HAL
open science

On the (non linear) foundations of Boussinesq approximation applicable to a thin layer of fluid

R. Perez Cordon, M. G. Velarde

To cite this version:

R. Perez Cordon, M. G. Velarde. On the (non linear) foundations of Boussinesq approximation applicable to a thin layer of fluid. Journal de Physique, 1975, 36 (7-8), pp.591-601. 10.1051/jphys:01975003607-8059100 . jpa-00208290

HAL Id: jpa-00208290

https://hal.science/jpa-00208290v1

Submitted on 4 Feb 2008

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Tome 36 N° 7-8 JUILLET-AOÛT 1975

LE JOURNAL DE PHYSIQUE

Classification
Physics Abstracts
6.315

ON THE (NON LINEAR) FOUNDATIONS OF BOUSSINESQ APPROXIMATION APPLICABLE TO A THIN LAYER OF FLUID

R. PEREZ CORDON

Departamento de Fisica
Universidad Autonoma de Madrid, Canto Blanco (Madrid), Spain

and M. G. VELARDE (*)

Service de Physique Théorique
C.E.N. de Saclay, BP 2, 91190 Gif-sur-Yvette, France

(*Reçu le 3 février 1975, accepté le 18 mars 1975*)

Résumé. — La description thermohydrodynamique d'une fine couche horizontale de fluide quelconque, chauffée par le bas (problème de Rayleigh-Bénard) est effectuée ici par une méthode perturbative à *deux* paramètres. Au premier ordre de perturbation on obtient les équations dites de Boussinesq-Oberbeck, en accord avec des résultats antérieurs de Mihaljan [*Astrophys. J.*, **136** (1962) 1126]. Les difficultés inhérentes à la méthode d'obtention des termes d'ordre supérieur sont ici (contrairement à la théorie de Mihaljan) exclues du développement. Ceci est rendu possible par un choix convenable d'un champ adiabatique hydrostatique de référence et de deux paramètres ayant le même ordre de grandeur. Dans une limite bien précise la théorie présentée ici recouvre d'anciens résultats obtenus par Malkus pour des couches de gaz dilué.

Abstract. — A two-parameter perturbation scheme for the thermohydrodynamic description of a horizontal layer of a single component arbitrary fluid heated from below (Rayleigh-Bénard problem) is presented here. The first approximation leads to the Boussinesq-Oberbeck equations. This agrees with previous results obtained by Mihaljan [*Astrophys. J.* **136** (1962) 1126]. Contrary to Mihaljan's theory however, the series expansion given here is free from *inherent* difficulties in obtaining higher order approximations *viz.* non-Boussinesq effects. This is done by choosing a suitable adiabatic hydrostatic reference field and two parameters of the *same* order of magnitude. In a well defined limit the theory presented here recovers earlier results obtained by Malkus (as yet unpublished) for dilute ideal gas layers.

1. **Introduction.** — The stability analysis of a thin horizontal fluid layer heated from below (Rayleigh-Bénard problem) is generally carried out within the so-called Boussinesq [1] (1903) or Oberbeck [2] (1888) approximation (see for details Chandrasekhar [3] chapter 2, and for remarks of historical interest, Joseph [4]). This approximation contains a number of approximations of varying importance. For instance viscous dissipation or compressibility effects are disregarded, as well as temperature variations of such parameters as viscosity, thermal conductivity or thermal expansion coefficient. However viscous dissipation, may be important on occasions. For if the body force is large or if the length scale of the problem is large, viscous heating plays a drastic role. Such might be the case for convection in the earth's mantle. On the other hand if compressibility effects are of importance they are comparable in magnitude to viscous heating effects when the Gruneisen's constant is of order unity. This happens to be the common situation with standard liquids and gases. There is yet another important feature of non Boussinesq effects of a different nature. A drastic qualitative difference

(*) *Permanent address* : Departamento de Fisica-C-3, Universidad Autonoma de Madrid, Canto Blanco (Madrid) Spain.

Article published online by EDP Sciences and available at http://dx.doi.org/10.1051/jphys:01975003607-8059100

occurs in the convective cell structure of fluids with a strong temperature dependence of the viscosity (see for details Hoard *et al.* [5]) (1).

Thus one is faced with the problem of assessing the role of the Boussinesq-Oberbeck approximation within the general thermohydrodynamic description of the fluid layer. Mihaljan [6] was the first author to start looking at a rigorous approach to the problem. Partially at least, he succeeded. He was able to define a two-parameter perturbative scheme for a rather general description of the thermohydrodynamics of fluid layers. Let L and θ be respectively the vertical depth of a horizontal fluid layer and the transverse temperature difference. He carried a regular change of variables to two new parameters, say ε_1 and ε_2 both smaller than unity (they are defined in section 2 below). The layer is assumed of small aspect ratio *viz.* horizontally of infinite extent. The Boussinesq equations are obtained at the $\varepsilon_1^0\,\varepsilon_2^0$ approximation *viz.* the first-order series expansion terms. Yet Mihaljan's scheme was ill defined. For ε_1 and ε_2 turn out to be parameters with greatly different values for a standard experimental situation of Rayleigh-Bénard convection. It is found than for standard fluid layers and thermal constraints ε_1 and ε_2 can be of respective order $\varepsilon_1 \sim 10^{-4}$, $\varepsilon_2 \sim 10^{-11}$ (see section 2 for more details). Thus one wonders about the meaning to ascribe to Mihaljan's second-order approximations. This should be given by the $\varepsilon_1^1\,\varepsilon_2^0$ and $\varepsilon_1^0\,\varepsilon_2^1$ terms. Yet $\varepsilon_1^2\,\varepsilon_2^0 \gg \varepsilon_1^0\,\varepsilon_2^1$.

The difficulty was discussed and solved by Malkus [7]. However Malkus's analysis is restricted to dilute ideal gas layers. As neither the restricted Malkus analysis nor its generalization has appeared in the literature we set ourselves the problem of assessing Boussinesq approximation on a firmer footing. Thus in section 2 we give a schematic and critical account of Mihaljan's [6] work. Section 3 is devoted to the discussion of the usefulness of defining a adiabatic hydrostatic reference field (a.h.f). A brief account of the Malkus analysis for a dilute ideal gas layer is given in section 4. In section 5 a straightforward generalization of Mihaljan's theory is presented although it still leads to an ill defined perturbative scheme. A general perturbative scheme, well defined at all orders, is presented in section 6. However no explicit consideration is given here to any non Boussinesq contributions. They will be the subject of a separate paper.

2. **Critique of Mihaljan's analysis (1962).** — The most rigorous exposition, available up to now in the literature of the general thermohydrodynamic description of a horizontal thin fluid layer leading to the Boussinesq approximation (see references [1, 2, 3 and 8]) is that of Mihaljan [6]. A straightforward generalization however will be provided in section 5. For the sake of completeness and for unity of exposition in this paper at present we need, a review, albeit schematic, of Mihaljan's work. This will aid us in the understanding of some inherent difficulties in Mihaljan's scheme when trying to account for non-Boussinesq effects.

Let us consider a horizontal, single component *isotropic* fluid layer of depth L and infinite horizontal extent. Mihaljan starts with the following assumptions:

i) The density of the fluid is a function of temperature T alone; and α, the volumetric expansion coefficient is considered constant. Thus we have an equation of state

$$\rho = \rho_0[1 - \alpha(T - T_0)]\,. \tag{2.1}$$

Here ρ is density. T_0 is some reference temperature to which a density ρ_0 corresponds. Eq. (2.1) precludes any pressure dependence. We shall find this restriction unnecessary in order to obtain the Boussinesq approximation (see section 5 below).

ii) The specific heat at constant volume, the thermal conductivity K and the shear and bulk viscosities, denoted respectively by μ_1 and μ_2 are functions of temperature only. Thus we have the equilibrium relations

$$c_p = c_v(T) - \frac{2\,\alpha^2\,TP}{\rho_0[1 - \alpha(T - T_0)]^3} \tag{2.2}$$

and

$$\mathcal{U} = \int_{T_0}^{T} c_v\,\mathrm{d}T - \frac{T\alpha P}{\rho_0[1 - \alpha(T - T_0)]^2}\,. \tag{2.3}$$

Here $\mathcal{U}$ denotes the internal energy, c_p the specific heat at constant pressure and P is the scalar thermodynamic pressure. We shall denote in the following $\theta \equiv T - T_0$. It may be useful to take T_0 as the temperature at the bottom of the layer but this need not necessarily be so.

With the restrictions imposed above the hydrodynamic equations read as follows :

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho = -\rho\frac{\partial}{\partial x_i}v_i \tag{2.4}$$

$$\rho\frac{\mathrm{d}v_i}{\mathrm{d}t} = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left\{\frac{\mu_2}{3}\frac{\partial v_\lambda}{\partial x_\lambda}\delta_{ij} + \mu_1\left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\frac{\partial v_\lambda}{\partial x_\lambda}\delta_{ij}\right]\right\} - \rho g\delta_{i3} \tag{2.5}$$

$$\rho c_p\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\partial}{\partial x_i}\left(K\frac{\partial T}{\partial x_i}\right) + \frac{T\alpha}{1 - \alpha(T - T_0)}\left(\frac{\mathrm{d}\pi}{\mathrm{d}t} - \rho g v_3\right) + \left[\frac{\mu_2}{3}\frac{\partial v_\lambda}{\partial x_\lambda}\delta_{ij} + \mu_1\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\frac{\partial v_\lambda}{\partial x_\lambda}\delta_{ij}\right)\right]\frac{\partial v_j}{\partial x_i}\,. \tag{2.6}$$

(1) A non linear steady temperature profile at rest amounts to the case of a temperature dependent conductivity.

Here a *subscript* denotes a cartesian component, subscript 3 represents the vertical direction. δ_{ij} is Kronecker's delta and Einstein's summation convention on repeated indices is used. For convenience we have introduced the quantity π

$$\pi \equiv P + \rho_0 gZ \tag{2.7}$$

where Z is the numerical value of the vertical coordinate.

Eq. (2.1) and the use of parameter θ allows us to rewrite the differential system in the following way

$$-\alpha \frac{d\theta}{dt} + (1 - \alpha\theta) \frac{\partial v_i}{\partial x_i} = 0 \tag{2.8}$$

$$\rho_0(1 - \alpha\theta) \frac{dv_i}{dt} = -\frac{\partial \pi}{\partial x_i} + \rho_0 g\alpha\theta\delta_{i3} + F_i \tag{2.9}$$

$$\rho_0(1 - \alpha\theta)\, c_p \frac{d\theta}{dt} = \frac{\partial}{\partial x_i}\left(K \frac{\partial \theta}{\partial x_i}\right) + \frac{T\alpha}{1 - \alpha\theta}\left(\frac{d\pi}{dt} - \rho_0 g v_3\right) + \Phi . \tag{2.10}$$

The following functions have being defined

$$F_i \equiv \frac{\partial}{\partial x_j}\left[\frac{\mu_2}{3}\frac{\partial v_\lambda}{\partial x_\lambda}\delta_{ij} + \mu_1\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\frac{\partial v_\lambda}{\partial x_\lambda}\delta_{ij}\right)\right] \tag{2.11}$$

$$\Phi \equiv \left[\frac{\mu_2}{3}\frac{\partial v_\lambda}{\partial x_\lambda}\delta_{ij} + \mu_1\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\frac{\partial v_\lambda}{\partial x_\lambda}\delta_{ij}\right)\right]\frac{\partial v_j}{\partial x_i} . \tag{2.12}$$

We shall now define scaling units : for both specific heats : $c_0 \equiv c_v(T_0)$. For both viscosities $\mu_0 \equiv \mu_1(T_0)$; for thermal conductivity : $K_0 = K(T_0)$. Thus one can express the material parameters in dimensionless form. We shall take them primed below. Notice that $\mu_1 \gg \mu_2$ and one has $\mu_2' \equiv (\mu_2/\mu_0) \ll 1$; $\mu_1' = 1$. The scaling unit for temperature is $\Theta \equiv T_0 - T_1$ where T_1 is the temperature at the top boundary. Height (the only length here considered) is scaled by the cell gap L. Velocities are scaled by $V \equiv \kappa_0/L$ where a reference thermometric conductivity *viz.* thermal diffusivity, is defined by $\kappa_0 \equiv K_0/\rho_0 c_0$. For the pressure we take $P_0 \equiv \rho_0 V^2$ and time is scaled according to $t_0 \equiv L/V$. Density is measured in units of ρ_0. Just to fix ideas for a $L = 1$ cm water layer and $\Theta = 1$ with T_0 around 300 K one has $\{\alpha \sim 10^{-4}$; $K_0 \sim 10^4$; $\mu_0 \sim 10^{-2}$; $c_0 \sim 10^7$; $\kappa_0 \sim 10^{-2}$; $V \sim 10^{-2}$; $P_0 \sim 10^{-4}$; $\nu_0 \equiv (\mu_0/\rho_0) \sim 10^{-2}\}$. Notice that a reference kinematic viscosity ν_0 has also been introduced.

It is important to observe that two scalings are superposed in the pressure term. One comes from the upper bound of the hydrostatic pressure variation along the vertical, this scaling being given by $P_0 \equiv \rho_0 gL$. The other scaling comes from Bernoulli's theorem. It gives a maximum variation of pressure of ordre $\frac{1}{2}\rho/v_{max}^2$ where v_{max} is the upper bound for the velocity field. Clearly both scales are of quite different order of magnitude. For water in the case referred to above one has $\rho_0 V^2 \sim 10^{-4}$ whereas $\rho_0 gL \sim 10^3$. We will see later that upon evaluating the pressure gradient the hydrostatic part can be dropped.

If we now incorporate the scaling quantities and look upon the primed (dimensionless) fields only, the original eq. (2.8) to (2.10) become :

$$-\varepsilon_1 \frac{d\theta'}{dt'} + (1 - \varepsilon_1\theta')\frac{\partial v_i'}{\partial x_i'} = 0 \tag{2.13}$$

$$(1 - \varepsilon_1\theta')\frac{dv_i'}{dt'} = -\frac{\partial \pi'}{\partial x_i'} + G\theta'\delta_{i3} + \sigma F_i' \tag{2.14}$$

$$(1 - \varepsilon_1\theta')\left[c_v' - \varepsilon_1^2\varepsilon_2\frac{2T'P'}{(1 - \varepsilon_1\theta')^3}\right] \times \frac{d\theta'}{dt'} = \frac{\partial}{\partial x_i'}\left(K'\frac{\partial}{\partial x_i}\theta'\right) + \varepsilon_2\left[\sigma\Phi' + \frac{T'}{1 - \varepsilon_1\theta'}\left(\varepsilon_1\frac{d\pi'}{dt'} - Gv_3'\right)\right] . \tag{2.15}$$

We have also introduced the following definitions :

$$\{\varepsilon_1 \equiv \alpha\Theta;\ \varepsilon_2 \equiv \kappa_0^2/c_0 L^2\Theta;\ \sigma \equiv \nu_0/\kappa_0;\ G \equiv R\sigma\}.$$

Thus we have *eight* parameters

$$\{\alpha, \Theta, L, \rho_0, g, c_0, \kappa_0, \nu_0\}.$$

σ is a reference Prandtl number and $R \equiv \alpha g\Theta L^3/\kappa_0\nu_0$ is the Rayleigh number. We have defined *four* pimonomials and by Buckingham's pi-theorem we only have *four* independent quantities. These can be taken ε_1, ε_2, G and σ. For the water layer referred to above one has the following estimates

$$\varepsilon_1 \sim 10^{-4};\ \varepsilon_2 \sim 10^{-11};\ G \sim 10^{-3};\ \sigma \sim 1 \tag{2.16}$$

for a Rayleigh number of value $R \sim 10^3$.

A perturbative scheme can be defined now. The obvious parameters to substitute for L and Θ, are ε_1 and ε_2. The transformation $\{ L, \Theta \} \to \{ \varepsilon_1, \varepsilon_2 \}$ is allowed as the Jacobian is non-vanishing. These are the parameters used by Mihaljan (1962) (2).

For easy of reference we shall, in expanding, take the following convention for a function $\Phi(\varepsilon_1, \varepsilon_2)$, and write

$$\Phi(\varepsilon_1, \varepsilon_2) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \Phi^{ij} \varepsilon_1^i \varepsilon_2^j . \tag{2.17}$$

For a function $\Psi(P', \theta')$ we shall write

$$\Psi(P', \theta') \equiv \Psi(P'^{00}, \theta'^{00}) + + \left(\frac{\partial \Psi}{\partial \theta'}\right)^{00} (\cdots) + \cdots \tag{2.18}$$

and so on.

Once all functions and parameters are formally expanded one collects the *zeroth-zeroth* order contribution, called here the first non trivial approximation. One gets

$$\frac{\partial v_i'^{00}}{\partial x_i'} = 0 \tag{2.19}$$

$$\frac{\partial v_i'^{00}}{\partial t'} + v_j'^{00} \frac{\partial v_i'^{00}}{\partial x_j'} = - \frac{\partial \pi'^{00}}{\partial x_i'} + + G\theta'^{00} \delta_{i3} + \sigma F_i'^{00} \tag{2.20}$$

$$c_v'^{00}\left(\frac{\partial \theta'^{00}}{\partial t} + v_j'^{00} \frac{\partial \theta'^{00}}{\partial x_j'}\right) = \frac{\partial}{\partial x_j'}\left(\kappa'^{00} \frac{\partial \theta'^{00}}{\partial x_j'}\right) . \tag{2.21}$$

We have used the definition

$$F_i'^{00} \equiv \frac{\partial}{\partial x_j'}\left[\mu_1'^{00}\left(\frac{\partial V_i'^{00}}{\partial x_j'} + \frac{\partial V_i'^{00}}{\partial x_i'}\right)\right] . \tag{2.22}$$

Notice that in the equations above the bulk viscosity does not show up. Thus in the first order approximation velocities are considered much smaller that the speed of sound in the fluid. Notice also that

$$T'^{00} = T_0' + \theta'^{00} \neq T_0' .$$

The eq. (2.19) to (2.22) are not yet the Boussinesq equations as commonly used in the literature. The parameters α, c_v, κ and μ_1 are not strict constants. To actually get the standard Boussinesq approximation one must impose $c_v \equiv c_0$, $K \equiv K_0$, $\mu_1 \equiv \mu_0$. Thus Milhaljan's scheme is not strictly consistent even at the level of Boussinesq approximation. This apparent gap in the logical framework of Mihaljan's analysis is overcome in section 6 below.

(2) Here and later on one of the parameters will basically be Θ, the temperature difference across the fluid layer. Thus one wonders about the relevance of discussing turbulent behaviour for high enough Rayleigh number in a given fluid layer within the Boussinesq approximation.

It is of interest to note that the equations used by Palm [9] and Segel and Stuart [10] can be obtained from eq. (2.19) to (2.22) here. It suffices to return to a description in terms of the dimensional fields and to incorporate a functional form $\mu_1 = \mu_0(1 + \gamma\theta)$ where γ is a parameter defined by these authors. Thus these authors' equations that were assumed to describe some non-Boussinesq effects arise in fact from Mihaljan's zeroth-zeroth order approximation.

It is clear that when carrying on Mihaljan's perturbative scheme to higher order approximations, no pressure dependence can be taken into account and yet this is not a negligible contribution as we show now. For, assuming an equation of state like

$$\rho = \rho_0\left[1 - \alpha\theta + \chi \Delta P + \left(\alpha^2 - \frac{\partial \alpha}{\partial T}\right)\frac{\theta^2}{2} + \cdots\right] \tag{2.23}$$

one has for a standard fluid for a range of values given above

$$\left(\frac{\partial \alpha}{\partial T}\right)\frac{\theta^2}{2} \sim 10^{-5} ; \quad \left(\frac{\partial^2 \alpha}{\partial T^2}\right)\frac{\theta^3}{3\,!} \sim 10^{-6}$$

whereas for a pressure drop of 10^{-3} C.G.S. units and $\chi \sim 10^{-11}$, $\chi \Delta P \sim 10^{-8}$. Such pressure dependent terms can in fact be disregarded at the first approximation. Yet for higher order corrections it may not be so anymore.

Lastly, one remarks on Mihaljan's unfortunate selection of parameters. For if $\varepsilon_1 \gg 1$, $\varepsilon_2 \gg 1$ one has however $\varepsilon_1 \gg \varepsilon_2$. Already a second order approximation does not appear very meaningful. This shows an inherent difficulty of Mihaljan's scheme to describe quasi-Boussinesq layers.

3. **Utility of a reference adiabatic hydrostatic field.** — Let us introduce the following a.h.f., P_a, T_a, ρ_a through the differential equations (3)

$$\frac{dP_a}{dz} = - g\rho_a \tag{3.1}$$

$$\frac{d}{dz}\left(K_a \frac{dT_a}{dz}\right) = 0 \tag{3.2}$$

$P_a = P_a(z)$, $\rho_a = \rho_a(T_a, P_a)$, $K_a = K_a(T_a, P_a)$ and $T_a = T_a(z)$. We have also assumed the fluid layer to be at rest. Define now a perturbation upon the a.h.f. The perturbations will be denoted with *tilded* quantities. One has :

$$T = T_a + \tilde{T} \tag{3.3}$$

$$P = P_a + \tilde{P} . \tag{3.4}$$

Assuming now for convenience in our reasonning,

(3) In this section the third cartesian coordinate x_3 will be sometimes denoted by z.

$K_a = K_0$ held constant, and $\rho_a = \rho_0[1 - \alpha(T_a - T_0)]$ we get from (3.1) and (3.2)

$$\frac{dP_a}{dz} = -g\rho_0[1 - \alpha(T_a - T_0)] \tag{3.1b}$$

$$K_0 \frac{d^2T_a}{dz^2} = 0 . \tag{3.2b}$$

Solutions of (3.1b) and (3.2b) are

$$T_a = T_0 - \beta z$$

(β appears as an arbitrary parameter) (3.5)

$$P_a = P_0 - \rho_0 g\left(z + \alpha\beta \frac{z^2}{2}\right) \tag{3.6}$$

and

$$\rho_a = \rho_0(1 + \alpha\beta z) .$$

We consider now the hydrodynamic eq. (2.4), (2.5) and the energy equation

$$\rho c_v\left(\frac{dT}{dt}\right) = -\rho \frac{\partial v_i}{\partial x_i} + \frac{\partial}{\partial x_i}\left(K \frac{\partial}{\partial x_i} T\right) + \Phi . \tag{2.6b}$$

Notice that eq. (2.6b) here differs from Mihaljan's eq. (2.6) in that we have not incorporated any change of internal energy due to volumetric variations and K is not K_0. Using eq. (2.4) (2.5) and (2.6b) we have for the perturbed fields (3.3) and (3.4)

$$\rho_0 \alpha\beta v_3 - \rho_0 \alpha \frac{d\tilde{T}}{dt} = -\rho_0(1 + \alpha\beta z - \alpha\tilde{T}) \frac{\partial}{\partial x_i} v_i \tag{3.7}$$

$$\rho_0(1 + \alpha\beta z - \alpha\tilde{T}) \frac{dv_i}{dt} = -\frac{\partial}{\partial x_i} \tilde{P} + F_i + g\rho_0 \alpha\tilde{T}\delta_{i3} \tag{3.8}$$

$$\rho_0(1 + \alpha\beta z - \alpha\tilde{T}) c_v\left(-\beta v_3 + \frac{d\tilde{T}}{dt}\right) = -\left[P_0 - g\rho_0\left(z + \frac{\alpha\beta z^2}{2}\right) + \tilde{P}\right] \frac{\partial v_i}{\partial x_i} + \frac{\partial}{\partial x_i}\left(K \frac{\partial}{\partial x_i} \tilde{T}\right) - \beta \frac{\partial K}{\partial z} + \Phi . \tag{3.9}$$

We shall now make use of the same scaling parameters introduced in the preceding section and give to β the value θ/L, constant. The two parameters ε_1 and ε_2 are also considered and a perturbative scheme developed for eq. (3.7) to (3.9). Up to the first non trivial approximation one gets

$$\frac{\partial}{\partial x_i} v_i = 0 \tag{3.10}$$

$$\frac{d}{dt} v_i = -\frac{1}{\rho_0} \frac{\partial}{\partial x_i} \tilde{P} + \frac{1}{\rho_0} F_i^0 + g\alpha\tilde{T}\delta_{i3} \tag{3.11}$$

$$\rho_0 c_v\left[-\beta v_3 + \frac{d\tilde{T}}{dt}\right] = -\beta \frac{\partial K}{\partial x_3} + \frac{\partial}{\partial x_i}\left(K \frac{\partial}{\partial x_i} \tilde{T}\right). \tag{3.12}$$

In eq. (3.11) we have introduced for F_i^0 the expression given in (2.22).

In retrospect eq. (3.10) to (3.12) are in fact Mihaljan's first order eq. (2.19) to (2.21). Yet a few remarks are of pertinent interest. In comparing with Mihaljan's approximation (1962) and here eqs. (2.19) to (2.21)) we notice that dropping the hydrostatic pressure gradient when computing pressure variations amounts to dropping the two scales referred to earlier. Thus all terms become of the same order of magnitude. On the other hand dropping the steady heat flux $K_0 \beta$ in the energy equation leaves the internal energy changes due to heat transfer only. Furthermore the contribution $P_0 \frac{\partial}{\partial x_i} v_i$ is eliminated from eq. (3.12) by using eq. (3.10). Also eq. (3.10) to (3.12) indeed refer to perturbations upon a given a.h.f. solution of the hydrodynamic equations. All terms in these equations are *numerically* alike. This is not so in Mihaljan's equations.

Also in order to understand now the utility of Mihaljan's scheme we proceed as follows. Let the a.h.f. be given by eq. (3.5) and (3.6). A straightforward perturbative analysis will now be given. Let tilded quantities again denote perturbation upon the a.h.f. of reference. A direct procedure is to proceed like Chandrasekhar (1961) (Chap. 2). Thus one again, gets up to first order in the tilded quantities, the first eq. (3.10), but for the Navier-Stokes equation one now gets

$$\rho_0 \frac{\partial}{\partial t} v_i = -\frac{\partial}{\partial x_i} \tilde{P} + F^* + g\rho_0 \alpha\tilde{T}\delta_{i3} \tag{3.11b}$$

where F^* represents the part of F that comes from expanding $\mu(T, P)$ in the perturbations $\tilde{T}$ and $\tilde{P}$. For the energy equation one gets

$$c_v \rho_0\left[-\beta v_3 + \frac{\partial\tilde{T}}{\partial t}\right] = -\beta \frac{\partial K}{\partial z} + \frac{\partial}{\partial x_i}\left(K \frac{\partial}{\partial x_i} \tilde{T}\right). \tag{3.12b}$$

One notices that with such a straightforward method we drop the convective terms from eq. (3.11b) and (3.12b). This is to be expected as for the example

given in section 2 above, one has the following estimates

$$v_i \frac{\partial}{\partial x_i} T \sim \beta v_3 \sim 10^{-2}\,; \quad v_i \frac{\partial}{\partial x_i} v_j \sim 10^{-4}\,.$$

However these convective terms remain in Mihaljan's first order approximation. On the other hand βv_3 appears in eq. (2.12b). Thus this latter method developed here is not consistent with dimensional analysis of the problem. Nor is it consistent with an actual numerical estimate of the contributions. From this point of view Mihaljan's analysis when adequately supplemented with a reference a.h.f. is a more suitable approach. The inherent difficulties referred to above still remain though. For this reason we now turn our attention to an analysis developed by Malkus [7].

4. **Analysis of a scheme proposed by Malkus (1964).** — In this section we discuss the approach developed by Malkus [7] for a dilute ideal gas layer. We shall again get the Boussinesq equations. But we shall also gain insight into the procedure to be followed in a general description given in section 6.

The starting point is again the set of hydrodynamic eq. (2.4) to (2.6) but based on the following equations of state

$$\rho = \frac{M}{R_0}\frac{P}{T} \qquad (M : \text{molecular weight}, \; R_0 : \text{gases universal constant}) \tag{4.1}$$

$$\mathcal{U} = c_v T\,. \tag{4.2}$$

Malkus proceeds to define a reference a.h.f. through the differential equations

$$\rho_a \frac{c_v}{M}\frac{dT_a}{dz} = \frac{P_a}{\rho_a}\frac{d\rho_a}{dz} \tag{4.3}$$

$$\frac{dP_a}{dz} = -\rho_a g\,. \tag{4.4}$$

This amounts to a local adiabatic condition imposed upon the equation of state. From eq. (4.3) and (4.4) one gets the relations

$$T_a = T_0\left(1 - \frac{Mgz}{c_p T_0}\right) \tag{4.5}$$

$$P_a = P_0\left(1 - \frac{Mgz}{c_p T_0}\right)^{c_p/R_0}. \tag{4.6}$$

The following relation holds

$$\left(\frac{\rho_a}{\rho_0}\right) = \left(1 - \frac{Mgz}{c_p T_0}\right)^{c_v/R_0}. \tag{4.7}$$

One now assumes that the unknown fields are local perturbed fields upon the a.h.f. of reference. Thus we consider again eq. (3.3) and (3.4) together with the corresponding equation for ρ. Upon substitution of the perturbed quantities into eq. (2.4) to (2.6) one gets a differential system for the *tilded* quantities in terms of the a.h.f. variables. One then introduces scaling units as done in section 2. The length are scaled with L; velocities with an upper bound for buoyant vertical velocity $V \equiv (gL\,\Delta\tilde{T}/T_0)^{1/2}$; time with L/V and viscosity with $\mu_0 \equiv \mu(T_0)$. Needless to say the only viscosity here considered is the shear viscosity !

As done before we denote the new dimensionless quantities with primes. With the a.h.f. the vertical temperature difference across the layer of height $z' = 1$, is

$$\Delta T_a = -T_0 \frac{gML}{c_p T_0} \equiv -\eta T_0\,. \tag{4.8}$$

Eq. (4.8) defines η. The value ΔT_a is independent of the actual temperature difference θ between boundaries. In fact this temperature difference can be thought of as a perturbation upon the a.h. temperature ; one has

$$\theta = \Delta T_a + \Delta\tilde{T}\,. \tag{4.9}$$

Already the a.h. fields can be recast into primed quantities by using the monomial η defined above. One has

$$\frac{T_a}{T_0} = 1 - \eta z' \tag{4.10}$$

$$\frac{P_a}{P_0} = (1 - \eta z')^{1/S} \tag{4.11}$$

$$\frac{\rho_a}{\rho_0} = (1 - \eta z')^{\frac{1-S}{S}} \tag{4.12}$$

where a second monomial has been introduced $S \equiv R/c_p$.

Pressure can be scaled with $\rho_0 V^2$; temperature with $T_0\,\varepsilon$ (where $\varepsilon \equiv \Delta\tilde{T}/T_0$) and density with $\rho_0\,\varepsilon$. The reason for these last three scales is to have all dimensionless numbers bounded by unity.

Thus we have *nine* parameters

$$\{ L, \theta, c_v, \mu_0, K_0, g, P_0, T_0, R_0 \}$$

and in the context here according to Buckingham's pi-theorem we are allowed to have *five* independent pi-monomials. Introduced already are η, s and ε. Two more are straightforwardly obtained. They are

$$R \equiv g\frac{L^3\,\varepsilon}{\kappa\nu} \qquad \text{(Rayleigh number)} \tag{4.13}$$

$$\sigma \equiv \frac{\nu}{\kappa} \qquad \text{(Prandtl number)}\,. \tag{4.14}$$

For later convenience one also defines

$$f \equiv 1 - \eta z' \tag{4.15}$$

$$m \equiv \frac{1-s}{s}\,. \tag{4.16}$$

In terms of these parameters and the dimensionless fields the hydrodynamic eq. (2.4) to (2.6) now become

$$\frac{\mathrm{d}}{\mathrm{d}t'}\left[m(f^m + \varepsilon\rho')\right] = -\frac{\partial}{\partial x'_i} v'_i \tag{4.17}$$

$$(f^m + \varepsilon\rho')\frac{\mathrm{d}}{\mathrm{d}t'} v'_i = -\rho'\,\delta_{i3} - \frac{\partial}{\partial x'_i} P' + \left(\frac{\sigma}{R}\right)^{1/2} R' \tag{4.18}$$

$$\varepsilon(f^m + \varepsilon\rho')\frac{\mathrm{d}T'}{\mathrm{d}t'} = \varepsilon\left(\frac{1}{\sigma R}\right)^{1/2}\frac{\partial}{\partial x'_i}\left(K'\frac{\partial}{\partial x'_i}T'\right) + \\ + \eta\left(\frac{1}{\sigma R}\right)^{1/2}\frac{\partial}{\partial z}K' + \varepsilon\eta\left\{v'_3\,\rho' + \frac{\mathrm{d}P'}{\mathrm{d}t'} + \left(\frac{\sigma}{R}\right)^{1/2}\Phi'\right\}. \tag{4.19}$$

Now Malkus uses ε and η as the two perturbative parameters. Thus up to the first non trivial approximation one gets

$$\frac{\partial}{\partial x'_i} v'^{00}_i = 0 \tag{4.20}$$

$$\frac{\partial}{\partial t} v'^{00} = T'^{00}\,\delta_{i3} + \left(\frac{\sigma}{R}\right)^{1/2} F^{00}_i - \frac{\partial}{\partial x'_i} P'_{00} \tag{4.21}$$

$$\frac{\partial}{\partial t'} T'^{00} = \left(\frac{1}{\sigma R}\right)^{1/2}\frac{\partial^2}{\partial x'^2_i} T'^{00}. \tag{4.22}$$

These are indeed the Boussinesq equations (see ref. [3] Chap. 2).

Now a few comments are pertinent here to differentiate Malkus's approach from that of Mihaljan.

Firstly we notice that for standard fluids say air, both parameters η and ε are of nearly the same order of magnitude. This fact, together with the fact that all primed quantities are bounded by unity, justifies consideration of higher order corrections. In this, Malkus's choice is fortunate.

Secondly from Malkus's scheme we get a deeper analysis of the Boussinesq approximation, restricted however to dilute ideal gases only. We see that dropping higher order terms we get equations for the perturbations to the a.h.f. alone. The solutions of these new equations added to the reference a.h.f. correspond to solutions of the Boussinesq approximation. This is just the scheme that we have generalized in section 6 below.

However we may note that the field defined by eq. (4.3) and (4.4) above is *not* the best picture for the physical description of a thin gas layer at rest under the adverse thermal constraint. Shortly we will introduce a different and more suitable a.h.f. Yet, however different the two a.h.f. may be both will lead to same first non trivial approximations. From the discussion that follows will also emerge the justification of selecting an arbitrary, perhaps unphysical, a.h.f. to generate these same equations together with relevant higher order corrections in a self-consistent scheme.

To fix ideas let us think in terms of a helium gas layer. This is indeed taken as an ideal gas under normal conditions. We introduce the most general a.h.f. through eq. (3.1) (3.2) and (4.1) and let

$$K \sim T^\alpha. \tag{4.23}$$

For a *hard-sphere* gas $\alpha = 0.5$. From eq. (3.1) and (3.2) we get

$$T = T_0^{\alpha+1} + (\alpha + 1)\,T^\alpha\left(\frac{\mathrm{d}T}{\mathrm{d}z}\right)_0 z^{1/\alpha+1} \tag{4.24}$$

$$P = P_0 \exp\left[-g\frac{M}{R_0}\frac{(T^\alpha - T_0^\alpha)}{\alpha T_0^\alpha\left(\frac{\mathrm{d}T}{\mathrm{d}z}\right)_0}\right]. \tag{4.25}$$

Notice that $\left(\frac{\mathrm{d}T_a}{\mathrm{d}z}\right)_0 < 0$. We shall denote

$$\left(\frac{\mathrm{d}T_a}{\mathrm{d}z}\right)_0 \equiv -U_0.$$

Now solutions (4.24) and (4.25) are incompatible with the system (4.3) (4.4) and (4.5). On the other hand comparing solutions (4.5) and (4.6) with solutions (4.24) and (4.25) shows that their second order terms are different. Thus in the first order approximation the perturbative scheme generated from (4.24)

(4) Using Malkus's parameters applied to the field (4.24) and (4.25) one gets at the first order approximation

$$\frac{\partial}{\partial x_i} v^{00}_i = 0 \tag{4.26}$$

$$\frac{\partial}{\partial t} v^{00}_i + v^{00}_j\frac{\partial}{\partial x_j} v^{00}_i = -\frac{\partial}{\partial x_i} P^{00} + \left(\frac{\sigma}{R}\right)^{1/2} F^{00}_i - \rho^{00}\,\delta_{i3} \tag{4.27}$$

$$\frac{\partial}{\partial t} T^{00} + v^{00}_j\frac{\partial}{\partial x_j} T^{00} - \frac{\eta}{\varepsilon}\frac{v^{00}_3}{s\zeta} = -\frac{\eta}{\varepsilon} v^{00}_3 + (\sigma R)^{-1/2}\frac{\partial^2}{\partial x_i^2} T^{00} \tag{4.28}$$

where a new monomial has been used $\zeta \equiv gM/RU_0$.

and (4.25) with (2.4) to (2.6) will be different from the similar approximation that arises from (4.20) (4.21) and (4.22) ([4]). Let us prove that they do in fact correspond to the same order of approximation.

Let us start by defining two different a.h.f. denoted by subscripts $i = 1,2$; T_{a_i}, P_{a_i} and ρ_{a_i}. Let T, P and ρ now be quantities satisfying the general hydrodynamic eq. (2.4) to (2.6). For sake of simplicity we shall concentrate our discussion on the temperature field only. Let tilded quantities be perturbations upon the a.h.f. of reference. We have the following identities

$$\frac{T}{T_0} = 1 + \frac{T_{a_1} - T_0}{T_0} + \frac{\tilde{T}_1}{T_0} = \\ = 1 + \frac{T_{a_2} - T_0}{T_0} + \frac{\tilde{T}_2}{T_0} . \quad (4.29)$$

Here T_0 is some arbitrary reference temperature held fixed. Let us now expand $\tilde{T}_1$ and $\tilde{T}_2$ in two parameters η and ε, assuming both to be of same order of magnitude. We get

$$\frac{T}{T_0} = 1 + \frac{T_{a_1} - T_0}{T_0} + \left(\frac{\tilde{T}_1}{T_0}\right)^{00} + \left(\frac{\tilde{T}_1}{T_0}\right)^{10} \varepsilon + \\ + \left(\frac{\tilde{T}_1}{T_0}\right)^{01} \eta + 0(2) \quad (4.30)$$

$$= 1 + \frac{T_{a_2} - T_0}{T_0} + \left(\frac{\tilde{T}_2}{T_0}\right)^{00} + \left(\frac{\tilde{T}_2}{T_0}\right)^{10} \varepsilon + \\ + \left(\frac{\tilde{T}_2}{T_0}\right)^{01} \eta + 0(2) . \quad (4.31)$$

Thus up to the first non trivial order of approximation one has :

$$\frac{T}{T_0} = 1 + \frac{T_{a_1} - T_0}{T_0} + \left(\frac{\tilde{T}_1}{T_0}\right)^{00} \quad (4.32)$$

together with

$$\frac{T}{T_0} = 1 + \frac{T_{a_2} - T_0}{T_0} + \left(\frac{\tilde{T}_2}{T_0}\right)^{00} . \quad (4.33)$$

Thus both (4.32) and (4.33) differ in first order terms in η or ε only. In order to get the Boussinesq approximation one neglects higher order terms therefore one concludes that no matter what a.h.f. is actually used, the first non trivial approximation gives the same answer.

5. **A first and straightforward generalization of Mihaljan's scheme and its inherent difficulties.** — In this section we clarify the role played by the *a priori* restrictions imposed by Mihaljan upon the equations of state and material parameters. Such restrictions are in fact unnecessary and even more destroyed the self-consistency of Mihaljan's scheme even at the level of the Boussinesq approximation. In the approach to be given now some of these restrictions are natural consequences of a well defined general perturbative scheme. We will however stay as close as we can to Mihaljan's line of reasoning.

Once more the starting equations are the general hydrodynamic equations to be taken now in the form (2.4) (2.5) and for the heat equation

$$\rho c_v \frac{dT}{dt} - \frac{T}{\rho} \frac{\alpha}{\chi} \frac{d\rho}{dt} = \frac{\partial}{\partial x_i} \left(K \frac{\partial}{\partial x_i} T \right) + \Phi . \quad (5.1)$$

No restrictions are imposed upon the temperature and/or pressure dependence of the material parameters. Reference values are P_0, the pressure at the bottom of the fluid layer, and some temperature value T_0, assuming for convenience an hypothetical isothermal hydrostatic field (i.h.f.) throughout the layer. Then we will refer the actual fluid temperature and pressure fields through the layer to this i.h.f. as

$$P = P_0 + \pi \quad (5.2)$$

$$T = T_0 + \theta . \quad (5.3)$$

Thus π and θ are perturbations upon the i.h.f. Units of scale are :

$$c_0 \equiv c_v(T_0, P_0) ; \; K_0 \equiv K(T_0, P_0) ; \; \mu_0 \equiv \mu_1(T_0, P_0) ; \\ \rho_0 \equiv \rho(T_0, P_0) ; \; \alpha_0 \equiv \alpha(T_0, P_0) ; \; \chi_0 \equiv \chi(T_0, P_0) .$$

We shall refer lengths to L and take as unit of temperature $\Theta = T_2 - T_1$, the temperature difference across the layer. Velocities are measured with $V \equiv \kappa_0/L$, pressure with $\pi_0 \equiv \rho_0 \kappa_0^2/L^2$ and times with $L^2/\kappa_0 \equiv L/V$. Here again $\kappa_0 \equiv k_0/\rho_0 c_0$ and

$$\nu_0 \equiv \mu_0/\rho_0$$

denote respectively thermal diffusivity and kinematic viscosity.

Thus we have *ten* parameters

$$\{ \rho_0, \nu_0, \kappa_0, c_0, T_0, \alpha_0, \chi_0, g, \Theta \text{ and } L \} .$$

Buckingham's pi-theorem predicts then *six* independent pi-monomials and *four* basic quantities. We take

$$\varepsilon_1 \equiv \Theta \alpha_0 \quad (5.4)$$

$$\varepsilon_2 \equiv gL/c_0 \Theta \quad (5.5)$$

$$\sigma \equiv \nu_0/\kappa_0 \quad (5.6)$$

$$\xi \equiv \Theta/T_0 \quad (5.7)$$

$$\zeta = gL^3/\kappa_0^2 = \sigma R/\varepsilon_1 \quad (5.8)$$

$$\psi \equiv \alpha_0 T_0/\rho_0 c_0 \Theta \chi_0 . \quad (5.9)$$

As in other occasions above R is Rayleigh dimensionless temperature difference. Again denoting the dimensionless fields with primed quantities the general hydrodynamic equations become :

$$\frac{d\rho'}{dt'} = -\rho' \frac{\partial v_i'}{\partial x_i'} \tag{5.10}$$

$$\rho' \frac{dv_i'}{dt'} = -\frac{\partial \hat{P}'}{\partial x_i'} + \sigma F_i' + \zeta(\rho' - 1)\,\delta_{i3} \tag{5.11}$$

$$\rho' c_v' \frac{d\theta'}{dt'} - \psi \frac{\alpha'}{\chi'} \frac{1}{\rho'} \frac{d\rho'}{dt'} - \psi\xi \frac{\alpha'}{\chi'} \frac{\theta'}{\rho'} \frac{d\rho'}{dt'} = \frac{\partial}{\partial x_i'}\left(K' \frac{\partial}{\partial x_i'} T'\right) + \frac{\sigma}{\zeta} \varepsilon_2 \Phi' \tag{5.12}$$

where

$$\hat{P} \equiv P + g\rho_0 x_3 . \tag{5.13}$$

To fix ideas relevant values of the parameters are given in table I :

TABLE I

Water layer under standard adverse thermal gradient. When unspecified the values of parameters are given in C.G.S. *units.*

$T_0 \sim 300$ K	$\Theta \sim 1$	$\kappa_0 \sim 10^{-2}$	$\sigma \sim 5$
$P_0 \sim 1$ atm	$\alpha_0 \sim 10^{-\zeta}$	$R \sim 10^3$	$\xi \sim 10^{-3}$
$L \sim 1$	$\chi_0 \sim 10^{-11}$	$\varepsilon_1 \sim 10^{-4}$	$\varphi \sim 10^8$
$g \sim 10^3$	$\nu_0 \sim 10^{-2}$	$\varepsilon_2 \sim 10^{-5}$	$\psi \sim 10$

Notice that $\varepsilon_1, \varepsilon_2 \ll 1$ and now $\varepsilon_1 \sim \varepsilon_2$, in contrast to the situation discussed earlier in this paper. The Jacobian of the transformation $\{ L, \Theta \} \to \{ \varepsilon_1, \varepsilon_2 \}$ is not vanishing. We thus choose ε_1 and ε_2 as perturbative parameters for the expansion. Notice also that derivatives of ρ' are of first order in both ε_1 and ε_2.

Again as developed before we obtain at the first non trivial approximation (*viz.* $\varepsilon_1^0\, \varepsilon_2^0$ terms) the Boussinesq-like equations, that we shall not write anymore (see section 2 above).

A few important remarks are pertinent. Firstly the term containing the bulk viscosity does not show up at the level of the first approximation. Thus there is no need of any *a priori* elimination of this parameter. This implies that the *a priori* assumption of *quasi-incompressibility* is not needed.

Secondly the scheme is valid irrespectively of whether the fluid be a liquid (incompressible) or a gas (ideal or not). In the limit of a dilute ideal gas one indeed recovers Malkus's results given in section 4 above.

Thirdly, as the equations obtained here are similar to those given in section 2 above we still have found the Boussinesq-like approximation but not strictly the Boussinesq equations (see ref. [3] or [8]). The material parameters must *arbitrarily* be set constant. Thus we need a condition ranging outside the proposed perturbative framework. Even though our approach does generalize Mihaljan's perturbative scheme and overcomes one of the inherent difficulties that arises from the choice of the expansion parameters we still have difficulties in properly accounting for the Boussinesq approximation. For this very reason we turn our attention to a generalization of the a.h.f. concept introduced in earlier sections.

6. **An appropriate a.h.f., the Boussinesq approximation and a general perturbative analysis.** — Let $\{ T_a, P_a, \rho_a \}$ define a suitable a.h.f. satisfying the differential system

$$K_0 \frac{d^2 T_a}{dz^2} = 0 \tag{6.1}$$

$$\frac{dP_a}{dz} = -g\rho_a \tag{6.2}$$

$$\rho_a c_0 \frac{dT_a}{dz} = \frac{1}{\rho_a} \frac{\alpha_0}{\rho_0} T_a \frac{d\rho_a}{dz} . \tag{6.3}$$

Here it is assumed that at $z = 0$ one has $T_a = T_0$, $P_a = P_0$, $\rho_a = \rho_0$. The subscript on the remaining quantities denote reference values taken at the T_0, P_0 thermodynamic state.

Solutions of (6.1) to (6.3) are

$$T_a = T_0\left(1 - \beta \frac{z}{T_0}\right) \tag{6.4}$$

$$\rho_a = \rho_0 \Big/ \left[1 - \frac{c_0 \chi_0}{\alpha_0} \rho_0 \ln\left(1 - \beta \frac{z}{T_0}\right)\right] \tag{6.5}$$

$$P_a = P_0 - \int_0^z dz g\rho_0 \Big/ \left[1 - \frac{c_0 \chi_0}{\alpha_0} \times \rho_0 \ln\left(1 - \beta \frac{z}{T_0}\right)\right] \tag{6.6}$$

where β denotes an unspecified but constant adverse thermal gradient. The easiest specification will be to choose β corresponding to the Rayleigh number for the onset of convective instability. But this need not necessarily be so.

Let us now define the following dimensionless monomials

$$\eta_1 \equiv \beta L / T_0 \tag{6.7}$$

$$\eta_2 \equiv \rho_0 g L / P_0 \tag{6.8}$$

$$\mathcal{C}_1 \equiv \alpha T_0 \tag{6.9}$$

$$\mathcal{C}_2 \equiv \chi_0 P_0 \tag{6.10}$$

$$\Phi \equiv P_0 / \rho_0 c_0 T_0 . \tag{6.11}$$

Thus eq. (6.4) to (6.6) for the primed fields become in dimensionless form

$$T'_a \equiv \frac{T_a}{T_0} = 1 - \eta_1 z' \tag{6.12}$$

$$\rho'_a \equiv \frac{\rho_a}{\rho_0} = 1 \Big/ 1 - \frac{\mathcal{C}_2}{\mathcal{C}_1} \ln (1 - \eta_1 z') \tag{6.13}$$

$$P'_a \equiv \frac{P_a}{P_0} = 1 - \eta_2 \mathcal{F}(z') . \tag{6.14}$$

Here as usual $z' = z/L$ and

$$\mathcal{F}(z') = \int_0^{z'} \times \times \left[1 - \frac{\mathcal{C}_2}{\mathcal{C}_1 \Phi} \ln (1 - \eta_1 z'')\right]^{-1} dz'' . \tag{6.15}$$

Now for the general fluid dynamics we write the equations in terms of perturbed fields (3.3) (3.4) upon the reference a.h.f.

$$\frac{d}{dt}(\rho_a + \tilde{\rho}) = - (\rho_a + \tilde{\rho}) \frac{\partial}{\partial x_i} v_i \tag{6.16}$$

$$(\rho_a + \tilde{\rho}) \frac{d}{dt} v_i = - \frac{\partial}{\partial x_i} \tilde{P} + F_i - g\tilde{\rho}\delta_{i3} \tag{6.17}$$

$$(\rho_a + \tilde{\rho}) c_v \frac{d}{dt}(T_a + \tilde{T}) - \frac{\alpha}{\chi} \frac{(T_a + \tilde{T})}{(\rho_a + \tilde{\rho})} \frac{d}{dt}(\rho_a + \tilde{\rho}) = = \frac{\partial}{\partial x_i}\left[K \frac{\partial}{\partial x_i}(T_a + \tilde{T})\right] + \Phi . \tag{6.18}$$

Let us now define new dimensionless quantities

$$\varepsilon \equiv \frac{\Delta\tilde{T}}{T_0} = \frac{\Delta\tilde{T}}{\Delta T_a} \frac{\Delta T_a}{T_0} \equiv - \omega\eta_1 \tag{6.19}$$

where $\Delta\tilde{T}$ denotes the difference in $\tilde{T}$ between higher and lower parts in the fluid layer. Eq. (6.19) also defines ω. One now writes

$$\tilde{T} = - T_0 \omega\eta_1 T' \tag{6.20}$$

and as usual the primed quantity is dimensionless. Again velocity is scaled with

$$V \equiv [gL \Delta\tilde{\rho}/\rho_0]^{1/2} . \tag{6.21}$$

Eq. (6.21) provides an estimated value of the velocity field for a maximum buoyancy effect taken at the upper boundary. Pressure is scaled with

$$\tilde{P} \equiv \rho_0 V^2 P' . \tag{6.22}$$

An estimate of $\tilde{\rho}$ can also be given. Let $\Delta\tilde{\rho}$ denote density contrast between higher and lower parts in the fluid layer. Then $\Delta\tilde{\rho}/\rho_0$ is of order η_1. Thus

$$\Delta\tilde{P} \sim \rho_0 V^2 \sim \rho_0 gL\eta_1 . \tag{6.23}$$

Now

$$\frac{\Delta\tilde{\rho}}{\rho_0} = \frac{\Delta\rho - \Delta\rho_a}{\rho_0} . \tag{6.24}$$

A Taylor series expansion gives

$$\frac{\Delta\rho}{\rho_0} = - \mathcal{C}_1(- \eta_1 - \omega\eta_1) + + \mathcal{C}_2[- \eta_2 \mathcal{F}(1) + \mathcal{O}(\eta_1 \eta_2)] + \text{higher order terms} \tag{6.25a}$$

and

$$\frac{\Delta\rho_a}{\rho_0} = - \frac{\mathcal{C}_2}{\mathcal{C}_1 \Phi} \eta_1 + \text{higher order terms} . \tag{6.25b}$$

Notice that under standard conditions both $\mathcal{C}_1$ and $\mathcal{C}_2$ are much smaller than unity. For instance for the water layer of Table I $\mathcal{C}_1 \sim 10^{-2}$ and $\mathcal{C}_2 \sim 10^{-5}$. Thus one is led to the following scale

$$\tilde{\rho} = \rho_0 \frac{\mathcal{C}_2}{\mathcal{C}_1 \Phi} \eta_1 \rho' . \tag{6.26}$$

With this unit one estimates V and $\rho_0 V^2$. One has

$$V \equiv \left[\frac{gL \Delta\tilde{\rho}}{\rho_0}\right]^{1/2} \sim \left(gL \frac{\mathcal{C}_2}{\mathcal{C}_1 \Phi} \eta_1\right)^{1/2} \tag{6.23b}$$

and

$$\rho_0 V^2 \sim P_0 \eta_1 \eta_2 \frac{\mathcal{C}_2}{\mathcal{C}_1} . \tag{6.27}$$

As in previous occasions L/V, K_0, c_0, α_0 and χ_0 define the remaining scaling factors. Thus we have *twelve* parameters

$$\{ \rho_0, \alpha_0, \chi_0, \mu_0, c_0, k_0, g, T_0, P_0, \beta, \Delta\tilde{T}, L \} .$$

Buckingham's pi-theorem leaves *eight* pi-monomials only. As yet we have defined $\{ \eta_1, \eta_2, \mathcal{C}_1, \mathcal{C}_2, \Phi, \omega \}$. We will also have σ (Prandtl number) and R (Rayleigh number). With the use of these parameters and the scaling units one has the following set of differential equations

$$v'_3 \frac{d\rho'_a}{dz'} + \frac{\mathcal{C}_2}{\mathcal{C}_1 \Phi} \eta_1 \frac{d\rho'}{dt'} = - \left(\rho'_a + \frac{\mathcal{C}_2}{\mathcal{C}_1} \eta_1 \rho'\right) \frac{\partial}{\partial x'_i} v'_i \tag{6.16b}$$

$$\left(\rho'_a + \frac{\mathcal{C}_2}{\mathcal{C}_1 \Phi} \eta_1 \rho'\right) \frac{dv'_i}{dt'} = - \frac{\partial}{\partial x'_i} P' + \left(\frac{\sigma}{R} \frac{\mathcal{C}_1^2 \Phi}{\mathcal{C}_2}\right)^{1/2} F'_i - \rho' \delta_{i3} \tag{6.17b}$$

$$\left(\rho'_a + \frac{\mathcal{C}_2}{\mathcal{C}_1 \Phi} \eta_1 \rho'\right) c'_v \left(- \eta_1 v'_3 - \omega\eta_1 \frac{dT'}{dt'}\right) - \frac{\alpha'}{\chi'} \left(\frac{1 - \eta_1 z' - \omega\eta_1 T'}{\rho'_a + \frac{\mathcal{C}_2}{\mathcal{C}_1 \Phi} \eta_1 \rho'}\right) \frac{\mathcal{C}_1 \Phi}{\mathcal{C}_2} \left(v'_3 \frac{d\rho'_a}{dz'} + \frac{\mathcal{C}_2}{\mathcal{C}_1 \Phi} \frac{d\rho'}{dt'}\right) =$$

$$= - \frac{\partial}{\partial x'_i} \frac{\left[k'\left(\omega\eta_1 \frac{\partial}{\partial x'_i} T' + \eta_1 \delta_{i3}\right)\right]}{\left(\sigma R \frac{\mathcal{C}_2}{\mathcal{C}_1 \Phi}\right)^{1/2}} + \eta_1 \eta_2 \left(\frac{\mathcal{C}_2 \Phi\sigma}{R}\right)^{1/2} \Phi' . \quad (6.18b)$$

In eq. (6.17b) and (6.18b) we have used $\nu_0 = \mu_0/\rho_0$, $\kappa_0 = k_0/\rho_0 c_0$, $R = gL^3 \mathcal{C}_1 \eta_1/\kappa_0 \nu_0$ and $\sigma = \nu_0/\kappa_0$.

A general and well defined two-parameter expansion is obtained in terms of η_1 and η_2. For the water layer of Table I above estimated values are $\eta_1 \sim \eta_2 \sim 10^{-3}$. Again the Jacobian of the transformation $(L, \beta) \to (\eta_1, \eta_2)$ does not vanish. Thus up to the first non trivial approximation one has in dimensionless form :

$$\frac{\partial}{\partial x'_i} v'^{00}_i = 0 \quad (6.28)$$

$$\frac{\partial}{\partial t} v'^{00}_i + v'^{00}_j \frac{\partial}{\partial x'_j} v'^{00}_i = - \frac{\partial}{\partial x'_i} P'^{00} + \left(\frac{\sigma \mathcal{C}_1^2 \Phi}{R\mathcal{C}_2}\right)^{1/2} F'^{00}_i - \rho'^{00} \delta_{i3} \quad (6.29)$$

$$\omega\left(\frac{\partial}{\partial t'} T'^{00} + v'^{00}_i \frac{\partial}{\partial x'_i} T'^{00}\right) + \left(\frac{\partial}{\partial t'} \rho'^{00} + v'^{00}_i \frac{\partial}{\partial x'_i} \rho'^{00}\right) = \left(\frac{\mathcal{C}_1^2 \Phi}{R\sigma\mathcal{C}_2}\right)^{1/2} \omega \frac{\partial^2}{\partial x'^2_i} T'^{00} . \quad (6.30)$$

Eq. (6.28) to (6.30) define the correct non linear Boussinesq approximation. Compare for instance the results of Mihaljan [6]. His eq. (4.6) corresponds term by term to our eq. (6.29) here. Each separate term does not however give the same physics in both papers. There is also an obvious difference between our eq. (6.30) and Mihaljan's eq. (4.6). The difference comes from our original definition of the reference a.h.f. The comparison with Chandrasekhar's [3] eq. (56), p. 19 is however almost straightforward. For if one linearizes out eq. (6.30) then goes back to the original fields and drops all pressure dependence in the equation of state, and also makes the transformation from the variables here to those of Chandrasekhar one gets his eq. (56), p. 19. It is thus clear that the only differences are matters of convention, approximation, and indeed of the physical meaning ascribed to the unperturbed reference fields.

The eq. (6.28) to (6.30) have been obtained as a first order perturbation upon the reference a.h.f. defined by eq. (6.4) to (6.6). Since the material parameters and the equations of state are almost unrestricted in our description the results obtained in this section constitute a non trivial generalization of Mihaljan's results [6] although at first sight it is purely formal. In our opinion it helps to understand the Boussinesq-Oberbeck approximation on a deeper basis. Our results also provide a natural generalization of results already obtained for dilute gases by Malkus [7]. It should finally be mentioned that an intuitive derivation of Boussinesq equations, though not a valid method to generate higher order corrections, has been given by Spiegel and Veronis [11].

Acknowledgments. — Part of this work was done while one of the authors (M.G.V.) was visiting the Institutt for Teoretisk Fysikk, University of Trondheim-N.T.H., Norway. He gratefully acknowledges the hospitality received there from Professor H. Wergeland and his group. He also wishes to acknowledge the hospitality at the Saclay Center. Both authors acknowledge correspondence with Professor W. V. R. Malkus and access to his unpublished lecture notes.

References

[1] Boussinesq, J., *Théorie analytique de la chaleur* (Paris : Gauthier-Villars) 1903, vol. 2, p. 172.

[2] Oberbeck, A., *Sitzb. K. Preuss. Akad. Wiss*, 1888, 383-95 and 1129-38. Translated by C. Abbe in Smithsonian Misc. Coll. (1891).

[3] Chandrasekhar, S., *Hydrodynamic and Hydromagnetic Stability* (Clarendon Press : Oxford) 1961, chap. 2.

[4] Joseph, D. D., *J. Fluid Mech.* **47** (1971) 257.

[5] Hoard, C. Q., Robertson, C. R. and Acrivos, A., *Int. J. Heat Mass Transfer* **13** (1970) 849.

[6] Mihaljan, J. A., *Astrophys. J.* **136** (1962) 1126.

[7] Malkus, W. V. R., 1964, Course Lectures, *Summer Study Program in Geophysical Fluid Dynamics*, Woods Hole Oceanographic Institution (unpublished).

[8] Velarde, M. G., in *Hydrodynamics*, Proceedings 1973. Les Houches Summer School, R. Balian, editor (Gordon and Breach, N. Y.) to appear in 1975.

[9] Palm, E., *J. Fluid Mech.* **8** (1960) 183.

[10] Segel, L. A. and Stuart, T. S., *J. Fluid Mech.* **13** (1962) 289.

[11] Spiegel, E. A. and Veronis, G., *Astrophys. J.* **131** (1961) 142. See also for corrections Veronis, G., *Astrophys. J.* **135** (1962) 655.