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CONVEX FUNCTIONALS OF NONLOCAL SINGULAR GRADIENTS AND REACTION-DIFFUSION PROBLEMS I: COMPARISON PRINCIPLES AND BOUNDED SOLUTIONS

OMAR ANZA HAFSA, JEAN-PHILIPPE MANDALLENA, AND GÉRARD MICHAILLE

ABSTRACT. We study nonlocal reaction-diffusion evolution equations of gradient flow type driven by singular nonlocal gradients and whose reaction term is CP-structured, a notion introduced in [AHMM19]. We characterise the subdifferentials of the associated nonlocal diffusion functionals, establish comparison principles, and prove the existence and uniqueness of bounded solutions. Applications to nonlocal cable theory and to the nonlocal dynamics of fake-news belief are given. This is the first part of a series of two papers: the second part will be devoted to convergence and stochastic homogenization.

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1. INTRODUCTION

Nonlocal evolution equations arise naturally in the modelling of phenomena involving finite-range interactions. Such phenomena appear in various contexts, including physics, mechanics, biology, and sociology. In the modelling of these problems, the classical local diffusion is replaced by a nonlocal one that accounts for interactions at a distance. This leads to nonlocal evolution equations.

The mathematical study of nonlocal evolution equations in the spirit of the present paper was initiated, without reaction term, by Andreu, Mazón, Rossi, and Toledo in [AMRT08, AMRT09] (see also [AVMRTM10] and the references therein). An important class of nonlocal evolution equations is that of gradient flows, in which the diffusion term is the subdifferential of a convex energy functional. This is what is studied in this paper, with the additional feature that the convex energy functional is built from a singular nonlocal gradient and that the reaction term is CP-structured, a notion introduced in [AHMM19] (see Definition 3.7). In the gradient flow framework, we also refer to the works of Braides and Piatnitski [BP22, BP21], of Piatnitski and Zhizhina [PZ20, PZ17] and to the book [AAB⁺23] (in particular [AAB⁺23, Chapter 9], which is devoted to parabolic problems).

The present paper is the first part of a series of two papers in which we continue our study of nonlocal evolution equations of gradient flow type developed in [AHMM24, AHMM26b, AHMM26a, AHMMP]. The main novelty with respect to these earlier works is that we consider nonlocal diffusion based on convex energy integrals whose density depends on singular nonlocal gradients (see §2.1–2.2). In this first part, we establish comparison principles and prove the existence of bounded solutions (see Section 3). The existence of bounded solutions is essential in many applications where the unknown represents a quantity that is intrinsically bounded: a proportion, a concentration, a voltage. The second part will be devoted to convergence and stochastic homogenization.

Let $O \subset \mathbb{R}^d$ be a bounded open set, representing the medium in which the quantity under consideration evolves, and let $J : [0, \infty[\rightarrow [0, \infty[$ be a Borel measurable function with $\text{supp}(J) \subset [0, \delta]$ for some $\delta > 0$. The singular nonlocal gradient of $u \in L^2(O)$ with respect to J is the function $D^J u : O \times O \rightarrow \mathbb{R}$ defined by

$$D^J u(x, y) := J(|x - y|)(u(x) - u(y)).$$

The function J , which models the spatial scale of the nonlocal effects, is allowed to be unbounded (singular). The natural domain of D^J is the space

$$\text{dom}(D^J) := \left\{ u \in L^2(O) : \int_O \int_O |D^J u(x, y)|^2 dx dy < \infty \right\},$$

which, under appropriate conditions on J , is related to fractional Sobolev spaces (see Proposition 2.3). Given an energy density $W : \mathbb{R}^d \times \mathbb{R} \rightarrow [0, \infty[$ that is convex in its second variable, which models the cost of nonlocal interactions, we consider the nonlocal energy

functional $\mathcal{F} : L^2(O) \rightarrow [0, \infty]$ defined by

$$\mathcal{F}(u) := \begin{cases} \int_O \int_O W(x, D^J u(x, y)) dx dy & \text{if } u \in \text{dom}(D^J) \\ \infty & \text{otherwise.} \end{cases}$$

For a nonlocal Dirichlet datum $g \in L^2(\partial^J O)$ (where $\partial^J O := (O + B_\delta(0)) \setminus \bar{O}$ is the nonlocal boundary of O), the boundary energy functional $\mathcal{D}^g : L^2(O) \rightarrow [0, \infty]$ is given by

$$\mathcal{D}^g(u) := \begin{cases} \int_O \int_{\partial^J O} [W(x, D_g^J u(x, y)) + W(y, D_g^J u(x, y))] dx dy & \text{if } u \in \text{dom}(D_g^J) \\ \infty & \text{otherwise,} \end{cases}$$

where

$$D_g^J u(x, y) := J(|x - y|)(u(x) - g(y))$$

and

$$\text{dom}(D_g^J) := \left\{ u \in L^2(O) : \int_O \int_{\partial^J O} |D_g^J u(x, y)|^2 dx dy < \infty \right\}.$$

The sum $\mathcal{F}^g := \mathcal{F} + \mathcal{D}^g$ then serves as the energy for the Dirichlet–Cauchy problem.

In the present paper, we consider evolution problems of the abstract form

$$\begin{cases} \frac{du}{dt}(t) + \partial \mathcal{E}(u(t)) \ni F(t, u(t)) \\ u(0) = u_0, \end{cases}$$

where $u(t) \in L^2(O)$, $\mathcal{E} \in \{\mathcal{F}, \mathcal{F}^g\}$ and $F : [0, \infty[\times L^2(O) \rightarrow L^2(O)$ is a reaction term, which models the creation or annihilation of the quantity $u(t, x)$ at each time and spatial position. When $\mathcal{E} = \mathcal{F}$, the problem corresponds to homogeneous Neumann–Cauchy boundary conditions, while when $\mathcal{E} = \mathcal{F}^g$, it corresponds to Dirichlet–Cauchy boundary conditions.

The contributions of this paper are threefold.

First, in Section 2, we develop the functional analytic foundations needed to handle singular nonlocal gradients. This includes the relationship between $\text{dom}(D^J)$ and fractional Sobolev spaces (see Proposition 2.3)¹, density results (see Propositions 2.5–2.9), the lower semicontinuity of the energy functionals (see Proposition 2.12), a nonlocal Green-type formula (see Lemma 2.16), and the characterisation of the subdifferentials $\partial \mathcal{F}$, $\partial \mathcal{D}^g$ and $\partial \mathcal{F}^g$ (see Propositions 2.17–2.18 and Corollary 2.19).

Second, in Section 3, we establish comparison principles for both the Neumann–Cauchy and Dirichlet–Cauchy nonlocal singular reaction–diffusion problems (see Theorem 3.6), together with stability estimates in $L^2(O)$. Building on these comparison principles and on the notion of CP-structured reaction functionals (see Definition 3.7), we prove the existence and uniqueness of bounded solutions (see Theorem 3.10).

Third, in Section 4, we present two applications to problems that are intrinsically nonlocal and for which the boundedness of the solution is a fundamental requirement: nonlocal cable

¹This result will be used in the part II of this paper, which will be devoted to convergence and stochastic homogenization.

theory (see §4.1 and Theorems 4.7–4.8) and nonlocal dynamics of fake-news belief (see §4.2 and Theorems 4.11–4.12).

Plan of the paper. In Section 2, we introduce the functional setting: singular nonlocal gradients, convex functionals of singular nonlocal gradients, and the characterisation of the subdifferentials of the energy functionals. In Section 3, we establish comparison principles for nonlocal singular reaction-diffusion problems and prove the existence and uniqueness of bounded solutions. In Section 4, we present two applications: nonlocal cable theory and nonlocal dynamics of fake-news belief. Finally, for the convenience of the reader, we state in Appendix ?? the version of Gronwall's lemma used in the paper.

Notation. Throughout the paper we will use the following notation.

- Given $x_0 \in \mathbb{R}^d$ we denote the open (resp. closed) ball of radius $r > 0$ centered at x_0 by $B_r(x_0)$ (resp. $\overline{B}_r(x_0)$).
- The closure (resp. interior) of a set $A \subset \mathbb{R}^d$ is denoted by \overline{A} (resp. $\text{int}(A)$).
- The Lebesgue measure on \mathbb{R}^d with $d \in \mathbb{N}^*$ is denoted by \mathcal{L}^d and for each Borel set $A \subset \mathbb{R}^d$, the measure of A with respect to \mathcal{L}^d is denoted by $\mathcal{L}^d(A)$.
- The class of bounded Borel subsets of \mathbb{R}^d is denoted by $\mathcal{B}_b(\mathbb{R}^d)$.
- The space of continuous functions from $[0, T]$ to $L^2(O)$ is denoted by $C([0, T]; L^2(O))$.
- The space of absolutely continuous functions from $[0, T]$ to $L^2(O)$ is denoted by $AC([0, T]; L^2(O))$.
- Given two open sets $U, V \subset \mathbb{R}^d$, the space of measurable functions from $U \times V$ to \mathbb{R} is denoted by $L^0(U \times V)$.
- Given $\{u_n\}_n \subset C([0, T]; L^2(O))$, by $u_n \rightarrow u$ in $C([0, T]; L^2(O))$ we mean that $\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \|u_n(t) - u(t)\|_{L^2(O)} = 0$.
- Given $\{u_n\}_n \subset L^2([0, T]; L^2(O))$, by $u_n \rightarrow u$ in $L^2([0, T]; L^2(O))$ we mean that for every $v \in L^2([0, T]; L^2(O))$, $\int_0^T \langle u_n(t), v(t) \rangle dt \rightarrow \int_0^T \langle u(t), v(t) \rangle dt$ as $n \rightarrow \infty$, where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L^2(O)$.
- Given $\{u_n\}_n \subset AC([0, T]; L^2(O))$, by $\frac{du_n}{dt} \rightarrow \frac{du}{dt}$ in $L^2([0, T]; L^2(O))$ we mean that for every $v \in L^2([0, T]; L^2(O))$, $\int_0^T \langle \frac{du_n}{dt}(t), v(t) \rangle dt \rightarrow \int_0^T \langle \frac{du}{dt}(t), v(t) \rangle dt$ as $n \rightarrow \infty$.
- The notation $\int_A \int_B f(x, y) dx dy$ means that we integrate the function f over the product set $A \times B$, with the variable x ranging in A and the variable y ranging in B .

2. FUNCTIONAL SETTING AND NONLOCAL CALCULUS

In what follows, $d \in \mathbb{N}^*$ and $O \subset \mathbb{R}^d$ is a bounded open set with sufficiently regular boundary.

2.1. Singular nonlocal gradients. Throughout the paper, $J : [0, \infty[\rightarrow [0, \infty[$ is a Borel measurable function for which there exists $\delta > 0$ with $\text{supp}(J) \subset [0, \delta]$.

Definition 2.1. For each $u \in L^2(O)$, the singular nonlocal gradient of u with respect to J is the function $D^J u : O \times O \rightarrow \mathbb{R}$ defined by

$$D^J u(x, y) := J(|x - y|)(u(x) - u(y)).$$

This gives rise to an operator $D^J : L^2(O) \rightarrow L^0(O \times O)$ with domain

$$\text{dom}(D^J) := \left\{ u \in L^2(O) : \int_O \int_O |D^J u(x, y)|^2 dx dy < \infty \right\}.$$

Remark 2.2. The fact that J can be unbounded justifies the term “singular” for the gradient D^J . If J is bounded then $\text{dom}(D^J) = L^2(O)$.

Let $s \in]0, 1[$ and define the fractional Sobolev space

$$W^{s,2}(O) := \left\{ u \in L^2(O) : \int_O \int_O \frac{|u(x) - u(y)|^2}{|x - y|^{d+2s}} dx dy < \infty \right\}.$$

Consider the following condition:

(S₀) there exist $\alpha > 0$ such that for every $r > 0$,

$$\frac{1}{r^\alpha} \mathbb{1}_{[0,\delta]}(r) \leq J(r).$$

The choice of the exponent α in (S₀) ensures that functions in $\text{dom}(D^J)$ possess the same regularity as functions in the fractional Sobolev space $W^{s,2}(O)$, as made precise in the following proposition.

Proposition 2.3. *Under (S₀), if $\alpha = \frac{d}{2} + s$ with $s \in]0, 1[$ then $\text{dom}(D^J) \subset W^{s,2}(O)$. Moreover, if $s \leq \frac{d}{2}$ then the embedding $\text{dom}(D^J) \hookrightarrow L^2(O)$ is compact.*

Proof of Proposition 2.3. Let $u \in \text{dom}(D^J)$. Since $2\alpha = d + 2s$ and by (S₀), we have

$$\begin{aligned} \int_O \int_O \frac{|u(x) - u(y)|^2}{|x - y|^{d+2s}} dx dy &= \int_O \int_O \frac{1}{(|x - y|^\alpha)^2} \mathbb{1}_{[0,\delta]}(|x - y|) |u(x) - u(y)|^2 dx dy \\ &\quad + \int_O \int_O \frac{1}{|x - y|^{\frac{d}{2}+s}} \mathbb{1}_{\mathbb{R} \setminus [0,\delta]}(|x - y|) |u(x) - u(y)|^2 dx dy \\ &\leq \int_O \int_O (J(|x - y|) |u(x) - u(y)|)^2 dx dy \\ &\quad + \frac{1}{\delta^{\frac{d}{2}+s}} \int_O \int_O |u(x) - u(y)|^2 dx dy \\ &\leq \int_O \int_O |D^J u(x, y)|^2 dx dy + \frac{4\mathcal{L}^d(O)}{\delta^{\frac{d}{2}+s}} \|u\|_{L^2(O)}^2. \end{aligned}$$

Since $u \in \text{dom}(D^J)$, both $\|u\|_{L^2(O)}$ and $\int_O \int_O |D^J u(x, y)|^2 dx dy$ are finite. Hence

$$\int_O \int_O \frac{|u(x) - u(y)|^2}{|x - y|^{d+2s}} dx dy < \infty,$$

which implies $u \in W^{s,2}(O)$.

Finally, when $s \leq \frac{d}{2}$, the Sobolev embedding $W^{s,2}(O) \hookrightarrow L^2(O)$ is compact². Since $\text{dom}(D^J) \subset W^{s,2}(O)$, the embedding $\text{dom}(D^J) \hookrightarrow L^2(O)$ is also compact. ■

²For $d > 2s$, the embedding $W^{s,2}(O) \hookrightarrow L^p(O)$ is compact for all $p \in]1, \frac{2d}{d-2s}[$, and for $d = 2s$ it is compact for all $p \in]1, \infty[$.

Remark 2.4. Proposition 2.3 is included here for the completeness of the section. It will be used in the part II of this paper, which is devoted to convergence and stochastic homogenization, in order to prove that Γ -convergence is in fact Mosco-convergence.

We also consider the following condition:

$$(S_1) \quad \int_{\mathbb{R}^d} |J(|\xi|)\xi|^2 d\xi < \infty.$$

This condition allows us to establish the following density result.

Proposition 2.5. *Under (S₁), $\text{dom}(D^J)$ is dense in $L^2(O)$ for the strong convergence in $L^2(O)$.*

Proof of Proposition 2.5. Let $C_c^{0,1}(O)$ denote the space of compactly supported Lipschitz functions from O to \mathbb{R} . As $C_c^{0,1}(O)$ is dense in $L^2(O)$ for the strong convergence in $L^2(O)$, it suffices to show that

$$C_c^{0,1}(O) \subset \text{dom}(D^J). \quad (2.1)$$

Let $u \in C_c^{0,1}(O)$. Using the Lipschitz property of u and (S₁), we have

$$\begin{aligned} \int_O \int_O |D^J u(x, y)|^2 dx dy &= \int_O \int_O |J(|x - y|)(u(x) - u(y))|^2 dx dy \\ &\leq L_u^2 \int_O \int_O |J(|x - y|)(x - y)|^2 dx dy \\ &\leq L_u^2 \mathcal{L}^d(O) \int_{\mathbb{R}^d} |J(|\xi|)\xi|^2 d\xi < \infty, \end{aligned}$$

where $L_u \geq 0$ is the Lipschitz constant of u . Hence $u \in \text{dom}(D^J)$, and (2.1) is proved. \blacksquare

To handle nonlocal Dirichlet boundary condition, let $O^J \subset \mathbb{R}^d$ be the bounded open set given by

$$O^J := O + B_\delta(0).$$

The nonlocal boundary of O with respect to J is then defined as the bounded open subset of \mathbb{R}^d

$$\partial^J O := O^J \setminus \overline{O}.$$

Definition 2.6. For each $u \in L^2(O)$ and each $g \in L^2(\partial^J O)$, the singular nonlocal gradient of u with respect to J and g is the function $D_g^J u : O \times \partial^J O \rightarrow \mathbb{R}$ defined by

$$D_g^J u(x, y) := J(|x - y|)(u(x) - g(y)).$$

This gives rise to an operator $D_g^J : L^2(O) \rightarrow L^0(O \times \partial^J O)$ with domain

$$\text{dom}(D_g^J) := \left\{ u \in L^2(O) : \int_O \int_{\partial^J O} |D_g^J u(x, y)|^2 dx dy < \infty \right\}.$$

Remark 2.7. It is easy to see that if $g \in L^2(\partial^J O)$ is such that

$$(S_2) \quad \int_O \int_{\partial^J O} |J(|x - y|)g(y)|^2 dx dy < \infty,$$

then $\text{dom}(D_g^J)$ is a vector space.

The following result is analogous to Proposition 2.5.

Proposition 2.8. *Under (S₁)–(S₂), $\text{dom}(D_g^J)$ is dense in $L^2(O)$ for the strong convergence in $L^2(O)$.*

Proof of Proposition 2.8. As in Proposition 2.5 it suffices to show that

$$C_c^{0,1}(O) \subset \text{dom}(D_g^J). \quad (2.2)$$

Let $u \in C_c^{0,1}(O)$ and define $\tilde{u} \in C_c^{0,1}(O^J)$ by

$$\tilde{u}(y) := \begin{cases} u(y) & \text{if } y \in O \\ 0 & \text{if } y \in O^J \setminus O. \end{cases}$$

Then $\tilde{u}(y) = 0$ for all $y \in \partial^J O$. Hence

$$\begin{aligned} \int_O \int_{\partial^J O} |D_g^J u(x, y)|^2 dx dy &= \int_O \int_{\partial^J O} |J(|x - y|)(u(x) - g(y))|^2 dx dy \\ &= \int_O \int_{\partial^J O} |J(|x - y|)(\tilde{u}(x) - \tilde{u}(y) - g(y))|^2 dx dy \\ &\leq 2 \int_O \int_{\partial^J O} |J(|x - y|)(\tilde{u}(x) - \tilde{u}(y))|^2 dx dy + 2K \end{aligned}$$

with $K := \int_O \int_{\partial^J O} |J(|x - y|)g(y)|^2 dx dy < \infty$ by (S₂). Using the Lipschitz property of \tilde{u} and (S₁), it follows that

$$\begin{aligned} \int_O \int_{\partial^J O} |D_g^J u(x, y)|^2 dx dy &\leq 2L_{\tilde{u}}^2 \int_O \int_{\partial^J O} |J(|x - y|)(x - y)|^2 dx dy + 2K \\ &\leq L_{\tilde{u}}^2 \mathcal{L}^d(O) \int_{\mathbb{R}^d} |J(|\xi|)\xi|^2 d\xi + 2K < \infty, \end{aligned}$$

where $L_{\tilde{u}} \geq 0$ is the Lipschitz constant of \tilde{u} . Hence $u \in \text{dom}(D_g^J)$, and (2.2) is proved. \blacksquare

From the proofs of Propositions 2.5 and 2.8 we deduce that $C_c^{0,1}(O) \subset \text{dom}(D^J) \cap \text{dom}(D_g^J)$. Since $C_c^{0,1}(O)$ is dense in $L^2(O)$, we obtain the following density result.

Proposition 2.9. *Under (S₁)–(S₂), $\text{dom}(D^J) \cap \text{dom}(D_g^J)$ is dense in $L^2(O)$ for the strong convergence in $L^2(O)$.*

2.2. Convex functionals of singular nonlocal gradients. Let $g \in L^2(\partial^J O)$ be such that (S₂) holds and let $\mathcal{D}^g : L^2(O) \rightarrow [0, \infty]$ be defined by

$$\mathcal{D}^g(u) := \begin{cases} \int_O \int_{\partial^J O} [W(x, D_g^J u(x, y)) + W(y, D_g^J u(x, y))] dx dy & \text{if } u \in \text{dom}(D_g^J) \\ \infty & \text{otherwise.} \end{cases} \quad (2.3)$$

Let $\mathcal{F}, \mathcal{F}^g : L^2(O) \rightarrow [0, \infty]$ be defined by:

$$\mathcal{F}(u) := \begin{cases} \int_O \int_O W(x, D^J u(x, y)) dx dy & \text{if } u \in \text{dom}(D^J) \\ \infty & \text{otherwise;} \end{cases} \quad (2.4)$$

$$\mathcal{F}^g(u) := \mathcal{F}(u) + \mathcal{D}^g(u), \quad (2.5)$$

where $W : \mathbb{R}^d \times \mathbb{R} \rightarrow [0, \infty[$ satisfies the following conditions:

(I₀) $W(\cdot, \xi)$ is $(\mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}))$ -measurable for all $\xi \in \mathbb{R}$ and $W(x, \cdot)$ is convex for all $x \in \mathbb{R}^d$;

(I₁) there exist $\alpha, \beta > 0$ such for every $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}$,

$$\alpha|\xi|^2 \leq W(x, \xi) \leq \beta(1 + |\xi|^2);$$

(I₂) for every $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}$, $W(x, 0) = 0$ and $W(x, \xi) = W(x, -\xi)$;

(I₃) for every $x \in \mathbb{R}^d$, $W(x, \cdot) : \mathbb{R} \rightarrow [0, \infty[$ is differentiable.

Remark 2.10. Clearly, the assumption $W(x, 0) = 0$ in (I₂) is not restrictive: if it is not satisfied, one can simply consider the shifted density $\widetilde{W}(x, \xi) := W(x, \xi) - W(x, 0)$ and note that all the conditions (I₀)–(I₃) remain fulfilled by \widetilde{W} .

Remark 2.11. For each $x \in \mathbb{R}^d$, let $\partial W(x, \cdot) : \mathbb{R} \rightrightarrows \mathbb{R}$ denote the subdifferential of $W(x, \cdot)$. From (I₀)–(I₁) it follows that the following additional conditions are verified:

(A₁) there exists $L > 0$ such that for every $x \in \mathbb{R}^d$ and every $(\xi, \xi') \in \mathbb{R} \times \mathbb{R}$,

$$|W(x, \xi) - W(x, \xi')| \leq L|\xi - \xi'| (1 + |\xi| + |\xi'|);$$

(A₂) there exists $C > 0$ such that for every $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}$,

$$|\partial_2 W(x, \xi)| \leq C(1 + |\xi|), \text{ i.e. } |\zeta| \leq C(1 + |\xi|) \text{ for all } \zeta \in \partial W(x, \cdot)(\xi).^3$$

If furthermore (I₃) holds, then:

(A₃) for every $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}$, $\partial W(x, \cdot)(\xi)$ is single-valued, i.e.

$$\partial W(x, \cdot)(\xi) = \{\partial_2 W(x, \xi)\},$$

where $\partial_2 W(x, \xi)$ is the derivative of $W(x, \cdot)$ at ξ ;

(A₄) for every $x \in \mathbb{R}^d$, $\partial_2 W(x, \cdot)$ is nondecreasing⁴.

Under (I₀)–(I₁), $W(x, \cdot)$ is convex and proper for all $x \in \mathbb{R}^d$. Consequently, the associated functionals \mathcal{F} and \mathcal{D}^g are convex and proper. Hence, the sum $\mathcal{F}^g = \mathcal{F} + \mathcal{D}^g$ is also a convex and proper functional. Moreover, we have

Proposition 2.12. *Assume that (I₀)–(I₂) hold. Then, the functionals \mathcal{F} and \mathcal{D}^g are lower semicontinuous with respect to the strong convergence in $L^2(O)$. In particular, $\mathcal{F} + \mathcal{D}^g$ is also lower semicontinuous.*

The proof of Proposition 2.12 requires the following lemmas (see Lemmas 2.13, 2.14 and 2.15). For each $M > 0$, set

$$J_M := \min\{J, M\}.$$

Lemma 2.13. (a) *For every $M > 0$, the linear operator $D^{J_M} : L^2(O) \rightarrow L^2(O \times O)$ is continuous, with operator norm satisfying*

$$\|D^{J_M}\| \leq 2M\sqrt{\mathcal{L}^d(O)}.$$

³For simplicity, any element of $\partial W(x, \cdot)(\xi)$ will be denoted by $\partial_2 W(x, \xi)$.

⁴This is equivalent to the monotonicity of $\partial_2 W(x, \cdot)$, i.e. $[\partial_2 W(x, \xi_1) - \partial_2 W(x, \xi_2)](\xi_1 - \xi_2) \geq 0$ for all $(\xi_1, \xi_2) \in \mathbb{R} \times \mathbb{R}$.

As a consequence, for every $M > 0$, every $\{u_\varepsilon\}_{\varepsilon>0} \subset L^2(O)$ and every $u \in L^2(O)$, if $u_\varepsilon \rightarrow u$ in $L^2(O)$ then $D^{J_M}u_\varepsilon \rightarrow D^{J_M}u$ in $L^2(O \times O)$.

(b) For every $M > 0$, the operator $D_g^{J_M} : L^2(O) \rightarrow L^2(O \times \partial^J O)$ is Lipschitz continuous with Lipschitz constant $M\sqrt{\mathcal{L}^d(\partial^J O)}$, i.e.

$$\|D_g^{J_M}u - D_g^{J_M}v\|_{L^2(O \times \partial^J O)} \leq M\sqrt{\mathcal{L}^d(\partial^J O)}\|u - v\|_{L^2(O)}$$

for all $u, v \in L^2(O)$. As a consequence, for every $M > 0$, every $\{u_\varepsilon\}_{\varepsilon>0} \subset L^2(O)$ and every $u \in L^2(O)$, if $u_\varepsilon \rightarrow u$ in $L^2(O)$ then $D_g^{J_M}u_\varepsilon \rightarrow D_g^{J_M}u$ in $L^2(O \times \partial^J O)$.

Proof of Lemma 2.13. (a) Fix $M > 0$. Given any $u \in L^2(O)$ we have

$$\begin{aligned} \int_O \int_O |D^{J_M}u(x, y)|^2 dx dy &= \int_O \int_O |J_M(|x - y|)(u(x) - u(y))|^2 dx dy \\ &\leq M^2 \int_O \int_O |u(x) - u(y)|^2 dx dy \\ &\leq 2M^2 \int_O \int_O (|u(x)|^2 + |u(y)|^2) dx dy \\ &= 4M^2 \mathcal{L}^d(O) \|u\|_{L^2(O)}^2, \end{aligned}$$

and the result follows.

(b) Fix $M > 0$. Given any $u \in L^2(O)$ we have

$$\begin{aligned} \|D_g^{J_M}u - D_g^{J_M}v\|_{L^2(O \times \partial^J O)}^2 &= \int_O \int_{\partial^J O} |D_g^{J_M}u(x, y) - D_g^{J_M}v(x, y)|^2 dx dy \\ &= \int_O \int_{\partial^J O} |J_M(|x - y|)(u(x) - v(x))|^2 dx dy \\ &\leq M^2 \int_O \int_{\partial^J O} |u(x) - v(x)|^2 dx dy \\ &= M^2 \mathcal{L}^d(\partial^J O) \|u - v\|_{L^2(O)}^2, \end{aligned}$$

and the result follows. ■

Lemma 2.14. (a) Let $u \in L^2(O)$ and let $\{u_\varepsilon\}_{\varepsilon>0} \subset L^2(O)$ be such that:

$$\begin{aligned} u_\varepsilon &\rightarrow u \text{ in } L^2(O); \\ c_1 &:= \sup_{\varepsilon>0} \|D^J u_\varepsilon\|_{L^2(O \times O)} < \infty. \end{aligned} \tag{2.6}$$

Then $\{u_\varepsilon\}_{\varepsilon>0} \subset \text{dom}(D^J)$ and $u \in \text{dom}(D^J)$.

(b) Let $u \in L^2(O)$ and let $\{u_\varepsilon\}_{\varepsilon>0} \subset L^2(O)$ be such that:

$$\begin{aligned} u_\varepsilon &\rightarrow u \text{ in } L^2(O); \\ c_2 &:= \sup_{\varepsilon>0} \|D_g^J u_\varepsilon\|_{L^2(O \times \partial^J O)} < \infty. \end{aligned} \tag{2.7}$$

Then $\{u_\varepsilon\}_{\varepsilon>0} \subset \text{dom}(D_g^J)$ and $u \in \text{dom}(D_g^J)$.

Proof of Lemma 2.14. (a) First, by (2.6), it is clear that $\{u_\varepsilon\}_{\varepsilon>0} \subset \text{dom}(D^J)$. Given any $M > 0$, since $0 \leq J_M \leq J$ we have $|D^{J_M}u_\varepsilon|^2 \leq |D^J u_\varepsilon|^2$ for all $\varepsilon > 0$, and thus

$$\|D^{J_M}u_\varepsilon\|_{L^2(O \times O)} \leq c_1 \text{ for all } \varepsilon > 0,$$

where $0 < c_1 < \infty$ is defined in (2.6). By Lemma 2.13(a) we have $\lim_{\varepsilon \rightarrow 0} \|D^{J_M}u_\varepsilon\|_{L^2(O \times O)} = \|D^{J_M}u\|_{L^2(O \times O)}$. Consequently

$$\|D^{J_M}u\|_{L^2(O \times O)} \leq c \text{ for all } M > 0. \quad (2.8)$$

On the other hand, it is easy to see that $|D^{J_M}u|^2 \uparrow |D^J u|^2$ as $M \rightarrow \infty$ pointwise in $O \times O$. Therefore, by (2.8) and the monotone convergence theorem, we deduce that

$$\lim_{M \rightarrow \infty} \|D^{J_M}u\|_{L^2(O \times O)}^2 = \|D^J u\|_{L^2(O \times O)}^2 \leq c_1,$$

which shows that $u \in \text{dom}(D^J)$.

(b) First, by (2.7), it is clear that $\{u_\varepsilon\}_{\varepsilon>0} \subset \text{dom}(D_g^J)$. Given any $M > 0$, since $0 \leq J_M \leq J$ we have $|D_g^{J_M}u_\varepsilon|^2 \leq |D_g^J u_\varepsilon|^2$ for all $\varepsilon > 0$, and thus

$$\|D_g^{J_M}u_\varepsilon\|_{L^2(O \times \partial^J O)} \leq c_2 \text{ for all } \varepsilon > 0,$$

where $0 < c_2 < \infty$ is defined in (2.7). By Lemma 2.13(b) we have $\lim_{\varepsilon \rightarrow 0} \|D_g^{J_M}u_\varepsilon\|_{L^2(O \times \partial^J O)} = \|D_g^{J_M}u\|_{L^2(O \times \partial^J O)}$. Consequently

$$\|D_g^{J_M}u\|_{L^2(O \times \partial^J O)} \leq c_2 \text{ for all } M > 0. \quad (2.9)$$

On the other hand, it is easy to see that $|D_g^{J_M}u|^2 \uparrow |D_g^J u|^2$ as $M \rightarrow \infty$ pointwise in $O \times \partial^J O$. Therefore, by (2.9) and the monotone convergence theorem, we deduce that

$$\lim_{M \rightarrow \infty} \|D_g^{J_M}u\|_{L^2(O \times \partial^J O)}^2 = \|D_g^J u\|_{L^2(O \times \partial^J O)}^2 \leq c_2,$$

which shows that $u \in \text{dom}(D_g^J)$. ■

Lemma 2.15. *Assume that (I₀)–(I₂) hold. For every $u \in L^2(O)$, one has:*

(a) $\{W(\cdot, D^{J_M}u)\}_{M>0}$ is nondecreasing, i.e. for every $M, M' > 0$,

$$\text{if } M < M' \text{ then } W(\cdot, D^{J_M}u) \leq W(\cdot, D^{J_{M'}}u);$$

(b) $W(\cdot, D^{J_M}u) \uparrow W(\cdot, D^J u)$ as $M \rightarrow \infty$ pointwise in $O \times O$, i.e.

$$W(\cdot, D^J u) = \sup_{M>0} W(\cdot, D^{J_M}u).$$

(c) $\{W(\cdot, D_g^{J_M}u)\}_{M>0}$ is nondecreasing, i.e. for every $M, M' > 0$,

$$\text{if } M < M' \text{ then } W(\cdot, D_g^{J_M}u) \leq W(\cdot, D_g^{J_{M'}}u);$$

(d) $W(\cdot, D_g^{J_M}u) \uparrow W(\cdot, D_g^J u)$ as $M \rightarrow \infty$ pointwise in $O \times \partial^J O$, i.e.

$$W(\cdot, D_g^J u) = \sup_{M>0} W(\cdot, D_g^{J_M}u).$$

Proof of Lemma 2.15. Let $u \in L^2(O)$.

(a) Let $M, M' > 0$ be such that $M < M'$. Fix $(x, y) \in O \times O$. Without loss of generality we can assume that $|x - y| < \delta$, where by assumption on J , the constant $\delta > 0$ satisfies $\text{supp}(J) \subset [0, \delta]$ (see §2.1). Then $\frac{J_M(|x-y|)}{J_{M'}(|x-y|)} \in]0, 1]$. By convexity, see (I₀), we have

$$\begin{aligned} W(x, J_M(|x-y|)(u(x) - u(y))) &\leq \frac{J_M(|x-y|)}{J_{M'}(|x-y|)} W(x, J_{M'}(|x-y|)(u(x) - u(y))) \\ &\quad + \left(1 - \frac{J_M(|x-y|)}{J_{M'}(|x-y|)}\right) W(x, 0). \end{aligned}$$

But $W(x, 0) = 0$ by (I₂), hence

$$W(x, D^{J_M}(x, y)) \leq \frac{J_M(|x-y|)}{J_{M'}(|x-y|)} W(x, D^{J_{M'}}(x, y)),$$

and the conclusion follows because $\frac{J_M(|x-y|)}{J_{M'}(|x-y|)} < 1$.

(b) Fix $(x, y) \in O \times O$. From (a) it follows that

$$\lim_{M \rightarrow \infty} W(x, D^{J_M}(x, y)) = \sup_{M > 0} W(x, D^{J_M}u(x, y)).$$

As $\lim_{M \rightarrow \infty} J_M(|x-y|) = J(|x-y|)$ we have

$$\lim_{M \rightarrow \infty} D^{J_M}u(x, y) = D^J u(x, y).$$

But $W(x, \cdot)$ is continuous by (I₀), hence

$$\lim_{M \rightarrow \infty} W(x, D^{J_M}(x, y)) = W(x, D^J u(x, y)),$$

which completes the proof of (b).

(c) Let $M, M' > 0$ be such that $M < M'$. Fix $(x, y) \in O \times \partial^J O$. Without loss of generality we can assume that $|x - y| < \delta$, where by assumption on J , the constant $\delta > 0$ satisfies $\text{supp}(J) \subset [0, \delta]$. Then $\frac{J_M(|x-y|)}{J_{M'}(|x-y|)} \in]0, 1]$. Let $z \in \{x, y\}$. By convexity, see (I₀), we have

$$\begin{aligned} W(z, J_M(|x-y|)(u(x) - g(y))) &\leq \frac{J_M(|x-y|)}{J_{M'}(|x-y|)} W(z, J_{M'}(|x-y|)(u(x) - g(y))) \\ &\quad + \left(1 - \frac{J_M(|x-y|)}{J_{M'}(|x-y|)}\right) W(z, 0). \end{aligned}$$

But $W(z, 0) = 0$ by (I₂), hence

$$W(z, D_g^{J_M}(x, y)) \leq \frac{J_M(|x-y|)}{J_{M'}(|x-y|)} W(z, D_g^{J_{M'}}(x, y)),$$

and desired conclusion follows because $\frac{J_M(|x-y|)}{J_{M'}(|x-y|)} < 1$.

(d) Fix $(x, y) \in O \times \partial^J O$ and let $z \in \{x, y\}$. From (c) it follows that

$$\lim_{M \rightarrow \infty} W(z, D_g^{J_M}(x, y)) = \sup_{M > 0} W(z, D_g^{J_M}u(x, y)).$$

As $\lim_{M \rightarrow \infty} J_M(|x - y|) = J(|x - y|)$ we have

$$\lim_{M \rightarrow \infty} D_g^{J_M} u(x, y) = D_g^J u(x, y).$$

But $W(z, \cdot)$ is continuous by (I₀), hence

$$\lim_{M \rightarrow \infty} W(z, D_g^{J_M}(x, y)) = W(z, D_g^J u(x, y)),$$

which completes the proof of (d). ■

Proof of Proposition 2.12. We begin by proving that \mathcal{F} is lower semicontinuous. Let $\{u_\varepsilon\}_{\varepsilon > 0} \subset L^2(O)$ and let $u \in L^2(O)$ be such that

$$u_\varepsilon \rightarrow u \text{ in } L^2(O). \quad (2.10)$$

Without loss of generality we can assume that $\lim_{\varepsilon \rightarrow 0} \mathcal{F}(u_\varepsilon) < \infty$ and so, up to a subsequence,

$$\sup_{\varepsilon > 0} \mathcal{F}(u_\varepsilon) < \infty. \quad (2.11)$$

Thus $\{u_\varepsilon\}_{\varepsilon > 0} \subset \text{dom}(D^J)$. Hence, for every $\varepsilon > 0$,

$$\mathcal{F}(u_\varepsilon) = \int_O \int_O W(x, D^J u_\varepsilon(x, y)) dx dy$$

for all $\varepsilon > 0$. Moreover, taking the left inequality in (I₁) into account, from (2.11) we see that

$$c_1 := \sup_{\varepsilon > 0} \|D^J u_\varepsilon\|_{L^2(O \times O)} < \infty, \quad (2.12)$$

so that by Lemma 2.14(a), $u \in \text{dom}(D^J)$. Thus

$$\mathcal{F}(u) = \int_O \int_O W(x, D^J u(x, y)) dx dy.$$

Consequently, we have to prove that

$$\liminf_{\varepsilon \rightarrow 0} \int_O \int_O W(x, D^J u_\varepsilon(x, y)) dx dy \geq \int_O \int_O W(x, D^J u(x, y)) dx dy. \quad (2.13)$$

Fix any $M > 0$. Then, Since $J_M \leq J$, it follows from (2.12) that

$$D^{J_M} u \in L^2(O \times O). \quad (2.14)$$

By Lemma 2.15(b) we have

$$W(\cdot, D^{J_M} v) \uparrow W(\cdot, D^J v) \text{ as } M \rightarrow \infty \text{ pointwise in } O \times O \text{ for all } v \in L^2(O). \quad (2.15)$$

Fix any $\varepsilon > 0$. Then, by using (2.15), $W(\cdot, D^J u_\varepsilon) \geq W(\cdot, D^{J_M} u_\varepsilon)$. Since W satisfies the Lipschitz condition ((A₁)) (see Remark 2.11), it follows that

$$W(\cdot, D^J u_\varepsilon) \geq W(\cdot, D^{J_M} u) - L |D^{J_M}(u_\varepsilon - u)| (1 + |D^{J_M} u_\varepsilon| + |D^{J_M} u|).$$

Hence, by (2.12) and Lemma 2.13(a),

$$\int_O \int_O W(x, D^J u_\varepsilon) dx dy \geq \int_O \int_O W(x, D^{J_M} u) dx dy - L_1 \|u_\varepsilon - u\|_{L^2(O)} (1 + \|D^{J_M} u\|_{L^2(O \times O)}),$$

where $L_1 := 2LM\sqrt{\mathcal{L}^d(O)}\max\{1, c_1, \sqrt{\mathcal{L}^{d \times d}(O \times O)}\}$. Letting $\varepsilon \rightarrow 0$, from (2.10) and (2.14) we deduce that for every $M > 0$,

$$\liminf_{\varepsilon \rightarrow 0} \int_O \int_O W(x, D^J u_\varepsilon) dx dy \geq \int_O \int_O W(x, D^{J_M} u) dx dy$$

Taking (2.15) into account and passing to the limit as $M \rightarrow \infty$ in the above inequality, we obtain (2.13).

We now prove that \mathcal{D}^g is lower semicontinuous. Let $\{u_\varepsilon\}_{\varepsilon > 0} \subset L^2(O)$ and let $u \in L^2(O)$ be such that

$$u_\varepsilon \rightarrow u \text{ in } L^2(O). \quad (2.16)$$

Without loss of generality we can assume that $\liminf_{\varepsilon \rightarrow 0} \mathcal{D}^g(u_\varepsilon) < \infty$ and so, up to a subsequence,

$$\sup_{\varepsilon > 0} \mathcal{D}^g(u_\varepsilon) < \infty. \quad (2.17)$$

Thus $\{u_\varepsilon\}_{\varepsilon > 0} \subset \text{dom}(D_g^J)$. Hence, for every $\varepsilon > 0$,

$$\mathcal{D}^g(u_\varepsilon) = \int_O \int_{\partial^J O} [W(x, D_g^J u_\varepsilon(x, y)) + W(y, D_g^J u_\varepsilon(x, y))] dx dy$$

for all $\varepsilon > 0$. Moreover, taking the left inequality in (I₁) into account, from (2.17) we see that

$$c_2 := \sup_{\varepsilon > 0} \|D_g^J u_\varepsilon\|_{L^2(O \times \partial^J O)} < \infty \quad (2.18)$$

and so, by Lemma 2.14(b), $u \in \text{dom}(D_g^J)$. Thus

$$\mathcal{D}^g(u) = \int_O \int_{\partial^J O} [W(x, D_g^J u(x, y)) + W(y, D_g^J u(x, y))] dx dy.$$

Consequently, we have to prove that

$$\begin{aligned} & \liminf_{\varepsilon \rightarrow 0} \int_O \int_{\partial^J O} [W(x, D_g^J u_\varepsilon(x, y)) + W(y, D_g^J u_\varepsilon(x, y))] dx dy \\ & \geq \int_O \int_{\partial^J O} [W(x, D_g^J u(x, y)) + W(y, D_g^J u(x, y))] dx dy. \end{aligned} \quad (2.19)$$

Fix any $M > 0$. Then, Since $J_M \leq J$, it follows from (2.18) that

$$D_g^{J_M} u \in L^2(O \times \partial^J O). \quad (2.20)$$

By Lemma 2.15(d) we have

$$W(\cdot, D_g^{J_M} v) \uparrow W(\cdot, D_g^J v) \text{ as } M \rightarrow \infty \text{ pointwise in } O \times \partial^J O \text{ for all } v \in L^2(O). \quad (2.21)$$

Fix any $\varepsilon > 0$. Then, by using (2.21), $W(\cdot, D_g^J u_\varepsilon) \geq W(\cdot, D_g^{J_M} u_\varepsilon)$. Since W satisfies the Lipschitz condition ((A₁)), it follows that

$$W(\cdot, D_g^J u_\varepsilon) \geq W(\cdot, D_g^{J_M} u) - L |D_g^{J_M} u_\varepsilon - D_g^{J_M} u| (1 + |D_g^{J_M} u_\varepsilon| + |D_g^{J_M} u|).$$

Hence, by (2.18) and Lemma 2.13(b),

$$\begin{aligned} & \int_O \int_{\partial^J O} [W(x, D_g^J u_\varepsilon(x, y)) + W(y, D_g^J u_\varepsilon(x, y))] dx dy \\ & \geq \int_O \int_{\partial^J O} [W(x, D_g^{J_M} u(x, y)) + W(y, D_g^{J_M} u(x, y))] dx dy \\ & \quad - L_2 \|u_\varepsilon - u\|_{L^2(O)} (1 + \|D_g^{J_M} u\|_{L^2(O \times \partial^J O)}), \end{aligned}$$

where $L_2 := 4LM\sqrt{\mathcal{L}^d(\partial^J O)} \max\{1, c_2, \sqrt{\mathcal{L}^{d \times d}(O \times \partial^J O)}\}$. Letting $\varepsilon \rightarrow 0$, from (2.16) and (2.20) we deduce that for every $M > 0$,

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_O \int_{\partial^J O} [W(x, D_g^J u_\varepsilon(x, y)) + W(y, D_g^J u_\varepsilon(x, y))] dx dy \\ & \geq \int_O \int_{\partial^J O} [W(x, D_g^{J_M} u(x, y)) + W(y, D_g^{J_M} u(x, y))] dx dy. \end{aligned}$$

Taking (2.21) into account and passing to the limit as $M \rightarrow \infty$ in the above inequality, we obtain (2.19), and the proof is complete. ■

In the sequel, after presenting a nonlocal Green formula in §2.3, we compute the subdifferentials of the functionals \mathcal{F} and \mathcal{F}^g in §2.4.

2.3. Nonlocal Green-type formula. Here we establish a nonlocal Green formula.

Lemma 2.16. *Assume that (I₀)–(I₃) hold and set*

$$\mathcal{D}_W^J := \left\{ u \in \text{dom}(D^J) : w_1^u \in L^2(O) \text{ and } w_2^u \in L^2(O) \right\} \quad (2.22)$$

with $w_1^u, w_2^u : O \rightarrow \mathbb{R}$ given by:

$$w_1^u(x) := \int_O J(|x - y|) \partial_2 W(x, D^J u(x, y)) dy; \quad (2.23)$$

$$w_2^u(x) := \int_O J(|x - y|) \partial_2 W(y, D^J u(x, y)) dy. \quad (2.24)$$

Then, for every $u \in \mathcal{D}_W^J$ and every $\varphi \in \text{dom}(D^J)$,

$$\begin{aligned} \int_O \int_O \partial_2 W(x, D^J u(x, y)) D^J \varphi(x, y) dx dy &= \int_O [w_1^u(x) + w_2^u(x)] \varphi(x) dx \\ &= \int_O \int_O J(|x - y|) [\partial_2 W(x, D^J u(x, y)) \\ & \quad + \partial_2 W(y, D^J u(x, y))] \varphi(x) dx dy. \end{aligned} \quad (2.25)$$

Proof of Lemma 2.16. Fix $u \in \mathcal{D}_W^J$ and $\varphi \in \text{dom}(D^J)$. Note that, by definition of \mathcal{D}_W^J , the right-hand side of (2.25) is well defined. We denote it by I . It is clear that

$$I = \int_O \int_O J(|x - y|) \partial_2 W(x, D^J u(x, y)) \varphi(x) dx dy + \int_O \int_O J(|x - y|) \partial_2 W(y, D^J u(x, y)) \varphi(x) dx dy.$$

As $D^J u(x, y) = -D^J u(y, x)$ we see that

$$\int_O \int_O J(|x - y|) \partial_2 W(y, D^J u(x, y)) \varphi(x) dx dy = \int_O \int_O J(|x - y|) \partial_2 W(y, -D^J u(y, x)) \varphi(x) dx dy.$$

Moreover, from (I₂) we deduce that $\partial_2 W(y, -D^J u(y, x)) = -\partial_2 W(y, D^J u(y, x))$. Hence

$$\begin{aligned} \int_O \int_O J(|x - y|) \partial_2 W(y, D^J u(x, y)) \varphi(x) dx dy &= \int_O \int_O J(|x - y|) \partial_2 W(y, D^J u(y, x)) [-\varphi(x)] dx dy \\ &= \int_O \int_O J(|y - x|) \partial_2 W(x, D^J u(x, y)) [-\varphi(y)] dy dx, \end{aligned}$$

and consequently

$$\begin{aligned} I &= \int_O \int_O J(|x - y|) \partial_2 W(x, D^J u(x, y)) \varphi(x) dx dy + \int_O \int_O J(|x - y|) \partial_2 W(x, D^J u(x, y)) [-\varphi(y)] dx dy \\ &= \int_O \int_O \partial_2 W(x, D^J u(x, y)) [J(|x - y|) (\varphi(x) - \varphi(y))] dx dy \\ &= \int_O \int_O \partial_2 W(x, D^J u(x, y)) D^J \varphi(x, y) dx dy, \end{aligned}$$

and the proof is complete. ■

2.4. Subdifferentials of convex functionals of singular nonlocal gradients. Let $\partial \mathcal{F} : L^2(O) \rightrightarrows L^2(O)$ denote the subdifferential of \mathcal{F} . The following proposition gives a characterization of $\partial \mathcal{F}$.

Proposition 2.17. *Assume that (S₁) and (I₀)–(I₃) hold. Set*

$$\mathcal{H}_W^J := \left\{ u \in \text{dom}(D^J) : \exists h \in L^2(O) \forall \varphi \in \text{dom}(D^J) \int_O \int_O \partial_2 W(x, D^J u) D^J \varphi dx dy = \int_O h \varphi dx \right\}.$$

- (a) *For every $u \in \mathcal{H}_W^J$, the corresponding function $h \in L^2(O)$ is unique. We denote it by h_u .*
- (b) *For every $(u, v) \in \text{dom}(D^J) \times \text{dom}(D^J)$, the functional \mathcal{F} is directionally differentiable at u in the direction v and*

$$\mathcal{F}'(u; v) := \lim_{t \rightarrow 0} \frac{1}{t} (\mathcal{F}(u + tv) - \mathcal{F}(u)) = \int_O \int_O \partial_2 W(x, D^J u(x, y)) D^J v(x, y) dx dy. \quad (2.26)$$

- (c) *For every $(u, v) \in \mathcal{H}_W^J \times \text{dom}(D^J)$, one has*

$$\mathcal{F}'(u; v) = \int_O h_u(x) v(x) dx. \quad (2.27)$$

For this reason, we denote h_u by $d^J \mathcal{F}(u)$.

- (d) *The subdifferential $\partial \mathcal{F}$ is characterized by*

$$\begin{cases} \text{dom}(\partial \mathcal{F}) = \mathcal{H}_W^J \\ \partial \mathcal{F}(u) = \{d^J \mathcal{F}(u)\}. \end{cases}$$

(e) One has $\mathcal{D}_W^J \subset \mathcal{H}_W^J$ and

$$d^J \mathcal{F}(u) = \int_O J(|\cdot - y|) [\partial_2 W(\cdot, D^J u(\cdot, y)) + \partial_2 W(y, D^J u(\cdot, y))] dy \text{ for all } u \in \mathcal{D}_W^J, \quad (2.28)$$

where \mathcal{D}_W^J is defined in (2.22).

Proof of Proposition 2.17. (a) The uniqueness of the function $h \in L^2(O)$ associated with $u \in \mathcal{H}_W^J$ follows directly from the density of $\text{dom}(D^J)$ in $L^2(O)$ (see Proposition 2.5).

(b) Fix $(u, v) \in \text{dom}(D^J) \times \text{dom}(D^J)$. It is easy to see that for every $t \in \mathbb{R}^*$,

$$\frac{1}{t} (\mathcal{F}(u + tv) - \mathcal{F}(u)) = \int_O \int_O f_t(x, y) dx dy. \quad (2.29)$$

with $f_t : O \times O \rightarrow \mathbb{R}$ defined by

$$f_t(x, y) := \frac{1}{t} [W(x, D^J u(x, y) + t D^J v(x, y)) - W(x, D^J u(x, y))].$$

By (I₃) we have

$$\lim_{t \rightarrow 0} f_t(x, y) = \partial_2 W(x, D^J u(x, y)) D^J v(x, y) \text{ for all } (x, y) \in O \times O.$$

Let $f : O \times O \rightarrow [0, \infty[$ be defined by

$$f(x, y) := L |D^J v(x, y)| (1 + 2 |D^J u(x, y)| + |D^J v(x, y)|)$$

with $L > 0$ given by (A₁) (see Remark 2.11). As $u \in \text{dom}(D^J)$ and $v \in \text{dom}(D^J)$ we have

$$f \in L^1(O \times O).$$

Moreover, by (A₁), we can assert that $\{f_t\}_{t \in]0, 1]}$ is dominated by f , i.e. for every $t \in]0, 1]$,

$$|f_t(x, y)| \leq f(x, y) \text{ for all } (x, y) \in O \times O.$$

Hence, letting $t \rightarrow 0$ in (2.29) and using the Lebesgue dominated convergence theorem, we obtain (2.26).

(c) If $(u, v) \in \mathcal{H}_W^J \times \text{dom}(D^J)$ then, by definition of \mathcal{H}_W^J , one has

$$\int_O \int_O \partial_2 W(x, D^J u(x, y)) D^J v(x, y) dx dy = \int_O h_u(x) v(x) dx,$$

and (2.27) follows from (2.26).

(d) It suffices to prove the following two assertions:

(d₁) for every $u \in \mathcal{H}_W^J$, $d^J \mathcal{F}(u) \in \partial \mathcal{F}(u)$;

(d₂) for every $u \in L^2(O)$, if $h \in \partial \mathcal{F}(u)$ then $u \in \mathcal{H}_W^J$ and $h = d^J \mathcal{F}(u)$.

Proof of (d₁). Fix $u \in \mathcal{H}_W^J$ and fix any $v \in \text{dom}(D^J)$. By (I₃), for every $(x, y) \in O \times O$, we have

$$W(x, D^J v(x, y)) \geq W(x, D^J u(x, y)) + \partial_2 W(x, D^J u(x, y)) (D^J v(x, y) - D^J u(x, y)).$$

Integrating over $O \times O$, we infer that

$$\mathcal{F}(v) \geq \mathcal{F}(u) + \int_O \int_O \partial_2 W(x, D^J u(x, y)) D^J (v - u)(x, y) dx dy.$$

As $u \in \mathcal{H}_W^J$ we can assert that

$$\int_O \int_O \partial_2 W(x, D^J u(x, y)) D^J (v - u)(x, y) dx dy = \int_O d^J \mathcal{F}(u)(x) (v(x) - u(x)) dx,$$

so that for every $v \in \text{dom}(D^J)$,

$$\mathcal{F}(v) \geq \mathcal{F}(u) + \int_O d^J \mathcal{F}(u)(x) (v(x) - u(x)) dx. \quad (2.30)$$

Since $\text{dom}(\mathcal{F}) = \text{dom}(D^J)$, the inequality (2.30) extends to all $v \in L^2(O)$, which shows that $d^J \mathcal{F}(u) \in \partial \mathcal{F}(u)$.

Proof of (d₂). Let $u \in L^2(O)$ and assume that $h \in \partial \mathcal{F}(u)$. Then, $\partial \mathcal{F}(u) \neq \emptyset$ and so $u \in \text{dom}(D^J)$ because $\text{dom}(\mathcal{F}) = \text{dom}(D^J)$. Fix any $v \in \text{dom}(D^J)$. Then, by using the definition of $\partial \mathcal{F}(u)$, one has

$$\mathcal{F}(u + tv) \geq \mathcal{F}(u) + t \int_O h(x)v(x) dx \text{ for all } t \in \mathbb{R}^*.$$

Hence:

$$\frac{1}{t} (\mathcal{F}(u + tv) - \mathcal{F}(u)) \leq \int_O h(x)v(x) dx \text{ for all } t < 0; \quad (2.31)$$

$$\frac{1}{t} (\mathcal{F}(u + tv) - \mathcal{F}(u)) \geq \int_O h(x)v(x) dx \text{ for all } t > 0. \quad (2.32)$$

Letting $t \rightarrow 0^-$ in (2.31) and $t \rightarrow 0^+$ in (2.32), and using (2.26), we deduce that

$$\int_O \int_O \partial_2 W(x, D^J u(x, y)) D^J v(x, y) dx dy = \int_O h(x)v(x) dx \text{ for all } v \in \text{dom}(D^J).$$

Consequently $u \in \mathcal{H}_W^J$ and $h = d^J \mathcal{F}(u)$ by (a) and (c).

(e) From Lemma 2.16 it is clear that if $u \in \mathcal{D}_W^J$ then $u \in \mathcal{H}_W^J$ with $h_u = w_1^u + w_2^u$, where $w_1^u \in L^2(O)$ and $w_2^u \in L^2(O)$ are given by (2.23) and (2.24) respectively. Thus $\mathcal{D}_W^J \subset \mathcal{H}_W^J$ and, by (c), (2.28) holds. ■

Let $\partial \mathcal{D}^g : L^2(O) \rightrightarrows L^2(O)$ denote the subdifferential of \mathcal{D}^g defined in (2.3). The following proposition characterizes $\partial \mathcal{D}^g$.

Proposition 2.18. *Assume that (S₁)–(S₂) and (I₀)–(I₃) hold. Let $g \in L^2(\partial^J O)$ be such that (S₂) holds and set*

$$\mathcal{H}_{g,W}^J := \left\{ u \in \text{dom}(D_g^J) : \exists h \in L^2(O) \forall \varphi \in \text{dom}(D_g^J) \int_O \int_{\partial^J O} J(|x - y|) [\partial_2 W(x, D_g^J u) + \partial_2 W(y, D_g^J u)] \varphi(x) dx dy = \int_O h \varphi dx \right\}.$$

(a) *For every $u \in \mathcal{H}_{g,W}^J$, the corresponding function $h \in L^2(O)$ is unique. We denote it by h_u^g .*

(b) For every $(u, v) \in \text{dom}(D_g^J) \times \text{dom}(D_g^J)$, the functional \mathcal{D}^g is directionally differentiable at u in the direction v and

$$\begin{aligned} (\mathcal{D}^g)'(u; v) &:= \lim_{t \rightarrow 0} \frac{1}{t} (\mathcal{D}^g(u + tv) - \mathcal{D}^g(u)) \\ &= \int_O \int_{\partial^J O} J(|x - y|) [\partial_2 W(x, D_g^J u(x, y)) + \partial_2 W(y, D_g^J u(x, y))] v(x) dx dy. \end{aligned} \quad (2.33)$$

(c) For every $(u, v) \in \mathcal{H}_{g,W}^J \times \text{dom}(D_g^J)$, one has

$$(\mathcal{D}^g)'(u; v) = \int_O h_u^g(x) v(x) dx. \quad (2.34)$$

For this reason, we denote h_u^g by $d^J \mathcal{D}^g(u)$.

(d) The subdifferential $\partial \mathcal{D}^g : L^2(O) \rightrightarrows L^2(O)$ of \mathcal{D}^g is characterized by

$$\begin{cases} \text{dom}(\partial \mathcal{D}^g) = \mathcal{H}_{g,W}^J \\ \partial \mathcal{D}^g(u) = \{d^J \mathcal{D}^g(u)\}. \end{cases}$$

(e) Set

$$\mathcal{D}_{g,W}^J := \left\{ u \in \text{dom}(D_g^J) : w_{g,1}^u \in L^2(O) \text{ and } w_{g,2}^u \in L^2(O) \right\}$$

with $w_{g,1}^u, w_{g,2}^u : O \rightarrow \mathbb{R}$ given by:

$$w_{g,1}^u(x) := \int_{\partial^J O} J(|x - y|) \partial_2 W(x, D_g^J u(x, y)) dy; \quad (2.35)$$

$$w_{g,2}^u(x) := \int_{\partial^J O} J(|x - y|) \partial_2 W(y, D_g^J u(x, y)) dy. \quad (2.36)$$

Then, $\mathcal{D}_{g,W}^J \subset \mathcal{H}_{g,W}^J$ and

$$d^J \mathcal{D}^g(u) = \int_{\partial^J O} J(|\cdot - y|) [\partial_2 W(\cdot, D_g^J u(\cdot, y)) + \partial_2 W(y, D_g^J u(\cdot, y))] dy \text{ for all } u \in \mathcal{D}_{g,W}^J. \quad (2.37)$$

Proof of Proposition 2.18. (a) The uniqueness of the function $h \in L^2(O)$ associated with $u \in \mathcal{H}_{g,W}^J$ follows directly from the density of $\text{dom}(D_g^J)$ in $L^2(O)$ (see Proposition 2.8).

(b) Fix $(u, v) \in \text{dom}(D_g^J) \times \text{dom}(D_g^J)$. It is easy to see that for every $t \in \mathbb{R}^*$,

$$\frac{1}{t} (\mathcal{D}^g(u + tv) - \mathcal{D}^g(u)) = \int_O \int_{\partial^J O} h_t(x, y) dx dy. \quad (2.38)$$

with $h_t : O \times \partial^J O \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} h_t(x, y) &:= \frac{1}{t} [W(x, D_g^J u(x, y) + tJ(|x - y|)v(x)) - W(x, D_g^J u(x, y))] \\ &\quad + \frac{1}{t} [W(y, D_g^J u(x, y) + tJ(|x - y|)v(x)) - W(y, D_g^J u(x, y))]. \end{aligned}$$

By (I₃), for every $(x, y) \in O \times \partial^J O$, we have

$$\begin{aligned} \lim_{t \rightarrow 0} h_t(x, y) &= \partial_2 W(x, D_g^J u(x, y)) J(|x - y|) v(x) + \partial_2 W(y, D_g^J u(x, y)) J(|x - y|) v(x) \\ &= J(|x - y|) [\partial_2 W(x, D_g^J u(x, y)) + \partial_2 W(y, D_g^J u(x, y))] v(x). \end{aligned}$$

Let $h : O \times \partial^J O \rightarrow [0, \infty[$ be defined by

$$h(x, y) := 2L |J(|x - y|)v(x)| (1 + 2|D_g^J u(x, y)| + |J(|x - y|)v(x)|)$$

with $L > 0$ given by (A₁). Moreover, it is easy to see that

$$\begin{aligned} h(x, y) &\leq 2L |D_g^J v(x, y)| (1 + 2|D_g^J u(x, y)|) \\ &\quad + 2L |J(|x - y|)g(y)| (1 + 2|D_g^J u(x, y)|) \\ &\quad + 4L (|D_g^J v(x, y)|^2 + |J(|x - y|)g(y)|^2). \end{aligned}$$

Taking (S₂) into account, since $u \in \text{dom}(D_g^J)$ and $v \in \text{dom}(D_g^J)$, we have

$$h \in L^1(O \times \partial^J O).$$

On the other hand, by (A₁), we can assert that $\{h_t\}_{t \in]0, 1]}$ is dominated by h , i.e. for every $t \in]0, 1]$,

$$|h_t(x, y)| \leq h(x, y) \text{ for all } (x, y) \in O \times \partial^J O.$$

Hence, letting $t \rightarrow 0$ in (2.38) and using the Lebesgue dominated convergence theorem, we obtain (2.33).

(c) If $(u, v) \in \mathcal{H}_{g,W}^J \times \text{dom}(D_g^J)$ then, by definition of $\mathcal{H}_{g,W}^J$, one has

$$\int_O \int_{\partial^J O} J(|x - y|) [\partial_2 W(x, D_g^J u) + \partial_2 W(y, D_g^J u)] v(x) dx dy = \int_O h_u^g(x) v(x) dx,$$

and (2.34) follows from (2.33).

(d) It suffices to prove the following two assertions:

(d₁) for every $u \in \mathcal{H}_{g,W}^J$, $d^J \mathcal{D}^g(u) \in \partial \mathcal{D}^g(u)$;

(d₂) for every $u \in L^2(O)$, if $h \in \partial \mathcal{D}^g(u)$ then $u \in \mathcal{H}_{g,W}^J$ and $h = d^J \mathcal{D}^g(u)$.

Proof of (d₁). Fix $u \in \mathcal{H}_{g,W}^J$ and fix any $v \in \text{dom}(D_g^J)$. By (I₃), for every $(x, y) \in O \times \partial^J O$, we have

$$\begin{aligned} W(z, D_g^J v(x, y)) &\geq W(z, D_g^J u(x, y)) + \partial_2 W(z, D_g^J u(x, y)) (D_g^J v(x, y) - D_g^J u(x, y)) \\ &= W(z, D_g^J u(x, y)) + J(|x - y|) \partial_2 W(z, D_g^J u(x, y)) (v(x) - u(x)) \end{aligned}$$

for $z \in \{x, y\}$. Hence

$$\begin{aligned} &W(x, D_g^J v(x, y)) + W(y, D_g^J v(x, y)) \\ &\geq W(x, D_g^J u(x, y)) + W(y, D_g^J u(x, y)) \\ &\quad + J(|x - y|) [\partial_2 W(x, D_g^J u(x, y)) + \partial_2 W(y, D_g^J u(x, y))] (v(x) - u(x)). \end{aligned}$$

Integrating over $O \times \partial^J O$, we infer that

$$\begin{aligned} \mathcal{D}^g(v) &\geq \mathcal{D}^g(u) \\ &\quad + \int_O \int_{\partial^J O} J(|x - y|) [\partial_2 W(x, D_g^J u(x, y)) + \partial_2 W(y, D_g^J u(x, y))] (v(x) - u(x)) dx dy. \end{aligned}$$

As $u \in \mathcal{H}_{g,W}^J$ we can assert that

$$\begin{aligned} & \int_O \int_{\partial^J O} J(|x-y|) [\partial_2 W(x, D_g^J u(x,y)) + \partial_2 W(y, D_g^J u(x,y))] (v(x) - u(x)) dx dy \\ &= \int_O d^J \mathcal{D}^g(u)(x)(v(x) - u(x)) dx, \end{aligned}$$

so that for every $v \in \text{dom}(D_g^J)$,

$$\mathcal{D}^g(v) \geq \mathcal{D}^g(u) + \int_O d^J \mathcal{D}^g(u)(x)(v(x) - u(x)) dx. \quad (2.39)$$

Since $\text{dom}(\mathcal{D}^g) = \text{dom}(D_g^J)$, the inequality (2.39) extends to all $v \in L^2(O)$, which shows that $d^J \mathcal{D}^g(u) \in \partial \mathcal{D}^g(u)$.

Proof of (d₂). Let $u \in L^2(O)$ and assume that $h \in \partial \mathcal{D}^g(u)$. Then, $\partial \mathcal{D}^g(u) \neq \emptyset$ and so $u \in \text{dom}(D_g^J)$ because $\text{dom}(\mathcal{D}^g) = \text{dom}(D_g^J)$. Fix any $v \in \text{dom}(D_g^J)$. Then, by using the definition of $\partial \mathcal{D}^g(u)$, one has

$$\mathcal{D}^g(u + tv) \geq \mathcal{D}^g(u) + t \int_O h(x)v(x) dx \text{ for all } t \in \mathbb{R}^*.$$

Hence:

$$\frac{1}{t} (\mathcal{D}^g(u + tv) - \mathcal{D}^g(u)) \leq \int_O h(x)v(x) dx \text{ for all } t < 0; \quad (2.40)$$

$$\frac{1}{t} (\mathcal{D}^g(u + tv) - \mathcal{D}^g(u)) \geq \int_O h(x)v(x) dx \text{ for all } t > 0. \quad (2.41)$$

Letting $t \rightarrow 0^-$ in (2.40) and $t \rightarrow 0^+$ in (2.41), and using (2.33), we deduce that

$$\begin{aligned} & \int_O \int_{\partial^J O} J(|x-y|) [\partial_2 W(x, D_g^J u(x,y)) + \partial_2 W(y, D_g^J u(x,y))] v(x) dx dy \\ &= \int_O h(x)v(x) dx \text{ for all } v \in \text{dom}(D_g^J). \end{aligned}$$

Consequently $u \in \mathcal{H}_{g,W}^J$ and $h = d^J \mathcal{D}^g(u)$ by (a) and (c).

(e) If $u \in \mathcal{D}_W^J$ then $u \in \mathcal{H}_{g,W}^J$ with $h_u^g = w_{g,1}^u + w_{g,2}^u$, where $w_{g,1}^u \in L^2(O)$ and $w_{g,2}^u \in L^2(O)$ are given by (2.35) and (2.36) respectively. Thus $\mathcal{D}_{g,W}^J \subset \mathcal{H}_{g,W}^J$ and, by (c), (2.37) holds. ■

Let $\partial \mathcal{F}^g : L^2(O) \rightrightarrows L^2(O)$ denote the subdifferential of $\mathcal{F}^g := \mathcal{F} + \mathcal{D}^g$. From Propositions 2.17 and 2.18 we deduce the following result characterizing $\partial \mathcal{F}^g$.

Corollary 2.19. *Under the hypotheses of Proposition 2.18, the subdifferential $\partial \mathcal{F}^g$ is characterized by*

$$\left\{ \begin{array}{l} \text{dom}(\partial \mathcal{F}^g) = \mathcal{H}_W^J \cap \mathcal{H}_{g,W}^J \\ \partial \mathcal{F}^g(u) = \left\{ d^J \mathcal{F}^g(u) := d^J \mathcal{F}(u) + d^J \mathcal{D}^g(u) \right\}. \end{array} \right.$$

Proof of Corollary 2.19. First, as $\mathcal{F}^g = \mathcal{F} + \mathcal{D}^g$ we have $\text{dom}(\partial \mathcal{F}^g) = \text{dom}(\partial \mathcal{F}) \cap \text{dom}(\partial \mathcal{D}^g)$. Hence, by Propositions 2.17(d) and 2.18(d), $\text{dom}(\partial \mathcal{F}^g) = \mathcal{H}_W^J \cap \mathcal{H}_{g,W}^J$.

On the other hand, since \mathcal{F} and \mathcal{D}^g are proper, convex and lower semicontinuous (see Remark 2.11 and Proposition 2.12), by [ABM14, Theorem 9.5.4, p. 358] we have

$$\partial\mathcal{F} + \partial\mathcal{D}^g \subset \partial(\mathcal{F} + \mathcal{D}^g) = \partial\mathcal{F}^g.$$

So, it remains to prove that for every $u \in \text{dom}(\partial\mathcal{F}^g) = \mathcal{H}_W^J \cap \mathcal{H}_{g,W}^J$,

$$\partial\mathcal{F}^g(u) \subset \left\{ d^J\mathcal{F}(u) + d^J\mathcal{D}^g(u) \right\}.$$

Given $u \in \mathcal{H}_W^J \cap \mathcal{H}_{g,W}^J$, let $h \in \partial\mathcal{F}^g(u)$. Fix any $v \in \text{dom}(\mathcal{F}^g) = \text{dom}(D^J) \cap \text{dom}(D_g^J)$. By definition of $\partial\mathcal{F}^g(u)$ we have

$$\mathcal{F}(u + tv) + \mathcal{D}^g(u + tv) \geq \mathcal{F}(u) + \mathcal{D}^g(u) + t \int_O h(x)v(x)dx \text{ for all } t \in \mathbb{R}.$$

Hence:

$$\frac{1}{t} \left(\mathcal{F}(u + tv) - \mathcal{F}(u) + \mathcal{D}^g(u + tv) - \mathcal{D}^g(u) \right) \geq \int_O h(x)v(x)dx \text{ for all } t > 0; \quad (2.42)$$

$$\frac{1}{t} \left(\mathcal{F}(u + tv) - \mathcal{F}(u) + \mathcal{D}^g(u + tv) - \mathcal{D}^g(u) \right) \leq \int_O h(x)v(x)dx \text{ for all } t < 0. \quad (2.43)$$

By Propositions 2.17(b)-(c) and 2.18(b)-(c) we have

$$\lim_{t \rightarrow 0} \frac{1}{t} \left(\mathcal{F}(u + tv) - \mathcal{F}(u) + \mathcal{D}^g(u + tv) - \mathcal{D}^g(u) \right) = \int_O (d^J\mathcal{F}(u)(x) + d^J\mathcal{D}^g(u)(x))v(x)dx.$$

Letting $t \rightarrow 0^+$ in (2.42) and $t \rightarrow 0^-$ in (2.43), we deduce that

$$\int_O (d^J\mathcal{F}(u)(x) + d^J\mathcal{D}^g(u)(x))v(x)dx = \int_O h(x)v(x)dx \text{ for all } v \in \text{dom}(D^J) \cap \text{dom}(D_g^J).$$

Since $\text{dom}(D^J) \cap \text{dom}(D_g^J)$ is dense in $L^2(O)$ for the strong convergence in $L^2(O)$ (see Proposition 2.9), the previous equality extends to all $v \in L^2(O)$. Consequently

$$h = d^J\mathcal{F}(u) + d^J\mathcal{D}^g(u),$$

which completes the proof. ■

3. COMPARISON PRINCIPLES AND BOUNDED SOLUTIONS OF NONLOCAL EVOLUTION PROBLEMS

3.1. Nonlocal reaction-diffusion problems of singular gradient flow type. Let $O \subset \mathbb{R}^d$ be a bounded open set. Given $T > 0$ and $\mathcal{E} : L^2(O) \rightarrow [0, \infty]$ a proper, convex and lower semicontinuous functional, we consider the following reaction-diffusion Cauchy problem of gradient flow type⁵:

$$(\mathcal{P}_{T,F,\mathcal{E}}^{u_0}) \begin{cases} \frac{du}{dt}(t) + \partial\mathcal{E}(u(t)) \ni F(t, u(t)) \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T] \\ u(0) = u_0 \in \overline{\text{dom}(\partial\mathcal{E})}, \end{cases}$$

⁵Implicitly, the problem is subject to the constraint $u(t) \in \text{dom}(\partial\mathcal{E})$ for \mathcal{L}^1 -a.a. $t \in [0, T]$, which is inherent in the differential inclusion $(\mathcal{P}_{T,F,\mathcal{E}}^{u_0})$ and therefore not stated explicitly.

where the reaction term $F : [0, \infty[\times L^2(O) \rightarrow L^2(O)$ is a Borel measurable map satisfying the following two conditions:

(R₁^T) there exists $L \in L^2([0, T])$ such that for every $(u, v) \in L^2(O) \times L^2(O)$ and every $t \in [0, T]$,

$$\|F(t, u) - F(t, v)\|_{L^2(O)} \leq L(t) \|u - v\|_{L^2(O)};$$

(R₂^T) $\|F(\cdot, 0)\|_{L^2(O)} \in L^2([0, T])$.

Definition 3.1. By a solution⁶ of $(\mathcal{P}_{T,F,\mathcal{E}}^{u_0})$ we mean $u \in AC([0, T]; L^2(O))$ such that:

- $u(0) = u_0$;
- $u(t) \in \text{dom}(\partial\mathcal{E})$ for \mathcal{L}^1 -a.a. $t \in [0, T]$;
- $\frac{du}{dt}(t) + \partial\mathcal{E}(u(t)) \ni F(t, u(t))$ for \mathcal{L}^1 -a.a. $t \in [0, T]$.

From now on, the class of Borel measurable maps $F : [0, \infty[\times L^2(O) \rightarrow L^2(O)$ verifying (R₁^T)–(R₂^T) is denoted by $\mathcal{F}_{(R_1^T)-(R_2^T)}$. The following general result ensures the existence and uniqueness of a (not necessarily bounded) solution of $(\mathcal{P}_{T,F,\mathcal{E}}^{u_0})$. For a proof we refer to [AHMM22, Theorem 2.2, p. 16].

Theorem 3.2. *Let $T > 0$. If $u_0 \in \overline{\text{dom}(\partial\mathcal{E})}$ and $F \in \mathcal{F}_{(R_1^T)-(R_2^T)}$ then $(\mathcal{P}_{T,F,\mathcal{E}}^{u_0})$ admits a unique solution $u \in AC([0, T]; L^2(O))$. Moreover, if $F(\cdot, u(\cdot)) \in AC([0, T]; L^2(O))$ then u admits a right derivative $\frac{d^+u}{dt}(t)$ at every $t \in [0, T[$ and satisfies*

$$\begin{cases} u(t) \in \text{dom}(\partial\mathcal{E}) \text{ for all } t \in]0, T] \\ \frac{d^+u}{dt}(t) + [\partial\mathcal{E}(u(t)) - F(t, u(t))]^0 = 0 \text{ for all } t \in [0, T[, \end{cases}$$

where $[\partial\mathcal{E}(u(t)) - F(t, u(t))]^0$ denotes the element of the minimal norm of the set $\partial\mathcal{E}(u(t)) - F(t, u(t))$, i.e. $\|[\partial\mathcal{E}(u(t)) - F(t, u(t))]^0\|_{L^2(O)} \leq \|v\|_{L^2(O)}$ for all $v \in \partial\mathcal{E}(u(t)) - F(t, u(t))$.

In this paper, we consider reaction-diffusion problems with nonlocal singular diffusion terms, i.e. $\mathcal{E} = \mathcal{J}$ or $\mathcal{E} = \mathcal{J}^g = \mathcal{J} + \mathcal{D}^g$, where $\mathcal{J}, \mathcal{J}^g : L^2(O) \rightarrow [0, \infty]$ are given by (2.4) and (2.3)–(2.5), respectively, and with CP-structured reaction terms (see Definition 3.7). When $\mathcal{E} = \mathcal{J}$, the problem $(\mathcal{P}_{T,F,\mathcal{J}}^{u_0})$ is called *homogeneous Neumann-Cauchy nonlocal singular reaction-diffusion problem*, while when $\mathcal{E} = \mathcal{J}^g$, the problem $(\mathcal{P}_{T,F,\mathcal{J}^g}^{u_0})$ is called *Dirichlet-Cauchy nonlocal singular reaction-diffusion problem*.

⁶For simplicity, we use the term “solution” instead of “strong solution”.

Taking Propositions 2.17–2.18 and Corollary 2.19 into account, the problems $(\mathcal{P}_{T,F,\mathcal{J}}^{u_0})$ and $(\mathcal{P}_{T,F,\mathcal{J}^g}^{u_0})$ can be written as follows⁷:

$$\begin{aligned} (\mathcal{P}_{T,F,\mathcal{J}}^{u_0}) & \begin{cases} \frac{du}{dt}(t) + d^J \mathcal{J}(u(t)) = F(t, u(t)) \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T] \\ u(t) \in \mathcal{H}_W^J \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T] \\ u(0) = u_0 \in \overline{\mathcal{H}_W^J}; \end{cases} \\ (\mathcal{P}_{T,F,\mathcal{J}^g}^{u_0}) & \begin{cases} \frac{du}{dt}(t) + d^J \mathcal{J}^g(u(t)) = F(t, u(t)) \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T] \\ u(t) \in \mathcal{H}_W^J \cap \mathcal{H}_{g,W}^J \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T] \\ u(0) = u_0 \in \overline{\mathcal{H}_W^J \cap \mathcal{H}_{g,W}^J}. \end{cases} \end{aligned}$$

Since \mathcal{J} and \mathcal{J}^g are proper, convex and lower semicontinuous functional (see Remark 2.11 and Proposition 2.12), Theorem 3.2 yields the following result concerning existence and uniqueness of (not necessarily bounded) solutions for the problems $(\mathcal{P}_{T,F,\mathcal{J}}^{u_0})$ and $(\mathcal{P}_{T,F,\mathcal{J}^g}^{u_0})$.

Corollary 3.3. *Assume that (S₁) and (I₀)–(I₃) hold and let $g \in L^2(\partial^J O)$ satisfy (S₂). Let $T > 0$. If $u_0 \in \overline{\mathcal{H}_W^J}$ (resp. $u_0 \in \overline{\mathcal{H}_W^J \cap \mathcal{H}_{g,W}^J}$) and $F \in \mathcal{F}_{(\mathbb{R}_1^T)-(\mathbb{R}_2^T)}$ then $(\mathcal{P}_{T,F,\mathcal{J}}^{u_0})$ (resp. $(\mathcal{P}_{T,F,\mathcal{J}^g}^{u_0})$) admits a unique solution $u \in AC([0, T]; L^2(O))$. Moreover, if $F(\cdot, u(\cdot)) \in AC([0, T]; L^2(O))$ then u admits a right derivative $\frac{d^+u}{dt}(t)$ at every $t \in [0, T[$ and satisfies*

$$\begin{cases} u(t) \in \mathcal{H}_W^J \text{ for all } t \in]0, T] \\ \text{(resp. } u(t) \in \mathcal{H}_W^J \cap \mathcal{H}_{g,W}^J \text{ for all } t \in]0, T]) \\ \frac{d^+u}{dt}(t) + d^J \mathcal{J}(u(t)) = F(t, u(t)) \text{ for all } t \in [0, T[\\ \text{(resp. } \frac{d^+u}{dt}(t) + d^J \mathcal{J}^g(u(t)) = F(t, u(t)) \text{ for all } t \in [0, T[). \end{cases}$$

To prove the existence and uniqueness of bounded solutions for the problems $(\mathcal{P}_{T,F,\mathcal{J}}^{u_0})$ and $(\mathcal{P}_{T,F,\mathcal{J}^g}^{u_0})$, we first need to establish suitable comparison principles (see §3.2). In addition, we introduce the class of CP-structured reaction functionals (see §3.3), which is crucial for applying the comparison principles. With these tools, we then establish the existence and uniqueness of bounded solutions (see §3.4).

3.2. Comparison principles. We begin by recalling the notions of sub-solution and super-solution.

Definition 3.4. By a sub-solution (resp. a super-solution) of $(\mathcal{P}_{T,F,\mathcal{E}}^{u_0})$ with $\mathcal{E} \in \{\mathcal{J}, \mathcal{J}^g\}$, we mean $u \in AC([0, T]; L^2(O))$ such that:

⁷In $(\mathcal{P}_{T,F,\mathcal{J}}^{u_0})$ (resp. $(\mathcal{P}_{T,F,\mathcal{J}^g}^{u_0})$), the condition “ $u(t) \in \mathcal{H}_W^J$ for \mathcal{L}^1 -a.a. $t \in [0, T]$ ” (resp. “ $u(t) \in \mathcal{H}_W^J \cap \mathcal{H}_{g,W}^J$ for \mathcal{L}^1 -a.a. $t \in [0, T]$ ”) corresponds to the constraint “ $u(t) \in \text{dom}(\partial \mathcal{J})$ for \mathcal{L}^1 -a.a. $t \in [0, T]$ ” (resp. “ $u(t) \in \text{dom}(\partial \mathcal{J}^g)$ for \mathcal{L}^1 -a.a. $t \in [0, T]$ ”).

- $u(0) = u_0$;
- $u(t) \in \text{dom}(\partial\mathcal{E}) \in \{\mathcal{H}_W^J, \mathcal{H}_W^J \cap \mathcal{H}_{g,W}^J\}$ for \mathcal{L}^1 -a.a. $t \in [0, T]$;
- $\frac{du}{dt}(t) + d^J\mathcal{E}(u(t)) \leq F(t, u(t))$ for \mathcal{L}^1 -a.a. $t \in [0, T]$
(resp. $\frac{du}{dt}(t) + d^J\mathcal{E}(u(t)) \geq F(t, u(t))$ for \mathcal{L}^1 -a.a. $t \in [0, T]$).

Remark 3.5. If u is both a sub-solution and a super-solution of $(\mathcal{P}_{T,F,\mathcal{E}}^{u_0})$, then u is a solution of $(\mathcal{P}_{T,F,\mathcal{E}}^{u_0})$.

The following result establishes comparison principles for the problems $(\mathcal{P}_{T,F,\mathcal{J}}^{u_0})$ and $(\mathcal{P}_{T,F,\mathcal{J}g}^{u_0})$. Roughly speaking, a comparison principle states that if u_1 is a sub-solution and u_2 is a super-solution of the problem, and if the corresponding data are ordered, then the ordering is preserved by the evolution, i.e. $u_1(t) \leq u_2(t)$ for all $t \in [0, T]$.

Theorem 3.6. *Assume that (S₁) and (I₀)–(I₃) hold. Let $(g_i)_{i=1,2}$ be such that for $i = 1, 2$, $g_i \in L^2(\partial^J O)$ and g_i verifies (S₂) with $g = g_i$. Let $(u_0^i)_{i=1,2}$ be such that for $i = 1, 2$, $u_0^i \in \overline{\mathcal{H}_W^J}$ (resp. $u_0^i \in \overline{\mathcal{H}_W^J \cap \mathcal{H}_{g_i,W}^J}$). For each $i \in \{1, 2\}$, let $F_i : [0, \infty[\times L^2(O) \rightarrow L^2(O)$ be defined by*

$$F_i(t, v)(x) := f_i(t, x, v(x)), \quad (3.1)$$

where $f_i : [0, \infty[\times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ is a $(\mathcal{B}([0, \infty[) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R}))$ -measurable function. We further assume that the function f_2 is lipschitz continuous uniformly with respect to (t, x) , i.e. there exists $L > 0$ such that for every $(\xi, \xi') \in \mathbb{R} \times \mathbb{R}$,

$$\sup_{(t,x) \in [0, \infty[\times \mathbb{R}^d} |f_2(t, x, \xi) - f_2(t, x, \xi')| \leq L |\xi - \xi'|. \quad (3.2)$$

Let $T > 0$ and let $(u_i)_{i=1,2}$ be such that u_1 is a sub-solution of $(\mathcal{P}_{T,F_1,\mathcal{J}}^{u_0^1})$ (resp. $(\mathcal{P}_{T,F_1,\mathcal{J}g_1}^{u_0^1})$) and u_2 is a super-solution of $(\mathcal{P}_{T,F_2,\mathcal{J}}^{u_0^2})$ (resp. $(\mathcal{P}_{T,F_2,\mathcal{J}g_2}^{u_0^2})$). Then

$$\left. \begin{array}{l} u_0^1 \leq u_0^2 \\ F_1 \leq F_2 \\ (\text{resp. } g_1 \leq g_2) \end{array} \right\} \implies u_1(t) \leq u_2(t) \text{ for all } t \in [0, T].$$

Furthermore, if $g_1 = g_2 = g$, $F_1 = F_2 = F$ and if for $i = 1, 2$, u_i is solution of $(\mathcal{P}_{T,F,\mathcal{J}}^{u_0^i})$ (resp. $(\mathcal{P}_{T,F,\mathcal{J}g}^{u_0^i})$) then, for every $t \in [0, T]$,

$$\|u_2(t) - u_1(t)\|_{L^2(O)} \leq 2e^{Lt} \|u_0^1 - u_0^2\|_{L^2(O)} \quad (3.3)$$

with $L > 0$ given by (3.2), i.e. $(\mathcal{P}_{T,F,\mathcal{J}}^{u_0})$ (resp. $(\mathcal{P}_{T,F,\mathcal{J}g}^{u_0})$) is stable in $L^2(O)$.

Proof of Theorem 3.6. We only give the proof in the Dirichlet-Cauchy case. (The homogeneous Neumann-Cauchy case follows by similar arguments.)

Set $u = u_2 - u_1$. As u_1 and u_2 are continuous it suffices to prove that

$$u^-(s) = 0 \text{ for } \mathcal{L}^1\text{-a.a. } s \in [0, T], \quad (3.4)$$

where $u^- := \max\{-u, 0\}$. The proof of (3.4) is divided into three steps.

Step 1: using u^- as a test function. As u_1 is a sub-solution of $(\mathcal{P}_{T,F_1,\mathcal{F}^{g_1}}^{u_0^1})$ and u_2 a super-solution of $(\mathcal{P}_{T,F_2,\mathcal{F}^{g_2}}^{u_0^2})$, we have:

$$u_1(0) = u_0^1 \quad (3.5)$$

$$u_2(0) = u_0^2 \quad (3.6)$$

$$u_1(t) \in \mathcal{H}_W^J \cap \mathcal{H}_{g_1,W}^J \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T]; \quad (3.7)$$

$$u_2(t) \in \mathcal{H}_W^J \cap \mathcal{H}_{g_2,W}^J \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T]; \quad (3.8)$$

$$\frac{du_1}{dt}(t) + d^J \mathcal{F}^{g_1}(u_1(t)) \leq F_1(t, u_1(t)) \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T]; \quad (3.9)$$

$$\frac{du_2}{dt}(t) + d^J \mathcal{F}^{g_2}(u_2(t)) \geq F_2(t, u_2(t)) \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T]. \quad (3.10)$$

Taking the definition of \mathcal{H}_W^J into account, from (3.7) and (3.8) we see that $(u_1(t), u_2(t)) \in \text{dom}(D^J) \times \text{dom}(D^J)$ for \mathcal{L}^1 -a.a. $t \in [0, T]$. As $\text{dom}(D^J)$ is a vector space, for \mathcal{L}^1 -a.e. $t \in [0, T]$, we have $u(t) = u_2(t) - u_1(t) \in \text{dom}(D^J)$, so that

$$\begin{aligned} \int_O \int_O |D^J |u(t)|(x, y)|^2 dx dy &= \int_O \int_O J(|x - y|)^2 ||u(t)(x)| - |u(t)(y)||^2 dx dy \\ &\leq \int_O \int_O J(|x - y|)^2 |u(t)(x) - u(t)(y)|^2 dx dy \\ &= \int_O \int_O |D^J u(t)(x, y)|^2 dx dy < \infty. \end{aligned}$$

Consequently $|u(t)| \in \text{dom}(D^J)$ for \mathcal{L}^1 -a.a. $t \in [0, T]$. Since $u^- = \frac{1}{2}(|u| - u)$, it follows that

$$u^-(t) \in \text{dom}(D^J) \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T]. \quad (3.11)$$

Let us show that

$$u(t) \in \text{dom}(D_{g_1}^J) \cap \text{dom}(D_{g_2}^J) \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T]. \quad (3.12)$$

Indeed, writing $u_2(t)(x) - u_1(t)(x) - g_1(y) = (u_2(t)(x) - g_2(y)) - (u_1(t)(x) - g_1(y)) - 2g_1(y) + g_2(y)$, we infer that

$$\begin{aligned} \int_O \int_{\partial^J O} |D_{g_1}^J u(t)(x, y)|^2 dx dy &= \int_O \int_{\partial^J O} |D_{g_1}^J (u_2(t) - u_1(t))(x, y)|^2 dx dy \\ &= \int_O \int_{\partial^J O} |J(|x - y|)(u_2(t)(x) - u_1(t)(x) - g_1(y))|^2 dx dy \\ &\leq 4 \int_O \int_{\partial^J O} |D_{g_1}^J u_1(t)(x, y)|^2 dx dy + 4 \int_O \int_{\partial^J O} |D_{g_2}^J u_2(t)(x, y)|^2 dx dy \\ &\quad + 16 \int_O \int_{\partial^J O} |J(|x - y|)g_1(y)|^2 dx dy \\ &\quad + 4 \int_O \int_{\partial^J O} |J(|x - y|)g_2(y)|^2 dx dy. \end{aligned} \quad (3.13)$$

For $i = 1, 2$, taking the definition of $\mathcal{H}_{g_i, W}^J$ into account, from (3.7), (3.8) and (S₂) with $g = g_i$, (3.13) yields

$$\int_O \int_{\partial^J O} |D_{g_1}^J u(t)(x, y)|^2 dx dy < \infty \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T].$$

In the same way, writing $u_2(t)(x) - u_1(t)(x) - g_2(y) = (u_2(t)(x) - g_2(y)) - (u_1(t)(x) - g_1(y)) - g_1(y)$, we have

$$\int_O \int_{\partial^J O} |D_{g_2}^J u(t)(x, y)|^2 dx dy < \infty \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T], \quad (3.14)$$

which proves (3.12).

We now establish

$$|u|(t) \in \text{dom}(D_{g_1}^J) \cap \text{dom}(D_{g_2}^J) \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T]. \quad (3.15)$$

Indeed, since $g_2 - g_1 \geq 0$, we have

$$\begin{aligned} |J(|x - y|)(|u|(t)(x) - g_1(y))|^2 &= |J(|x - y|)|^2 |u|(t)(x) - (g_2(y) - g_1(y)) - 2g_1(y) + g_2(y)|^2 \\ &\leq 3|J(|x - y|)(u(t) - g_2(y))|^2 + 27|J(|x - y|)g_1(y)|^2 \\ &\quad + 3|J(|x - y|)g_2(y)|^2 \end{aligned}$$

and

$$\begin{aligned} |J(|x - y|)(|u|(t)(x) - g_2(y))|^2 &= |J(|x - y|)|^2 |u|(t)(x) - (g_2(y) - g_1(y)) - g_1(y)|^2 \\ &\leq 2|J(|x - y|)(u(t) - g_2(y))|^2 + 8|J(|x - y|)g_1(y)|^2. \end{aligned}$$

Hence, by using (3.14) and (S₂) with $g = g_1$ and $g = g_2$, we obtain

$$\begin{aligned} \int_O \int_{\partial^J O} |D_{g_1}^J |u|(t)(x, y)|^2 dx dy &\leq 3 \int_O \int_{\partial^J O} |D_{g_2}^J u(t)(x, y)|^2 dx dy \\ &\quad + 27 \int_O \int_{\partial^J O} |J(|x - y|)g_1(y)|^2 dx dy \\ &\quad + 3 \int_O \int_{\partial^J O} |J(|x - y|)g_2(y)|^2 dx dy \\ &< \infty \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T] \end{aligned}$$

and

$$\begin{aligned} \int_O \int_{\partial^J O} |D_{g_2}^J |u|(t)(x, y)|^2 dx dy &\leq 2 \int_O \int_{\partial^J O} |D_{g_2}^J u(t)(x, y)|^2 dx dy \\ &\quad + 8 \int_O \int_{\partial^J O} |J(|x - y|)g_1(y)|^2 dx dy \\ &< \infty \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T]. \end{aligned}$$

This proves (3.15).

Since $u^- = \frac{1}{2}(|u| - u)$ and $\text{dom}(D_{g_1}^J)$ and $\text{dom}(D_{g_2}^J)$ are vector spaces (see Remark 2.7), from (3.12) and (3.15) we deduce that

$$u^-(t) \in \text{dom}(D_{g_1}^J) \cap \text{dom}(D_{g_2}^J) \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T]. \quad (3.16)$$

Fix any $t \in [0, T]$ satisfying (3.7), (3.8), (3.9), (3.10), (3.11) and (3.16). To simplify the notation, we omit the explicit dependence on t in u_1 , u_2 and u^- . Combining (3.9) with (3.10) we have

$$\frac{du}{dt} + d^J \mathcal{F}^{g_2}(u_2) - d^J \mathcal{F}^{g_1}(u_1) \geq F_2(t, u_2) - F_1(t, u_1).$$

Multiplying this inequality by the nonnegative function u^- and integrating over O , we obtain

$$\int_O \frac{du}{dt} u^- dx + \int_O d^J \mathcal{F}^{g_2}(u_2) u^- dx - \int_O d^J \mathcal{F}^{g_1}(u_1) u^- dx \geq \int_O [F_2(t, u_2) - F_1(t, u_1)] u^- dx.$$

Taking (3.7) and (3.8), together with (3.11) and (3.16) into account, and using Propositions 2.17(b)–(c) and 2.18(b)–(c), as well as Corollary 2.19, we deduce that for $i = 1, 2$,

$$\begin{aligned} \int_O d^J \mathcal{F}^{g_i}(u_i) u^- dx &= \int_O d^J \mathcal{F}(u_i) u^- dx + \int_O d^J \mathcal{D}^g(u_i) u^- dx \\ &= \int_O \int_O \partial_2 W(x, D^J u_i(x, y)) D^J u^-(x, y) dx dy \\ &\quad + \int_O \int_{\partial^J O} J(|x - y|) [\partial_2 W(x, D_{g_i}^J u_i(x, y)) + \partial_2 W(y, D_{g_i}^J u_i(x, y))] u^-(x) dx dy. \end{aligned}$$

Hence, by using (3.1),

$$\begin{aligned} &\int_O \frac{du}{dt} u^- dx + \int_O \int_O [\partial_2 W(x, D^J u_2(x, y)) - \partial_2 W(x, D^J u_1(x, y))] D^J u^-(x, y) dx dy \\ &+ \int_O \int_{\partial^J O} J(|x - y|) [\partial_2 W(x, D_{g_2}^J u_2(x, y)) - \partial_2 W(x, D_{g_1}^J u_1(x, y))] u^-(x) dx dy \\ &+ \int_O \int_{\partial^J O} J(|x - y|) [\partial_2 W(y, D_{g_2}^J u_2(x, y)) - \partial_2 W(y, D_{g_1}^J u_1(x, y))] u^-(x) dx dy \\ &\geq \int_O (f_2(t, x, u_2(x)) - f_1(t, x, u_1(x))) u^-(x) dx. \end{aligned}$$

Step 2: using the nondecreasing property of $\partial_2 W(x, \cdot)$. The goal of this step is to show that

$$\int_O \frac{du}{dt} u^- dx \geq \int_O (f_2(t, x, u_2(x)) - f_1(t, x, u_1(x))) u^-(x) dx. \quad (3.17)$$

Set

$$S := S_1 + S_2 + S_3$$

with:

$$\begin{aligned} S_1 &:= \int_O \int_O [\partial_2 W(x, D^J u_2(x, y)) - \partial_2 W(x, D^J u_1(x, y))] D^J u^-(x, y) dx dy; \\ S_2 &:= \int_O \int_{\partial^J O} J(|x - y|) [\partial_2 W(x, D_{g_2}^J u_2(x, y)) - \partial_2 W(x, D_{g_1}^J u_1(x, y))] u^-(x) dx dy; \\ S_3 &:= \int_O \int_{\partial^J O} J(|x - y|) [\partial_2 W(y, D_{g_2}^J u_2(x, y)) - \partial_2 W(y, D_{g_1}^J u_1(x, y))] u^-(x) dx dy. \end{aligned}$$

To establish (3.17) it suffices to prove that $S \leq 0$. For this, it is enough to prove that

$$S_1 \leq 0, S_2 \leq 0 \text{ and } S_3 \leq 0.$$

Step 2-1: proving that $S_1 \leq 0$. Set:

$$\begin{aligned} P_{-+} &:= \left\{ (x, y) \in O \times O : u(x) \leq 0 \text{ and } u(y) \geq 0 \right\}; \\ P_{--} &:= \left\{ (x, y) \in O \times O : u(x) \leq 0 \text{ and } u(y) < 0 \right\}; \\ P_{++} &:= \left\{ (x, y) \in O \times O : u(x) > 0 \text{ and } u(y) \geq 0 \right\}; \\ P_{+-} &:= \left\{ (x, y) \in O \times O : u(x) > 0 \text{ and } u(y) < 0 \right\}. \end{aligned}$$

Then

$$S_1 = \sum_{i,j \in \{-,+\}} \iint_{P_{ij}} \Psi(x, y) dx dy.$$

with

$$\Psi(x, y) := [\partial_2 W(x, D^J u_2(x, y)) - \partial_2 W(x, D^J u_1(x, y))] D^J u^-(x, y).$$

Therefore, to prove that $S_1 \leq 0$, it is sufficient to prove that for every $i, j \in \{-, +\}$,

$$\iint_{P_{ij}} \Psi(x, y) dx dy \leq 0. \quad (3.18)$$

Case where $(i, j) = (-, +)$. Fix any $(x, y) \in P_{-+}$. Then $u_2(x) \leq u_1(x)$ and $u_2(y) \geq u_1(y)$, hence $u_2(x) - u_2(y) \leq u_1(x) - u_1(y)$. Therefore

$$D^J u_2(x, y) \leq D^J u_1(x, y).$$

From (I₀) and (I₃) (see Remark 2.11) we can assert that (A₄) holds, i.e. for every $x \in \mathbb{R}^d$, the derivative $\partial_2 W(x, \cdot)$ of $W(x, \cdot)$ is nondecreasing. It follows that

$$\partial_2 W(x, D^J u_2(x, y)) - \partial_2 W(x, D^J u_1(x, y)) \leq 0. \quad (3.19)$$

On the other hand, as $u(x) \leq 0$ and $u(y) \geq 0$ we have $u^-(x) = -u(x)$ and $u^-(y) = 0$. Hence

$$D^J u^-(x, y) = -J(|x - y|)u(x) \geq 0. \quad (3.20)$$

From (3.19) and (3.20) we deduce that $\Psi(x, y) \leq 0$ for all $(x, y) \in P_{-+}$, and (3.18) follows for $(i, j) = (-, +)$.

Case where $(i, j) = (-, -)$. Fix any $(x, y) \in P_{--}$. Then $u^-(x) = -u(x)$ and $u^-(y) = -u(y)$. Hence

$$\begin{aligned} D^J u^-(x, y) &= -J(|x - y|)(u(x) - u(y)) \\ &= -J(|x - y|)(u_2(x) - u_1(x) - u_2(y) + u_1(y)) \\ &= -(J(|x - y|)(u_2(x) - u_2(y)) - J(|x - y|)(u_1(x) - u_1(y))) \\ &= -(D^J u_2(x, y) - D^J u_1(x, y)). \end{aligned}$$

But $[\partial_2 W(x, D^J u_2(x, y)) - \partial_2 W(x, D^J u_1(x, y))](D^J u_2(x, y) - D^J u_1(x, y)) \geq 0$ since $\partial_2 W(x, \cdot)$ is monotone. Consequently, for every $(x, y) \in P_{--}$,

$$\Psi(x, y) = -[\partial_2 W(x, D^J u_2(x, y)) - \partial_2 W(x, D^J u_1(x, y))](D^J u_2(x, y) - D^J u_1(x, y)) \leq 0,$$

and (3.18) follows for $(i, j) = (-, -)$.

Case where $(i, j) = (+, +)$. Fix any $(x, y) \in P_{++}$. Then $u^-(x) = 0$ and $u^-(y) = 0$. Hence $D^J u^-(x, y) = 0$. Consequently $\Psi(x, y) = 0$ for all $(x, y) \in P_{++}$. Thus (3.18) is satisfied for $(i, j) = (+, +)$.

Case where $(i, j) = (+, -)$. Fix any $(x, y) \in P_{+-}$. Then $u_2(x) > u_1(x)$ and $u_2(y) < u_1(y)$, hence $u_2(x) - u_2(y) > u_1(x) - u_1(y)$. Therefore

$$D^J u_2(x, y) > D^J u_1(x, y).$$

As $\partial_2 W(x, \cdot)$ is nondecreasing, it follows that

$$\partial_2 W(x, D^J u_2(x, y)) - \partial_2 W(x, D^J u_1(x, y)) \geq 0. \quad (3.21)$$

On the other hand, as $u(x) > 0$ and $u(y) < 0$ we have $u^-(x) = 0$ and $u^-(y) = -u(y)$. Hence

$$D^J u^-(x, y) = J(|x - y|)u(y) \leq 0. \quad (3.22)$$

From (3.21) and (3.22) we deduce that $\Psi(x, y) \leq 0$ for all $(x, y) \in P_{+-}$, and (3.18) follows for $(i, j) = (+, -)$.

Step 2-2: proving that $S_2 \leq 0$ and $S_3 \leq 0$. Fix any $(x, y) \in O \times \partial^J O$ and let us introduce the following formal notation:

$$\Delta_{(w,h)}^{(v,g)}(x, y) := [\partial_2 W(x, D_g^J v(x, y)) - \partial_2 W(x, D_h^J w(x, y))].$$

Without loss of generality, we can suppose that $u(x) \leq 0$. Then $u^-(x) = -u(x) = u_1(x) - u_2(x)$. First of all, we have

$$\begin{aligned} J(|x - y|)\Delta_{(u_1, g_1)}^{(u_2, g_2)}(x, y)u^-(x) &= -\Delta_{(u_1, g_2)}^{(u_2, g_2)}(x, y)J(|x - y|)(u_2(x) - u_1(x)) \\ &\quad + \Delta_{(u_1, g_1)}^{(u_1, g_2)}(x, y)J(|x - y|)u(x) \\ &= -\Delta_{(u_1, g_2)}^{(u_2, g_2)}(x, y)(D_{g_2}^J u_2(x, y) - D_{g_2}^J u_1(x, y)) \\ &\quad - \Delta_{(u_1, g_1)}^{(u_1, g_2)}(x, y)J(|x - y|)u(x). \end{aligned}$$

On the other hand, as $\partial_2 W(x, \cdot)$ is monotone we have

$$-\Delta_{(u_1, g_2)}^{(u_2, g_2)}(x, y)(D_{g_2}^J u_2(x, y) - D_{g_2}^J u_1(x, y)) \leq 0. \quad (3.23)$$

Moreover, $D_{g_2}^J u_1 \leq D_{g_1}^J u_1$ because $g_1 \leq g_2$, and so, since $\partial_2 W(x, \cdot)$ is nondecreasing, $\Delta_{(u_1, g_1)}^{(u_1, g_2)}(x, y) \leq 0$. Therefore

$$-\Delta_{(u_1, g_1)}^{(u_1, g_2)}(x, y)J(|x - y|)u(x) \leq 0. \quad (3.24)$$

From (3.23) and (3.24) it follows that for every $(x, y) \in O \times \partial^J O$,

$$J(|x - y|)\Delta_{(u_1, g_1)}^{(u_2, g_2)}(x, y)u^-(x) \leq 0,$$

which implies that $S_2 \leq 0$.

By the same arguments, we obtain $S_3 \leq 0$.

Step 3: applying Gronwall's lemma. From (3.17) we have

$$\begin{aligned} \int_O \frac{du}{dt}(t)u^-(t)dx &\geq \int_O (f_2(t, x, u_2(t)(x)) - f_2(t, x, u_1(t)))u^-(t)dx \\ &\quad + \int_O (f_2(t, x, u_1(t)(x)) - f_1(t, x, u_1(t)(x)))u^-(t)dx \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T]. \end{aligned}$$

As $F_1 \leq F_2$ we have $f_2(t, x, u_1(t)(x)) - f_1(t, x, u_1(t)(x)) \geq 0$ for all $(t, x) \in [0, T] \times O$. Therefore

$$\int_O \frac{du}{dt}(t)u^-(t)dx \geq \int_O (f_2(t, x, u_2(t)(x)) - f_2(t, x, u_1(t)(x)))u^-(t)dx \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T].$$

Since, by assumption, f_2 is lipschitz continuous uniformly with respect to (t, x) , there exists $L > 0$ such that

$$f_2(t, x, u_2(t)(x)) - f_2(t, x, u_1(t)(x)) \geq -L|u(t)(x)| \text{ for all } (t, x) \in [0, T] \times O.$$

Hence

$$\int_O \frac{du}{dt}(t)u^-(t)dx \geq -L \int_O |u(t)|u^-(t)dx \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T]. \quad (3.25)$$

Clearly, we have

$$\begin{aligned} \int_O \frac{du}{dt}(t)u^-(t)dx &= - \int_O \frac{du^-}{dt}(t)u^-(t)dx \\ &= -\frac{1}{2} \frac{d}{dt} \int_O |u^-(t)|^2 dx \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T]. \end{aligned} \quad (3.26)$$

Moreover, it is easy to see that

$$\int_O |u(t)|u^-(t)dx = \int_O |u^-(t)|^2 dx \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T]. \quad (3.27)$$

Combining (3.25) with (3.26) and (3.27) it follows that

$$\frac{d}{dt} \int_O |u^-(t)|^2 dx \leq 2L \int_O |u^-(t)|^2 dx \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T],$$

and consequently

$$\int_O |u^-(s)|^2 dx \leq \int_O |u^-(0)|^2 dx + \int_0^s 2L \int_O |u^-(t)|^2 dt \text{ for all } s \in [0, T].$$

Applying Gronwall's lemma we deduce that

$$\int_O |u^-(s)|^2 dx \leq e^{2Ls} \int_O |u^-(0)|^2 dx \text{ for all } s \in [0, T]. \quad (3.28)$$

As $u_1(0) = u_0^1$, $u_2(0) = u_0^2$ and $u_0^2 - u_0^1 \geq 0$ we have $u(0) = u_2(0) - u_1(0) \geq 0$, which implies that $u^-(0) = 0$. Hence $\int_O |u^-(0)|^2 dx = 0$. Therefore, by (3.28) we obtain

$$\int_O |u^-(s)|^2 dx = 0 \text{ for all } s \in [0, T],$$

which yields (3.4).

Step 4: stability. From (3.28), for any $s \in [0, T]$, we have

$$\int_O |(u_2 - u_1)^-(s)|^2 dx \leq e^{2Ls} \int_O |(u_2 - u_1)^-(0)|^2 dx. \quad (3.29)$$

Since $g_1 = g_2 = g$, $F_1 = F_2 = F$, we can invert the roles of u_2 and u_1 , which gives

$$\int_O |(u_2 - u_1)^+(s)|^2 dx \leq e^{2Ls} \int_O |(u_2 - u_1)^+(0)|^2 dx. \quad (3.30)$$

Collecting (3.29) and (3.30) we infer that

$$\begin{aligned} \int_O |(u_2 - u_1)(s)|^2 dx &= \int_O |(u_2 - u_1)^+(s) - (u_2 - u_1)^-(s)|^2 dx \\ &\leq 2 \int_O |(u_2 - u_1)^+(s)|^2 dx + 2 \int_O |(u_2 - u_1)^-(s)|^2 dx \\ &\leq 2e^{2Ls} \int_O \left(|(u_2 - u_1)^+(0)|^2 + |(u_2 - u_1)^-(0)|^2 \right) dx. \end{aligned}$$

As $|(u_2 - u_1)^+| \leq |u_2 - u_1|$ and $|(u_2 - u_1)^-| \leq |u_2 - u_1|$ we deduce that

$$\int_O |(u_2 - u_1)(s)|^2 dx \leq 4e^{2Ls} \int_O |(u_2 - u_1)(0)|^2 dx,$$

and (3.3) follows. ■

3.3. The class of CP-structured reaction functionals. The following class of reaction functionals, called the class of CP-structured reaction functionals on $[0, T]$ with $T > 0$ and denoted by $\mathcal{F}_{\text{CP}}^T$, was introduced in [AHMM19] (see also [AHMM22, §2.2.2, p. 27]).

Definition 3.7. Let $T > 0$. Given $F : [0, \infty[\times L^2(O) \rightarrow L^2(O)$ we say that F is a CP-structured reaction functional if

$$F(t, u)(x) = f(t, x, u(x))$$

for all $(t, x, u) \in [0, \infty[\times O \times L^2(O)$ where $f : [0, \infty[\times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ is a Borel measurable function satisfying the following three properties:

- (CP₁^T) $f(t, x, \zeta)$ is locally Lipschitz continuous in ζ uniformly with respect to $(t, x) \in [0, T] \times \mathbb{R}^d$;
- (CP₂^T) $f(\cdot, \cdot, 0) \in L^2([0, T]; L^2(O))$;
- (CP₃^T) there exist $\underline{f}, \bar{f} : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ with $\underline{f} \leq 0 \leq \bar{f}$ and $(\underline{\rho}, \bar{\rho}) \in \mathbb{R}^2$ with $\underline{\rho} \leq \bar{\rho}$ such that each of the two following ordinary differential equations

$$\begin{aligned} (\underline{\text{ODE}}^T) &\begin{cases} \frac{dz}{dt}(t) = \underline{f}(t, z(t)) \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T] \\ z(0) = \underline{\rho} \end{cases} \\ (\overline{\text{ODE}}^T) &\begin{cases} \frac{dz}{dt}(t) = \bar{f}(t, z(t)) \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T] \\ z(0) = \bar{\rho} \end{cases} \end{aligned}$$

admits at least a solution, \underline{z} for $(\underline{\text{ODE}})$ and \bar{z} for $(\overline{\text{ODE}})$, satisfying

$$\begin{cases} \underline{f}(t, \underline{z}(t)) \leq f(t, x, \underline{z}(t)) \\ \bar{f}(t, \bar{z}(t)) \geq f(t, x, \bar{z}(t)) \end{cases} \quad (3.31)$$

for $\mathcal{L}^1 \otimes \mathcal{L}^d$ -a.a. $(t, x) \in [0, T] \times \mathbb{R}^d$.

Remark 3.8. Note that if in (CP_1^T) we replace “locally Lipschitz continuous” by “Lipschitz continuous” then $\mathcal{F}_{\text{CP}}^T \subset \mathcal{F}_{(\mathbb{R}_1^T)-(\mathbb{R}_2^T)}$.

Remark 3.9. From (CP_3^T) we see that \underline{z} and \bar{z} are decreasing and increasing respectively, and so $\underline{z}(T) \leq \underline{z}(t) \leq \underline{z}(0) = \underline{\rho} \leq \bar{\rho} \leq \bar{z}(0) \leq \bar{z}(t) \leq \bar{z}(T)$ for all $t \in [0, T]$.

For $T > 0$ and $(a, b) \in \mathbb{R}^2$ with $a \leq b$, we consider the following problems:

$$\begin{aligned} (\mathcal{P}_{T,F,\mathcal{F}}^{u_0,[a,b]}) & \left\{ \begin{array}{l} \frac{du}{dt}(t) + d^J \mathcal{F}(u(t)) = F(t, u(t)) \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T] \\ u(t) \in \mathcal{H}_W^J \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T] \\ u(0) = u_0 \in \overline{\mathcal{H}_W^J} \cap L^2(O; [a, b]); \end{array} \right. \\ (\mathcal{P}_{T,F,\mathcal{F}^g}^{u_0,[a,b]}) & \left\{ \begin{array}{l} \frac{du}{dt}(t) + d^J \mathcal{F}^g(u(t)) = F(t, u(t)) \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T] \\ u(t) \in \mathcal{H}_W^J \cap \mathcal{H}_{g,W}^J \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T] \\ u(0) = u_0 \in \overline{\mathcal{H}_W^J \cap \mathcal{H}_{g,W}^J} \cap L^2(O; [a, b]). \end{array} \right. \end{aligned}$$

In the next paragraph, we show that, by applying the comparison principle, if the reaction functional F is CP-structured, one can prove the existence of bounded solutions for the problems $(\mathcal{P}_{T,F,\mathcal{F}}^{u_0,[a,b]})$ and $(\mathcal{P}_{T,F,\mathcal{F}^g}^{u_0,[a,b]})$, see Theorem 3.10.

3.4. Existence and uniqueness of bounded solutions. From Corollary 3.3 and Theorem 3.6 we can establish the existence and uniqueness of bounded solutions.

Theorem 3.10. *Assume that (S_1) and (I_0) – (I_3) . Let $T > 0$ and let $F \in \mathcal{F}_{\text{CP}}^T$ with (\underline{f}, \bar{f}) , $(\underline{\rho}, \bar{\rho})$ and (\underline{z}, \bar{z}) given by (CP_3^T) . Let $g \in L^2(\partial^J O)$ satisfy (S_2) and suppose further that*

$$(S_3) \quad \int_{\mathbb{R}^d} (J(|\xi|))^2 d\xi < \infty.$$

Set:

$$\underline{R}_g := \left\{ r \in \mathbb{R} : d^J \mathcal{D}^g(r)(x) \leq 0 \text{ for } \mathcal{L}^d\text{-a.a. } x \in O \right\}; \quad (3.32)$$

$$\bar{R}_g := \left\{ r \in \mathbb{R} : d^J \mathcal{D}^g(r)(x) \geq 0 \text{ for } \mathcal{L}^d\text{-a.a. } x \in O \right\}. \quad (3.33)$$

If $u_0 \in \overline{\mathcal{H}_W^J} \cap L^2(O; [\underline{\rho}, \bar{\rho}])$ (resp. if $u_0 \in \overline{\mathcal{H}_W^J \cap \mathcal{H}_{g,W}^J} \cap L^2(O; [\underline{\rho}, \bar{\rho}])$) and if $\underline{\rho} \in \underline{R}_g$ and $\bar{\rho} \in \bar{R}_g$, then $(\mathcal{P}_{T,F,\mathcal{F}}^{u_0,[\underline{\rho}, \bar{\rho}]})$ (resp. $(\mathcal{P}_{T,F,\mathcal{F}^g}^{u_0,[\underline{\rho}, \bar{\rho}]})$) admits a unique solution $u \in AC([0, T]; L^2(O))$ such that

$$\underline{z}(T) \leq \underline{z}(t) \leq u(t) \leq \bar{z}(t) \leq \bar{z}(T) \text{ for all } t \in [0, T].$$

Moreover, if $F(\cdot, u(\cdot)) \in AC([0, T]; L^2(O))$ then u admits a right derivative $\frac{d^+u}{dt}(t)$ at every $t \in [0, T[$ and satisfies

$$\begin{cases} u(t) \in \mathcal{H}_W^J \cap L^2(O; [\underline{z}(T), \bar{z}(T)]) \text{ for all } t \in]0, T[\\ (\text{resp. } u(t) \in \mathcal{H}_W^J \cap \mathcal{H}_{g,W}^J \cap L^2(O; [\underline{z}(T), \bar{z}(T)]) \text{ for all } t \in]0, T[) \\ \frac{d^+u}{dt}(t) + d^J \mathcal{F}(u(t)) = F(t, u(t)) \text{ for all } t \in [0, T[\\ (\text{resp. } \frac{d^+u}{dt}(t) + d^J \mathcal{F}^g(u(t)) = F(t, u(t)) \text{ for all } t \in [0, T[). \end{cases}$$

Remark 3.11. When $W(x, \xi) = a(x)|\xi|^2$ with $a(x) > 0$, since $\partial_2 W(x, \xi) = 2a(x)\xi$, a direct computation shows that for every $r \in \mathbb{R}$,

$$d^J \mathcal{D}^g(r)(x) = 2 \int_{\partial^J O} [a(x) + a(y)] J(|x - y|)^2 (r - g(y)) dy.$$

Therefore, we have:

$$\begin{aligned} \underline{\rho} \in \underline{R}_g &\iff \underline{\rho} \leq \operatorname{ess\,inf}_{x \in O} \frac{\int_{\partial^J O} [a(x) + a(y)] J(|x - y|)^2 g(y) dy}{\int_{\partial^J O} [a(x) + a(y)] J(|x - y|)^2 dy}; \\ \bar{\rho} \in \bar{R}_g &\iff \bar{\rho} \geq \operatorname{ess\,sup}_{x \in O} \frac{\int_{\partial^J O} [a(x) + a(y)] J(|x - y|)^2 g(y) dy}{\int_{\partial^J O} [a(x) + a(y)] J(|x - y|)^2 dy}. \end{aligned}$$

These conditions are the natural counterparts, in the singular nonlocal gradient setting, of the conditions introduced in [AHMM24, Corollary 2.11].

Proof of Theorem 3.10. We only give the proof in the Dirichlet-Cauchy case. (The homogeneous Neumann-Cauchy case follows by similar arguments).

Firstly, let $f : [0, T] \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ be given by Definition 3.7. Taking (CP_1^T) into account, from McShane extension's theorem we can assert that there exists $\hat{f} : [0, T] \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ such that:

- $\hat{f}(t, x, \zeta) = f(t, x, \zeta)$ for all $(t, x, \zeta) \in [0, T] \times \mathbb{R}^d \times [\underline{z}(T), \bar{z}(T)]$; (3.34)
- $\hat{f}(t, x, \zeta)$ is Lipschitz continuous in ζ uniformly with respect to $(t, x) \in [0, T] \times \mathbb{R}^d$.

Let $\hat{F} : [0, T] \times L^2(O) \rightarrow L^2(O)$ be given by $\hat{F}(t, u)(x) := \hat{f}(t, x, u(x))$. Then, it is clear that $\hat{F} \in \mathcal{F}_{(\mathbb{R}_1^T)-(\mathbb{R}_2^T)}$. Hence, by Corollary 3.3, $(\mathcal{P}_{T, \hat{F}, \mathcal{F}^g}^{u_0, [\underline{\rho}, \bar{\rho}]})$ admits a unique solution $u \in AC([0, T]; L^2(O))$.

Secondly, by (CP_3^T) (see Remark 3.9) we have

$$\underline{z}(T) \leq \underline{z}(t) \leq \underline{z}(0) = \underline{\rho} \leq \bar{\rho} \leq \bar{z}(0) \leq \bar{z}(t) \leq \bar{z}(T) \text{ for all } t \in [0, T], \quad (3.35)$$

so that

$$[\underline{z}(t), \bar{z}(t)] \subset [\underline{z}(T), \bar{z}(T)] \text{ for all } t \in [0, T]. \quad (3.36)$$

Thus, by (3.34) and (3.36),

$$\widehat{F}(t, \underline{z}(t)) = F(t, \underline{z}(t)) = f(t, \cdot, \underline{z}(t)) \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T].$$

Therefore, using (3.31) in $(\mathbb{C}\mathbb{P}_3^T)$, we have

$$\frac{d\underline{z}}{dt}(t) = \underline{f}(t, \underline{z}(t)) \leq \widehat{F}(t, \underline{z}(t)) \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T]. \quad (3.37)$$

As \underline{z} does not depend on the space variable, using (S_3) it is easy to see that $\underline{z} \in \mathcal{D}_W^J \cap \mathcal{D}_{g,W}^J \subset \mathcal{H}_W^J \cap \mathcal{H}_{g,W}^J$. Hence, by Proposition 2.17(e) and (I_2) – (I_3) , $d^J \mathcal{F}(\underline{z}(t)) = 0$ for \mathcal{L}^1 -a.a. $t \in [0, T]$. Hence $d^J \mathcal{F}^g(\underline{z}) = d^J \mathcal{D}^g(\underline{z})$, i.e.

$$d^J \mathcal{F}^g(\underline{z}(t)) = d^J \mathcal{D}^g(\underline{z}(t)) \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T].$$

Since $\underline{z} \in \mathcal{D}_{g,W}^J$, by Proposition 2.18(e) we have

$$d^J \mathcal{D}^g(\underline{z}) = \int_{\partial^J \mathcal{O}} J(|\cdot - y|) [\partial_2 W(\cdot, J(|\cdot - y|)(\underline{z} - g(y))) + \partial_2 W(y, J(|\cdot - y|)(\underline{z} - g(y)))] dy.$$

But, by (A_4) (see Remark 2.11), $\partial_2 W$ is nondecreasing with respect to its second variable, hence

$$\begin{aligned} \partial_2 W(\cdot, J(|\cdot - y|)(\underline{z} - g(y))) + \partial_2 W(y, J(|\cdot - y|)(\underline{z} - g(y))) &\leq \partial_2 W(\cdot, J(|\cdot - y|)(\underline{\rho} - g(y))) \\ &\quad + \partial_2 W(y, J(|\cdot - y|)(\underline{\rho} - g(y))) \end{aligned}$$

because $\underline{z} \leq \underline{\rho}$ by (3.35). As $\underline{\rho} \in \underline{R}_g$ it follows that

$$d^J \mathcal{D}^g(\underline{z}) \leq d^J \mathcal{D}^g(\underline{\rho}) \leq 0.$$

Therefore

$$d^J \mathcal{F}^g(\underline{z}(t)) \leq 0 \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T]. \quad (3.38)$$

From (3.37) and (3.38) we deduce that

$$\frac{d\underline{z}}{dt}(t) + d^J \mathcal{F}^g(\underline{z}(t)) \leq \widehat{F}(t, \underline{z}(t)) \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T].$$

Since $\underline{z}(0) = \underline{\rho}$, this shows that \underline{z} is a sub-solution of $(\mathcal{P}_{T, \widehat{F}, \mathcal{F}^g}^{\underline{\rho}})$. On the other hand, u is (a solution and so) a super-solution of $(\mathcal{P}_{T, \widehat{F}, \mathcal{F}^g}^{u_0, [\underline{\rho}, \bar{\rho}]})$ where $u(0) = u_0 \geq \underline{\rho}$. Then, applying Theorem 3.6 (with $u_0^1 = \underline{\rho}$, $u_0^2 = u_0$, $F_1 = F_2 = \widehat{F}$ and $g_1 = g_2 = g$), and taking (3.35) into account, we deduce that

$$\underline{z}(T) \leq \underline{z}(t) \leq \underline{\rho} \leq u(t) \text{ for all } t \in [0, T]. \quad (3.39)$$

Similarly, considering \bar{z} and using the fact that $\bar{\rho} \in \bar{R}_g$, we show that \bar{z} is a supersolution of $(\mathcal{P}_{T, \widehat{F}, \mathcal{F}^g}^{\bar{\rho}})$. Comparing it with u , viewed as a subsolution of $(\mathcal{P}_{T, \widehat{F}, \mathcal{F}^g}^{u_0, [\underline{\rho}, \bar{\rho}]})$ with $u(0) = u_0 \leq \bar{\rho}$, Theorem 3.6 (applied with $u_0^1 = u_0$, $u_0^2 = \bar{\rho}$, $F_1 = F_2 = \widehat{F}$, and $g_1 = g_2 = g$), together with (3.35), yields

$$u(t) \leq \bar{\rho} \leq \bar{z}(t) \leq \bar{z}(T) \text{ for all } t \in [0, T]. \quad (3.40)$$

From (3.39) and (3.40) it follows that

$$u(t) \in [\underline{z}(t), \bar{z}(t)] \subset [\underline{z}(T), \bar{z}(T)] \text{ for all } t \in [0, T]. \quad (3.41)$$

Finally, from (3.34) and (3.41) we see that $\widehat{F}(t, u(t)) = F(t, u(t))$ for all $t \in [0, T]$ so that u is the unique solution of $(\mathcal{P}_{T, \widehat{F}, \mathcal{F}^g}^{u_0, [\underline{\rho}, \overline{\rho}]})$, and the proof is complete. ■

Remark 3.12 (nonhomogeneous Neumann-Cauchy boundary conditions). In applications, one is often led to consider a prescribed flux entering or leaving the domain O . Indeed, for the homogeneous Neumann-Cauchy problem without a reaction term ($F = 0$), integrating the equation $\frac{du}{dt}(t) + d^J \mathcal{F}(u(t)) = 0$ over O yields

$$\frac{d}{dt} \int_O u(t, x) dx = 0.$$

This conservation implies that the domain O is isolated. To model external exchange, one must prescribe a nonlocal flux across the boundary. In the classical local setting, a nonhomogeneous Neumann condition $\frac{\partial u}{\partial \nu} = h$ on ∂O adds the linear term $-\int_{\partial O} hu \, d\sigma$ to the energy. In our nonlocal setting, let $G : [0, \infty[\rightarrow [0, \infty[$ be a Borel measurable function with $\text{supp}(G) \subset [0, \eta]$ for some $\eta > 0$, and set $\partial^G O := (O + B_\eta(0)) \setminus \overline{O}$. (The external exchange G can differ from the internal diffusion J to reflect different physical mechanisms). Given $h \in L^2(\partial^G O)$ satisfying

$$\int_O \int_{\partial^G O} |G(|x - y|)h(y)|^2 dx dy < \infty, \quad (3.42)$$

the nonhomogeneous Neumann condition is incorporated by adding to \mathcal{F} the linear functional $\mathcal{N}^h : L^2(O) \rightarrow \mathbb{R}$ defined by

$$\mathcal{N}^h(u) := - \int_O \left(\int_{\partial^G O} G(|x - y|)h(y) dy \right) u(x) dx.$$

Under (3.42), the function $\mathcal{N}^h : O \rightarrow \mathbb{R}$ defined by

$$\mathcal{N}^h(x) := \int_{\partial^G O} G(|x - y|)h(y) dy$$

belongs to $L^2(O)$, so that

$$\mathcal{N}^h(u) = - \int_O \mathcal{N}^h(x)u(x) dx \text{ for all } u \in L^2(O).$$

Hence, \mathcal{N}^h is a linear and continuous functional on $L^2(O)$. Therefore

$$\nabla \mathcal{N}^h(u) = -\mathcal{N}^h \text{ for all } u \in L^2(O).$$

Consequently $\partial(\mathcal{F} + \mathcal{N}^h)(u) = \{d^J \mathcal{F}(u) - \mathcal{N}^h\}$ and the corresponding nonhomogeneous Neumann-Cauchy problem can be rewritten as a homogeneous one by considering the modified reaction term

$$\widetilde{F}(t, u)(x) := F(t, u)(x) + \mathcal{N}^h(x).$$

Thus, the existence, uniqueness, and comparison results with h fixed (see Corollary 3.3 and Theorem 3.6) apply directly. For the existence of bounded solutions (see Theorem 3.10) one additionally needs the modified reaction term \widetilde{F} to remain CP-structured, which requires extra assumptions on \mathcal{N}^h , notably to satisfy (CP_3^T) , for instance $\mathcal{N}^h \in L^\infty(O)$. Indeed, if $(\underline{\rho}, \underline{f}, t \mapsto \underline{z}(t))$ and $(\overline{\rho}, \overline{f}, t \mapsto \overline{z}(t))$ are the triples for which F satisfies (CP_3^T) , then it is

easy to check that $(\underline{\rho}, \underline{f} - \|\mathcal{N}^h\|_{L^\infty(O)}, t \mapsto \underline{z}(t) - t\|\mathcal{N}^h\|_{L^\infty(O)})$ and $(\overline{\rho}, \overline{f} + \|\mathcal{N}^h\|_{L^\infty(O)}, t \mapsto \overline{z}(t) + t\|\mathcal{N}^h\|_{L^\infty(O)})$ are the triples for which \tilde{F} satisfies (CP_3^T) .

4. APPLICATIONS

Here, we present applications to problems that exhibit a nonlocal nature. The general modelling strategy is as follows. We first identify the quantity $u(t, x)$ that describes the state of the system at time $t \in [0, T]$ and spatial position $x \in O$, and whose evolution we wish to characterise. The corresponding nonlocal evolution equation is then obtained by applying a nonlocal balance principle. This principle is formulated for $u(t, x)$ (or for a related quantity depending on $u(t, x)$) and reads schematically:

$$\left[\begin{array}{l} \text{variation of the quantity} \\ \text{over a small time} \\ \text{interval } \Delta t \text{ at position } x \end{array} \right] = \left[\begin{array}{l} \text{exchange of the quantity} \\ \text{between } x \text{ to other} \\ \text{positions } y \in O \end{array} \right] + \left[\begin{array}{l} \text{creation or} \\ \text{annihilation} \\ \text{of the quantity at } x \end{array} \right]. \quad (4.1)$$

Expressed mathematically, this principle yields

$$\frac{\partial u}{\partial t}(t, x) = \int_O \mathcal{J}(x, y, u(t, x), u(t, y)) dy + f(t, x, u(t, x)), \quad (4.2)$$

where \mathcal{J} encodes the nonlocal diffusion or transport of the quantity and f is a reaction term accounting for its creation or annihilation. The integral

$$\int_O \mathcal{J}(x, y, u(t, x), u(t, y)) dy \quad (4.3)$$

is called the nonlocal diffusion term. Viewing u as a map from $[0, T]$ into $L^2(O)$ defined by $u(t)(x) := u(t, x)$, we can rewrite (4.2) in abstract form as

$$\frac{du}{dt}(t) = \mathcal{A}(u(t)) + F(t, u(t)) \quad (4.4)$$

with $\mathcal{A}(u(t))(x) := \int_O \mathcal{J}(x, y, u(t, x), u(t, y)) dy$ and $F(t, u(t))(x) := f(t, x, u(t, x))$. In general, \mathcal{A} may have a more involved structure than (4.3). In particular, it can arise as a (variational) derivative or as the subdifferential of a nonlocal energy functional. In the latter case, the nonlocal evolution equation (4.4) is multivalued.

4.1. Nonlocal cable theory. Cable theory describes the propagation of the membrane potential along a dendrite or axon. As in the classical theory, two physical ingredients are retained: the lipid membrane, characterised by its capacitance $c_m > 0$ and its ionic current density I_{ion} , and the intracellular cytoplasm, which mediates axial conduction. Let $O \subset \mathbb{R}^d$ ($d = 3$) be a bounded open set, representing the dendritic or axon, and let $u(t, x)$ denote the membrane potential at time $t \in [0, T]$ and spatial position $x \in O$. From physiological considerations, the membrane potential is bounded, i.e. $u(t, x) \in [V_{\min}, V_{\max}]$ where $V_{\min}, V_{\max} \in \mathbb{R}$ depend only on the considered dendrite or axon.

4.1.1. *From classical to nonlocal cable theory.* In order to derive a nonlocal cable evolution equation, we follow the phenomenological approach leading to the classical local cable equation, based on Kirchhoff's current law and Ohm's law. We first derive an elementary nonlocal cable equation from these physical principles, and then generalize it in §4.1.2 by means of a nonlocal balance principle.

We begin by deriving a nonlocal counterpart of Kirchhoff's current law expressed in terms of the singular nonlocal gradient. We first recall the classical local derivation before extending it to the nonlocal setting. Let $i_a(t, x)$ denote the axial current and $i_m(t, x)$ the membrane current per unit length at time t and spatial position x . By Kirchhoff's current law, the axial current satisfies

$$i_a(t, x) = i_a(t, x + \Delta x) + i_m(t, x)\Delta x,$$

which, in the local case $\Delta x \rightarrow 0$, gives

$$\frac{\partial i_a}{\partial x}(t, x) = -i_m(t, x). \quad (4.5)$$

For the nonlocal derivation, we apply the same reasoning between two points x and y ($x \neq y$) of the dendrite or axon. Kirchhoff's law gives $i_a(t, x) = i_a(t, y) + |x - y|i_m(t, x)$ and, symmetrically, $i_a(t, y) = i_a(t, x) + |x - y|i_m(t, y)$. Subtracting these two equalities yields

$$\frac{1}{|x - y|}(i_a(t, x) - i_a(t, y)) = -\frac{1}{2}(i_m(t, x) - i_m(t, y)).$$

Assuming that the conservation of currents occurs at distance $\delta > 0$ and defining $J_a : [0, \infty[\rightarrow [0, \infty[$ by $J_a(r) := \frac{1}{r}\mathbb{1}_{[0, \delta]}(r)$, the above equality can be written in terms of the singular nonlocal gradient as

$$D^{J_a}i_a(t, x, y) = -i_m(t, x, y), \quad (4.6)$$

where $i_m(t, x, y) := \frac{1}{2}\mathbb{1}_{]0, \delta[}(|x - y|)(i_m(t, x) - i_m(t, y))$. This is the nonlocal counterpart of (4.5).

We now derive a nonlocal counterpart of Ohm's law expressed in terms of the singular nonlocal gradient. As before, we first recall the classical local derivation. Let r_a denote the axial resistance per unit length. By Ohm's law, the voltage drop satisfies $u(t, x + \Delta x) - u(t, x) = -r_a i_a(t, x) \Delta x$, which, in the local case $\Delta x \rightarrow 0$, gives

$$\frac{\partial u}{\partial x}(t, x) = -r_a i_a(t, x). \quad (4.7)$$

Combining (4.5) and (4.7), we infer that

$$i_m(t, x) = \frac{1}{r_a} \frac{\partial^2 u}{\partial x^2}(t, x). \quad (4.8)$$

For the nonlocal derivation, we apply the same reasoning between two points x and y ($x \neq y$) of the dendrite or axon. Ohm's law gives $u(t, x) - u(t, y) = -r_a i_a(t, x) |x - y|$ and, symmetrically, $u(t, y) - u(t, x) = -r_a i_a(t, y) |x - y|$. Subtracting these two equalities yields

$$\frac{1}{|x - y|}(u(t, x) - u(t, y)) = \frac{r_a}{2}(i_a(t, x) - i_a(t, y)).$$

Assuming that Ohm's law holds at distance $\delta > 0$, the above equality can be written in terms of the singular nonlocal gradient as

$$D^{J_a}u(t, x, y) = -r_a i_a(t, x, y), \quad (4.9)$$

where $i_a(t, x, y) := \frac{1}{2} \mathbb{1}_{]0, \delta[}(|x - y|)(i_a(t, y) - i_a(t, x))$. This is the nonlocal counterpart of (4.7).

By combining (4.6) and (4.9), we obtain a second-order nonlocal identity comparable to (4.8). To this end, consider the nonlocal functional $\mathcal{F}_a : L^2(O) \rightarrow [0, \infty]$ defined by

$$\mathcal{F}_a(v) := \begin{cases} \int_O \int_O |D^{J_a}v(x, y)|^2 dx dy & \text{if } v \in \text{dom}(D^{J_a}) \\ \infty & \text{otherwise.} \end{cases}$$

Definition 4.1. The nonlocal Laplacian associated with J_a is the subdifferential of \mathcal{F}_a , denoted by Δ^{J_a} .

Remark 4.2. By Proposition 2.17 with $W(x, \xi) = |\xi|^2$, when $x \mapsto \int_O J_a(|x - y|) D^{J_a}u(t, x, y) dy$ belongs to $L^2(O)$, one has

$$\Delta^{J_a}u(t, x) = 2 \int_O J_a(|x - y|) D^{J_a}u(t, x, y) dy.$$

Equation (4.10) in the following proposition is the nonlocal counterpart of (4.8).

Proposition 4.3. *If $x \mapsto \int_O J_a(|x - y|) D^{J_a}u(t, x, y) dy$ belongs to $L^2(O)$, then for \mathcal{L}^d -a.e. $x \in O$,*

$$\int_O i_m(t, y, x) dy = \frac{1}{r_a} \Delta^{J_a}u(t, x). \quad (4.10)$$

Proof of Proposition 4.3. By Remark 4.2, (4.9) and (4.6), we have

$$\begin{aligned} \Delta^{J_a}u(t, x) &= 2 \int_O J_a(|x - y|) D^{J_a}u(t, x, y) dy \\ &= -2r_a \int_O J_a(|x - y|) i_a(t, x, y) dy \\ &= -r_a \int_O D^{J_a}i_a(t, y, x) dy \\ &= r_a \int_O i_m(t, y, x) dy, \end{aligned}$$

and (4.10) follows. ■

We now derive an elementary nonlocal cable equation. As before, we first recall the classical local derivation. Assuming that the membrane behaves like an RC circuit (Resistance and Capacitance in parallel), the membrane current has two components: the capacitive current $c_m \frac{\partial u}{\partial t}(t, x)$, where c_m is the membrane capacitance, and the leak current $\frac{u(t, x)}{r_m}$, where r_m is the membrane resistance per unit length. This gives

$$i_m(t, x) = c_m \frac{\partial u}{\partial t}(t, x) + \frac{u(t, x)}{r_m}, \quad (4.11)$$

which, combined with (4.8), yields the classical cable equation:

$$\frac{\partial u}{\partial t}(t, x) = \frac{1}{c_m r_a} \frac{\partial^2 u}{\partial x^2}(t, x) - \frac{1}{c_m r_m} u(t, x). \quad (4.12)$$

For the nonlocal derivation, we apply the same reasoning to the integrated membrane current $\int_O i_m(t, y, x) dy$. By Proposition 4.3, this yields the following elementary nonlocal cable equation:

$$\frac{\partial u}{\partial t}(t, x) = \frac{1}{c_m r_a} \Delta^{J_a} u(t, x) - \frac{1}{c_m r_m} u(t, x). \quad (4.13)$$

This is the nonlocal counterpart of (4.12) in which the local second-order axial diffusion $\frac{1}{c_m r_a} \frac{\partial^2}{\partial x^2}$ is replaced by the nonlocal term $\frac{1}{c_m r_a} \Delta^{J_a}$.

In the next paragraph, we generalize this elementary nonlocal equation by employing a more fundamental nonlocal balance principle, as described in the preamble of Section 4.

4.1.2. *Generalization via nonlocal balance principle.* To generalize the elementary nonlocal cable equation (4.13) derived above, we apply the nonlocal balance principle (4.1) to the membrane charge $q_m(t, x) := c_m u(t, x)$, the relevant conserved quantity, which gives

$$\frac{\partial q_m}{\partial t}(t, x) = \int_O \tilde{\mathcal{J}}_a(x, y, q_m(t, x), q_m(t, y)) dy + \tilde{f}_m(t, x, q_m(t, x)), \quad (4.14)$$

where $\tilde{\mathcal{J}}_a$ encodes the nonlocal axial transport of charge along the length of the dendrite or axon, and \tilde{f}_m represents the transmembrane currents (sources or sinks), such as ion channel activity or external stimuli, which exchange charge with the surrounding environment. Dividing (4.14) by c_m and setting

$$\begin{cases} \mathcal{J}_a(x, y, \xi, \zeta) = \tilde{\mathcal{J}}_a(x, y, c_m \xi, c_m \zeta) \\ f_m(t, x, \xi) = \frac{1}{c_m} \tilde{f}_m(t, x, c_m \xi), \end{cases}$$

we obtain the evolution equation for the membrane potential:

$$\frac{\partial u}{\partial t}(t, x) = \frac{1}{c_m} \int_O \mathcal{J}_a(x, y, u(t, x), u(t, y)) dy + f_m(t, x, u(t, x)). \quad (4.15)$$

Viewing u as a map $u : [0, T] \rightarrow L^2(O; [V_{\min}, V_{\max}])$ defined by $u(t)(x) := u(t, x)$, (4.15) takes the following abstract form:

$$\frac{du}{dt}(t) = \frac{1}{c_m} \mathcal{A}_a(u(t)) + F_m(t, u(t)). \quad (4.16)$$

with $\mathcal{A}_a(u(t))(x) := \int_O \mathcal{J}_a(x, y, u(t, x), u(t, y)) dy$ and $F_m(t, u(t))(x) := f_m(t, x, u(t, x))$

We assume that the nonlocal axial diffusion term \mathcal{A}_a is the variational derivative of a nonlocal energy functional \mathcal{F}_a , i.e.

$$\mathcal{A}_a := d^{J_a} \mathcal{F}_a \quad (4.17)$$

with $\mathcal{F}_a : L^2(O) \rightarrow [0, \infty]$ defined by

$$\mathcal{F}_a(v) := \begin{cases} \int_O \int_O W_a(x, D^{J_a} v(x, y)) dx dy & \text{if } v \in \text{dom}(D^{J_a}) \\ \infty & \text{otherwise,} \end{cases} \quad (4.18)$$

where W_a is an energy density and $D^{J_a}v(x, y)$ is the nonlocal gradient associated with an “interaction kernel” $J_a : [0, \infty[\rightarrow [0, \infty[$ satisfying $\text{supp}(J_a) \subset [0, \delta_a]$ for some $\delta_a > 0$. This condition models finite-range nonlocal axial diffusion, meaning that only points whose mutual distance does not exceed δ_a interact.

The interaction radius δ_a reflects the underlying microstructure of the dendrite or axon and determines the spatial scale of the nonlocal effects. The dependence on x in W_a models the fact that the dendrite or axon has a heterogeneous microstructure. A canonical choice for \mathcal{J}_a is the heterogeneous nonlocal Dirichlet (Laplacian-type) energy, obtained by taking

$$W_a(x, \xi) = c_a(x)|\xi|^2 \quad (4.19)$$

with $c_a(x) > 0$ representing the axial conduction coefficient of the dendrite or axon at spatial position x . (In the homogeneous case relevant to the elementary equation (4.13), $c_a(x)$ corresponds to $\frac{1}{r_a}$.)

The term $F_m : [0, T] \times L^2(O) \rightarrow L^2(O)$ collects all transmembrane currents, i.e.

$$F_m(t, v)(x) := \frac{1}{c_m} [I_{\text{app}}(t, x) - I_{\text{ion}}(t, x, v(x))], \quad (4.20)$$

where $I_{\text{app}}(t, x)$ is a prescribed applied stimulus and $I_{\text{ion}}(t, x, \xi)$ is the ionic current density (leak, voltage-gated, and ligand-gated components). The negative sign follows the usual convention: a positive outward ionic current depolarises the membrane potential. In the passive-leak case

$$I_{\text{ion}}(t, x, \xi) = g_\ell(t, x)(\xi - E_\ell(t, x)), \quad (4.21)$$

where $g_\ell(t, x) > 0$ is the leak conductance and $E_\ell(t, x) \in \mathbb{R}$ the reversal potential. (Note that conductance is simply the inverse of resistance. The elementary equation (4.13) corresponds to the special case where the conductance is a constant $g_\ell = \frac{1}{r_m}$ and the reversal potential is $E_\ell = 0$.) The dependence on x allows for spatial heterogeneity of the membrane properties, while the dependence on t accounts for possible temporal variations of the ionic environment. With (4.21), (4.20) reduces to

$$F_m(t, v)(x) = \frac{1}{c_m} [I_{\text{app}}(t, x) - g_\ell(t, x)(v(x) - E_\ell(t, x))]. \quad (4.22)$$

Combining (4.17) and (4.18) with the transmembrane term (4.22) in the passive-leak setting, (4.16) becomes

$$\frac{du}{dt}(t) = \frac{1}{c_m} d^{J_a} \mathcal{J}_a(u(t)) + \frac{1}{c_m} [I_{\text{app}}(t, \cdot) - g_\ell(t, \cdot)(u(t) - E_\ell(t, \cdot))]. \quad (4.23)$$

This is a nonlocal counterpart of the classical cable evolution equation in which the local second-order axial diffusion $\frac{1}{c_m r_a} \frac{\partial^2}{\partial x^2}$ is replaced by the nonlocal term $\frac{1}{c_m} d^{J_a} \mathcal{J}_a$. (Note that this general nonlocal evolution equation (4.23) exactly recovers the elementary nonlocal cable equation (4.13) derived in §4.1.1 under the following simplifying assumptions: no external stimulus ($I_{\text{app}} \equiv 0$), a zero reversal potential ($E_\ell \equiv 0$), a constant leak conductance ($g_\ell \equiv \frac{1}{r_m}$), a homogeneous axial conduction ($c_a \equiv \frac{1}{r_a}$) and a nonlocal diffusion term equal to $\frac{1}{r_a} \Delta^{J_a}$.)

4.1.3. *Existence of bounded solutions for the nonlocal cable evolution equation.* We now apply Theorem 3.10 to the nonlocal cable evolution equation (4.23) derived in §4.1.2. The relevance of Theorem 3.10 in this context lies in the fact that it provides the existence of bounded solutions, a property that is fundamental in nonlocal cable theory since, from physiological considerations, the membrane potential u must remain bounded. To this end, we consider the following assumptions:

- (CT₁) $c_m > 0$;
- (CT₂) $g_\ell \in L_{\text{loc}}^\infty([0, \infty[\times \mathbb{R}^d;]0, \infty[)$;
- (CT₃) $E_\ell \in L_{\text{loc}}^\infty([0, \infty[\times \mathbb{R}^d)$;
- (CT₄) $I_{\text{app}} \in L_{\text{loc}}^\infty([0, \infty[\times \mathbb{R}^d)$;
- (CT₅) there exist $V_{\min}, V_{\max} \in \mathbb{R}$ with $V_{\min} \leq V_{\max}$ such that for $\mathcal{L}^1 \otimes \mathcal{L}^d$ -a.e. $(t, x) \in [0, \infty[\times \mathbb{R}^d$,

$$g_\ell(t, x)(V_{\min} - E_\ell(t, x)) \leq I_{\text{app}}(t, x) \leq g_\ell(t, x)(V_{\max} - E_\ell(t, x)); \quad (4.24)$$

- (CT₆) $J_a : [0, \infty[\rightarrow [0, \infty[$ satisfies (S₁) and (S₃) with $J = J_a$;
- (CT₇) $W_a : \mathbb{R}^d \times \mathbb{R} \rightarrow [0, \infty[$ satisfies (I₀)–(I₃) with $W = W_a$.

Remark 4.4. If W_a is given by (4.19), (CT₇) is equivalent to the existence of $\bar{c}_a \geq \underline{c}_a > 0$ such that $c_a(x) \in [\underline{c}_a, \bar{c}_a]$ for all $x \in \mathbb{R}^d$.

Remark 4.5. Dividing 4.24 by $g_\ell(t, x) > 0$, (CT₅) is equivalent to

$$V_{\min} \leq E_\ell(t, x) + \frac{I_{\text{app}}(t, x)}{g_\ell(t, x)} \leq V_{\max} \text{ for } \mathcal{L}^1 \otimes \mathcal{L}^d\text{-a.e. } (t, x) \in [0, \infty[\times \mathbb{R}^d. \quad (4.25)$$

In the absence of applied stimulus ($I_{\text{app}} = 0$), (4.25) reduces to

$$V_{\min} \leq E_\ell(t, x) \leq V_{\max} \text{ for } \mathcal{L}^1 \otimes \mathcal{L}^d\text{-a.e. } (t, x) \in [0, \infty[\times \mathbb{R}^d.$$

Let $f_m : [0, \infty[\times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f_m(t, x, \zeta) := \frac{1}{c_m} [I_{\text{app}}(t, x) - g_\ell(t, x)(\zeta - E_\ell(t, x))]$$

and, given a bounded open set $O \subset \mathbb{R}^d$, let $F_m : [0, \infty[\times L^2(O) \rightarrow L^2(O)$ be defined by

$$F_m(t, v)(x) := f_m(t, x, v(x)).$$

Remark 4.6. The condition (CT₅) is equivalent to requiring that for $\mathcal{L}^1 \otimes \mathcal{L}^d$ -a.a. $(t, x) \in [0, \infty[\times \mathbb{R}^d$,

$$\begin{cases} 0 \leq f_m(t, x, V_{\min}) \\ 0 \geq f_m(t, x, V_{\max}). \end{cases} \quad (4.26)$$

Let $T > 0$ and consider the homogeneous Neumann-Cauchy nonlocal cable problem:

$$(\mathcal{P}_T^{\text{NC-CT}}) \left\{ \begin{array}{l} \frac{du}{dt}(t) + \frac{1}{c_m} d^{J_a} \mathcal{J}_a(u(t)) = F_m(t, u(t)) \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T] \\ u(t) \in \mathcal{H}_{W_a}^{J_a} \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T] \\ u(0) = u_0 \in \overline{\mathcal{H}_{W_a}^{J_a}} \cap L^2(O; [V_{\min}, V_{\max}]). \end{array} \right.$$

Note that $(\mathcal{P}_T^{\text{NC-CT}})$ is a particular case of the general problem $(\mathcal{P}_{T,F,\mathcal{J}}^{u_0,[a,b]})$ corresponding to $\mathcal{J} = \frac{1}{c_m} \mathcal{J}_a$, $F = F_m$, $a = V_{\min}$ and $b = V_{\max}$, and can therefore be treated by means of Theorem 3.10. This yields the following result.

Theorem 4.7. *Assume that (CT₁)–(CT₇) hold. Then, $(\mathcal{P}_T^{\text{NC-CT}})$ admits a unique solution $u \in AC([0, T]; L^2(O))$ satisfying $u(t) \in [V_{\min}, V_{\max}]$ for all $t \in [0, T]$.*

Proof of Theorem 4.7. The proof consists in applying Theorem 3.10 in the homogeneous Neumann–Cauchy case with $J = J_a$, $W = \frac{1}{c_m} W_a$ (so that $\mathcal{J} = \frac{1}{c_m} \mathcal{J}_a$), $F = F_m$, $\underline{\rho} = V_{\min}$ and $\bar{\rho} = V_{\max}$. It is clear that $\frac{1}{c_m} W_a$ satisfies (I₀)–(I₃) since W_a does by (CT₇) and $c_m > 0$ by (CT₁). So it remains to verify that F_m is CP-structured, i.e. that the conditions (CP₁^T)–(CP₃^T) of Definition 3.7 hold with $f = f_m$.

First, for every $(\zeta, \zeta') \in \mathbb{R} \times \mathbb{R}$ and every $(t, x) \in [0, T] \times \mathbb{R}^d$,

$$\begin{aligned} |f_m(t, x, \zeta) - f_m(t, x, \zeta')| &= \frac{g_\ell(t, x)}{c_m} |\zeta - \zeta'| \\ &\leq L_m |\zeta - \zeta'| \end{aligned}$$

with $L_m := \frac{\|g_\ell\|_{L^\infty([0, T] \times O)}}{c_m} < \infty$ by (CT₁)–(CT₂). Hence $f_m(t, x, \zeta)$ is Lipschitz continuous in ζ uniformly with respect to $(t, x) \in [0, T] \times \mathbb{R}^d$, which shows (CP₁^T).

Second, we have $f_m(t, x, 0) = \frac{1}{c_m} [I_{\text{app}}(t, x) + g_\ell(t, x) E_\ell(t, x)]$, and so $f_m(\cdot, \cdot, 0) \in L^\infty([0, T] \times O)$ by (CT₁)–(CT₄). This shows that (CP₂^T) is satisfied.

Third, define $\underline{f}_m \equiv 0$ and $\bar{f}_m \equiv 0$, so that $\underline{f}_m \leq 0 \leq \bar{f}_m$. Then, the corresponding ordinary differential equations

$$\begin{cases} \frac{dz_m}{dt}(t) = \underline{f}_m(t, z_m(t)) \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T] \\ z_m(0) = V_{\min} \end{cases}$$

and

$$\begin{cases} \frac{d\bar{z}_m}{dt}(t) = \bar{f}_m(t, \bar{z}_m(t)) \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T] \\ \bar{z}_m(0) = V_{\max} \end{cases}$$

admit the unique constant solutions $\underline{z}_m \equiv V_{\min}$ and $\bar{z}_m \equiv V_{\max}$. By (CT₅) we have (4.26), which shows that (3.31) is verified with $(\underline{f}, \bar{f}) = (\underline{f}_m, \bar{f}_m)$ and $(\underline{z}, \bar{z}) = (\underline{z}_m, \bar{z}_m)$. Thus (CP₃^T) is satisfied with $(\underline{f}, \bar{f}) = (\underline{f}_m, \bar{f}_m)$, $(\underline{\rho}, \bar{\rho}) = (V_{\min}, V_{\max})$ and $(\underline{z}, \bar{z}) = (\underline{z}_m, \bar{z}_m)$.

By Theorem 3.10 we deduce that $(\mathcal{P}_T^{\text{NC-CT}})$ admits a unique solution $u \in AC([0, T]; L^2(O))$ such that

$$\underline{z}_m(T) \leq \underline{z}_m(t) \leq u(t) \leq \bar{z}_m(t) \leq \bar{z}_m(T) \text{ for all } t \in [0, T]. \quad (4.27)$$

Since $\underline{z}_m \equiv V_{\min}$ and $\bar{z}_m \equiv V_{\max}$, (4.27) means that $u(t) \in [V_{\min}, V_{\max}]$ for all $t \in [0, T]$, and the proof is complete. ■

Let $g \in L^2(\partial^{J_a}O)$. In the same way, we can also treat the Dirichlet–Cauchy nonlocal cable problem:

$$(\mathcal{P}_T^{\text{DC-CT}}) \begin{cases} \frac{du}{dt}(t) + \frac{1}{c_m} d^{J_a} \mathcal{F}_a^g(u(t)) = F_m(t, u(t)) \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T] \\ u(t) \in \mathcal{H}_{W_a}^{J_a} \cap \mathcal{H}_{g, W_a}^{J_a} \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T] \\ u(0) = u_0 \in \overline{\mathcal{H}_{W_a}^{J_a} \cap \mathcal{H}_{g, W_a}^{J_a}} \cap L^2(O; [V_{\min}, V_{\max}]) \end{cases}$$

with $\mathcal{F}_a^g := \mathcal{F}_a + \mathcal{D}^g$, where $\mathcal{D}^g : L^2(O) \rightarrow [0, \infty]$ is defined by

$$\mathcal{D}^g(v) := \begin{cases} \int_O \int_{\partial^{J_a}O} [W_a(x, D_g^{J_a}v(x, y)) + W_a(y, D_g^{J_a}v(x, y))] dx dy & \text{if } v \in \text{dom}(D_g^{J_a}) \\ \infty & \text{otherwise.} \end{cases}$$

With suitable additional assumptions, by applying Theorem 3.10 we obtain the Dirichlet–Cauchy counterpart of Theorem 4.7.

Theorem 4.8. *Under the hypotheses of Theorem 4.7, assume further that:*

(CT₈) g satisfies (S₂) with $J = J_a$;

(CT₉) $V_{\min} \in \underline{R}_g$ and $V_{\max} \in \overline{R}_g$, where \underline{R}_g and \overline{R}_g are given by (3.32)–(3.33).

Then, $(\mathcal{P}_T^{\text{DC-CT}})$ admits a unique solution $u \in AC([0, T]; L^2(O))$ satisfying $u(t) \in [V_{\min}, V_{\max}]$ for all $t \in [0, T]$.

4.2. Nonlocal dynamics of fake-news belief. Fake news propagation is a nonlocal phenomenon: the adoption or rejection of misinformation by an individual depends on interactions with a finite range of peers (physically or via social networks). Let $O \subset \mathbb{R}^d$ be a bounded open set, representing a social space (e.g. a city or a social network) in which individuals are distributed, and let $u(t, x) \in [0, 1]$ be the proportion of individuals at time $t \in [0, T]$ and spatial position $x \in O$ who believe a given fake news.⁸ In fact, u represents the “intensity of belief” in the fake news: the value $u = 0$ corresponds to total disbelief, while $u = 1$ corresponds to complete belief.

4.2.1. A first naive model. To build intuition, consider the simple case where the rate of change of the belief intensity (or information) $\frac{\partial u}{\partial t}(t, x)$ is simply the sum of all the information arriving at x minus all the information leaving x , i.e.

$$\frac{\partial u}{\partial t}(t, x) = \text{incoming} - \text{outgoing},$$

and where the transition rate at which information moves from y to x or x to y is proportional to $\frac{1}{|x-y|}$ and acts only for $|x-y| < \delta$ (meaning that the further apart individuals are, the less information circulates). Then

$$\frac{\partial u}{\partial t}(t, x) = \int_O \mathbb{1}_{]0, \delta[}(|x-y|) \frac{1}{|x-y|} u(t, y) dy - \int_O \mathbb{1}_{]0, \delta[}(|x-y|) \frac{1}{|x-y|} u(t, x) dy,$$

⁸Although the social space is intrinsically discrete, as the population of individuals is assumed to be very large, O can be considered as an open bounded subset of \mathbb{R}^d , and u can be treated as a real function.

which can be rewritten as

$$\frac{\partial u}{\partial t}(t, x) + \int_O J(|x - y|) D^J u(t)(x, y) dy = 0 \quad \text{for } u(t) \in \text{dom}(D^J)$$

with $J(r) = \frac{1}{\sqrt{r}} \mathbb{1}_{]0, \delta[}(r)$. Note that, from Definition 4.1, this nonlocal diffusion equation can be written as

$$\frac{\partial u}{\partial t}(t, x) + \frac{1}{2} \Delta^J u(t, x) = 0,$$

which is the nonlocal counterpart of the classical local diffusion equation $\frac{\partial u}{\partial t} + \frac{1}{2} \Delta u = 0$. However, noticing that $\int_O \left(\int_O J(|x - y|) D^J u(t)(x, y) dy \right) dx = 0$, we infer that $\int_O \frac{\partial u}{\partial t}(t, x) dx = 0$. This means there is conservation of information, which is not realistic. A more realistic model must incorporate reaction mechanisms that account for fact-checking, debunking, or loss of interest. The next paragraph develops a more general framework that includes such reaction mechanisms.

4.2.2. Derivation of a nonlocal fake-news belief evolution equation via nonlocal balance principle. The nonlocal balance principle ((4.1)) applies to $u(t, x)$ as follows:

$$\begin{aligned} \frac{\partial u}{\partial t}(t, x) &= \text{“nonlocal belief exchange”} + \text{“debunking, decay or reinforcement”} \\ &= \int_O \mathcal{J}_f(x, y, u(t, x), u(t, y)) dy + f_f(t, x, u(t, x)), \end{aligned} \quad (4.28)$$

where \mathcal{J}_f encodes the nonlocal structure of the social space governing the spread of the fake news and f_f accounts for external effects such as debunking efforts (e.g. fact-checking campaigns or official corrections), natural decay of belief (e.g. progressive forgetting or loss of interest over time), or automated reinforcement mechanisms (e.g. algorithmic recommendation systems that repeatedly expose users to the same content). Viewing u as a map $u : [0, T] \rightarrow L^2(O; [0, 1])$ defined by $u(t)(x) := u(t, x)$, (4.28) translates to

$$\frac{du}{dt}(t) = \mathcal{A}_f(u(t)) + F_f(t, u(t)) \quad (4.29)$$

with $\mathcal{A}_f(u(t))(x) := \int_O \mathcal{J}_f(x, y, u(t, x), u(t, y)) dy$ and $F_f(t, u(t))(x) := f_f(t, x, u(t, x))$

We assume that the nonlocal diffusion term \mathcal{A}_f is the variational derivative of a nonlocal energy functional \mathcal{J}_f , i.e.

$$\mathcal{A}_f := d^{\mathcal{J}_f} \mathcal{J}_f \quad (4.30)$$

with $\mathcal{J}_f : L^2(O) \rightarrow [0, \infty]$ is defined by

$$\mathcal{J}_f(v) := \begin{cases} \int_O \int_O W_f(x, D^{\mathcal{J}_f} v(x, y)) dx dy & \text{if } v \in \text{dom}(D^{\mathcal{J}_f}) \\ \infty & \text{otherwise,} \end{cases} \quad (4.31)$$

where W_f is an energy density and $D^{\mathcal{J}_f} v(x, y)$ is the nonlocal gradient associated with an “interaction kernel” $\mathcal{J}_f : [0, \infty[\rightarrow [0, \infty[$ satisfying $\text{supp}(\mathcal{J}_f) \subset [0, \delta_f]$ for some $\delta_f > 0$. This condition models finite-range social interaction, meaning that only individuals whose mutual distance (physical or via a social network) does not exceed δ_f influence one another.

The interaction radius δ_f determines the range of social influence, while the dependence of W_f on x allows for spatial heterogeneity in the connectivity or susceptibility of individuals. Specifically, W_f quantifies how strongly the system tends to align beliefs among connected agents. A canonical choice is

$$W_f(x, \xi) = c_f(x) |\xi|^2, \quad (4.32)$$

where $c_f(x) > 0$ represents, at spatial position x , the strength of social ties or the degree of susceptibility of individuals to the influence of others.

The term $F_f : [0, T] \times L^2(O) \rightarrow L^2(O)$ models the reaction mechanisms triggered by the introduction of the fake news into the social space, i.e. debunking, decay of belief, and reinforcement, and is given by

$$F_f(t, v)(x) := [r_f(t, x) - d_f(t, x)]v(x)(1 - v(x)) - c_f(t, x)v(x), \quad (4.33)$$

where $d_f(t, x) \geq 0$ is the debunking rate (fact-checking, official corrections), $c_f(t, x) \geq 0$ is the natural decay rate (forgetting, loss of interest) and $r_f(t, x) \geq 0$ is the reinforcement rate (algorithmic amplification, echo chambers). These rates depend on time t and spatial position x in order to account for temporal variations in intervention efforts and spatial heterogeneity in the social structure.

The factor $v(1 - v)$ represents the interaction between the proportion v of believers and the proportion $1 - v$ of non-believers. Both debunking and reinforcement act through this interaction, and therefore vanish at the extreme states $v = 0$ and $v = 1$. The linear term $-c_f v$ models the natural decay of belief, independently of social interaction.

Combining (4.30) and (4.31) with (4.33), (4.29) becomes

$$\frac{du}{dt}(t) = d^{J_f} \mathcal{F}_f(u(t)) + [r_f(t, \cdot) - d_f(t, \cdot)]u(t)(1 - u(t)) - c_f(t, \cdot)u(t). \quad (4.34)$$

This is the nonlocal fake-news belief evolution equation, i.e. the nonlocal evolution of the proportion of individuals who believe the fake news.

4.2.3. Existence of bounded solutions for the nonlocal fake-news belief evolution equation. We now apply Theorem 3.10 to the nonlocal fake-news belief evolution equation (4.34) derived in §4.2.2. The relevance of Theorem 3.10 in this context lies in the fact that it provides the existence of bounded solutions, a property that is fundamental since the belief intensity u , being a proportion, must remain in $[0, 1]$. To this end, we consider the following assumptions:

- (FN₁) $r_f \in L_{\text{loc}}^\infty([0, \infty[\times \mathbb{R}^d; [0, \infty[)$;
- (FN₂) $d_f \in L_{\text{loc}}^\infty([0, \infty[\times \mathbb{R}^d; [0, \infty[)$;
- (FN₃) $c_f \in L_{\text{loc}}^\infty([0, \infty[\times \mathbb{R}^d; [0, \infty[)$;
- (FN₄) $J_f : [0, \infty[\rightarrow [0, \infty[$ satisfies (S₁) and (S₃) with $J = J_f$;
- (FN₅) $W_f : \mathbb{R}^d \times \mathbb{R} \rightarrow [0, \infty[$ satisfies (I₀)–(I₃) with $W = W_f$.

Remark 4.9. If W_f is given by (4.32), (FN₅) is equivalent to the existence of $\bar{c}_f \geq \underline{c}_f > 0$ such that $c_f(x) \in [\underline{c}_f, \bar{c}_f]$ for all $x \in \mathbb{R}^d$.

Let $f_f : [0, \infty[\times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f_f(t, x, \zeta) := (r_f(t, x) - d_f(t, x))\zeta(1 - \zeta) - c_f(t, x)\zeta$$

and, given a bounded open set $O \subset \mathbb{R}^d$, let $F_f : [0, \infty[\times L^2(O) \rightarrow L^2(O)$ be defined by

$$F_f(t, v)(x) := f_f(t, x, v(x)).$$

Remark 4.10. Under (FN_3) , for $\mathcal{L}^1 \otimes \mathcal{L}^d$ -a.e. $(t, x) \in [0, \infty[\times \mathbb{R}^d$, we have

$$\begin{cases} 0 = f_f(t, x, 0) \\ 0 \geq -c_f(t, x) = f_f(t, x, 1). \end{cases} \quad (4.35)$$

Let $T > 0$ and consider the homogeneous Neumann-Cauchy nonlocal fake-news belief problem:

$$(\mathcal{P}_T^{\text{NC-FN}}) \begin{cases} \frac{du}{dt}(t) + d^{J_f} \mathcal{J}_f(u(t)) = F_f(t, u(t)) \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T] \\ u(t) \in \mathcal{H}_{W_f}^{J_f} \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T] \\ u(0) = u_0 \in \overline{\mathcal{H}_{W_f}^{J_f}} \cap L^2(O; [0, 1]). \end{cases}$$

Note that $(\mathcal{P}_T^{\text{NC-FN}})$ is a particular case of the general problem $(\mathcal{P}_{T,F,\mathcal{J}}^{u_0,[a,b]})$ corresponding to $\mathcal{J} = \mathcal{J}_f$, $F = F_f$, $a = 0$ and $b = 1$, and can therefore be treated by means of Theorem 3.10. This yields the following result.

Theorem 4.11. *Assume that (FN_1) – (FN_5) hold. Then, $(\mathcal{P}_T^{\text{NC-FN}})$ admits a unique solution $u \in AC([0, T]; L^2(O))$ satisfying $u(t) \in [0, 1]$ for all $t \in [0, T]$.*

Proof of Theorem 4.11. The proof consists in applying Theorem 3.10 in the homogeneous Neumann-Cauchy case with $J = J_f$, $W = W_f$ (so that $\mathcal{J} = \mathcal{J}_f$), $F = F_f$, $\rho = 0$ and $\bar{\rho} = 1$. It is clear that W_f satisfies (I_0) – (I_3) by (FN_5) . So it remains to verify that \bar{F}_f is CP-structured, i.e. that the conditions (CP_1^T) – (CP_3^T) of Definition 3.7 hold with $f = f_f$.

First, for every compact interval $K = [m, M] \subset \mathbb{R}$, every $(\zeta, \zeta') \in K \times K$ and every $(t, x) \in [0, T] \times \mathbb{R}^d$,

$$\begin{aligned} |f_f(t, x, \zeta) - f_f(t, x, \zeta')| &= |(r_f(t, x) - d_f(t, x))[\zeta(1 - \zeta) - \zeta'(1 - \zeta')] - c_f(t, x)(\zeta - \zeta')| \\ &= |(r_f(t, x) - d_f(t, x))(\zeta - \zeta')(1 - \zeta - \zeta') - c_f(t, x)(\zeta - \zeta')| \\ &= |(r_f(t, x) - d_f(t, x))(1 - \zeta - \zeta') - c_f(t, x)| |\zeta - \zeta'| \\ &\leq L_f^K |\zeta - \zeta'| \end{aligned}$$

with $L_f^K := (\|r_f\|_{L^\infty([0, T] \times O)} + \|d_f\|_{L^\infty([0, T] \times O)})[1 + 2 \max(|m|, |M|)] + \|c_f\|_{L^\infty([0, T] \times O)} < \infty$ by (FN_1) – (FN_3) . Hence $f_f(t, x, \zeta)$ is locally Lipschitz continuous in ζ uniformly with respect to $(t, x) \in [0, T] \times \mathbb{R}^d$, which shows (CP_1^T) .

Second, we have $f_f(t, x, 0) = 0$, and so $f_f(\cdot, \cdot, 0) \in L^2([0, T]; L^2(O))$. This shows that (CP_2^T) is satisfied.

Third, define $\underline{f}_f \equiv 0$ and $\bar{f}_f \equiv 0$, so that $\underline{f}_f \leq 0 \leq \bar{f}_f$. Then the corresponding ordinary differential equations

$$\begin{cases} \frac{dz_f}{dt}(t) = \underline{f}_f(t, z_f(t)) \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T] \\ z_f(0) = 0 \end{cases}$$

and

$$\begin{cases} \frac{d\bar{z}_f}{dt}(t) = \bar{f}_f(t, \bar{z}_f(t)) \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T] \\ \bar{z}_f(0) = 1 \end{cases}$$

admit the unique constant solutions $\underline{z}_f \equiv 0$ and $\bar{z}_f \equiv 1$. By (FN₃) we have (4.35), which shows that (3.31) is verified with $(\underline{f}, \bar{f}) = (\underline{f}_f, \bar{f}_f)$ and $(\underline{z}, \bar{z}) = (\underline{z}_f, \bar{z}_f)$. Thus (CP₃^T) is satisfied with $(\underline{f}, \bar{f}) = (\underline{f}_f, \bar{f}_f)$, $(\underline{\rho}, \bar{\rho}) = (0, 1)$ and $(\underline{z}, \bar{z}) = (\underline{z}_f, \bar{z}_f)$.

By Theorem 3.10 we deduce that $(\mathcal{P}_T^{\text{NC-FN}})$ admits a unique solution $u \in AC([0, T]; L^2(O))$ such that

$$\underline{z}_f(T) \leq \underline{z}_f(t) \leq u(t) \leq \bar{z}_f(t) \leq \bar{z}_f(T) \text{ for all } t \in [0, T]. \quad (4.36)$$

Since $\underline{z}_f \equiv 0$ and $\bar{z}_f \equiv 1$, (4.36) means that $u(t) \in [0, 1]$ for all $t \in [0, T]$, and the proof is complete. ■

Let $g \in L^2(\partial^{J_f}O)$. In the same way, we can also treat the Dirichlet–Cauchy nonlocal fake-news belief problem:

$$(\mathcal{P}_T^{\text{DC-FN}}) \begin{cases} \frac{du}{dt}(t) + d^{J_f} \mathcal{F}_f^g(u(t)) = F_f(t, u(t)) \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T] \\ u(t) \in \mathcal{H}_{W_f}^{J_f} \cap \mathcal{H}_{g, W_f}^{J_f} \text{ for } \mathcal{L}^1\text{-a.a. } t \in [0, T] \\ u(0) = u_0 \in \overline{\mathcal{H}_{W_f}^{J_f} \cap \mathcal{H}_{g, W_f}^{J_f}} \cap L^2(O; [0, 1]) \end{cases}$$

with $\mathcal{F}_f^g := \mathcal{F}_f + \mathcal{D}^g$, where $\mathcal{D}^g : L^2(O) \rightarrow [0, \infty]$ is defined by

$$\mathcal{D}^g(v) := \begin{cases} \int_O \int_{\partial^{J_f}O} [W_f(x, D_g^{J_f} v(x, y)) + W_f(y, D_g^{J_f} v(x, y))] dx dy & \text{if } v \in \text{dom}(D_g^{J_f}) \\ \infty & \text{otherwise.} \end{cases}$$

With suitable additional assumptions, by applying Theorem 3.10 we obtain the Dirichlet–Cauchy counterpart of Theorem 4.11.

Theorem 4.12. *Under the hypotheses of Theorem 4.11, assume further that:*

(FN₆) g satisfies (S₂) with $J = J_f$;

(FN₇) $0 \in \underline{R}_g$ and $1 \in \bar{R}_g$, where \underline{R}_g and \bar{R}_g are given by (3.32)–(3.33).

Then $(\mathcal{P}_T^{\text{DC-FN}})$ admits a unique solution $u \in AC([0, T]; L^2(O))$ satisfying $u(t) \in [0, 1]$ for all $t \in [0, T]$.

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