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## IDEAL BOSE EINSTEIN CONDENSATION AND DISORDER EFFECTS

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**Résumé.** — La condensation de Bose Einstein, considérée comme une transition de phase, est étudiée dans un espace de dimension arbitraire. La correspondance, avec la limite  $n \rightarrow \infty$  du modèle  $n$ -vecteur pour la condensation à volume constant et avec la limite  $n = -2$  pour la condensation à pression constante, est discutée et précisée. L'influence d'un type particulier de désordre (sources et puits de bosons aléatoires) est étudiée; la transition de phase subsiste, avec de nouveaux exposants critiques qui ne satisfont pas certaines lois d'échelle.

**Abstract.** — Ideal Bose Einstein condensation is studied as a cooperative phase transition for arbitrary dimensionality. The correspondence with the  $n \rightarrow \infty$  limit of the  $n$ -vector model in the constant volume case and with the  $n = -2$  limit in the constant pressure case is discussed and precised. The influence of a special type of disorder (random sources and sinks) is studied; a sharp phase transition occurs with new critical exponents which are calculated and shown to violate some scaling laws.

Ideal Bose Einstein (B. E.) condensation is one of the few phase transition models which are exactly soluble in any dimensionality  $d$ . The cooperation inside the system leading to the phase transition comes only from the statistics, there being no interaction potential between particles.

It has been recognized for some time [1] that the ideal B. E. condensation at constant volume was similar to the spherical model and corresponded to the  $n \rightarrow \infty$  limit of the  $n$ -vector model, as far as critical behaviour is concerned. This correspondence shows that the properly chosen order parameter for condensation at constant volume has an infinite number of degrees of freedom. On the other hand, condensation at constant pressure is equivalent to the gaussian model and, in some sense, to the case  $n = -2$  [2]. This, *a priori* surprising, equivalence deserves some detailed discussion since ideal B. E. condensation appears then to contain two limits  $n = -2$  and  $n = \infty$  of the physical interval of values for  $n$ .

In many respects, B.E. condensation presents also analogies with the ordinary liquid-gas transition, corresponding to  $n = 1$ . Similarities and differences, and the question of the definition of the order parameter will be discussed. The motivations for the study of B.E. condensation in the presence of disorder

are two fold: i) a first motivation is the attempt to *squeeze* the infinite number of degrees of freedom of the order parameter for condensation at constant volume by introducing an inhomogeneous field; however the hope of being able to span continuously the range  $-2 < n < +\infty$  is not realized in practice, as the disorder introduced leads to values for the critical exponents which do not correspond to any homogeneous problem described by  $-2 < n < \infty$ ,

ii) a second motivation is the study of an exactly soluble model of disorder, which turns out to be different from both the *mobile* impurities and *frozen* impurities models; in the presence of the type of disorder considered here, a sharp second-order phase transition subsists, but some basic scaling laws are violated.

1. **The homogeneous ideal Bose-Einstein gas.** —  
 1.1 **BASIC FORMALISM.** — The Hamiltonian of the ideal B.E. gas is

$$\mathcal{H} = \sum_k \varepsilon(k) n_k$$

where  $n_k = a_k^+ a_k$ ,  $a_k^+$  and  $a_k$  are creation and annihilation boson operators. We take  $\varepsilon(k) = a |k|^\sigma$ . The standard case  $\sigma = 2$  corresponds to short range forces in the corresponding  $n$  vector model, whereas

the case  $\sigma < 2$  corresponds to long range forces decreasing as  $1/r^{d+\sigma}$  [3]. The case  $\sigma > 2$  appears as somewhat academic for the moment, because in general such  $k^\sigma$  terms will appear in conjunction with  $k^2$  terms which will be dominant.

Let us define as usual the order parameter  $M$  as  $\langle a_0^+ \rangle N^{-1/2}$ , where  $N$  is the number of particles.

$$N = \sum_k \langle n_k \rangle$$

and the brackets denote thermal average.

The square of the order parameter is thus the relative number of bosons in the condensate.

Introduction of a symmetry breaking term in the Hamiltonian

$$\mathcal{H} = \sum_k \varepsilon(k) n_k - N^{-1/2} (a_0 B^* + a_0^+ B) \quad (1)$$

where  $B$  is the complex field conjugate to  $M$ , allows to obtain the equation of state  $f(T, M, B) = 0$ .

The grand potential  $\Omega$  is obtained as

$$\Omega = T \frac{V}{(2\pi)^d} \times \int d^d k \text{Log} (1 - e^{-\beta(\varepsilon(k) + \bar{\mu})}) - \frac{1}{N} \frac{B^* B}{\bar{\mu}} \quad (2)$$

where  $\beta$  is the inverse temperature and  $\bar{\mu}$  is, for simplicity, the opposite of the chemical potential, and therefore non-negative. The pressure  $p$  and the number of particles  $N$  are then derived:

$$p = - \frac{\partial \Omega}{\partial V} = - \frac{T}{(2\pi)^d} \int d^d k \text{Log} (1 - e^{-\beta(\varepsilon(k) + \bar{\mu})}) \quad (3)$$

$$N = \frac{\partial \Omega}{\partial \bar{\mu}} = \frac{V}{(2\pi)^d} \int d^d k \frac{1}{e^{\beta(\varepsilon(k) + \bar{\mu})} - 1} + \frac{1}{N} \frac{B^* B}{\bar{\mu}^2}. \quad (4)$$

The order parameter is found to be

$$M = - \frac{\partial \Omega}{\partial B} = \frac{1}{N} \frac{B^*}{\bar{\mu}} \quad (5)$$

so that we have

$$N = \frac{V}{(2\pi)^d} \int d^d k \frac{1}{e^{\beta(\varepsilon(k) + \bar{\mu})} - 1} + NM^* M. \quad (6)$$

In this last formula, the dependence is only on the modulus of the complex number  $M$ , so that for simplicity, we will take hereafter  $M$  and  $B$  real.

The correlation function is

$$g_k = \frac{1}{N} \langle a_k^+ a_k \rangle = \frac{1}{N} \frac{1}{e^{\beta(\varepsilon(k) + \bar{\mu})} - 1} \quad (7)$$

and the susceptibility is

$$\chi = \frac{\partial M}{\partial B} = \frac{1}{N\bar{\mu}} \quad (8)$$

**1.2 THE PHASE DIAGRAM.** — Let us consider first the case  $B = 0$  of zero applied conjugate field and let us determine the phase diagram  $(p, V)$  with its isotherms. At constant volume and number of particles, for example,  $\bar{\mu}$  can always be calculated at high enough temperature from equation 4. If the temperature is lowered,  $\bar{\mu}$  decreases. For a certain critical temperature  $T_c$ ,  $\bar{\mu}$  is zero and the susceptibility diverges. For  $T$  smaller than  $T_c$ ,  $\bar{\mu}$  is zero, but there is a non zero order parameter, so that one must use equation 6 with  $M \neq 0$ . The transition line in the  $p - V$  plane at constant  $N$  is thus defined by

$$\begin{cases} p = p(T, \bar{\mu} = 0) \\ V = V(T, \bar{\mu} = 0). \end{cases}$$

The isotherms are defined by

$$\begin{aligned} T > T_c & \begin{cases} p = p(T, \bar{\mu}) \\ V = V(T, \bar{\mu}) \end{cases} \\ T < T_c & \quad p = p(T, 0). \end{aligned}$$

The integrals defining  $p$  and  $V$  are easily evaluated, and one finds:

$$p = CT^{d/\sigma+1} F\left(e^{-\beta\bar{\mu}}, \frac{d}{\sigma} + 1\right) \quad (9)$$

$$\frac{N}{V} = CT^{d/\sigma} F(e^{-\beta\bar{\mu}}, d/\sigma) \quad (10)$$

where  $C = \frac{d}{\sigma} (4\pi)^{-d/2} \frac{\Gamma(d/\sigma)}{\Gamma\left(\frac{d}{2} + 1\right)} a^{-d/\sigma}$ , and

$$F(x, \alpha) = \sum_{n=1}^{\infty} \frac{x^n}{n^\alpha}.$$

From these formulas, the equation of the transition line is easily derived:

— for  $d < \sigma$ , we obtain  $V \rightarrow 0$  if  $\mu \rightarrow 0$  so that the transition line collapses on the  $p$ -axis;

— and for  $d > \sigma$ , we have

$$p = K \left(\frac{V}{N}\right)^{-(1+\sigma/d)}$$

where  $K = C^{-\sigma/d} \zeta(d/\sigma + 1) (\zeta(d/\sigma))^{-(1+\sigma/d)}$ .

For the isotherms:

— for large  $\bar{\mu}$ , that is small  $p$  and small density, we have  $pV = NT$  as expected;

— and for small  $\bar{\mu}$ , we obtain for the slope on the coexistence curve:

$$\left. \frac{dp}{dV} \right|_{T=\tau_c} = -\infty \quad \text{for } d/\sigma < 1/2,$$

non zero and finite for  $d/\sigma = 1/2$   
and  $d/\sigma > 2$ ,

$$= 0 \quad \text{for } 1/2 < d/\sigma \leq 2.$$

There are consequently four typical kinds of phase diagrams (plus the special case  $d/\sigma = 1/2$ , which has a collapsed diagram with finite critical compressibility).

- i) a normal diagram, with finite critical compressibility for  $d/\sigma > 2$  (Fig. 1a),
- ii) a normal diagram, with infinite critical compressibility for  $1 < d/\sigma \leq 2$  (Fig. 1b),
- iii) a collapsed diagram, with infinite critical compressibility for  $1/2 < d/\sigma \leq 1$  (Fig. 2a),
- iv) a collapsed diagram with zero critical compressibility for  $0 < d/\sigma < 1/2$  (Fig. 2b).

When  $d/\sigma > 1$ , there is a transition for all values of  $p$  and  $V$ . On the contrary, when  $d/\sigma \leq 1$ , there is a transition at constant  $p$ , but not at constant  $V$ .

There exists an obvious analogy between this phase diagram and the ordinary liquid-gas phase diagram. If the order parameter had been chosen similarly as being the difference in specific volume between the condensate (liquid) and the rest (gas), then the B.E. phase transition would have to be considered as a first order phase transition (this is the view point adopted, for instance, by K. Huang [4]). In comparison with the ordinary liquid gas transition, some peculiar features of the B.E. condensation, considering for specificity the case ( $d = 3, \sigma = 2$ ), are the zero specific volume of the condensate, the particular position ( $p = \infty, V = 0$ ) of the *critical point* and the horizontal slope of the isotherms on the coexistence curve, excluding the possibility of metastability effects (this last feature is valid in a restricted dimensionality range  $\sigma/2 < d \leq 2\sigma$ ).

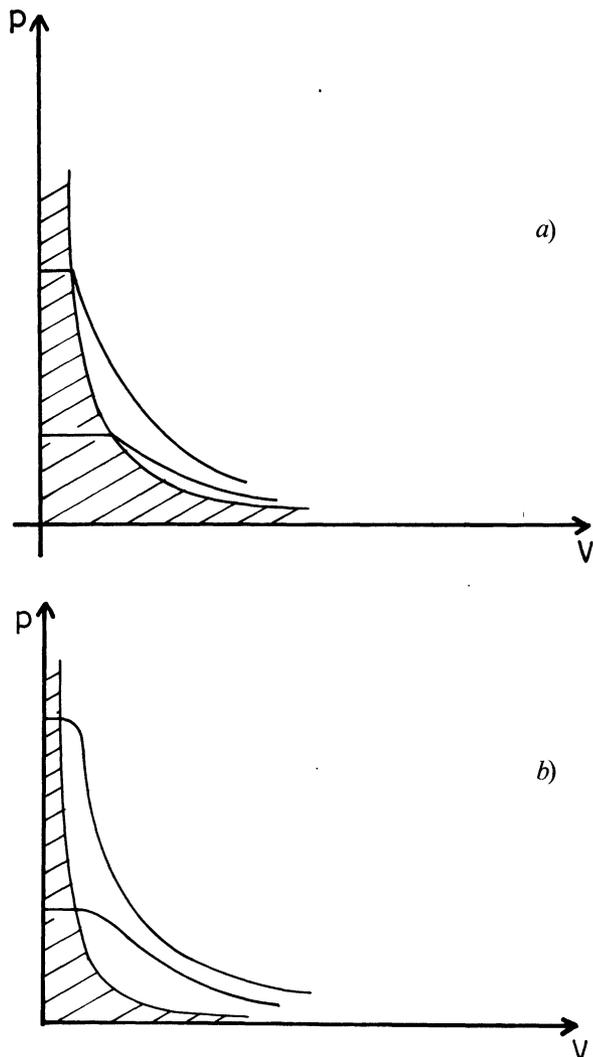


FIG. 1. — Normal phase diagram in the  $(p, V)$  plane: the low temperature phase area is hatched.  
a)  $d/\sigma = 3$ ,  
b)  $d/\sigma = 1.5$ .

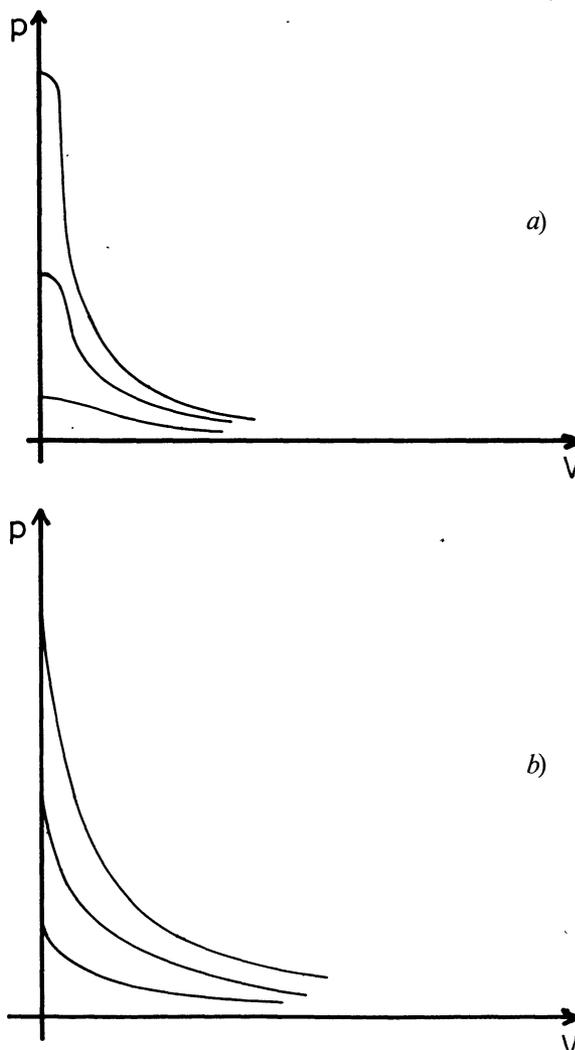


FIG. 2. — Collapsed phase diagram in the  $(p, V)$  plane.  
a)  $d/\sigma = 1$ ,  
b)  $d/\sigma = 0.4$ .

Proceeding now with the choice of order parameter made in section 1.1 we shall consider the constant volume and constant pressure situations, determine the critical behaviour and discuss the order of the transition.

1.3 B. E. CONDENSATION AT CONSTANT VOLUME. — At constant volume and number of particles, the equation of state  $f(M, B, T) = 0$  is

$$\frac{N}{V} = CT^{d/\sigma} F(e^{-\beta\bar{\mu}}, d/\sigma) + \frac{N}{V} M^2$$

where  $\bar{\mu} = B/NM$ .

A finite  $T_c$  exists only for  $d > \sigma$ .  $T_c$  is then defined by

$$\frac{N}{V} = CT_c^{d/\sigma} F(1, d/\sigma)$$

so that

$$T_c = \left( \frac{V}{N} C \zeta(d/\sigma) \right)^{-\sigma/d} \quad (d > \sigma).$$

From the equation of state

$$1 = \left( \frac{T}{T_c} \right)^{d/\sigma} \frac{F(e^{-\beta\bar{\mu}}, d/\sigma)}{F(1, d/\sigma)} + M^2$$

the critical exponents  $\beta$ ,  $\delta$ ,  $\gamma$  are easily derived.

At  $B = 0$ , we find  $M \sim (T - T_c)^{1/2}$ .

At  $T = T_c$ , we have

$$\begin{cases} d > 2\sigma, & B \sim M^3 \\ \sigma < d < 2\sigma, & B \sim M^{(d+\sigma)/(d-\sigma)}. \end{cases}$$

At  $T > T_c$ , we have  $\chi^{-1} = N\bar{\mu}$

$$\begin{cases} \sim (T - T_c) & \text{if } d \geq 2\sigma \\ \sim (T - T_c)^{\sigma/(d-\sigma)} & \text{if } \sigma < d < 2\sigma. \end{cases}$$

The exponents  $\eta$  and  $\nu$  are derived from the correlation function at small  $k$  and  $\bar{\mu}$

$$g_k \sim \frac{T}{N(\varepsilon(k) + \bar{\mu})}$$

— At  $T = T_c$  ( $\bar{\mu} = 0$ ),  $g_k \sim \frac{1}{k^\sigma}$ .

— The correlation length is

$$\xi \sim \bar{\mu}^{-1/\sigma} \begin{cases} \sim (T - T_c)^{-1/\sigma} & \text{if } d \geq \sigma \\ \sim (T - T_c)^{-1/(d-\sigma)} & \text{if } \sigma < d < 2\sigma. \end{cases}$$

The exponent  $\alpha$  for the specific heat can be calculated from

$$C_v = -T \left. \frac{\partial^2 F}{\partial T^2} \right|_{V,N}$$

The values of the critical exponents are given in Table I

TABLE I

*Critical exponents in the constant volume case*

| $d < \sigma$           | No B.E. condensation at finite temperature |                       |                                    |                                      |                     |                            |
|------------------------|--|-----------------------|------------------------------------|--------------------------------------|---------------------|----------------------------|
| $\sigma < d < 2\sigma$ | $\alpha = \frac{d-2\sigma}{d-\sigma}$      | $\beta = \frac{1}{2}$ | $\gamma = \frac{\sigma}{d-\sigma}$ | $\delta = \frac{d+\sigma}{d-\sigma}$ | $\eta = 2 - \sigma$ | $\nu = \frac{1}{d-\sigma}$ |
| $d < 2\sigma$          | $\alpha = 0$                               | $\beta = \frac{1}{2}$ | $\gamma = 1$                       | $\delta = 3$                         | $\eta = 2 - \sigma$ | $\nu = \frac{1}{\sigma}$   |

The phase transition is second order in Landau sense, with  $N_0$ , the number of particles in the condensate, square of the order parameter, rising linearly in  $T_c - T$  below  $T_c$ .

For  $d < \sigma$ , the only singular temperature is the zero temperature. It is possible to define critical exponents around  $T = 0$  but, to avoid excess listing, only finite temperature transitions are considered in this paper.

As mentioned before, the critical exponents both for  $\sigma < d < 2\sigma$  and for  $d > 2\sigma$  are identical to those of the  $n \rightarrow \infty$  limit of the  $n$ -vector model. This infinite dimensionality of the order parameter is due to the fact that, at constant volume, the wave function of the condensate can actually be distributed anywhere in space: there is a considerable degeneracy

of the condensed state. In presence of a repulsive interaction between particles, however small, the degeneracy of the condensed state is considerably reduced; thus, for  $\text{He}_4$ , the number of degrees of freedom is  $n = 2$ . In the ideal case, at constant pressure, there is a postulate of spatial homogeneity which reduces the effective number of degrees of freedom even further to...  $n = -2$ , as discussed now.

1.4 B. E. CONDENSATION AT CONSTANT PRESSURE. — If the pressure is held constant, when the temperature decreases,  $\bar{\mu}$  decreases again and the critical temperature is reached when  $\bar{\mu}$  is zero. If  $d > \sigma$ , the phase diagram being of normal type, when  $T$  reaches  $T_c$ , the system collapses under the external pressure, with the order parameter  $M$  jumping from 0 to 1.

If  $d$  is less than  $\sigma$ , when  $T$  approaches  $T_c$ , the volume of the system tends to zero, and the collapse occurs continuously. This collapse is due to the fact that the bosons in the ground state do not contribute to the internal pressure.

In contradistinction with the constant volume situation, the critical temperature at fixed pressure, as function of the dimensionality  $d$ , does not go to zero for  $d < \sigma$  but instead is a continuous, monotonously decreasing function of  $d$  (Fig. 3). As in the gaussian model, the low temperature phase is not thermodynamically well defined, and the transition is first order in the Landau sense; however there is a critical regime because the correlation length does diverge at  $T_c$ , and some critical indices can be calculated.

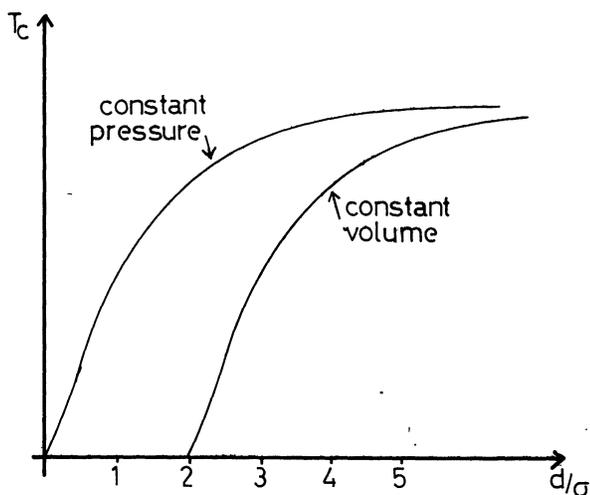


FIG. 3. — Critical temperature versus dimensionality for constant pressure and constant volume B.E. condensation. Arbitrary temperature scale.

From equation 3 and 9, we obtain

$$\begin{aligned} \bar{\mu} &\sim (T - T_c) & \text{for } d > \sigma, \\ \bar{\mu} &\sim (T - T_c)^{\sigma/d} & \text{for } d < \sigma, \end{aligned}$$

which determines  $\gamma$  from  $\chi^{-1} = N\bar{\mu}$ .  $\eta$  and  $\nu$  are derived from  $g_k$  just as in the constant volume case, and  $\alpha$  is calculated from the specific heat at constant pressure

$$C_p = -TN \frac{\partial^2 \bar{\mu}}{\partial T^2}$$

The results are collected in Table II

TABLE II

Critical exponents in the constant pressure case

|                    |                         |                     |                     |                  |
|--------------------|-------------------------|---------------------|---------------------|------------------|
| $d/\sigma < 1$     | $\alpha = 2 - \sigma/d$ | $\gamma = \sigma/d$ | $\eta = 2 - \sigma$ | $\nu = 1/d$      |
| $1 < d/\sigma < 2$ | $\alpha = 2 - d/\sigma$ | $\gamma = 1$        | $\eta = 2 - \sigma$ | $\nu = 1/\sigma$ |
| $d/\sigma > 2$     | $\alpha = 0$            | $\gamma = 1$        | $\eta = 2 - \sigma$ | $\nu = 1/\sigma$ |

Although Fisher scaling law  $\gamma = (2 - \eta) \nu$  is obeyed in all cases, Josephson law  $d\nu = 2 - \alpha$  is violated for  $d/\sigma < 1$  [5].

For  $d/\sigma > 1$ , the behaviour of  $T_c$ , the values of the exponents ( $\alpha, \gamma, \eta, \nu$ ) are identical to those of the  $n$ -vector model for  $n = -2$ . Although for  $n = -2$  the low temperature phase is well defined, it has been proved [6], for  $d = 1$  and  $\sigma = 2$ , that the critical point for  $n = -2$  is a triple point in the  $(T, n)$  diagram. The pathologies of the constant pressure B.E. condensation, plus the connection with the case  $n = -2$ , enhance the plausibility of the  $n = -2$  line being a boundary of the continuity domain in the  $(n, d)$  plane.

The description of the low temperature phase may actually be done in two ways at least; either  $p$  is considered as being the internal pressure, then the temperature cannot be decreased below  $T_c$  and there is indeed no low temperature phase; or, as done above,  $p$  is considered as being the external pressure and the collapse of the system lowers its dimensionality to zero.

2. The inhomogeneous B.E. gas. — To study the influence of disorder on this transition, we introduce a field of random sources and sinks of bosons

$$\mathcal{H}^1 = \mathcal{H} + \sum_k (h_k a_k^+ + h_k^* a_k)$$

where  $\mathcal{H}$  is defined by (1) and the  $h$ 's are random variables the correlation function of which is taken as

$$\overline{h_k h_{k'}} = h^2 k^\theta \delta_{kk'} \quad (11)$$

where the bar denotes statistical average over  $h$ .

This type of disorder is a disorder due to a non uniform field, and therefore different from usual models of impurity disorder. In a magnet, it would be equivalent to a non uniform stray magnetic field. Here, the effect of the coupling term simply produces a displacement of the boson oscillators.

To calculate the thermodynamic properties of this model we first perform the thermal average as in the homogeneous case, then, we take the statistical average over the random  $h$ -field on physical quantities. We obtain for  $\Omega'$ , the grand potential,

$$\begin{aligned} \Omega' &= -T \overline{\text{Log} \langle e^{-\beta(\mathcal{H}' + \bar{\mu}N)} \rangle} \\ \Omega' &= \Omega - \frac{V}{(2\pi)^d} \int \frac{\overline{h_k h_k^*}}{\varepsilon(k) + \bar{\mu}} d^d k \end{aligned} \quad (12)$$

from which  $p'$  and  $\left(\frac{N}{V}\right)'$  are easily obtained

$$p' = p + \frac{1}{(2\pi)^d} \int \frac{\overline{h_k h_k^*}}{\varepsilon(k) + \bar{\mu}} d^d k \quad (13)$$

$$\left(\frac{N}{V}\right)' = \frac{N}{V} + \frac{1}{(2\pi)^d} \int \frac{\overline{h_k h_k^*}}{(\varepsilon(k) + \bar{\mu})^2} d^d k \quad (14)$$

The correlation function is

$$g'_k = g_k + \frac{1}{N} \frac{\overline{h_k h_k^*}}{(\varepsilon(k) + \bar{\mu})^2} \quad (15)$$

2.1 THE PHASE DIAGRAM. — As in the homogeneous case, we first evaluate  $p'$  and  $\left(\frac{N}{V}\right)'$

$$p' = p + DG \left( \frac{d + \theta}{\sigma}, \bar{\mu} \right)$$

$$\left(\frac{N}{V}\right)' = \frac{N}{V} - D \frac{\partial G}{\partial \mu} \left( \frac{d + \theta}{\sigma}, \bar{\mu} \right)$$

where  $D = \frac{d}{\sigma} (4\pi)^{-d/2} h^2 \frac{a^{-(d+\theta)}}{\Gamma(d/2 + 1)}$  and

$$G(\alpha, x) = \int_0^1 \frac{t^{\alpha-1} dt}{t+x}$$

A high momentum cut-off has been used in the  $k$ -integration in the last integral of  $p'$  to avoid unphysical divergences due to (11). There still exists, for  $\frac{d + \theta}{\sigma} > 1$ , a transition line defined by  $\bar{\mu} = 0$ :

— for  $1 < \frac{d + \theta}{\sigma} < 2$ ,

$$\left. \begin{aligned} p' &= p(\mu = 0) + \frac{D}{\frac{d + \theta}{\sigma} - 1} \text{ collapsed diagram} \\ \left(\frac{N}{V}\right)' &= 0 \end{aligned} \right\}$$

— for  $\frac{d + \theta}{\sigma} > 2$

$$\left. \begin{aligned} p' &= p(\mu = 0) + \frac{D}{\frac{d + \theta}{\sigma} - 1} \text{ normal diagram} \\ \left(\frac{N}{V}\right)' &= \frac{N}{V}(\mu = 0) + \frac{D}{\frac{d + \theta}{\sigma} - 2} \end{aligned} \right\}$$

On figure 4, the diagram is of normal type in regions I, II, III, IV and of collapsed type in regions V and VI.

For the isotherms we have:

— for large  $\mu$ ,  $p^2 V = \frac{\sigma DN}{d + \theta}$  (the gas does not obey the ideal gas law),  
 — for small  $\mu$ ,

$\left. \frac{dp}{dV} \right|_{T=T_c}$  is — finite, non zero, in region C of figure 5,  
 — zero in region B,  
 — infinite negative in region A.

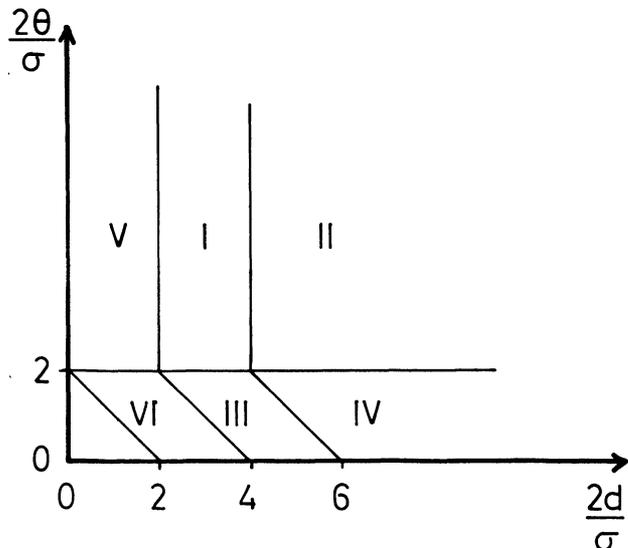


FIG. 4. — Definition of regions I, II, III, IV, V, VI discussed in the text for the inhomogeneous case.

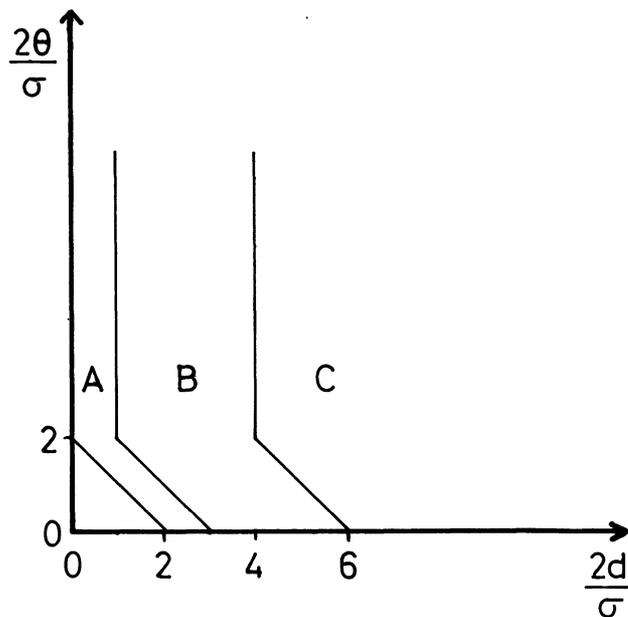


FIG. 5. — Definition of regions A, B, C discussed in the text for the inhomogeneous case.

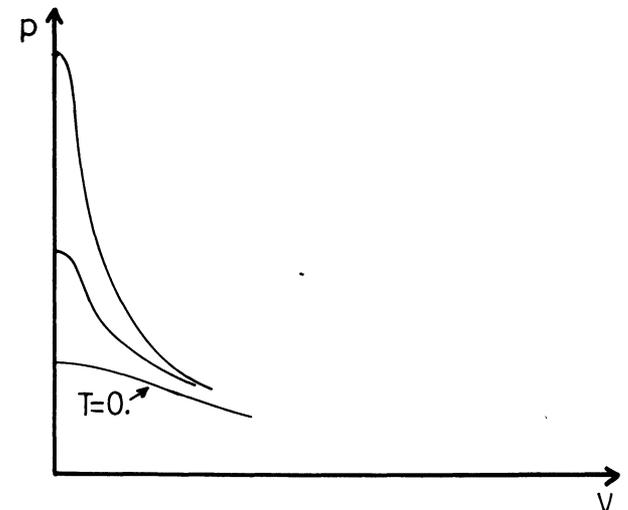


FIG. 6. — Collapsed phase diagram in the  $(p, V)$  plane for the inhomogeneous case, drawn for  $d/\sigma = 1.5$  and  $\theta/\sigma = .25$ .

Figure 6 shows an example of a collapsed diagram, and figure 7 is an example of the normal type. It can be seen that, in the latter case, B.E. condensation occurs at constant volume only if this volume is less than a given volume, and that, in all types of  $(p, V)$  diagrams, there is a forbidden region below the  $T = 0$  isotherm.

2.2 CRITICAL INDICES IN THE CONSTANT VOLUME AND CONSTANT PRESSURE CASES. — The calculation of the critical indices in the inhomogeneous case runs as in the homogeneous case, with careful comparison of the two types of terms coming from disorder or otherwise.

Results are shown in Tables III and IV

On these tables,  $\eta$  is seen to be smaller in the presence of disorder and this goes in opposite direction to what would be expected from an *effective* number of degrees of freedom  $-2 < n < +\infty$ . Also both the Fisher scaling law  $\gamma = (2 - \eta) \nu$ , and Josephson scaling law  $d\nu = 2 - \alpha$  are violated in the regions where disorder is relevant.

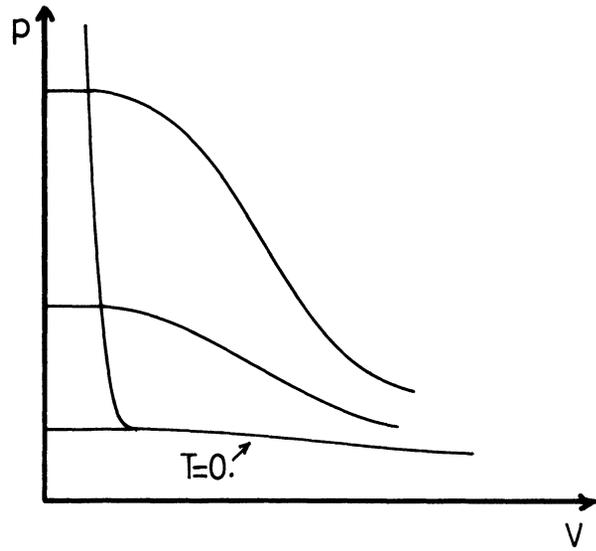


FIG. 7. — Normal phase diagram in the  $(p, V)$  plane for the inhomogeneous case, drawn for  $d/\sigma = 1.5$  and  $\theta/\sigma = .75$ .

TABLE III

Critical indices in the inhomogeneous constant volume case (Region numbers refer to figure 4)

| Region | $\alpha$  | $\beta$       | $\gamma$                              | $\delta$                                  | $\eta$                 | $\nu$                            |
|--------|---|---------------|---------------------------------------|---|------------------------|----------------------------------|
| I      | $\frac{d - 2\sigma}{d - \sigma}$                    | $\frac{1}{2}$ | $\frac{\sigma}{d - \sigma}$           | $\frac{d + \sigma}{d - \sigma}$           | $2 - \sigma$           | $\frac{1}{d - \sigma}$           |
| II     | 0   | $\frac{1}{2}$ | 1                                     | 3   | $2 - \sigma$           | $\frac{1}{\sigma}$               |
| III    | $\frac{\theta + d - 3\sigma}{\theta + d - 2\sigma}$ | $\frac{1}{2}$ | $\frac{\sigma}{\theta + d - 2\sigma}$ | $\frac{d + \theta}{d + \theta - 2\sigma}$ | $2 + \theta - 2\sigma$ | $\frac{1}{\theta + d - 2\sigma}$ |
| IV     | 0   | $\frac{1}{2}$ | 1                                     | 3   | $2 + \theta - 2\sigma$ | $\frac{1}{\sigma}$               |
| V, VI  | No B.E. condensation at finite temperature          |               |                                       |   |                        |                                  |

TABLE IV

Critical indices in the inhomogeneous constant pressure case (Region numbers refer to figure 4)

| Region | $\alpha$   | $\gamma$                             | $\eta$                 | $\nu$                           |
|--------|--|--------------------------------------|------------------------|---------------------------------|
| I      | $2 - \frac{d}{\sigma}$                               | 1                                    | $2 - \sigma$           | $\frac{1}{\sigma}$              |
| II     | 0  | 1                                    | $2 - \sigma$           | $\frac{1}{\sigma}$              |
| III    | $3 - \frac{d + \theta}{\sigma}$                      | 1                                    | $2 + \theta - 2\sigma$ | $\frac{1}{\sigma}$              |
| IV     | 0  | 1                                    | $2 + \theta - 2\sigma$ | $\frac{1}{\sigma}$              |
| V      | $2 - \frac{\sigma}{d}$                               | $\frac{\sigma}{d}$                   | $2 - \sigma$           | $\frac{1}{d}$                   |
| VI     | $\frac{2d + 2\theta - 3\sigma}{d + \theta - \sigma}$ | $\frac{\sigma}{d + \theta - \sigma}$ | $2 + \theta - 2\sigma$ | $\frac{1}{d + \theta - \sigma}$ |

3. **Conclusion.** — In conclusion, we think we have shown that there is more in the ideal Bose-Einstein condensation than one might think at first. Dimensionality effects in the phase diagram and in the shape of the isotherms have been discussed. The consequent drastic differences between the condensations at constant volume and at constant pressure are closely related to the differences between the  $n$  infinite and  $n = -2$  limits of the physical range of values for  $n$  in the  $n$ -vector model. The introduction of a particular type of disorder, yielding to exact solution,

although inappropriate to span the domain

$$-2 < n < +\infty$$

as first hoped, has the merit of extending our typology of disorder effects by providing a model which differs qualitatively from both the *mobile* impurities and the *frozen* impurities models.

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