotopy theory to classify the possible configurations of several ordered systems in two and three space dimensions, since the analysis yields information on metastable configurations and topological conservation laws. But we saw that our analysis was only as good as the continuum approximation to the system. Thus a static configuration of size d_0 anticipated by the continuum theory had any bearing on the real system only if $d_0 \gg l_0$, which in turn depended on the b -term in the hamiltonian. Similarly the topological index was seen to be conserved only as long as the system did not wander off into a rapidly varying configuration where the continuum description failed. The general analysis was thus interesting only if the excursion into such states was either inhibited by an energy barrier or for some other reason, slow. Although we studied only nematics and $^3\text{He-A}$, our topological analysis can be applied verbatim to any system in two or three space dimensions for which $Y = P^n$ or S^n , $n = 2, 3$. For example we can expect a Hopf map in a ferromagnet if X is two dimensional. Even for a different Y , once the group $\Pi_n(Y)$ is found (say by referring to a book) the rest of the analysis is the same.

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Appendix. — We shall prove here that $Y[\text{SO}(3)] = P^2$. Let us begin by considering $Y[\text{SU}(2)]$. If we start with an arbitrary 2×2 complex matrix

$$\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}$$

and require that it be unitary ($UU^+ = I$) and has $\det = +1$ we get the general element of

$$\text{SU}(2) = \begin{pmatrix} z_1 & z_2 \\ -z_2^* & z_1 \end{pmatrix} \quad \text{with} \quad |z_1|^2 + |z_2|^2 = 1.$$

Thus the elements of $\text{SU}(2)$ can be visualized as points on a unit sphere S^3 . Now, to every two elements of $\text{SU}(2)$ differing by a sign, there corresponds one element of $\text{SO}(3)$. A simple example is given by the $\text{SU}(2)$ matrices representing 2π and 4π rotations about, say, the z -axis

$$e^{i2\pi\sigma_z/2} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad e^{i4\pi\sigma_z/2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

both of which correspond to the identity element of $SO(3)$. Thus antipodal points (z_1, z_2) and $(-z_1, -z_2)$ on S^3 correspond to the same element of $SO(3)$, and $Y[SO(3)] = P^3$. As this identification of antipodal points has no impact on the topological classification or stability of maps in which the sphere X wraps around the sphere Y m times, we set $Y = S^3$ in our analysis.

Although in this appendix it was more convenient

to visualize $Y[SU(2)] = S^3$ as points in a four dimensional space constrained by

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1,$$

we found it convenient, for obtaining the maps, to use the alternate equivalent description of S^3 : as a solid sphere in 3-space with points on the surface identified, since X itself has that form.

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