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# NON-ALGEBRAIC DECAY FOR SOLUTIONS TO THE NAVIER–STOKES EQUATIONS

LORENZO BRANDOLESE, MATTHIEU PAGEARD, AND CILON F. PERUSATO

ABSTRACT. Around forty years ago, Michael Wiegner provided, in a seminal paper, sharp algebraic decay rates for solutions of the Navier–Stokes equations, showing that these solutions behave asymptotically like the solutions of the heat equation with the same data as  $t \rightarrow +\infty$ , in the  $L^2$ -norm, up to some critical decay rate. In the present paper, we close a gap that appears in the conclusion of Wiegner’s theorem in the 2D case, for solutions with non-algebraic decay rate.

## 1. INTRODUCTION

This study is motivated by the recent paper [3], where it is proved that, in the class of Leray solutions of the unforced Navier–Stokes equations, the subclass of solutions with  $L^2$ -algebraic decay is negligible, in an appropriate topological sense. This leads us to study solutions with non-algebraic decay rates.

The Navier–Stokes equations in the whole space  $\mathbb{R}^d$  ( $d = 2, 3$ ) read

$$(1.1) \quad \begin{cases} \partial_t u + u \cdot \nabla u = \Delta u - \nabla p + f, \\ \nabla \cdot u = 0, \\ u(x, 0) = u_0(x), \end{cases} \quad x \in \mathbb{R}^d, t > 0,$$

where  $u = (u_1, \dots, u_d)$  is the velocity field, and  $p$  is the (scalar) pressure field. In this work, we will always assume  $u_0 \in L^2_\sigma(\mathbb{R}^d)$ , the space of solenoidal vector fields with components in  $L^2$ , and that the given external force  $f = f(x, t)$  satisfies  $f \in L^1((0, \infty), L^2(\mathbb{R}^d))$ .

We are interested in the Leray solutions to system (1.1), namely the functions  $u$  verifying  $u \in C_w([0, \infty), L^2_\sigma(\mathbb{R}^d)) \cap L^2((0, \infty), \dot{H}^1(\mathbb{R}^d))$  with  $u(\cdot, 0) = u_0$ , satisfying (1.1) in the weak sense, as well as the energy estimate in its strong form:

$$(1.2) \quad \|u(t)\|^2 + 2 \int_s^t \|\nabla u(\tau)\|^2 d\tau \leq \|u(s)\|^2 + 2 \int_s^t \int_{\mathbb{R}^d} |u(x, \tau) \cdot f(x, \tau)| dx d\tau,$$

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for  $s = 0$  and almost all  $s > 0$ , and every  $t \geq s$  (here,  $\|\cdot\|$  denotes the  $L^2$ -norm). It is well known (see [16]) that such solutions satisfy, for some constant  $C > 0$ ,

$$(1.3) \quad \|u(t)\|^2 + 2 \int_s^t \|\nabla u(\tau)\|^2 d\tau \leq \|u(s)\|^2 + C \int_s^t \|f(\tau)\| d\tau,$$

for  $s = 0$  and almost all  $s > 0$ , and all  $t \geq s$ . When  $d = 2$ , the Leray solutions are unique and inequality (1.3) is known to hold for all  $0 \leq s \leq t$ .

We now provide a quick overview of the literature concerning the asymptotic behavior and decay rates of the energy  $\|u(t)\|^2$  for solutions to the Navier–Stokes equations. This problem dates back to Leray, who in his pioneering paper [9] left open the question of whether the weak solutions he constructed, in the absence of external forcing, vanish to zero in  $L^2$  as  $t \rightarrow +\infty$ . This question was positively answered by Kato [7] and Masuda [10] only fifty years later. Subsequently, Schonbek [12] obtained the first explicit decay rates by using her well-known *Fourier-Splitting* technique, under an additional  $L^p$ -condition,  $1 \leq p < 2$ , on the initial data. Optimal algebraic decay rates for the  $L^2$ -norm of Leray solutions were obtained by Wiegner [16], with or without forcing. Moreover, the conditions required on  $u_0$  to achieve an algebraic decay prescribed in [16] are also optimal, as later shown in [13]. Energy dissipation to zero in the presence of forcing satisfying conditions more general than  $f \in L^1((0, \infty), L^2(\mathbb{R}^d))$  is obtained in [11]. The large time behavior of these solutions was subsequently extensively investigated in various directions, both for the Navier–Stokes equations and many of their generalizations. We refer to [1] for a detailed review of this topic and to [5, 14, 15] for a sample of the most recent results.

However, energy dissipation at algebraic rates is far from being the generic scenario; rather, it represents an exceptional situation. To illustrate this fact, let us consider the class  $\mathcal{L}$  of all Leray solutions to the unforced Navier–Stokes equations. We can endow  $\mathcal{L}$  with the initial topology induced by the map  $I: \mathcal{L} \rightarrow L^2_\sigma$ , where  $I$  is the map  $u \mapsto u_0$ , that associates, to a solution  $u$ , the corresponding initial datum. From this, in [3] the authors established the following result.

**Theorem 1.1** (L. Brandolese, C. Perusato, and P. Zingano, 2024 [3]). *With the above topology,  $\mathcal{L}$  is a Baire space and the set of unforced Leray solutions with algebraic decay is meager in  $\mathcal{L}$ .*

Motivated by the above result, the purpose of this work is to extend Wiegner’s theorem [16], to cover a considerably *larger class of decay profiles*, encompassing the Leray solutions with non-algebraic decay. As we will see, such an extension is especially relevant in the two-dimensional case, as the classical Wiegner theorem provides little information in this case. As a byproduct of our analysis, we will be able to close a gap in the 2D case.

To provide a deeper background for the present study, let us recall the already mentioned Wiegner result [16]. For the external force, Wiegner’s assumptions are the following:

for all  $t > 0$ ,

$$(1.4) \quad \begin{cases} f \in L^1((0, \infty), L^2(\mathbb{R}^d)), \\ \|f(t)\| \lesssim (1+t)^{-\alpha-1}, \\ \|f(t)\|_{L^d(\mathbb{R}^d)} \lesssim t^{-\alpha-(d+2)/4}. \end{cases}$$

We also consider the solution  $v(t) \in C([0, \infty), L^2_\sigma(\mathbb{R}^d))$  of the associated Stokes flow, that in the whole space agrees with the solution of the heat equation:

$$(1.5) \quad \partial_t v = \Delta v + f, \quad v(\cdot, 0) = u_0.$$

**Theorem 1.2** (M. Wiegner, 1987 [16]). *Let  $d \geq 2$ ,  $u_0 \in L^2_\sigma(\mathbb{R}^d)$ , and  $f \in L^1((0, \infty), L^2(\mathbb{R}^d))$ . Then,  $\|u(t)\| \rightarrow 0$  as  $t \rightarrow +\infty$ . Moreover, if  $f$  satisfies (1.4) for some  $0 \leq \alpha \leq (d+2)/4$  and if*

$$(1.6) \quad \|v(t)\| = O(1+t)^{-\alpha}, \quad \text{for some } 0 \leq \alpha \leq (d+2)/4,$$

then  $\|u(t)\| = O(1+t)^{-\alpha}$ . In addition,

$$(1.7) \quad \|u(t) - v(t)\| = \begin{cases} o(1+t)^{-(d-2)/4} & \text{if } \alpha = 0, \\ O(1+t)^{-2\alpha-(d-2)/4} & \text{if } 0 < \alpha < \frac{1}{2}, \\ O(1+t)^{-(d+2)/4} \log(2+t) & \text{if } \alpha = \frac{1}{2}, \\ O(1+t)^{-(d+2)/4} & \text{if } \frac{1}{2} < \alpha \leq \frac{d+2}{4}. \end{cases}$$

Notice that, when  $0 \leq \alpha < (d+2)/4$  and  $\|v(t)\| \approx t^{-\alpha}$ , we have  $\|u(t) - v(t)\| = o(t^{-\alpha})$  as  $t \rightarrow +\infty$  by (1.7). Thus, Theorem 1.2 *essentially* states that, up to the critical decay rate  $t^{-(d+2)/4}$ , the solution  $u$  behaves asymptotically, in the  $L^2$ -norm, like the solution  $v$  of the corresponding linear problem, at the first order. However, a closer look at the theorem reveals that there is a gap in this interpretation.

Indeed, let us consider the particular case  $d = 2$ ,  $f \equiv 0$  and  $\alpha = 0$ : in this case, Theorem 1.2 asserts that  $\|u(t) - v(t)\| = o(1)$ , as  $t \rightarrow +\infty$ . But for the linear part, it is well known that  $\|v(t)\| = \|e^{t\Delta}u_0\| = o(1)$  as  $t \rightarrow +\infty$ , so we cannot conclude in this case that  $u(t)$  behaves asymptotically, in the  $L^2$ -norm, like  $v(t)$ . For example, if  $u_0$  is such that  $\|e^{t\Delta}u_0\|$  decays at a logarithmic rate, then one might expect that  $\|u(t)\|$  itself would decay at the same logarithmic rate, but such a conclusion is *not* a consequence of Wiegner's theorem in 2D; yet, it will be a consequence of our results, namely Assertion i) of Theorem 1.3 below.

The above difficulty is specific to the 2D case: in three or higher spatial dimensions, the information about the decay of  $\|u(t) - v(t)\|$  provided by (1.7) is more precise, and the above gap no longer appears. See Remark 1.6 for additional explanation on this point. For this reason, in the present paper we will only focus on the 2D case.

**Statement of the main result.** We collect here the different assumptions on the external force that we will need in this paper. The (already mentioned) first condition that will always be required is

$$(1.8a) \quad f \in L^1((0, \infty), L^2(\mathbb{R}^2)).$$

A second assumption, that will be of constant use, is the following:

$$(1.8b) \quad t \mapsto \sqrt{t}\|f(t)\| \in L^2([T, +\infty)), \quad \text{for some } T \geq 0.$$

Our last condition, though not systematically required, but useful to strengthen our conclusions, is

$$(1.8c) \quad \|f(t)\| = O(t^{-1})\Phi(t), \quad \text{as } t \rightarrow +\infty,$$

for a function  $\Phi$ , defined in  $[0, +\infty)$ , such that  $\lim_{t \rightarrow +\infty} \Phi(t) = 0$ . This latter condition is more stringent than the previous one (1.8b): indeed, if  $f$  satisfies both (1.8a) and (1.8c), then condition (1.8b) follows by interpolation. Indeed, by (1.8c) and the fact that  $\Phi(t) \rightarrow 0$ , we have  $t\|f(t)\|^2 \lesssim \Phi(t)\|f(t)\| \lesssim \|f(t)\|$  for all large  $t$ , and the integrability condition (1.8b) follows after integration thanks to (1.8a). We will prescribe below a few additional conditions on  $\Phi$ .

Our next assumptions concern the large-time decay of  $v$  in the  $L^2$ -norm. First of all, it is well known, and easily checked using the Fourier transform and the dominated convergence theorem, that under the sole condition (1.8a), the solution of the heat equation (1.5) with  $L^2$ -initial data  $u_0$  satisfies

$$\|v(t)\| \rightarrow 0, \quad \text{as } t \rightarrow +\infty.$$

We will make this fact more precise by *assuming* that

$$\|v(t)\| \lesssim \Phi(t),$$

with  $\Phi$  decaying to zero “without fluctuations”, in a sense formalized by conditions (1.9a)-(1.9b) below:

$$(1.9a) \quad \Phi \text{ is monotonically decreasing on } [0, \infty), \text{ with } \Phi(0+) > 0 \text{ and } \lim_{t \rightarrow +\infty} \Phi(t) = 0,$$

and

$$(1.9b) \quad \exists \kappa \geq 1, \forall t > 0, \quad \frac{1}{t} \int_0^t \Phi \leq \kappa \Phi(t).$$

Notice that functions satisfying (1.9a) are necessarily locally (square-)integrable on  $[0, +\infty)$ , so the integral  $\int_0^t \Phi$  is well-defined.

In the second assertion of our main result, we will require a stronger condition on  $\Phi$ , namely

$$(1.9c) \quad \exists \kappa \geq 1, \forall t > 0, \quad \frac{1}{t} \int_0^t \Phi(s)^2 ds \leq \kappa \Phi(t)^2.$$

By the Cauchy-Schwarz inequality, one easily checks that (1.9c) implies (1.9b). Typical examples of functions satisfying (1.9a)-(1.9b) are  $\Phi(t) = (1+t)^{-a} \ln(e+t)^{-b}$ , with  $0 \leq a < 1$  and  $b \in \mathbb{R}$  (and  $b > 0$  if  $a = 0$ ). Typical examples of functions satisfying (1.9a)-(1.9c) are the same functions as before, but with the smaller range  $0 \leq a < 1/2$ . Slower decaying functions, obtained *e.g.* by taking negative powers of iterated logarithms, such as  $\Phi(t) = (\log(\log(2e+t)))^{-b}$  with  $b > 0$ , are also possible and satisfy the most stringent version of our conditions. Let us also observe that conditions (1.9a)-(1.9b) exclude  $\Phi \in L^1(\mathbb{R}^+)$  (see (2.3) below), whereas conditions (1.9a)-(1.9c) exclude  $\Phi \in L^2(\mathbb{R}^+)$ . In particular, cases ii) and iii) in Theorem 1.3 below are mutually exclusive.

We now state our main result.

**Theorem 1.3.** *Let  $u_0 \in L^2_\sigma(\mathbb{R}^2)$  and  $f$  satisfy (1.8a)-(1.8b). Let  $u$  be the Leray solution of the Navier-Stokes equations (1.1) with  $d = 2$ , and  $v$  be the solution of the heat equation (1.5), starting from  $u_0$ . Suppose that*

$$\|v(t)\| = O(\Phi(t)), \quad \text{as } t \rightarrow +\infty.$$

with  $\Phi$  satisfying conditions (1.9a)-(1.9b).

Then, the following properties hold:

i) We have

$$(1.10) \quad \|u(t)\| = O(\Phi(t)) \quad \text{and} \quad \|u(t) - v(t)\| = o(\Phi(t)), \quad \text{as } t \rightarrow +\infty.$$

ii) Under the more stringent condition (1.9c) on  $\Phi$  (which excludes  $\Phi \in L^2(\mathbb{R}^+)$ ) and (1.8c) on  $f$ , we have

$$\|u(t) - v(t)\| = O(\Phi(t)^2), \quad \text{as } t \rightarrow +\infty.$$

iii) Under the alternative condition  $\Phi \in L^2(\mathbb{R}^+)$  (which excludes (1.9c)) and condition (1.8c) on  $f$ , we have

$$\|u(t) - v(t)\| = O(t^{-1}), \quad \text{as } t \rightarrow +\infty.$$

As an illustration of our result, let us consider  $u_0$  and  $f$  as in Example A.1 below. We have  $\|v(t)\| \approx \Phi(t)$ , where  $\Phi(t) := (\ln(e+t))^{-1/2}$ . As  $\|v(t)\| \lesssim \Phi(t)$  and  $\Phi$  satisfies conditions (1.9a)-(1.9c), Conclusion ii) of Theorem 1.3 applies, and we obtain  $\|u(t) - v(t)\| \lesssim \Phi(t)^2$ . But since we also have  $\Phi(t) \lesssim \|v(t)\|$ , we deduce that  $u(t)$  behaves asymptotically like  $v(t)$  in the  $L^2$ -norm: we have  $\|u(t) - v(t)\| = o(\|v(t)\|)$  as  $t \rightarrow +\infty$ . In particular,  $\|u(t)\| \approx \|v(t)\|$ . Notice that the application of Theorem 1.2 in the 2D case would only give the weaker result  $\|u(t)\| \rightarrow 0$  as  $t \rightarrow +\infty$ , without a priori excluding that  $\|u(t)\|$  may decay slower or faster than  $\|v(t)\|$ .

*Remark 1.4.* We point out that (1.8b) is purely an  $L^2$ -integrability condition on  $t^{1/2}\|f(t)\|$ , with no reference to algebraic decay. In particular, Condition (1.8b) is considerably more general than Wiegner's condition (1.4) on the external force, even when  $\alpha = 0$ . Indeed, condition (1.4) implies  $\|f(t)\| \lesssim (1+t)^{-1}$ , so  $t\|f(t)\|^2 \lesssim \|f(t)\|$ . But (1.4) also implies  $f \in L^1((0, \infty), L^2(\mathbb{R}^2))$ , and we get (1.8b).

*Remark 1.5.* Conclusion iii) allows to achieve Wiegner's optimal 2D-decay rate  $O(t^{-(d+2)/4}) = O(t^{-1})$  for  $\|u(t) - v(t)\|$ , which is the same as in Theorem 1.2, but under weaker conditions. It is easy to prescribe conditions on  $u_0$  and  $f$  implying that  $v \in L^2(\mathbb{R}^+L^2(\mathbb{R}^2))$ , which in turn ensures the required condition  $\Phi \in L^2(\mathbb{R}^+)$  for the application of Conclusion iii): indeed,  $\|v\|_{L^2(\mathbb{R}^+, L^2(\mathbb{R}^2))} \leq \|u_0\|_{\dot{H}^{-1}(\mathbb{R}^2)} + \|f\|_{L^2(\mathbb{R}^+, \dot{H}^{-2}(\mathbb{R}^2))}$  (see [4, Lemma 4.10]).

*Remark 1.6* (The higher dimensional case). The main interest of Conclusions i)-ii) is that the function  $\Phi$  can be chosen to decay arbitrarily slowly as  $t \rightarrow +\infty$ . In dimension  $d \geq 3$ , the results of the previous theorems can be essentially covered, at least for slowly decaying profiles  $\Phi$ , as a direct consequence of Wiegner's theorem. The present observation does not apply in the 2D case, because we do not have any information on the decay rate of  $\|u(t) - v(t)\|$

from Wiegner's Theorem 1.2 in 2D when  $\alpha = 0$ . More precisely, when  $d \geq 3$ , under the sole assumption  $u_0 \in L^2_\sigma(\mathbb{R}^d)$  (assuming here  $f \equiv 0$  for simplicity), one has the following key property:

$$(1.11) \quad \|u(t) - e^{t\Delta}u_0\|_{L^2(\mathbb{R}^d)} = o(t^{-(d-2)/4}),$$

by (1.7). In particular, if we have

$$(1.12) \quad \|e^{t\Delta}u_0\|_{L^2(\mathbb{R}^d)} \leq \Phi(t) \quad (t \gg 1),$$

then, as soon as  $\Phi$  satisfies a condition of mild decay, such as *e.g.*  $\liminf_{t \rightarrow \infty} t^{-(d-2)/4}\Phi(t) > 0$ , we will automatically have

$$(1.13) \quad \|u(t)\|_{L^2(\mathbb{R}^d)} = O(\Phi(t)) \quad \text{and} \quad \|u(t) - e^{t\Delta}u_0\|_{L^2(\mathbb{R}^d)} = o(\Phi(t)),$$

which agrees with Conclusion i) of Theorem 1.3. Similarly, when  $d \geq 3$ , it readily follows by Wiegner's theorem that

$$\|u(t)\|_{L^2(\mathbb{R}^d)} = O(\Phi(t)) \quad \text{and} \quad \|u(t) - e^{t\Delta}u_0\|_{L^2(\mathbb{R}^d)} = o(\Phi(t)^2),$$

by only assuming the slightly more stringent slow decay condition  $\liminf_{t \rightarrow \infty} t^{-(d-2)/4}\Phi(t)^2 > 0$ .

**Notation.** For a function  $w = (w_1, \dots, w_d)$ , writing  $w \in L^2(\mathbb{R}^d)$  means that  $w_i \in L^2(\mathbb{R}^d)$  for all  $1 \leq i \leq d$ , while  $w \in L^2_\sigma(\mathbb{R}^d)$  means that  $w \in L^2(\mathbb{R}^d)$  and  $\operatorname{div} w = 0$  in the distributional sense. The orthogonal projection from  $L^2(\mathbb{R}^d)$  onto  $L^2_\sigma(\mathbb{R}^d)$  is denoted by  $\mathbb{P}$ . Classical results on the Riesz transform imply that this map extends as a bounded linear operator on  $L^p(\mathbb{R}^d)$ , for  $1 < p < +\infty$ . For brevity, we will often simply write  $\|\cdot\|$  instead of  $\|\cdot\|_{L^2(\mathbb{R}^d)}$ .

We use  $o(1)$  to denote a function that vanishes at infinity,  $O(1)$  for a bounded function and, similarly:  $O(g(t)) = O(1)g(t)$  and  $o(g(t)) = o(1)g(t)$ . The notation  $A(t) \lesssim B(t)$  means that there exists a constant  $c > 0$ , independent of  $t$ , such that  $A(t) \leq cB(t)$ . The notation  $A(t) \approx B(t)$  means that we have both  $A(t) \lesssim B(t)$  and  $B(t) \lesssim A(t)$ .

We denote by  $\mathcal{S}'(\mathbb{R}^d)$  the space of tempered distributions. The Fourier transform of a function (or of a tempered distribution)  $f$  is denoted by  $\widehat{f}$ . We denote by  $e^{t\Delta}$  the heat semigroup.

The integrals over the whole space will be denoted simply by  $\int$ , instead of  $\int_{\mathbb{R}^d}$ , unless the explicit indication of  $\mathbb{R}^d$  is more convenient.

Finally, for two vectors  $a, b \in \mathbb{R}^d$ , we denote by  $a \otimes b \in \operatorname{Mat}_{d \times d}(\mathbb{R})$  the matrix given by  $(a \otimes b)_{ij} = a_i b_j$ . For a matrix  $A = A(x) \in \operatorname{Mat}_{d \times d}(\mathbb{R})$ , we define its divergence by  $(\operatorname{div} A)_i := \sum_j \partial_j A_{ji}$ .

## 2. PROOF OF THE MAIN THEOREMS

Let us start by discussing some elementary consequences of (1.9a)-(1.9b). These conditions imply that, for all  $t > 0$ ,

$$(2.1) \quad \Phi(t/2) \leq \frac{2}{t} \int_0^{t/2} \Phi(s) \, ds \leq \frac{2}{t} \int_0^t \Phi(s) \, ds \leq 2\kappa\Phi(t).$$

Moreover, for all  $t \geq 1$ ,

$$(2.2) \quad t\Phi(t) \geq \frac{1}{\kappa} \int_0^t \Phi(s) \, ds \geq \frac{1}{\kappa} \int_0^1 \Phi =: c^{-1} > 0.$$

Notice that the condition  $\Phi(0) > 0$  would not be sufficient to imply that  $c^{-1} > 0$ , so the stronger condition  $\Phi(0+) > 0$  is needed in (1.9a).

Another consequence of (1.9a)-(1.9b) is that

$$(2.3) \quad \lim_{t \rightarrow +\infty} t\Phi(t) = +\infty.$$

Indeed, otherwise there would exist a constant  $C > 0$  and a sequence  $t_n \rightarrow +\infty$  such that  $\int_0^{t_n} \Phi(s) \, ds \leq \kappa t_n \Phi(t_n) \leq C$ , and by the monotone convergence theorem we would obtain  $\Phi \in L^1(\mathbb{R}^+)$ . But then  $t\Phi(t) \geq \frac{1}{2\kappa} \int_0^\infty \Phi(s) \, ds$  for sufficiently large  $t$ , which in turn contradicts that  $\Phi \in L^1(\mathbb{R}^+)$  when  $\Phi \not\equiv 0$ .

The first ingredient is the following lemma.

**Lemma 2.1.** *Let  $u_0 \in L^2_\sigma(\mathbb{R}^2)$  and  $f$  satisfy (1.8a)-(1.8b). Let  $u$  be the Leray solution of the Navier-Stokes equations (1.1) with  $d = 2$ , starting from  $u_0$ . Then,*

$$\|\nabla u(t)\| = o(t^{-1/2}), \quad \text{as } t \rightarrow +\infty.$$

*Proof.* Let  $\sigma > 0$  and  $t_0 > 0$ . We multiply the first equation in the Navier-Stokes system (1.1) by  $2(s - t_0)^\sigma u$  and integrate the result over  $\mathbb{R}^2 \times [t_0, t]$ . After a few computations, we get

$$\begin{aligned} (t - t_0)^\sigma \|u(t)\|^2 + 2 \int_{t_0}^t (s - t_0)^\sigma \|\nabla u(s)\|^2 \, ds \\ \leq \sigma \int_{t_0}^t (s - t_0)^{\sigma-1} \|u(s)\|^2 \, ds + 2 \int_{t_0}^t (s - t_0)^\sigma \|u(s)\| \|f(s)\| \, ds. \end{aligned}$$

Let  $\epsilon > 0$ . According to Theorem 1.2,  $\|u(t)\| \rightarrow 0$  as  $t \rightarrow +\infty$  (recalling that assumption (1.8a) alone ensures this). From this fact and assumption (1.8a) on  $f$ , we have, for  $t_0$  sufficiently large and all  $t \geq t_0$ ,

$$\begin{aligned} (t - t_0)^\sigma \|u(t)\|^2 + 2 \int_{t_0}^t (s - t_0)^\sigma \|\nabla u(s)\|^2 \, ds &\leq \epsilon(t - t_0)^\sigma + 2\sqrt{\epsilon} \int_{t_0}^t (s - t_0)^\sigma \|f(s)\| \, ds \\ &\leq 2\epsilon(t - t_0)^\sigma. \end{aligned}$$

So in particular,

$$(2.4) \quad \int_{t_0}^t (s - t_0)^\sigma \|\nabla u(s)\|^2 \, ds \leq \epsilon(t - t_0)^\sigma.$$

Let us denote by  $\omega := \text{curl } u = \partial_1 u_2 - \partial_2 u_1$  the vorticity of  $u$ , which satisfies the equation

$$(2.5) \quad \partial_t \omega + u \cdot \nabla \omega = \Delta \omega + \text{curl } f.$$

Multiplying equation (2.5) by  $2(s - t_0)^{\sigma+1}\omega$ , we obtain, after space-time integration,

$$\begin{aligned} (t - t_0)^{1+\sigma} \|\omega(t)\|^2 + 2 \int_{t_0}^t (s - t_0)^{1+\sigma} \|\nabla\omega(s)\|^2 ds \\ = (1 + \sigma) \int_{t_0}^t (s - t_0)^\sigma \|\omega(s)\|^2 ds + 2 \int_{t_0}^t (s - t_0)^{1+\sigma} \int_{\mathbb{R}^2} (\operatorname{curl} f) \omega dx ds \\ \leq (1 + \sigma) \int_{t_0}^t (s - t_0)^\sigma \|\omega(s)\|^2 ds + 2 \int_{t_0}^t (s - t_0)^{1+\sigma} \|f(s)\| \|\nabla\omega(s)\| ds. \end{aligned}$$

Applying the Young inequality  $2ab \leq a^2 + b^2$  to the last integrand, and absorbing one of the two terms in the integral on the left-hand side, we deduce

$$\begin{aligned} (t - t_0)^{1+\sigma} \|\omega(t)\|^2 + \int_{t_0}^t (s - t_0)^{1+\sigma} \|\nabla\omega(s)\|^2 ds \\ \leq (1 + \sigma) \int_{t_0}^t (s - t_0)^\sigma \|\omega(s)\|^2 ds + \int_{t_0}^t (s - t_0)^{1+\sigma} \|f(s)\|^2 ds. \end{aligned}$$

But as  $\nabla \cdot u = 0$ , we have  $\|\omega(s)\| = \|\nabla u(s)\|$ , and we can apply (2.4) to gather

$$(t - t_0)^{1+\sigma} \|\omega(t)\|^2 + \int_{t_0}^t (s - t_0)^{1+\sigma} \|\nabla\omega(s)\|^2 ds \leq (1 + \sigma) \epsilon (t - t_0)^\sigma + \int_{t_0}^t (s - t_0)^{1+\sigma} \|f(s)\|^2 ds.$$

Now, in view of condition (1.8b), we obtain, by possibly increasing  $t_0$ ,

$$\int_{t_0}^t (s - t_0)^{1+\sigma} \|f(s)\|^2 ds \leq (t - t_0)^\sigma \int_{t_0}^t s \|f(s)\|^2 ds \leq \epsilon (t - t_0)^\sigma.$$

Therefore, for some constant  $C > 0$  and all  $t > t_0$ ,

$$\|\omega(t)\|^2 = \|\nabla u(t)\|^2 \leq C \epsilon (t - t_0)^{-1}.$$

So  $\|\nabla u(t)\| = o(t^{-1/2})$  as  $t \rightarrow +\infty$  as desired.  $\square$

*Remark 2.2.* It is worth noting that the above argument improves upon the result presented in [6] in the following sense: there, in order to establish the simplest case of algebraic decay for  $\nabla u$ , the authors assumed stronger conditions (see *e.g.* [6, Theorem 1.9]), for instance involving the first derivative of  $f$ . By contrast, our argument requires on  $f$  only to satisfy conditions (1.8a)-(1.8b). The previous argument can also be applied to obtain the following fact: if, for some  $\gamma > 0$ , we have  $\|u(t)\| = O(t^{-\gamma})$  and  $\|f(t)\| = O(t^{-1-\gamma})$  as  $t \rightarrow +\infty$ , then

$$(2.6) \quad \|\nabla u(t)\| = O(t^{-\gamma-1/2}), \text{ as } t \rightarrow +\infty.$$

In order to obtain (2.6), it suffices to multiply the first equation in the Navier–Stokes system (1.1) by  $2(s - t_0)^{\sigma+2\gamma}u$  instead of  $2(s - t_0)^\sigma u$ , and then to repeat the same steps performed in the proof of Lemma 2.1 above.

With the preceding results at hand, we are ready to prove the first conclusion of Theorem 1.3.

*Proof of Assertion i) of Theorem 1.3.* Let us write the Duhamel formula

$$u(t) = v(t) - \int_0^t e^{(t-s)\Delta} \mathbb{P}(u \cdot \nabla) u(s) \, ds.$$

As  $\nabla \cdot u = 0$ , we have  $(u \cdot \nabla)u = \nabla \cdot (u \otimes u)$ , hence we can decompose

$$u(t) = v(t) - \int_0^{t/2} e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (u \otimes u)(s) \, ds - \int_{t/2}^t e^{(t-s)\Delta} \mathbb{P}(u \cdot \nabla) u(s) \, ds.$$

Since  $\mathbb{P}$  is bounded in  $L^2$  and by the usual  $L^1$ - $L^2$  estimates for  $e^{t\Delta}$  and  $e^{t\Delta} \nabla$ , and using also that  $\|v(t)\| \lesssim \Phi(t)$ , we have, for some absolute constant  $C_0 > 0$  and all  $t > 0$ ,

$$(2.7) \quad \|u(t)\| \leq C_0 \left[ \Phi(t) + \int_0^{t/2} (t-s)^{-1} \|u(s)\|^2 \, ds + \int_{t/2}^t (t-s)^{-1/2} \|u(s)\| \|\nabla u(s)\| \, ds \right].$$

Assumptions (1.9a)-(1.9b) imply that  $\Phi > 0$ . Then, inspired by [8], we can introduce, for all  $t \geq 0$ , the quantities

$$E(t) := \frac{\|u(t)\|}{\Phi(t)} \quad \text{and} \quad E_{\max}(t) := \sup_{s \in [0, t]} E(s).$$

From (2.7), we have

$$(2.8) \quad E(t) \leq C_0(1 + I_1(t) + I_2(t)),$$

where

$$I_1(t) := \frac{1}{\Phi(t)} \int_0^{t/2} (t-s)^{-1} \|u(s)\|^2 \, ds \quad \text{and} \quad I_2(t) := \frac{1}{\Phi(t)} \int_{t/2}^t (t-s)^{-1/2} \|u(s)\| \|\nabla u(s)\| \, ds.$$

Notice that the integral defining  $I_2(t)$  is finite for every  $t > 0$  because  $\|\nabla u(s)\| = o(s^{-1/2})$  by Lemma 2.1 and  $\|u(s)\|$  is bounded. Let us first estimate  $I_1$ . From (1.3), we have, for all  $t \geq 0$ ,

$$\|u(t)\|^2 \leq \|u_0\|^2 + C \int_0^\infty \|f(\tau)\| \, d\tau =: M.$$

As previously observed, we have  $\|u(t)\| \rightarrow 0$  as  $t \rightarrow +\infty$ , hence there exists  $t_1 \geq 1$  such that  $\|u(s)\| \leq (8C_0\kappa)^{-1}$  for all  $s \geq t_1$ , where  $\kappa$  is given by (1.9b). Then, for all  $t \geq 2t_1$ ,

$$\begin{aligned} I_1(t) &= \frac{1}{\Phi(t)} \left[ \int_0^{t_1} (t-s)^{-1} \|u(s)\|^2 \, ds + \int_{t_1}^{t/2} (t-s)^{-1} \|u(s)\|^2 \, ds \right] \\ &\leq \frac{1}{\Phi(t)} \left[ Mt_1(t-t_1)^{-1} + \frac{2(8C_0\kappa)^{-1}}{t} \int_{t_1}^{t/2} \|u(s)\| \, ds \right] \\ &\leq \frac{1}{\Phi(t)} \left[ Mt_1(t-t_1)^{-1} + \frac{2(8C_0\kappa)^{-1}}{t} \int_{t_1}^t E(s)\Phi(s) \, ds \right]. \end{aligned}$$

So, applying (1.9b) and recalling (2.2),

$$\begin{aligned} I_1(t) &\leq \frac{2Mt_1}{t\Phi(t)} + \frac{2(8C_0\kappa)^{-1}}{t\Phi(t)} \left( \int_{t_1}^t \Phi(s) ds \right) E_{\max}(t) \\ &\leq \frac{2Mt_1}{t\Phi(t)} + 2\kappa(8C_0\kappa)^{-1} E_{\max}(t) \\ &\leq 2cMt_1 + \frac{E_{\max}(t)}{4C_0}. \end{aligned}$$

We now estimate  $I_2$ . By Lemma 2.1,  $s^{1/2}\|\nabla u(s)\| \rightarrow 0$  as  $s \rightarrow +\infty$ . Therefore,

$$(2.9) \quad \varepsilon(t) := \int_{t/2}^t (t-s)^{-1/2} \|\nabla u(s)\| ds \rightarrow 0, \quad \text{as } t \rightarrow +\infty.$$

We deduce that there exists  $t_2 \geq 2t_1$  such that, for all  $t \geq t_2$ ,

$$\varepsilon(t) \leq (8C_0\kappa)^{-1}.$$

Applying this and using (2.1), we have, for all  $t \geq t_2$ ,

$$\begin{aligned} I_2(t) &= \frac{1}{\Phi(t)} \int_{t/2}^t (t-s)^{-1/2} \|u(s)\| \|\nabla u(s)\| ds \\ &\leq \frac{1}{\Phi(t)} \int_{t/2}^t (t-s)^{-1/2} E(s)\Phi(s) \|\nabla u(s)\| ds \\ &\leq \frac{\Phi(t/2)}{\Phi(t)} E_{\max}(t) \int_{t/2}^t (t-s)^{-1/2} \|\nabla u(s)\| ds \\ &\leq 2\kappa E_{\max}(t) \varepsilon(t) \\ &\leq \frac{E_{\max}(t)}{4C_0}. \end{aligned}$$

Going back to (2.8), we deduce that

$$E(t) \leq C_0(1 + 2cMt_1) + \frac{E_{\max}(t)}{2}, \quad \forall t \geq t_2.$$

This implies

$$(2.10) \quad \|u(t)\| \leq 2C_0(1 + 2cMt_1)\Phi(t), \quad \forall t \geq t_2.$$

We now estimate the difference  $\theta(t) = u(t) - v(t)$ . For this purpose, observe that

$$\theta(t) = - \int_0^t e^{(t-s)\Delta} \mathbb{P}(u \cdot \nabla) u(s) ds,$$

hence

$$(2.11) \quad \|\theta(t)\| \leq C_0 \int_0^{t/2} (t-s)^{-1} \|u(s)\|^2 ds + C_0 \int_{t/2}^t (t-s)^{-1/2} \|u(s)\| \|\nabla u(s)\| ds.$$

Let us estimate the first integral. Using the decay property (2.10), we obtain, for some constant  $C > 0$  and all  $t > 0$ ,

$$(2.12) \quad \int_0^{t/2} (t-s)^{-1} \|u(s)\|^2 ds \leq 2t^{-1} \int_0^{t/2} \|u(s)\|^2 ds \leq Ct^{-1} \int_0^t \Phi(s)^2 ds.$$

Let  $M' := \Phi(0)$ . From assumption (1.9a), we have  $\Phi(t) \rightarrow 0$  as  $t \rightarrow +\infty$ , hence there exists  $t_3 \geq t_2$  such that  $\Phi(t) \leq 1$  for all  $t \geq t_3$ . Define  $\lambda(t) := t\Phi(t)$ . Then, for all  $t \geq t_3$ , we have  $0 < \lambda(t) \leq t$ , and

$$\begin{aligned} t^{-1} \int_0^t \Phi(s)^2 ds &\leq t^{-1} \int_0^{\lambda(t)} \Phi(s)^2 ds + t^{-1} \int_{\lambda(t)}^t \Phi(s)^2 ds \\ &\leq M' t^{-1} \frac{\lambda(t)}{\lambda(t)} \int_0^{\lambda(t)} \Phi(s) ds + t^{-1} \Phi(\lambda(t)) \int_{\lambda(t)}^t \Phi(s) ds. \end{aligned}$$

Applying the Cesàro condition (1.9a) at the argument  $\lambda(t)$  to the first average and at  $t$  to the second (after extending the integral to  $[0, t]$ ) gives

$$t^{-1} \int_0^t \Phi(s)^2 ds \leq M' \kappa t^{-1} \lambda(t) \Phi(\lambda(t)) + \kappa \Phi(t) \Phi(\lambda(t)) \leq \kappa(M' + 1) \Phi(t) \Phi(t\Phi(t)).$$

Let us now estimate the second integral in (2.11). From the decay (2.10), property (2.1) and the definition (2.9) of  $\varepsilon(t)$ , we have, for all  $t \geq t_3$ ,

$$(2.13) \quad \int_{t/2}^t (t-s)^{-1/2} \|u(s)\| \|\nabla u(s)\| ds \lesssim \Phi(t) \int_{t/2}^t (t-s)^{-1/2} \|\nabla u(s)\| ds \lesssim \Phi(t) \varepsilon(t).$$

We finally get

$$\|\theta(t)\| \lesssim \Phi(t) \Phi(t\Phi(t)) + \Phi(t) \varepsilon(t) = o(\Phi(t)), \quad \text{as } t \rightarrow +\infty,$$

because  $t\Phi(t) \rightarrow +\infty$  by (2.3), and  $\Phi(t) \rightarrow 0$  as  $t \rightarrow +\infty$ .  $\square$

Before proving the other assertions of Theorem 1.3, we need another lemma, which is a refinement of Lemma 2.1.

**Lemma 2.3.** *Let  $u_0 \in L^2_\sigma(\mathbb{R}^2)$  and  $f$  satisfy (1.8a) and (1.8c). Let  $u$  be the Leray solution of the Navier-Stokes equations (1.1) with  $d = 2$ , and  $v$  be the solution of the heat equation (1.5), starting from  $u_0$ . Let  $\Phi$  satisfy conditions (1.9a)-(1.9b), and be such that*

$$\|v(t)\| = O(\Phi(t)), \quad \text{as } t \rightarrow +\infty.$$

Then,

$$\|\nabla u(t)\| = O(t^{-1/2} \Phi(t)), \quad \text{as } t \rightarrow +\infty.$$

*Proof.* We use once more the vorticity equation (2.5): multiplying it by  $2(s - \frac{t}{2}) \omega(y, s)$  and integrating with respect to  $(y, s) \in \mathbb{R}^2 \times [t/2, t]$ , we get

$$\frac{t}{2} \|\omega(t)\|^2 + 2 \int_{t/2}^t (s - \frac{t}{2}) \|\nabla \omega(s)\|^2 ds = \int_{t/2}^t \|\omega(s)\|^2 ds + 2 \int_{t/2}^t \int (s - \frac{t}{2}) \text{curl } f(\cdot, s) \omega(\cdot, s) ds.$$

From this, we deduce

$$\frac{t}{2} \|\omega(t)\|^2 + 2 \int_{t/2}^t (s - \frac{t}{2}) \|\nabla \omega(s)\|^2 ds \leq \int_{t/2}^t \|\omega(s)\|^2 ds + 2 \int_{t/2}^t (s - \frac{t}{2}) \|f(s)\| \|\nabla \omega(s)\| ds.$$

Hence, applying the Young inequality in the last integrand, we gather

$$\frac{t}{2} \|\omega(t)\|^2 + \int_{t/2}^t (s - \frac{t}{2}) \|\nabla \omega(s)\|^2 ds \leq \int_{t/2}^t \|\omega(s)\|^2 ds + \int_{t/2}^t (s - \frac{t}{2}) \|f(s)\|^2 ds.$$

Recalling that  $\|\omega\| = \|\nabla u\|$  and using the energy inequality (1.2), we obtain

$$\begin{aligned} \frac{t}{2} \|\nabla u(t)\|^2 &\leq \int_{t/2}^t \|\nabla u(s)\|^2 ds + \int_{t/2}^t (s - \frac{t}{2}) \|f(s)\|^2 ds \\ &\leq \|u(t/2)\|^2 + 2 \int_{t/2}^t \|u(s)\| \|f(s)\| ds + \int_{t/2}^t (s - \frac{t}{2}) \|f(s)\|^2 ds. \end{aligned}$$

Now, the conditions (1.8a) and (1.8c) on  $f$  imply (1.8b) by interpolation. Therefore, Conclusion i) of Theorem 1.3 applies, and we deduce that  $\|u(t)\| \lesssim \Phi(t)$ . From this, condition (1.8c) and (2.1), we deduce that  $\frac{t}{2} \|\nabla u(t)\|^2 \lesssim \Phi(t)^2$ , for all  $t > 0$ . This establishes Lemma 2.3.  $\square$

The remaining parts of Theorem 1.3 are now easily established.

*Proof of Assertions ii) and iii) of Theorem 1.3.* As observed in the proof of Lemma 2.3, our assumptions imply the existence of some constant  $C > 0$  such that  $\|u(t)\| \leq C\Phi(t)$ , for all  $t \geq 0$ .

We go back to inequalities (2.11)-(2.13) for  $\theta(t) = u(t) - v(t)$  and deduce

$$\|\theta(t)\| \leq Ct^{-1} \int_0^t \Phi(s)^2 ds + C\Phi(t) \int_{t/2}^t (t-s)^{-1/2} \|\nabla u(s)\| ds.$$

The second term is treated by applying Lemma 2.3 and (2.1), that gives

$$(2.14) \quad \|\theta(t)\| \lesssim t^{-1} \int_0^t \Phi(s)^2 ds + \Phi(t)^2 \lesssim t^{-1} \int_0^t \Phi(s)^2 ds,$$

where we used the monotonicity of  $\Phi$  in the last inequality. Indeed,  $(t/2)\Phi(t)^2 \leq \int_{t/2}^t \Phi^2 \leq \int_0^t \Phi^2$ . For estimating the right-hand side, we distinguish two cases. Under the assumptions of Part ii) of Theorem 1.3, we can directly apply condition (1.9c) and conclude that  $\|\theta(t)\| \lesssim \Phi(t)^2$ . Otherwise, under the assumptions of Part iii) of Theorem 1.3, we have  $\Phi \in L^2(\mathbb{R}^+)$ , and we readily get  $\|\theta(t)\| \lesssim t^{-1}$ .  $\square$

*Remark 2.4.* Let us discuss further the conditions that we put on  $\Phi$ . Estimate (2.14) holds assuming only (1.9a)-(1.9b). Condition (1.9c) is mainly useful to state Theorem 1.3 in a more attractive way, but such an assumption could be simply dropped by replacing Conclusion ii) with the conclusion

$$(2.15) \quad \|u(t) - v(t)\| = O\left(\frac{1}{t} \int_0^t \Phi(s)^2 ds\right) \quad \text{as } t \rightarrow +\infty.$$

In the case of algebraic decays, i.e., when  $\Phi(t) = C(1+t)^{-\alpha}$ , Wiegner's theorem, for  $d = 2$ , matches Conclusion ii) of Theorem 1.3 when  $0 < \alpha < 1/2$  and Conclusion iii) when  $1/2 < \alpha$ . On the other hand, the assertion of Wiegner's theorem in the borderline case  $\alpha = 1/2$  is not covered by Theorem 1.3 the way it is stated. However, we can recover Wiegner's conclusion also when  $\alpha = 1/2$  directly from (2.15), that provides the same logarithmic correction in the decay rate as in (1.7).

The Cesàro type condition (1.9b) plays a deeper role in our arguments. There is no evidence that such a condition is *necessary* for the validity of Conclusion i) of Theorem 1.3, but we do not expect (1.9b) can be relaxed considerably. For example, a weaker condition like  $t\Phi(t) \rightarrow +\infty$ , instead of (1.9b), does not seem to be enough. Indeed, this larger class would encompass decreasing functions  $\Phi$  “fluctuating” between two positive decreasing profiles  $\Phi_1$  and  $\Phi_2$ , in the sense that  $\Phi_1(t) \leq \Phi(t) \leq \Phi_2(t)$  for all nonnegative  $t$ , with  $\Phi(t_k) = \Phi_1(t_k)$  and  $\Phi(t'_k) = \Phi_2(t'_k)$  for some sequences  $(t_k)$  and  $(t'_k)$  diverging to  $+\infty$ . We could also take  $\Phi_1$  and  $\Phi_2$  such that  $\Phi_1(t) = O(\Phi_2(t)^2)$  as  $t \rightarrow +\infty$ . In such a situation, one can hope for a bound of the form  $\|u(t) - v(t)\| = O(\Phi_2(t)^2)$ , for example if  $\Phi_2$  satisfies (1.9c); but because of the time average that genuinely appears when one writes  $u(t) - v(t)$  using Duhamel formula, it seems impossible to get the stronger bound  $\|u(t) - v(t)\| = o(\Phi_1(t))$  and so the bound  $\|u(t) - v(t)\| = o(\Phi(t))$ , as  $t \rightarrow +\infty$ . For this reason, Conclusion i) of Theorem 1.3, does not seem to be valid in such a more general setting. An explicit construction of an initial data such that the solution of the heat equation  $v$  decays with similar fluctuations in the  $L^2$ -norm, is provided in [2, Example 3.1].

*Remark 2.5.* Let us point out that, once the bound  $\|u(t)\| = O(\Phi(t))$  is available, the estimate for  $\|\theta(t)\|$  in the proof of Assertion i) of Theorem 1.3 relies solely on this bound and on the decay of  $\|\nabla u(t)\|$  without invoking the hypothesis on  $\|v(t)\|$ . Hence the implication  $\|u(t)\| = O(\Phi(t)) \implies \|\theta(t)\| = o(\Phi(t))$  is self-contained. This fact is key to establish the following “inverse form” of Theorem 1.3, in the same spirit as [3] (see also [13]), by using the triangle inequality.

**Corollary 2.6.** *Let  $u_0 \in L^2_\sigma(\mathbb{R}^2)$  and  $f$  satisfy (1.8a)-(1.8b). Let  $u$  be the Leray solution of the Navier-Stokes equations (1.1) with  $d = 2$ , and  $v$  be the solution of the heat equation (1.5), starting from  $u_0$ . Suppose that*

$$\|u(t)\| = O(\Phi(t)), \quad \text{as } t \rightarrow +\infty,$$

*with  $\Phi$  satisfying conditions (1.9a)-(1.9b). Then,  $\|v(t)\| = O(\Phi(t))$  as  $t \rightarrow +\infty$ .*

An immediate consequence is the following equivalence between the two-sided estimates: under the hypotheses of Corollary 2.6,

$$\|u(t)\| \approx \Phi(t) \iff \|v(t)\| \approx \Phi(t).$$

#### APPENDIX A. AN ILLUSTRATION OF THEOREM 1.3

In this Appendix, we provide an example motivated by the discussion following the statement of Theorem 1.3.

*Example A.1.* Let  $\chi \in C_c^\infty(\mathbb{R}^2)$  be a non-radial function, identically equal to 1 in  $\{\xi \in \mathbb{R}^2: |\xi| \leq 1/4\}$ , supported in the disc  $\{\xi \in \mathbb{R}^2: |\xi| \leq 1/2\}$ . Let  $\widehat{\omega}_0(\xi) = |\ln |\xi||^{-1} \chi(\xi)$ . Let  $\widehat{u}_0(\xi) = |\xi|^{-2} (i\xi_2, -i\xi_1) \widehat{\omega}_0(\xi)$ . Then,  $u_0 \in L_\sigma^2(\mathbb{R}^2)$ , and  $\omega_0$  is the corresponding initial vorticity, namely  $\omega_0 = \text{curl } u_0$ . We choose  $f \equiv 0$ . The non-radiality of  $\chi$  is just useful to avoid the somewhat trivial case  $\mathbb{P}(u \cdot \nabla \omega) \equiv 0$ , corresponding to the situation where the solution of the Navier–Stokes equations boils down to the solution of the heat equation. Let us prove that

$$(A.1) \quad \|v(t)\| \approx (\ln(e+t))^{-1/2}.$$

We apply Schonbek’s Fourier-Splitting method [12]. Performing an energy estimate on the heat equation (1.5) yields

$$\frac{1}{2} \frac{d}{dt} \|v(t)\|^2 + \frac{1}{t} \int_{|\xi| \geq (\sqrt{t})^{-1}} |\widehat{v}(\xi, t)|^2 d\xi \leq - \int_{|\xi| \leq (\sqrt{t})^{-1}} |\xi|^2 |\widehat{v}(\xi, t)|^2 d\xi.$$

From this, we deduce

$$(A.2) \quad \frac{1}{2} \frac{d}{dt} \|v(t)\|^2 + \frac{1}{t} \|v(t)\|^2 \leq \frac{1}{t} \int_{|\xi| \leq (\sqrt{t})^{-1}} |\widehat{v}(\xi, t)|^2 d\xi.$$

But for  $t \geq 16$ ,

$$\begin{aligned} \int_{|\xi| \leq (\sqrt{t})^{-1}} |\widehat{v}(\xi, t)|^2 d\xi &= \int_{|\xi| \leq (\sqrt{t})^{-1}} e^{-2t|\xi|^2} |\xi|^{-2} |\ln |\xi||^{-2} d\xi \\ &\approx \int_{(\sqrt{t})^{-1}}^{+\infty} \rho^{-1} |\ln \rho|^{-2} d\rho \\ &\approx (\ln t)^{-1}. \end{aligned}$$

So in particular, for all  $t \geq 2$ ,  $\|v(t)\| \gtrsim (\ln t)^{-1/2}$ . But multiplying (A.2) by  $t$  and integrating over  $[2, t]$ , we get, for all  $t \geq 2$ ,

$$t \|v(t)\|^2 \lesssim \int_e^t (\ln s)^{-1} ds \lesssim t (\ln t)^{-1}.$$

These calculations imply (A.1) and the application of Theorem 1.3 shows that  $\|u(t)\|$  has the same large-time behavior.

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