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# A Non-Standard Semantics for Inexact Knowledge with Introspection

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## Abstract

Standard Kripke models are inadequate to model situations of inexact knowledge with introspection, since positive and negative introspection force the relation of epistemic indiscernibility to be transitive and euclidian. Correlatively, Williamson's margin for error semantics for inexact knowledge invalidates axioms 4 and 5. We state a non-standard semantics for modal logic which is shown to be complete for **K45**, without constraining the accessibility relation to be transitive or euclidian. The semantics corresponds to a system of modular knowledge, in which iterated modalities and simple modalities are not on a par. We show how the semantics helps to solve Williamson's luminosity paradox, and argue that it corresponds to an integrated model of perceptual and introspective knowledge that is psychologically more plausible than the one defended by Williamson. A generalized version of the semantics is formulated, in which modalities are iteration-sensitive up to  $n$  and insensitive beyond  $n$ .

## 1 Inexact knowledge with introspection

Standard modal models for knowledge are commonly **S5** models in which the epistemic accessibility relation is an equivalence relation, namely a relation that is reflexive, symmetric and transitive. From an axiomatic point of view, reflexivity corresponds to the fact that knowledge is veridical, symmetry to the idea that if something is true, one knows one will not exclude it, and transitivity to the idea that knowledge is positively introspective, that is the property that whenever I know some proposition, I know that I know it. **S5** models can also be described as reflexive models that are euclidian, which also makes them symmetric and transitive. Euclidianity corresponds to the property of negative introspection, namely to the property that whenever I don't know, I know that I don't know. **S5** models are commonly used to represent situations of social knowledge, for instance in game theory, due to their well-known correspondence with partition models of information (Osborne & Rubinstein 1994).

An important feature of these models is the fact that they represent a notion of precise or exact knowledge in the following sense: whenever an agent fails to discriminate between two worlds or situations  $w$  and  $w'$ , any other situation which he fails to discriminate from  $w$  is also a situation which he fails to discriminate from  $w'$ , and vice versa. In other words, even though one's knowledge is not necessarily as fine-grained as it should be, it is at least clear cut, since one's uncertainty is partitional. This contrasts with situations of imprecise knowledge, in which the relation of epistemic indiscernibility can fail to be transitive, as in cases of perceptual knowledge in which I can't discriminate

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between any two adjacent shades of color, and yet such that I can distinguish between shades of color that are non-adjacent. Such situations are equivalently described as situations in which one’s knowledge fails to be euclidian, if it is assumed that one’s failure to discriminate between worlds is at least symmetrical. In cases like these, one’s uncertainty is no longer partitional, but rather fuzzy. Situations of this kind have been described as situations of *inexact* knowledge (Williamson 1992a), although the term “inexact” has also been used to refer to situations of false belief (failure of reflexivity), and the term “imprecise” preferred to talk of vague or fuzzy or approximate knowledge (Mongin 2002). In this paper, we shall use the terms “imprecise” and “inexact” interchangeably, and we shall focus on situations of vague knowledge for which one’s accessibility relation, although reflexive and symmetric, fails to be transitive and euclidian.

The representation of situations of inexact knowledge is not as straightforward as one might suppose. Indeed, how should we model situations in which one’s knowledge is imprecise, and yet in which one wants to maintain properties like negative and positive introspection? A good indication that this is not obvious is provided by the existence of a general argument by T. Williamson against the idea that knowledge is positively introspective in general, based on the description of situations of approximate knowledge.

Consider, for instance, a situation of visual knowledge in which I am asked to distinguish objects by their sizes. From where I am, I can’t discriminate between objects that differ from each other only by less than one centimeter. However, I can discriminate between objects that differ from each other by more than one centimeter. This is a situation where I can’t discriminate between 10 and 11, nor between 11 and 12, but in which I can nevertheless discriminate between 10 and 12, so the relation of visual indiscriminability is reflexive and symmetric but non-transitive. Suppose further that I am asked to make judgements about whether the objects are small enough to fit in a certain box. Let us suppose that objects with size 10 and 11 can fit in, but that objects with size 12 and more cannot. This situation can be represented by the following Kripke model, where worlds are named by numbers, and where  $p$  represents the property of fitting in the box.

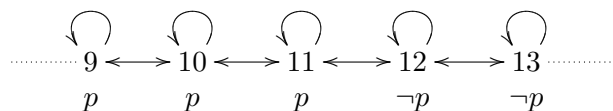


Figure 1: A structure of inexact knowledge

Let us represent  $\Box\phi$  the proposition “I know that  $\phi$ ”. By giving the modal operator its usual semantics, it is easily seen that  $10 \models \Box p$ , since all the worlds that I can’t discriminate from 10 also satisfy  $p$ . This means that when I look at an object of size 10, I know that it will fit in the box. Likewise,  $13 \models \Box\neg p$ : looking at an object of size 13, I know it won’t fit in the box. Intermediate cases, however, are cases for which I fail to know whether they fit in the box, namely  $11 \models \neg\Box p$  and  $11 \models \neg\Box\neg p$ , and similarly for 12. Statements of higher-order knowledge, however, become problematic. Indeed, since  $11 \models \neg\Box p$ , it follows that  $10 \models \neg\Box\Box p$ . Therefore, although I know that an object of size 10 will fit in the box, I don’t know that I know it.

This consequence is problematic, for one could insist that knowledge about one’s visual knowledge is not constrained in the way one’s visual knowledge is. In his account of inexact knowledge, by contrast, Williamson (1992a) turned this limitation of the standard semantics for knowledge into a negative argument against the principle of positive introspection. A model like the one we gave is a particular instance of what Williamson calls a *fixed margin model*. A fixed margin model, relative to a monomodal propositional language, is a quadruple  $\langle W, d, \alpha, V \rangle$ , where  $W$  is a set of worlds,  $\alpha$

is a non-negative real number,  $d$  a metric over  $W$  (namely a function from  $W \times W$  to  $\mathbb{R}^+$ , such that  $d(w, w') = 0$  if and only if  $w = w'$ ,  $d(w, w') = d(w', w)$ , and  $d(w, z) \leq d(w, x) + d(x, z)$ ), and  $V$  is a valuation function over the atoms. The satisfaction clause for the  $\Box$  is the expected one, namely  $M, w \models \Box\phi$  iff for every  $w'$  such that  $d(w, w') \leq \alpha$ ,  $M, w' \models \phi$ . The fixed parameter  $\alpha$  corresponds to the notion of margin for error: at a world  $w$ , one knows  $\phi$  if and only if  $\phi$  holds throughout the worlds that are within the margin  $\alpha$ , that is at all the worlds that are not discriminable from  $w$ . As Williamson shows, validity in fixed margin models is axiomatized by the normal logic **KT**, namely the logic of reflexive-symmetric frames, and neither axiom 4 nor axiom 5 is valid in fixed margin models, by obvious failures of transitivity and euclidianity for the distance function.<sup>1</sup>

Whether this consequence is welcome or unwelcome should depend on the notion of knowledge the semantics is intended to capture. For Williamson, the failure of positive and negative introspection in margin models is actually an important lesson that we should draw concerning the notion of self-knowledge in general. Indeed, Williamson insists that “where one has only a limited capacity to discriminate between cases in which  $p$  is true and cases in which  $p$  is false, knowledge requires a margin for error” (2000: 18). If knowledge obeys a margin for error principle, then to suppose that positive introspection is valid is likely to give rise to paradox. In the previous scenario, for instance, in which one’s margin for error is of 1 centimeter,  $0 \models \Box p$ , that is I do know that an object of size 0 will fit in the box. But if one assumes positive introspection to be valid, it also holds that  $0 \models \Box\Box p$ , and so  $1 \models \Box p$ . By repeated applications of the same rule, it follows that  $i \models p$  for every  $i \geq 0$ , a plain contradiction if  $p$  does not hold universally in the model. Thus, it should follow from my knowledge that an object of size 0 will fit in the box that any object, whatever its size, will fit in the box. Putting together margin for error and positive introspection, we thus end up with a form of epistemic sorites, on the basis of which Williamson argues that positive introspection does not hold. Exactly the same reasoning can be performed if one assumes negative, instead of positive introspection.

This result does not depend on the model, and is even more general, since for every formula  $\phi$ ,  $\phi \rightarrow \Box\phi$  is valid in a fixed margin model if and only if either  $\phi$  is valid or  $\neg\phi$  is valid (Williamson 1992b, 1994). In particular, positive introspection (and likewise negative introspection) is valid in fixed margin models if and only if either  $\Box\phi$  is valid, or  $\neg\Box\phi$  is valid. Thus, I know that I know  $\phi$  (resp. don’t know  $\phi$ ) at some world only if either I know  $\phi$  at every world, or if I don’t know  $\phi$  at every world. As a result, if I know a proposition to hold at some world, and if that proposition is contingent, it is inconsistent to assume positive introspection in full generality (and similarly for negative introspection). This result is fairly dramatic, for it seems to show that whenever knowledge obeys a margin for error principle that applies non-trivially, knowledge can’t be introspective unconditionally. The problem may be summarized in the following rough terms: *knowledge can’t be vague while obeying positive or negative introspection at the same time.*

Contrary to Williamson, we don’t consider this conclusion to be sound. From a psychological point of view, as argued by Dokic & Égré (2004) and Égré (forthcoming), Williamson’s argument rests on the controversial assumption that all levels of knowledge obey the same kinds of margin for error. In particular, it presupposes that my visual knowledge and the knowledge I have about my visual knowledge both rest on the same discriminative capacities, an assumption we can’t take for granted. From a logical point of view, moreover, the result is relative to the semantics given to  $\Box$ . As Williamson notes, the failure of positive and negative introspection in margin models reflects the non-transitive and non-euclidian character of the perceptual indiscriminability relation. Conversely, imposing positive and negative introspection “from the outside” forces knowledge to hold universally

<sup>1</sup>Williamson also presents a *variable* margin semantics, relative to which the logic **KT** is sound and complete. For lack of space, we do not consider it here, although the results of the following sections would carry over to it.

or nowhere throughout the model. But if one wishes both to model a notion of inexact knowledge, obeying margin for error principles, and yet to preserve the introspective properties, a possibility is to revise the standard semantics for knowledge. We are therefore led to following question:

**Question 1.** *Is there a non-standard semantics suitable to validate introspection (either positive or negative) and which would still be adequate to model the notion of inexact knowledge?*

We give a positive answer to this question in the next section. More precisely, what we are looking for is a semantics for knowledge based on the non-transitive and non-euclidian property of indiscriminability, but nevertheless adequate to support introspection (whether positive or negative). We argue that the semantics is also plausible from a cognitive point of view, namely that it corresponds to a system of modular knowledge, in which higher-order knowledge is not necessarily on a par with knowledge at the low level.

## 2 A Centered Semantics for knowledge

### 2.1 The new semantics

In the standard semantics, it takes 2 steps from a given world to check whether an iterative formula of the form  $\Box\Box p$  holds at that world, and more generally it takes  $n$  transitions within a model to check for the satisfiability of a formula with  $n$  nested operators. In a situation of perceptual knowledge like the one pictured in Figure 1, this property is at odds with our intuition: looking at an object of size 10, I know it will fit in the box, and yet the semantics predicts that I don't know that I know it, since an object of size 12 doesn't fit in the box. However, it seems that one's reflective knowledge should not depend on such remote epistemic alternatives. To restore that intuition, we define a “centered semantics” in which the epistemic alternatives relevant for iterated modalities remain the worlds accessible in one transition from the world of evaluation. In other words, every fact concerning the knowledge of the agent should be decided solely on the basis of worlds that are not distinguishable from that world, without having to move further along the accessibility relation. Given a model  $\mathcal{M} = \langle W, R, V \rangle$ , we first define the notion of satisfaction for couples of worlds, and extract the definition of satisfaction for single worlds:

**Definition 1.** *Satisfaction for couples of worlds:*

- (i)  $\mathcal{M}, (w, w') \models_{CS} p$  iff  $w' \in V(p)$ .
- (ii)  $\mathcal{M}, (w, w') \models_{CS} \neg\phi$  iff  $\mathcal{M}, (w, w') \not\models_{CS} \phi$ .
- (iii)  $\mathcal{M}, (w, w') \models_{CS} (\phi \wedge \psi)$  iff  $\mathcal{M}, (w, w') \models_{CS} \phi$  and  $\mathcal{M}, (w, w') \models_{CS} \psi$ .
- (iv)  $\mathcal{M}, (w, w') \models_{CS} \Box\phi$  iff for all  $w''$  such that  $wRw''$ ,  $\mathcal{M}, (w, w'') \models_{CS} \phi$ .

**Definition 2.**  $\mathcal{M}, w \models_{CS} \phi$  iff  $\mathcal{M}, (w, w) \models_{CS} \phi$

Clause (iv) of the definition accounts for the “centered” feature of the semantics, for it entails that for every  $w$  and  $w'$ :  $\mathcal{M}, (w, w') \models_{CS} \Box\phi$  iff  $\mathcal{M}, (w, w) \models_{CS} \Box\phi$  iff  $\mathcal{M}, w \models_{CS} \Box\phi$ . This ensures that instead of looking at worlds that are two steps away to check whether  $\Box\Box\phi$  is satisfied, one backtracks to the actual world to see whether  $\Box\phi$  already holds there. For instance, relative to the model  $\mathcal{M}$  of Figure 1, it holds that  $\mathcal{M}, 10 \models_{CS} \Box p$ ,  $\mathcal{M}, 11 \models_{CS} \neg\Box p$ , and  $\mathcal{M}, 12 \models_{CS} \Box\neg p$ , just as with the standard semantics. However, we now have:  $\mathcal{M}, 10 \models_{CS} \Box\Box p$ , and likewise for any further level of iteration. Interestingly, it also holds that  $\mathcal{M}, 11 \models_{CS} \Box\neg\Box p$ . More generally, the semantics validates both positive and negative introspection, and we can prove the following completeness theorem:

## 2.2 K45 is sound and complete with respect to (CS)

Given a Kripke model  $\mathcal{M}$ , we say that  $\mathcal{M}$  CS-validates  $\phi$ , and we write  $\mathcal{M} \models_{\text{CS}} \phi$  if and only if for every world  $w$  of the model,  $\mathcal{M}, w \models_{\text{CS}} \phi$ . We call a formula  $\phi$  CS-valid, and we write  $\models_{\text{CS}} \phi$ , if every model  $\mathcal{M}$  CS-validates  $\phi$ .

**Theorem 1.** *K45 is sound with respect to (CS).*

*Proof.* It is straightforward to check that (CS) validates axioms K, 4 and 5, and that modus ponens and uniform substitution preserve validity. The only non-trivial case concerns the rule of necessitation. Suppose  $\models_{\text{CS}} \phi$ , but  $\not\models_{\text{CS}} \Box\phi$ . So there is a model  $\mathcal{M} = \langle W, R, V \rangle$  and a couple of worlds  $(w, w')$  such that  $\mathcal{M}, (w, w') \not\models_{\text{CS}} \phi$ . Consider any model  $\mathcal{M}' = \langle W, R', V \rangle$  with the same domain and valuation as  $\mathcal{M}$ , but in which  $R'(w') = R(w)$ . We first show by induction that for any formula  $\phi$ ,  $\mathcal{M}, (w, x) \models_{\text{CS}} \phi$  iff  $\mathcal{M}', (w', x) \models_{\text{CS}} \phi$ . The atomic and boolean cases are straightforward. Consider  $\phi := \Box\psi$ .  $\mathcal{M}, (w, x) \models_{\text{CS}} \Box\psi$  iff for every  $v$  such that  $wRv$ ,  $\mathcal{M}, (w, v) \models_{\text{CS}} \psi$  iff for every  $v$  such that  $w'R'v$ ,  $\mathcal{M}, (w, v) \models_{\text{CS}} \psi$  (by definition of  $R'$ ), iff for every  $v$  such that  $w'R'v$ ,  $\mathcal{M}', (w', v) \models_{\text{CS}} \psi$  (by induction hypothesis), iff  $\mathcal{M}', (w', x) \models_{\text{CS}} \Box\psi$ . From this, it follows that  $\mathcal{M}, (w, w') \models_{\text{CS}} \phi$  iff  $\mathcal{M}', (w', w') \models_{\text{CS}} \phi$ . So if we suppose that  $\phi$  is CS-valid but nevertheless such that  $\mathcal{M}, (w, w') \not\models_{\text{CS}} \phi$ , then we should have  $\mathcal{M}', (w', w') \not\models_{\text{CS}} \phi$ , that is  $\mathcal{M}', w' \not\models_{\text{CS}} \phi$ , and  $\phi$  could not be valid.  $\square$

The proof that the rule of necessitation preserves validity is slightly more complicated than the usual proof given for the standard semantics. The reason is that in the standard semantics, the rule of necessitation also holds within models: given a model  $\mathcal{M}$ , if  $\mathcal{M} \models \phi$ , then it follows that  $\mathcal{M} \models \Box\phi$ . Another way to put it is to say that necessitation is not only *frame-valid*, but also *model-valid* for the standard semantics. Relative to CS, however, necessitation is only frame-valid. Consider, for instance, a model  $\mathcal{M}$  with three worlds  $w, w', w''$  such that  $V(q) = \{w, w''\}$  and  $V(p) = \{w'\}$ , and in which  $wRw'$  and  $w'Rw''$ . Clearly  $\mathcal{M} \models_{\text{CS}} \Box p \rightarrow q$  but  $\mathcal{M} \not\models_{\text{CS}} \Box(\Box p \rightarrow q)$ .

**Theorem 2.** *K45 is complete with respect to (CS).*

*Proof.* We rely on the standard completeness proof for **K45**: **K45** is sound and complete w.r.t. the class of transitive and euclidean frames. Let us assume for contradiction that **K45** is not complete for (CS). This means that there is a sentence  $\phi$  such that  $\models_{\text{CS}} \phi$  but  $\not\models_{\text{K45}} \phi$ . By completeness, there is a transitive and euclidean model  $\mathcal{M}$  and a world  $w_0$  such that  $\mathcal{M}, w_0 \not\models \phi$ . We show that this model contradicts  $\models_{\text{CS}} \phi$ , by showing that  $\mathcal{M}, w_0 \not\models_{\text{CS}} \phi$ .

**Lemma 1.** *Let  $\mathcal{M}$  be a transitive and euclidean model of modal logic (ML) and  $\phi$  an ML-sentence. Then for every world  $w_0$ ,  $\mathcal{M}, w_0 \models \phi$  iff  $\mathcal{M}, w_0 \models_{\text{CS}} \phi$*

We show by induction on the length of the formula that for every world  $w$  in the submodel generated from  $w_0$ ,  $\mathcal{M}, w \models \phi$  iff  $\mathcal{M}, (w_0, w) \models_{\text{CS}} \phi$ . The lemma follows immediately, since  $\mathcal{M}, w_0 \models_{\text{CS}} \phi$  iff  $\mathcal{M}, (w_0, w_0) \models_{\text{CS}} \phi$ . The only non-trivial case is  $\phi = \Box\psi$ . Since  $R$  is euclidean and transitive, it holds that  $wRw'$  iff  $w_0Rw'$  (Assume  $wRw'$ . By transitivity  $w$  is reachable from  $w_0$ , and  $wRw'$ , so by transitivity again we have that  $w_0Rw'$ . Now assume  $w_0Rw'$ . Since  $w$  is reachable from  $w_0$  and  $R$  is transitive,  $w_0Rw$ . By euclideanity,  $wRw'$  ensues). We then have:

$$\begin{aligned} \mathcal{M}, w \models \Box\psi & \text{ iff for all } w' \text{ such that } wRw', \mathcal{M}, w' \models \psi, \text{ by definition of } \models \\ & \text{ iff for all } w' \text{ such that } wRw', \mathcal{M}, (w_0, w') \models_{\text{CS}} \psi, \text{ by induction hypothesis.} \\ & \text{ iff for all } w' \text{ such that } w_0Rw', \mathcal{M}, (w_0, w') \models_{\text{CS}} \psi, \text{ by the property of } R. \\ & \text{ iff } \mathcal{M}, (w_0, w) \models_{\text{CS}} \Box\psi, \text{ by definition of } \models_{\text{CS}}. \end{aligned}$$

□

The lemma shows that the shift from the standard semantics to the centered semantics preserves satisfaction on the class of transitive and euclidean models. This does not mean that (CS) is just a trivial rewording of the definition of satisfaction, because  $\models$  and  $\models_{\text{CS}}$  do not match in general: the previous proof rests in an essential way on the assumption that the accessibility relation is transitive and euclidean. The important fact is thus that our stock of models is now bigger: we have at our disposal not only the transitive euclidean models, but the full class of models, without having to relinquish the introspection principles. This includes, in particular, non-transitive and non-euclidian models like the model of Figure 1.

The model of Figure 1, it may be recalled, may also be seen as fixed-margin model  $\langle W, d, \alpha, V \rangle$  with margin of error  $\alpha = 1$ . As a matter of fact, the completeness theorem we stated for **K45** with respect to the centered semantics can be turned into a completeness theorem for **S5** with respect to Williamson's fixed-margin semantics. Given a fixed margin model  $\mathcal{M} = \langle W, d, \alpha, V \rangle$ , we define a centered fixed-margin semantics (CMS), paralleling the definition of (CS). The definition of satisfaction (for couple of worlds) is the same for the atomic and boolean cases, and becomes, for the

**Definition 3.**  $\mathcal{M}, (w, w') \models_{\text{CMS}} \Box\phi$  iff for every  $v$  such that  $d(w, v) \leq \alpha$ ,  $\mathcal{M}, (w, v) \models_{\text{CMS}} \phi$

As before, we define  $\mathcal{M}, w \models_{\text{CMS}} \phi$ , iff  $\mathcal{M}, (w, w) \models_{\text{CMS}} \phi$ . A fixed-margin model  $\mathcal{M}$  (CMS)-validates  $\phi$  iff every world of the model (CMS)-satisfies  $\phi$ ;  $\models_{\text{CMS}} \phi$  iff every margin model (CMS)-validates  $\phi$ . We know that Williamson's fixed margin semantics (FM) is sound and complete for **KTB**, and that (CS) is sound and complete for **K45**. Putting together the results, we get:

**Theorem 3.** **S5** is sound and complete with respect to (CMS)

*Proof.* Every fixed margin model with parameter  $\alpha$  can be seen as a reflexive symmetric standard model such that  $wRv$  iff  $d(w, v) \leq \alpha$ . From this, it follows that if  $\models_{\text{CS}} \phi$  then  $\models_{\text{CMS}} \phi$ , and so **K45** is sound w.r.t (CMS). Moreover, (CMS) validates T (and B), so (CMS) is sound for **S5**. Completeness is just an adaptation of Lemma 1: given a reflexive euclidian model  $\mathcal{M}$  of **S5**, one can see it as a fixed margin model  $\mathcal{M}^*$  with parameter  $\alpha = 0$ , setting  $d(w, v) = 0$  iff  $wRv$ ; from Lemma 1 it can be checked that  $\mathcal{M}, w \models \phi$  iff  $\mathcal{M}^*, w \models_{\text{CMS}} \phi$ .

□

From the standpoint of epistemic logic, **K45** can be seen as a system of introspective *belief*. With the inclusion of axiom T, **S5** is a system of introspective *knowledge* properly so called, and we take the completeness of **S5** with respect to the centered margin semantics to give a positive answer to the Question raised in the previous section.

### 3 Luminosity and knowledge iterations

#### 3.1 Luminosity

Williamson's argument against positive (as well as negative) introspection is part of a more general argument against the so-called *luminosity* of mental states. Williamson calls a mental state luminous if and only if the occurrence of the state entails the knowledge that one is in that state (Williamson 2000, chap. 4). According to Williamson, no non-trivial mental state is luminous, a non-trivial state being defined as a state that lasts for some time, but not all the time (2000: 107). This psychological

claim rests on the idea that knowledge about one’s mental states, in order to be reliable, obeys a margin of error, and is backed up by Williamson’s result that  $\phi \rightarrow \Box\phi$  is valid in (fixed or variable) margin semantics if and only if either  $\phi$  is valid or  $\neg\phi$  is valid. In centered semantics, however, it no longer holds that  $\models_{CS} \phi \rightarrow \Box\phi$  iff either  $\models_{CS} \phi$  or  $\models_{CS} \neg\phi$ , as shown by the fact that  $Kp \rightarrow KKp$  is CS-valid, but neither  $Kp$  nor  $\neg Kp$  is CS-valid in the model of Figure 1. If, like Williamson, we admit that knowing can be a mental state, then this suggests that at least states of knowledge may be luminous without being trivial.

To be sure, consider close situations in which I’m asked whether I feel cold or not. Let the model of Figure 1 now represent a thermometric scale, where  $p$  stand for “I feel cold”, with the assumption that I cannot perceptually discriminate between any two situations that differ only by  $1^\circ\text{C}$ . The model depicts a case in which I feel cold up to  $11^\circ\text{C}$ , and start not to feel cold from  $12^\circ\text{C}$  onward. Standard Kripke semantics, like Williamson’s fixed margin semantics (with a margin of  $1^\circ\text{C}$ ), predicts that at the world where the temperature is  $12^\circ\text{C}$ , I start not to feel cold, but don’t know yet that I no longer feel cold. At the world where the temperature is  $13^\circ\text{C}$ , I know I don’t feel cold, but don’t know that I know this, due to the standard semantics: hence neither my feeling cold, nor my knowing that I feel cold is luminous. Using the centered semantics, it still holds that at  $12^\circ\text{C}$  I don’t know yet that I start not to feel cold, but at  $13^\circ\text{C}$  I know it, and know that I know it. Relative to the centered semantics, the model of Figure 1 depicts a situation in which feeling cold is not a luminous condition, but in which my knowing that I feel cold is luminous. Thus we may agree with Williamson that *not all* mental states are luminous, but nevertheless disagree on the idea that *no* non-trivial mental state is luminous.

Which semantics is more plausible from a psychological point of view is a question that should be decided upon empirical grounds and that we shall leave open for psychological investigation. That being said, it may be argued that our centered semantics is too quick to make knowledge insensitive to iterations. There may be situations, for instance, in which my knowing  $p$  is not sufficient to warrant my knowing that I know  $p$ , even though my knowing that I know  $p$  is sufficient to warrant any further level of iteration. We could imagine, for instance, that at  $13^\circ\text{C}$  I just become aware that I am not cold, but that in order for me to become aware of this awareness, the temperature should reach at least  $14^\circ\text{C}$ . More generally, we may conceive like Williamson that the higher-order awareness we have of our perceptual states comes in degrees which co-vary with the intensity of the perceptual stimulus, but only up to a point, from which iterations become insensitive. To close this paper, we state a semantics which makes room for that possibility, and which actually allows to set the collapse between modalities at any arbitrary level.

### 3.2 Centered semantics with a parameter

The semantics, which we call “token semantics” for short, is a parameterized version of centered semantics. Satisfaction is defined with respect to a sequence of worlds and a number of tokens:  $q$  is short for an arbitrary sequence of worlds,  $qw$  for an arbitrary sequence with last item  $w$ , and  $n$  is an arbitrary number of tokens.

**Definition 4.** *Token satisfaction:*

- (i)  $\mathcal{M}, qw \models_{TS} p [n]$  iff  $w \in V(p)$ .
- (ii)  $\mathcal{M}, qw \models_{TS} \neg\phi [n]$  iff  $\mathcal{M}, qw \not\models_{TS} \phi [n]$ .
- (iii)  $\mathcal{M}, qw \models_{TS} (\phi \wedge \psi) [n]$  iff  $\mathcal{M}, qw \models_{CS} \phi$  and  $\mathcal{M}, qw \models_{CS} \psi [n]$ .
- (iv)  $\mathcal{M}, qw \models_{TS} \Box\psi [n]$  iff
  - $n \neq 0$  and for all  $w'$  such that  $wRw'$ ,  $qw w' \models_{TS} \psi [n - 1]$
  - Or  $n = 0$  and  $q \models_{TS} \psi [1]$ .



**Definition 5.** Let  $n$  be such that  $1 \leq n \leq \omega$  (we assume  $\omega - 1 = \omega$ ).  $TS(n)$  is the modal semantics defined by the following satisfaction relation:  $\mathcal{M}, w \models_{TS(n)} \phi$  iff  $\mathcal{M}, w \models_{TS} \phi [n]$ .

Thus, a token is spent for each move along the accessibility relation. When all tokens have been spent, accessible worlds are those which were accessible when there was no token left but one. As intended, centered semantics and standard semantics come out as special cases:

**Fact 1.** •  $\mathcal{M}, w \models_{CS} \phi$  iff  $\mathcal{M}, \langle w \rangle \models_{TS} \phi [1]$   
 •  $\mathcal{M}, w \models \phi$  iff  $\mathcal{M}, \langle w \rangle \models_{TS} \phi [\omega]$

Now, what happens with the introspection principles? (CS) validates 4 and 5, but this is not true for every  $TS(n)$  semantics: it is easy to see that 4 and 5, though  $TS(1)$ -valid, are not  $TS(n)$ -valid for  $n \geq 2$ . But there is a generalized form of these principles which is correlated to our family of semantics. Let  $\Box_n$  be short for  $\Box \dots \Box$ ,  $n$  times. We consider:

$$(4n) \quad \Box_n p \rightarrow \Box_n \Box p$$

Intuitively  $4n$  is like just like 4, but it works only at a minimum distance of  $n$  steps from the starting point.  $4n$  can be used to capture the cognitive limitation corresponding to the inability to distinguish meta-representations involved in self-knowledge at levels beyond  $n$ : it says that knowing that one knows etc...(n times) implies knowing that one knows etc...(n + 1 times). Similarly, 5 can be generalized to:

$$(5n) \quad \Diamond_n p \rightarrow \Diamond_{n-1} \Box \Diamond p$$

$TS(n)$  has a CS-like behaviour, but only at distance  $n$ , this is precisely what is captured by  $4n$  and  $5n$ :

**Theorem 4.** For every integer  $n$ , **K4n5n** is sound and complete with respect to  $TS(n)$ .

*Proof.* We just sketch the completeness proof, which generalizes the method of Theorem 2. First, note that  $4n$  defines the class of  $n$ -transitive frames, i.e. the class of frames satisfying:

$$\forall x_1, \dots, x_{n+2} ((x_1 R x_2 \wedge \dots \wedge x_{n+1} R x_{n+2}) \rightarrow x_n R x_{n+2})$$

Similarly,  $5n$  defines the class of  $n$ -euclidean frames, i.e. the class of frames satisfying:

$$\forall x_1, \dots, x_{n+2} ((x_1 R x_2 \wedge \dots \wedge x_n R x_{n+1} \wedge x_n R x_{n+2}) \rightarrow x_{n+2} R x_{n+1})$$

$4n$  and  $5n$  are Sahlqvist formulas. Therefore, by the Sahlqvist completeness theorem (see Blackburn & al. 1999, c. 4), **K4n5n** is complete for the class of  $n$ -euclidean and  $n$ -transitive frames. Now, it is sufficient to check that the following generalization of Lemma 1 holds:

**Lemma 2.** Let  $\mathcal{M}$  be an  $n$ -transitive and  $n$ -euclidean model and  $\phi$  an arbitrary sentence. Then for every world  $w_0$ ,  $\mathcal{M}, w_0 \models \phi$  iff  $\mathcal{M}, w_0 \models_{TS(n)} \phi$

□

The present generalization of centered semantics thus offers a compromise between the standard semantics (up to degree  $n$ ) and the non-standard semantics (beyond  $n$ ). The following yields the corresponding triviality results for the modalities:

**Theorem 5.** Let  $\phi$  be any basic modal sentence: for all  $n \geq 1$ , there is a sentence  $\phi_n$  of degree at most  $n$ , such that  $\phi$  is  $TS(n)$ -equivalent to  $\phi_n$ .

*Proof.*  $\phi_n$  is obtained from  $\phi$  by syntactic transformation. Let  $\psi$  be a maximal subformula of  $\phi$  embedded under  $n - 1$  modalities (*i.e.* all subformulas containing  $\psi$  are embedded under fewer than  $n - 1$  modalities). By well-known results,  $\psi$  can be turned into a formula  $\psi'$  of degree at most 1 which is **K45**-equivalent to  $\psi$ . Replace  $\psi$  by  $\psi'$ . We get  $\phi_n$  by applying this operation to all maximal subformulas embedded under  $n - 1$  modalities. It follows from the definition of  $TS(n)$  that  $\phi_n$  is  $TS(n)$ -equivalent to  $\phi$ . □

In the same way in which we formulated a centered version of Williamson's margin semantics with parameter 1, for any  $n$  we could formulate a centered margin semantics with parameter  $n$ , yielding an analogous completeness result for **KT<sub>B</sub>4n5n**, corresponding to a modular logic of inexact knowledge with introspection, standard up to  $n$ , and non-standard beyond. The results of this paper should convince us that logics of introspective belief and knowledge like **K45** and **S5** are not tied intrinsically to the representation of a notion of *exact* knowledge. From a conceptual point of view, they show that Williamson's luminosity paradox can be solved without abandoning the introspective principles, nor the original motivation for margin of error principles.

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