



HAL
open science

Proof Systems Supporting Qualitative Reasoning

Daniel Crowley, Yakoub Salhi

► **To cite this version:**

Daniel Crowley, Yakoub Salhi. Proof Systems Supporting Qualitative Reasoning. ACM/SIGAPP Symposium on Applied Computing, Mar 2026, Thessaloniki, Greece. pp.980-982, <10.1145/3748522.3779770>. <hal-05590531>

HAL Id: hal-05590531

<https://hal.science/hal-05590531v1>

Submitted on 13 Apr 2026

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons CC BY 4.0 - Attribution - International License

Proof Systems Supporting Qualitative Reasoning

Daniel Crowley and Yakoub Salhi

{crowley,salhi}@cril.fr

Univ. Artois, CNRS, UMR 8188, Centre de Recherche en Informatique de Lens (CRIL)

Lens, France

Abstract

Integrating classical propositional logic with qualitative spatial and temporal calculi yields an interesting framework for representing and reasoning about complex systems. Automated verification in this enriched setting requires proof systems that extend traditional propositional techniques with specialized spatial and temporal inference mechanisms. Here we first present a resolution proof system for establishing the unsatisfiability of formulas expressed in a normal form analogous to conjunctive normal form (CNF). This system augments the classical resolution rule with additional inference rules tailored to qualitative reasoning. We then introduce a tableau proof system similarly extended to qualitative contexts. Finally, we generalize our framework to support reasoning across multiple qualitative calculi.

CCS Concepts

• **Theory of computation** → **Automated reasoning**.

Keywords

Structural proof systems, Qualitative reasoning, Knowledge representation, AI foundations

ACM Reference Format:

Daniel Crowley and Yakoub Salhi. 2026. Proof Systems Supporting Qualitative Reasoning. In *The 41st ACM/SIGAPP Symposium on Applied Computing (SAC '26)*, March 23–27, 2026, Thessaloniki, Greece. ACM, New York, NY, USA, 3 pages. <https://doi.org/10.1145/3748522.3779770>

1 Introduction

Qualitative spatial and temporal reasoning provides symbolic calculi for modeling relational aspects of time and space [7]. Well-known examples include Allen’s Interval Algebra [1] and the Region Connection Calculus, in particular RCC5 and RCC8 [10].

Propositional logic remains a simple but powerful formalism, supported by highly optimized SAT solvers [3]. Extending propositional logic while preserving access to SAT-style reasoning is one of the motivations behind Satisfiability Modulo Theories (SMT) [2]. In a similar spirit, we investigate propositional logics enriched with qualitative spatial/temporal constraints.

Our aim is to provide proof systems for automated verification tasks such as satisfiability and validity checking and counter-model generation in a propositional language enriched with qualitative constraints. The main contributions are: (1) a resolution proof system \mathcal{RS}_q extending classical resolution [11] with qualitative rules

and complete for calculi satisfying a standard path-consistency property; (2) a tableau proof system \mathcal{TS}_q that extends classical propositional tableaux [13] with qualitative rules; (3) an extension of both proof systems to combinations of qualitative calculi.

The closest related work is a tableau system for a logic combining linear temporal logic with RCC8 [5, 12].

2 Motivation

Integrating classical propositional logic with qualitative spatio-temporal calculi yields a versatile specification language. Propositional variables describe high-level modes, events or resources, while qualitative constraints $(x r y)$ describe spatial or temporal relations between entities.

This combination supports, in particular:

- *Conditional qualitative constraints*: formulas of the form $\phi \rightarrow (x r y)$, where ϕ is propositional, express that only in certain contexts must a qualitative relation hold.
- *Interdependencies between qualitative relations*: implications between constraints, such as $((x r y) \rightarrow (x' r' y')) \rightarrow (x'' r'' y'')$, capture higher-level rules.
- *Mixed discrete/qualitative requirements*: real systems often combine discrete decisions (modes, on/off choices) with relational spatial or temporal conditions (containment, adjacency, precedence).

For verification, we need to detect unsatisfiability and, ideally, produce either machine-checkable proofs or explicit counterexamples. Our resolution proof system extends propositional resolution with qualitative inference to obtain refutations, while the tableau system incrementally constructs and prunes candidate scenarios, yielding either models or structured explanations of inconsistency.

3 Background

We recall basic notions of qualitative spatial and temporal reasoning (QSTR) [7].

A qualitative calculus \mathcal{F} is defined over an infinite domain D and a finite set B of jointly exhaustive and pairwise disjoint base relations. In the case of RCC5, the set of base relations is $\{DR, PP, PPI, PO, EQ\}$ [10]. The identity relation Id (e.g. EQ) belongs to B , and B is closed under converse $(\cdot)^{-1}$.

Indefinite information is represented by subsets $r \subseteq B$. We write $(x r y)$ for a constraint over qualitative variables x, y and a relation r . A constraint is *atomic* if $|r| = 1$.

Besides union and intersection, we use the converse $r^{-1} = \cup\{b^{-1} \mid b \in r\}$ and *weak composition* $r \diamond r'$ [7], defined from true composition (\circ) of base relations: $r \diamond r' = \cup\{b \circ b' \mid b \in r, b' \in r'\}$, where $b \circ b' = \{b'' \in B \mid b'' \cap (b \circ b') \neq \emptyset\}$.

Let S be a finite set of qualitative constraints over a finite set V of variables. For each ordered pair $(x, y) \in V \times V$, we write $S(x, y)$ for



This work is licensed under a Creative Commons Attribution 4.0 International License. [SAC'26, Thessaloniki, Greece](https://creativecommons.org/licenses/by/4.0/)

© 2026 Copyright held by the owner/author(s).

ACM ISBN 979-8-4007-2294-3/2026/03

<https://doi.org/10.1145/3748522.3779770>

the relation associated with (x, y) (defaulting to B when no explicit constraint on (x, y) is present, and to $\{Id\}$ when $x = y$). If several constraints on the same pair appear, we replace them by a single constraint whose relation is the intersection of the originals, so we may assume that there is at most one constraint per pair.

Definition 3.1. A set S of qualitative constraints is *path consistent* if for all $x, y, z \in V$ we have $S(x, y) \subseteq S(x, z) \diamond S(z, y)$.

The *path-consistency closure* $PC(S)$ of S is obtained by iteratively tightening constraints so that the above inclusion holds for all triples x, y, z . This closure can be computed in $O(|V|^3)$ time [9, 15]. If for some x, y we obtain $S(x, y) = \emptyset$, we say that S is *trivially inconsistent*.

We work with calculi satisfying the following property [8]:

Property RPC. For every set S of atomic constraints, S is inconsistent if and only if $PC(S)$ is trivially inconsistent, i.e., there exist variables x, y with $PC(S)(x, y) = \emptyset$.

4 Combining Propositional Logic and QSTR

We extend propositional logic by allowing qualitative constraints as atomic formulas.

Let Prop be a finite set of propositional variables and QVar a finite set of qualitative variables. Formulas of $\text{PL}_{\mathcal{F}}$ are given by:

$$\Phi ::= p \mid (x r y) \mid \Phi \wedge \Phi \mid \Phi \vee \Phi \mid \neg \Phi,$$

where $p \in \text{Prop}$, $x, y \in \text{QVar}$ and $r \subseteq B$.

An interpretation is a pair $\langle I, \sigma \rangle$, where I maps each $p \in \text{Prop}$ to 0 or 1, and $\sigma : \text{QVar} \rightarrow D$ assigns domain elements to qualitative variables. Satisfaction is defined in the obvious way for propositional connectives and by

$$\langle I, \sigma \rangle \models (x r y) \quad \text{iff} \quad \exists b \in r : (\sigma(x), \sigma(y)) \in b.$$

Normal form. For our resolution method we use a CNF-like normal form, QCNF, consisting of finite conjunctions (sets) of q-clauses, which are finite disjunctions (sets) of q-literals. A q-literal is (1) a propositional variable p , (2) a qualitative constraint $(x r y)$, or (3) the negation of 1 or 2.

Two $\text{PL}_{\mathcal{F}}$ formulas are *equi-satisfiable* if either both admit a model or both do not. Every formula can be converted into an equi-satisfiable QCNF formula of linear size by a Tseitin-style transformation [14], introducing fresh variables q_{ψ} for subformulas ψ and clauses enforcing $q_{\psi} \leftrightarrow \psi$.

5 A Resolution Method for $\text{PL}_{\mathcal{F}}$

We now present a resolution proof system \mathcal{RS}_q for showing the unsatisfiability of QCNF formulas. It extends classical propositional resolution [11] with inference rules for qualitative constraints.

The rules of \mathcal{RS}_q are given in Figure 1. Rule $[E]$ removes trivially false literals $(x \emptyset y)$. Rule $[N]$ replaces a negated constraint $\neg(x r y)$ with the positive constraint $(x (B \setminus r) y)$. Rule $[D]$ decomposes non-atomic q-literals, $[I]$ exploits converses, $[J]$ tightens constraints via intersection, and $[WC]$ propagates information via weak composition. Rule $[R]$ is standard propositional resolution on a pair of complementary q-literals.

$$\begin{array}{c} \frac{\alpha \cup \{(x \emptyset y)\}}{\alpha} [E] \qquad \frac{\alpha \cup \{\neg(x r y)\}}{\alpha \cup \{(x (B \setminus r) y)\}} [N] \\ \frac{\alpha \cup \{(x (r \cup r') y)\}}{\alpha \cup \{(x r y)\} \cup \{(x r' y)\}} [D] \qquad \frac{\alpha \cup \{(x r y)\}}{\alpha \cup \{(y r^{-1} x)\}} [I] \\ \frac{\alpha \cup \{I\} \quad \beta \cup \{\bar{I}\}}{\alpha \cup \beta} [R] \\ \frac{\alpha \cup \{(x r y)\} \quad \beta \cup \{(x r' y)\}}{\alpha \cup \beta \cup \{(x (r \cap r') y)\}} [J] \\ \frac{\alpha \cup \{(x r y)\} \quad \beta \cup \{(y r' z)\}}{\alpha \cup \beta \cup \{(x (r \diamond r') z)\}} [WC] \end{array}$$

Figure 1: Resolution Proof System \mathcal{RS}_q .

A *refutation* of a QCNF formula Φ in \mathcal{RS}_q is a derivation of the empty q-clause \emptyset from

$$\Phi \cup \bigcup_{\substack{x, y \in V \\ x \neq y}} \{(x B y)\} \cup \bigcup_{x \in V} \{(x \{Id\} x)\},$$

where V is the set of qualitative variables in Φ .

For instance, a refutation of $(\neg p \vee (x \{PP\} y)) \wedge p \wedge (y \{PP\} z) \wedge (x \{PP\} y)$ in PL_{RCC5} can be obtained as follows:

$$\frac{\frac{\neg p \vee (x \{PP\} y) \quad p}{(x \{PP\} y)} [R] \quad (y \{PP\} z)}{(x \{PP\} z)} [WC] \quad (x \{EQ\} z)}{\frac{(x \emptyset z)}{\emptyset} [E]} [J]$$

THEOREM 5.1 (SOUNDNESS). *If Φ admits a refutation in \mathcal{RS}_q , then Φ is unsatisfiable.*

THEOREM 5.2 (COMPLETENESS). *Let Φ be a QCNF formula and \mathcal{F} a calculus satisfying Property RPC. If Φ is unsatisfiable, then there is a refutation of Φ in \mathcal{RS}_q .*

The completeness proof has two main steps. First, we treat the base case where all qualitative literals are atomic and unnegated. In this situation, the qualitative rules of \mathcal{RS}_q simulate path consistency on the underlying set of atomic constraints, and Property RPC ensures that any inconsistency eventually shows up as a trivial constraint $(x \emptyset y)$, from which a refutation is obtained using rule $[E]$. Second, for general QCNF formulas we argue by induction on a simple notion of complexity of qualitative literals: literals are more complex when they mention several base relations or appear under negation. Rules $[N]$ and $[D]$ are designed so that they systematically replace more complex literals by simpler ones, until only atomic (and non-negated) constraints remain. Applying the first step to this atomic situation yields a refutation, so every unsatisfiable QCNF admits a derivation of the empty clause in \mathcal{RS}_q .

Extended resolution (ER) [4, 14] can be integrated into \mathcal{RS}_q . Fresh propositional variables can abbreviate qualitative literals, e.g. $q_{xy} \leftrightarrow (x r y)$, or derived constraints, e.g. $q_{xz} \leftrightarrow (x (r \diamond r') z)$. In the latter case we may also add $(q_{xy} \wedge q_{yz}) \rightarrow q_{xz}$ to record qualitative dependencies. This avoids recomputing the same compositions in many places and can yield shorter refutations.

6 A Tableau Method for $PL_{\mathcal{F}}$

We now introduce a tableau proof system \mathcal{TS}_q for $PL_{\mathcal{F}}$, extending classical propositional tableaux [13] with qualitative rules.

A *signed formula* is an expression $S : \Phi$ where $S \in \{T, F\}$ and Φ is a $PL_{\mathcal{F}}$ formula. We interpret $T : \Phi$ as “ Φ is true” and $F : \Phi$ as “ Φ is false”. An interpretation $\langle I, \sigma \rangle$ satisfies $T : \Phi$ iff it satisfies Φ , and satisfies $F : \Phi$ iff it does not satisfy Φ .

The rules of \mathcal{TS}_q are shown in Figure 2. In addition to the usual rules for \wedge , \vee , and \neg , we have rules for qualitative literals: $[D_T]$ splits relation sets using union, $[C_T]$ handles complements, $[I_T]$ uses converses, $[J_T]$ intersects compatible constraints, and $[WC_T]$ propagates constraints via weak composition.

$\frac{T : \Phi \wedge \Psi}{\frac{T : \Phi}{T : \Psi}} \quad [\wedge_T]$	$\frac{F : \Phi \wedge \Psi}{F : \Phi \mid F : \Psi} \quad [\wedge_F]$
$\frac{F : \Phi \vee \Psi}{\frac{F : \Phi}{F : \Psi}} \quad [\vee_F]$	$\frac{T : \Phi \vee \Psi}{T : \Phi \mid T : \Psi} \quad [\vee_T]$
$\frac{T : \neg \Phi}{F : \Phi} \quad [\neg_T]$	$\frac{F : \neg \Phi}{T : \Phi} \quad [\neg_F]$
$\frac{T : (x (r \cup r') y)}{T : (x r y) \mid T : (x r' y)} \quad [D_T]$	$\frac{F : (x (B \setminus r) y)}{T : (x r y)} \quad [C_T]$
$\frac{T : (x r y)}{T : (y r^{-1} x)} \quad [I_T]$	$\frac{T : (x r y) \quad T : (x r' y)}{T : (x (r \cap r') z)} \quad [J_T]$
$\frac{T : (x r y) \quad T : (y r' z)}{T : (x (r \circ r') z)} \quad [WC_T]$	

Figure 2: Tableau Proof System \mathcal{TS}_q for $PL_{\mathcal{F}}$.

A branch is *closed* if it contains a pair of conjugate signed formulas ($T : \Phi$ and $F : \Phi$) or a literal $T : (x \emptyset y)$. Otherwise it is *open*. We say that a tableau derivation is *completed* if, for every open branch, no additional signed formula can be added; that is, no further rule instance is applicable to any open branch.

THEOREM 6.1 (SOUNDNESS). *If there is a closed tableau for $F : \Phi$ in \mathcal{TS}_q , then Φ is valid.*

THEOREM 6.2 (COMPLETENESS). *If a signed formula $X : \Phi$ is inconsistent, then every completed tableau derivation for $\{S : \Phi\} \cup \{T : (x \{Id\} x) : x \in V\} \cup \{T : (x B y) : x, y \in V, x \neq y\}$ must be closed, where V is the set of qualitative variables occurring in Φ .*

The completeness proof uses a standard Hintikka-style argument: each open completed branch forms a Hintikka set, from which we construct a satisfying model using Property RPC for the qualitative part.

7 Combinations of QSTR Calculi

We finally consider combinations of several calculi.

Disjoint variable sets. Let $\Sigma = \{\mathcal{F}_1, \dots, \mathcal{F}_k\}$ be calculi with disjoint variable sets. The language PL_{Σ} allows atoms from any \mathcal{F}_i , and interpretations are tuples $\langle I, \sigma_1, \dots, \sigma_k \rangle$. Because constraints from different calculi never share variables, our resolution and tableau systems lift directly: qualitative rules apply componentwise to each calculus.

Shared variable sets and compatibility. More interesting is the case where all calculi share the same qualitative variables, representing orthogonal aspects (e.g., topological relations with relative size [6]). For each pair (i, j) we assume a compatibility function $f_{ij} : B_i \rightarrow 2^{B_j}$ mapping each base relation of \mathcal{F}_i to the set of \mathcal{F}_j -relations it permits. These functions must preserve satisfiability: any model of a set of \mathcal{F}_i -constraints induces compatible \mathcal{F}_j -constraints via f_{ij} .

We work with *quasi-models*, consisting of: (1) a propositional assignment I , (2) one complete atomic scenario \mathcal{S}_i per calculus \mathcal{F}_i , (3) a compatibility condition correlating \mathcal{S}_i and \mathcal{S}_j via f_{ij} .

A formula is satisfiable iff it is satisfied in a quasi-model, a condition met for example by the combination of RCC8 and Point Algebra [6].

To integrate compatibility into proof search we add simple rules, for each (i, j) :

$$\frac{\alpha \cup (x r y)}{\alpha \cup (x (\bigcup_{b \in r} f_{ij}(b)) y)} \quad [CR_{ij}^R], \quad \frac{T : (x r y)}{T : (x (\bigcup_{b \in r} f_{ij}(b)) y)} \quad [CR_{ij}^T].$$

Soundness follows from the soundness of the base systems together with the preservation property of the compatibility functions f_{ij} , and completeness holds because the additional cross-calculus rules ensure that every inconsistency arising from the interaction between calculi is made visible during proof search.

Acknowledgments

This work was funded by the project *Enhancing B Language Reasoners Using SAT and SMT (BLaSST)*, ANR-21-CE25-0010.

References

- [1] James F. Allen. 1983. Maintaining Knowledge about Temporal Intervals. *Commun. ACM*, 26, 832–843.
- [2] Armin Biere, Marijn Heule, Hans van Maaren, and Toby Walsh, (Eds.) 2021. *Handbook of Satisfiability. Frontiers in Artificial Intelligence and Applications*. Vol. 336. IOS Press.
- [3] Ronald Brachman and Hector Levesque. 2004. *Knowledge Representation and Reasoning*. Morgan Kaufmann Publishers Inc., San Francisco, CA, USA.
- [4] Stephen A. Cook. 1976. A Short Proof of the Pigeon Hole Principle using Extended Resolution. *SIGACT News*, 8, 4, (Oct. 1976), 28–32.
- [5] David Gabelaia, Roman Kontchakov, Agi Kurucz, Frank Wolter, and Michael Zakharyashev. 2003. On the Computational Complexity of Spatio-Temporal Logics. In *FLAIRS, USA*. AAAI Press, 460–464.
- [6] Alfonso Gerevini and Jochen Renz. 2002. Combining Topological and Size Information for Spatial Reasoning. *Artif. Intell.*, 137, 1–42.
- [7] Gérard Ligozat. 2013. *Qualitative Spatial and Temporal Reasoning. Iste Series*. Wiley.
- [8] Zhiguo Long and Sanjiang Li. 2015. On Distributive Subalgebras of Qualitative Spatial and Temporal Calculi. In *COSIT*.
- [9] Alan K. Mackworth and Eugene C. Freuder. 1985. The Complexity of Some Polynomial Network Consistency Algorithms for Constraint Satisfaction Problems. *Artif. Intell.*, 25, 65–74.
- [10] David A. Randell, Zhan Cui, and Anthony Cohn. 1992. A Spatial Logic Based on Regions and Connection. In *KR*.
- [11] J. A. Robinson. 1965. A Machine-Oriented Logic Based on the Resolution Principle. *Journal ACM*, 12, 1, (Jan. 1965), 23–41.
- [12] Michael Sioutis, Jean-François Condotta, Yakoub Salhi, and Bertrand Mazure. 2015. Generalized Qualitative Spatio-Temporal Reasoning: Complexity and Tableau Method. In *TABLEAUX*.
- [13] Raymond M. Smullyan. 1968. *First-Order Logic*. Springer Verlag, New York.
- [14] Grigori S Tseitin. 1983. On the Complexity of Derivation in Propositional Calculus. *Automation of reasoning: 2: Classical papers on computational logic 1967–1970*, 466–483.
- [15] Peter van Beek. 1992. Reasoning about Qualitative Temporal Information. *Artif. Intell.*, 58, 297–326.