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► **To cite this version:**

Slim Jmal, Mattéo Tacchi-Bénard, Emmanuel Witrant. Lagrangian Duality for Measure-Valued Entropy Solutions of Nonlinear Hyperbolic PDEs. 2026. <hal-05580991v2>

**HAL Id: hal-05580991**

**<https://hal.science/hal-05580991v2>**

Preprint submitted on 29 Apr 2026

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# Lagrangian Duality for Measure-Valued Entropy Solutions of Nonlinear Hyperbolic PDEs

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## Abstract

We study nonlinear hyperbolic conservation laws on compact domains through a convex lifting to measures. The entropy measure-valued formulation of the PDE yields weak equations and entropy inequalities over occupation and boundary measures, which can be recast as the feasibility of an infinite-dimensional linear program over measures. Existing approaches approximate this primal problem through semidefinite programming (SDP) relaxations based on Putinar-type positivity certificates, and then leverage the Christoffel-Darboux kernel to recover the graph of the solution. In contrast, we develop and exploit the Lagrangian dual of the measure formulation, expressed over test functions and nonnegative entropy multipliers, again represented with Putinar’s sum-of-squares (SoS) positivity certificates. The dual provides a PDE-structured certificate of entropy admissibility and supports reconstruction principles based on the approximate saturation of dual inequalities. This dual-first viewpoint brings a new insight into the problem and allows us to recover solutions without the explicit use of Christoffel-Darboux kernels, while still allowing SoS/SDP tools to be used on the dual side, when the problem data are polynomial.

**Keywords:** Polynomial sums of squares, Moment-SoS hierarchy, Convex optimization, Nonlinear hyperbolic PDEs, Burgers equation, Measure-valued solutions, Entropy solutions, Lagrangian duality.

## 1 Introduction

Nonlinear hyperbolic conservation laws of the form

$$\frac{\partial}{\partial t}y(t, x) + \frac{\partial}{\partial x}f(y(t, x)) = 0 \tag{1}$$

constitute a fundamental class of models for transport, wave propagation, and shock formation. Even for smooth fluxes, discontinuities may develop in finite time, so weak solutions are generally not unique [1]. The standard selection mechanism is provided by entropy inequalities, systematized by Kruzhkov [2], which ensure uniqueness of entropy weak solutions. Measure-valued formulations, which naturally approximate weak solutions, are particularly well suited to convex formulations of nonlinear PDE constraints [3, 4]. In the scalar case, entropy measure-valued solutions are expected to concentrate on the graph of the entropy solution, with a single entropy condition in the case of a strictly convex flux [5].

A key idea underlying several recent developments is to lift nonlinear dynamics to linear constraints over measures, transforming nonlinear equation constraints into linear equalities over Borel measures, and leading to infinite-dimensional linear programs known as generalized moment problems (GMPs). When the data are polynomial (or semi-algebraic) and the supports are compact semi-algebraic sets, a systematic numerical approximation is provided by Lasserre’s moment-SoS hierarchy, i.e. sequences of convex SDP relaxations with convergence guarantees under standard assumptions [6]. In the context of scalar conservation laws, Marx *et al.* [7, 8]

showed how the weak balance law and entropy inequalities can be encoded as linear constraints on space-time occupation measures and boundary measures, yielding a convex relaxation scheme for entropy solutions. Related methodologies have been used to address transport problems [9], derive certified bounds for nonlinear dynamical systems and PDEs [10, 11, 12], and to treat broader classes of (quasi-dissipative) evolution equations within moment-SOS paradigms [13]. Finally, once a moment description is available, reconstructing minimizers or supports hinges on structural properties of moment matrices; recent work uses Christoffel-Darboux kernels to approximate supports and recover solution graphs from moments [14, 15].

In this paper, we instead develop a mirror viewpoint based on Lagrangian duality: keeping the primal measure formulation as the conceptual starting point, we focus on its dual problem over continuous test functions and nonnegative entropy multipliers. This duality is naturally grounded in convex analysis [16] and conic duality between Borel measures and nonnegative continuous functions [17]. The resulting dual inequalities retain the transport structure of the PDE and can be interpreted as PDE-structured certificates of admissibility. Moreover, they suggest reconstruction principles based on minimizing dual polynomials directly, hence bypassing of Christoffel-Darboux analysis. Although our focus is the forward problem, the same dual-certificate viewpoint is compatible with control-oriented extensions where Lyapunov functionals can be used for the estimation of coupled first-order hyperbolic PDEs [18], yielding convex feasibility conditions that are manageable through linear matrix inequalities (LMIs), as in this contribution.

The paper is organized as follows: Section 2 introduces the original problem, as well as its generalized moment problem (GMP) formulation over measure-valued solutions, and a new interpretation of entropy conditions, then Section 3 derives the Lagrangian dual of this formulation, with a special emphasis on strong duality properties. Section 4 next develops the methodology for both computing finite dimensional approximations of the infinite dimensional dual GMP (namely, the moment-SoS hierarchy), and using the obtained optimizers to approximate the solution to the original conservation law. Finally, Section 5 presents the numerical results and experimentations, before Section 6 concludes with perspectives for future work.

## 1.1 Notation

$\mathbb{N}$  and  $\mathbb{R}$  denote the sets of natural numbers and real numbers, respectively. For  $\mathbf{X} \subset \mathbb{R}^n$  and  $k \in \mathbb{N} \cup \{\infty\}$ ,  $\mathcal{C}^k(\mathbf{X})$ ,  $\mathcal{C}^k(\mathbf{X})_+$ , and  $\mathcal{C}_c^k(\mathbf{X})$  respectively denote the sets of  $k$ -times continuously differentiable functions, globally nonnegative functions, and functions with compact support in  $\mathbf{X}$ . We denote by  $\mathcal{B}(\mathbf{X})$  the  $\sigma$ -algebra of Borel subsets of  $\mathbf{X}$ , and by  $\mathcal{M}(\mathbf{X})$ ,  $\mathcal{M}(\mathbf{X})_+$ , and  $\mathcal{P}(\mathbf{X})$  the sets of signed, nonnegative, and probability Borel measures on  $\mathbf{X}$ , respectively. Finally, for  $p \in [1, \infty)$ ,  $\mathcal{L}^p(\mathbf{X})$  denotes the space of measurable functions  $f$  such that  $|f|^p$  is Lebesgue integrable, while  $\mathcal{L}^\infty(\mathbf{X})$  denotes the space of essentially bounded functions.

## 2 Primal Problem over Measures

Let  $\mathbf{T} := [0, T]$  and  $\mathbf{X} := [L, R]$  be compact sets. Consider the Cauchy-Dirichlet problem on  $\Omega := \mathbf{T} \times \mathbf{X}$

$$\frac{\partial y}{\partial t}(t, x) + \frac{\partial(f \circ y)}{\partial x}(t, x) = 0, \quad (t, x) \in \Omega \quad (1)$$

$$y(0, x) = y_0(x), \quad x \in \mathbf{X} \quad (1a)$$

$$y(t, L) = y_L(t), \quad t \in \mathbf{T} \quad (1b)$$

where (1) is a scalar hyperbolic conservation law,  $f \in \mathcal{C}^1(\mathbb{R})$ , (1a) is an initial condition with  $y_0 \in \mathcal{L}^1(\mathbf{X}) \cap \mathcal{L}^\infty(\mathbf{X})$ , and (1b) a boundary condition with  $y_L \in \mathcal{L}^1(\mathbf{T}) \cap \mathcal{L}^\infty(\mathbf{T})$ .

**Theorem 1** (Maximum Principle). *Let  $y$  be a solution of (1),*

$$\begin{aligned}\underline{y} &= \operatorname{ess\,inf}_{(t,x) \in \mathbf{T} \times \mathbf{X}} \min(y_0(x), y_L(t)), \\ \bar{y} &= \operatorname{ess\,sup}_{(t,x) \in \mathbf{T} \times \mathbf{X}} \max(y_0(x), y_L(t)),\end{aligned}$$

and  $\mathbf{Y} = [\underline{y}, \bar{y}]$ . Then, for almost all  $(t, x) \in \mathbf{T} \times \mathbf{X}$ , it holds

$$\underline{y} \leq y(t, x) \leq \bar{y} \quad \text{i.e.,} \quad y(t, x) \in \mathbf{Y}. \quad (2)$$

**Remark 2.** *In order for the problem with the left boundary condition to be well posed, we further assume  $y \geq 0$ .*

## 2.1 Entropy Weak Solutions

**Definition 3** (Weak Solution). *A function  $y \in \mathcal{L}^\infty(\Omega)$  is a weak solution to (1) if, for all test functions  $\varphi \in \mathcal{C}^1(\Omega)$ , it satisfies*

$$\begin{aligned}0 &= \int_{\Omega} \left( \frac{\partial \varphi}{\partial t}(t, x) y(t, x) + \frac{\partial \varphi}{\partial x}(t, x) f(y(t, x)) \right) dx dt \\ &\quad + \int_{\mathbf{X}} \varphi(0, x) y_0(x) dx - \int_{\mathbf{X}} \varphi(T, x) y(T, x) dx \\ &\quad + \int_{\mathbf{T}} \varphi(t, L) f(y_L(t)) dt - \int_{\mathbf{T}} \varphi(t, R) f(y(t, R)) dt\end{aligned} \quad (3)$$

**Definition 4** (Entropy Solution).

(i)  $(\eta, q) \in \mathcal{C}^1(\mathbb{R})^2$  is called an entropy pair for (1) if  $\eta$  is strictly convex and  $q' = f'\eta'$ .

(ii) A weak solution  $y \in \mathcal{L}^\infty(\Omega)$  of (3) is an entropy solution if, for all entropy pairs and all non-negative test functions  $\psi \in \mathcal{C}^1(\Omega)_+$ , it satisfies

$$\begin{aligned}0 &\leq \int_{\Omega} \left( \frac{\partial \psi}{\partial t}(t, x) \eta(y(t, x)) + \frac{\partial \psi}{\partial x}(t, x) q(y(t, x)) \right) dx dt \\ &\quad + \int_{\mathbf{X}} \psi(0, x) \eta(y_0(x)) dx - \int_{\mathbf{X}} \psi(T, x) \eta(y(T, x)) dx \\ &\quad + \int_{\mathbf{T}} \psi(t, L) q(y_L(t)) dt - \int_{\mathbf{T}} \psi(t, R) q(y(t, R)) dt\end{aligned} \quad (4)$$

## 2.2 Entropy Measure-Valued Solutions

**Definition 5** (Young Measure). *A Young measure  $\mu$  on a Euclidean space  $\mathcal{X}$  is a map  $\mu : \mathcal{X} \rightarrow \mathcal{M}(\mathbb{R})_+$ ,  $\xi \mapsto \mu_\xi$ , such that  $\forall g \in \mathcal{C}_c^0(\mathbb{R})$ ,  $\xi \mapsto \int_{\mathbb{R}} g(y) \mu_\xi(dy)$  is measurable.*

The standard PDE (1) can then be rewritten in a measure-valued form using Young measures:

$$\frac{\partial}{\partial t} \langle \mu_{(t,x)}, y \rangle + \frac{\partial}{\partial x} \langle \mu_{(t,x)}, f \rangle = 0, \quad (t, x) \in \Omega \quad (5)$$

$$\mu_{(0,x)} = \sigma_0(x), \quad x \in \mathbf{X} \quad (5a)$$

$$\mu_{(t,L)} = \sigma_L(t), \quad t \in \mathbf{T} \quad (5b)$$

where  $\sigma_0(x)$  and  $\sigma_L(t)$  are probability measures in  $\mathcal{P}(\mathbf{Y})$  concentrated on the graphs of the initial and boundary data, i.e.  $\sigma_0(x) = \delta_{y_0(x)}$ ,  $\sigma_L(t) = \delta_{y_L(t)}$ , and  $\langle \cdot, \cdot \rangle$  denotes integration of a function  $g \in \mathcal{C}(\mathbb{R})$  against a measure  $\mu \in \mathcal{M}(\mathbb{R})$ :

$$\langle \mu, g \rangle := \int_{\mathbb{R}} g(y) \mu(dy) \quad (6)$$

**Definition 6** (Entropy Measure-Valued Solution). *A Young measure  $\mu : (t, x) \mapsto \mu_{(t,x)} \in \mathcal{P}(\mathbf{Y})$  is an entropy MV solution to (1) if, for all test functions  $\varphi \in \mathcal{C}^1(\Omega)$ , it satisfies*

$$\begin{aligned} 0 &= \int_{\Omega} \left( \frac{\partial \varphi}{\partial t} \langle \mu_{(t,x)}, y \rangle + \frac{\partial \varphi}{\partial x} \langle \mu_{(t,x)}, f \rangle \right) dx dt \\ &+ \int_{\mathbf{X}} \varphi(0, x) \langle \sigma_0(x), y \rangle dx - \int_{\mathbf{X}} \varphi(T, x) \langle \mu_{(T,x)}, y \rangle dx \\ &+ \int_{\mathbf{T}} \varphi(t, L) \langle \sigma_L(t), f \rangle dt - \int_{\mathbf{T}} \varphi(t, R) \langle \mu_{(t,R)}, f \rangle dt \end{aligned} \quad (7)$$

and, for all entropy pairs  $(\eta, q)$  and all non-negative test functions  $\psi \in \mathcal{C}^1(\Omega)_+$ , it satisfies

$$\begin{aligned} 0 &\leq \int_{\Omega} \left( \frac{\partial \psi}{\partial t} \langle \mu_{(t,x)}, \eta \rangle + \frac{\partial \psi}{\partial x} \langle \mu_{(t,x)}, q \rangle \right) dx dt \\ &+ \int_{\mathbf{X}} \psi(0, x) \langle \sigma_0(x), \eta \rangle dx - \int_{\mathbf{X}} \psi(T, x) \langle \mu_{(T,x)}, \eta \rangle dx \\ &+ \int_{\mathbf{T}} \psi(t, L) \langle \sigma_L(t), q \rangle dt - \int_{\mathbf{T}} \psi(t, R) \langle \mu_{(t,R)}, q \rangle dt \end{aligned} \quad (8)$$

**Remark 7.** *Entropy solutions are entropy MV solutions via the embedding  $y(t, x) \mapsto \mu_{(t,x)} = \delta_{y(t,x)}$ .*

**Remark 8.** *For the sake of simplicity, throughout the remainder of the paper we assume that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is strictly convex, so that a single entropy pair  $(\eta, q)$  with  $\eta(y) = \frac{y^2}{2}$  is enough to enforce uniqueness of entropy (MV) solutions.*

### 2.3 Linear Optimization Problem over Measures

Let  $\mathbf{K} := \Omega \times \mathbf{Y}$  and  $\nu := \lambda_{\mathbf{T}} \lambda_{\mathbf{X}} \mu_{(t,x)} \in \mathcal{M}(\mathbf{K})_+$  such that  $\nu(dt, dx, dy) = dt dx \mu_{(t,x)}(dy)$ , where  $\mu$  is an entropy measure-valued solution satisfying (8). Introduce the boundary measures supported respectively on  $\mathbf{K}_0 = \mathbf{K}_T := \mathbf{X} \times \mathbf{Y}$  and on  $\mathbf{K}_L = \mathbf{K}_R := \mathbf{T} \times \mathbf{Y}$ :

$$\nu_0(dx, dy) := dx \sigma_0(dy | x) \quad (9a)$$

$$\nu_T(dx, dy) := dx \mu_{(T,x)}(dy) \quad (9b)$$

$$\nu_L(dt, dy) := dt \sigma_L(dy | t) \quad (9c)$$

$$\nu_R(dt, dy) := dt \mu_{(t,R)}(dy) \quad (9d)$$

**Lemma 9** (Slack Measure for Entropy Condition). *Adopting the above  $\nu$  notations (9), condition (8) is equivalent to the existence of a Borel measure  $\sigma \in \mathcal{M}(\Omega)_+$  such that*

$$\begin{aligned} &\left[ -\frac{\partial}{\partial t} (\eta \nu) + \eta \cdot (\delta_{t=0} \otimes \nu_0 - \delta_{t=T} \otimes \nu_T) \right. \\ &\left. - \frac{\partial}{\partial x} (q \nu) + q \cdot (\delta_{x=L} \otimes \nu_L - \delta_{x=R} \otimes \nu_R) \right]^{(t,x)} = \sigma \end{aligned}$$

*Proof.* Riesz-Haviland theorem [19] proves the polynomial case  $\psi \in \mathbb{R}[t, x]$ , and Stone-Weierstrass theorem completes the proof by density of polynomial functions in  $\mathcal{C}^1(\Omega)$ .  $\square$

Defining the objective functional

$$\mathcal{J}(\nu, \nu_T, \nu_R) := \int_{\mathbf{K}} h d\nu + \int_{\mathbf{K}_T} h_T d\nu_T + \int_{\mathbf{K}_R} h_R d\nu_R$$

given any continuous cost integrands  $h, h_T, h_R$ , the entropy measure-valued solution is the unique feasible point of the following infinite-dimensional LP on measures:

$$\begin{aligned}
& \min_{\substack{\nu, \nu_T, \nu_R \\ \sigma_1, \dots, \sigma_N}} \mathcal{J}(\nu, \nu_T, \nu_R) & (10) \\
& \text{s.t. } \mathcal{D}(\nu, \nu_T, \nu_R) = \rho_{1/2} \\
& \mathcal{E}_k(\nu, \nu_T, \nu_R, \sigma_k) = \rho_k \quad k = 1, \dots, N \\
& \nu \in \mathcal{M}(\mathbf{K})_+, \quad \sigma_k \in \mathcal{M}(\Omega)_+ \quad k = 1, \dots, N \\
& \nu_T \in \mathcal{M}(\mathbf{K}_T)_+, \quad \nu_R \in \mathcal{M}(\mathbf{K}_R)_+ \\
& \nu^{(t,x)} = \lambda_\Omega, \quad \nu_T^{(x)} = \lambda_{\mathbf{X}}, \quad \nu_R^{(t)} = \lambda_{\mathbf{T}}
\end{aligned}$$

where  $\nu^{(t,x)} : \mathcal{B}(\Omega) \ni \mathbf{B} \mapsto \nu(\mathbf{B} \times \mathbf{Y})$  denotes the marginal of  $\nu$  in  $(t, x)$ ,  $\rho_k = [\eta_k \cdot \delta_{t=0} \otimes \nu_0 + q_k \cdot \delta_{x=L} \otimes \nu_L]^{(t,x)}$ ,  $\eta_k(y) = \frac{y^{2k}}{2k}$  and  $q'_k = \eta'_k f'$ , for  $k \in \{1/2\} \cup \{1, \dots, N\}$ , and  $\mathcal{D}$  defines the weak PDE operator:

$$\mathcal{D}(\nu, \nu_T, \nu_R) := \left( \frac{\partial}{\partial t}(y \nu) + \frac{\partial}{\partial x}(f \nu) + y \cdot \delta_{t=T} \otimes \nu_T + f \cdot \delta_{x=R} \otimes \nu_R \right)^{(t,x)} \quad (11a)$$

and for  $k = 1, \dots, N$ ,  $\mathcal{E}_k$  defines the entropy operator:

$$\mathcal{E}_k(\nu, \nu_T, \nu_R, \sigma) := \sigma_k + \left( \frac{\partial}{\partial t}(\eta_k \nu) + \frac{\partial}{\partial x}(q_k \nu) + \eta_k \cdot \delta_{t=T} \otimes \nu_T + q_k \cdot \delta_{x=R} \otimes \nu_R \right)^{(t,x)} \quad (11b)$$

**Remark 10.** *The above LP is written in standard form, with linear mappings of the decision variables on the left-hand side of the constraints and fixed parameters on the right-hand side.*

**Remark 11.**  *$k = 1/2$  corresponds to  $\eta(y) = y$  and  $q = f$  (i.e. the weak PDE formulation) and is introduced only for conciseness of notations. As stated above, in infinite dimension and with strictly convex flux  $f(y)$ ,  $N = 1$  is enough to ensure uniqueness of the MV solution, but in the finite dimensional representations, [7] suggests to add as many of these constraints as possible to improve the numerics.*

### 3 Dual Problem over Test Functions

Like their finite dimensional counterparts, infinite dimensional LPs are also subject to Lagrangian duality, with Lagrange multipliers taken in the dual of the space where equality constraints hold, with respect to the appropriate topology (namely the weak-\* topology). One can either use an augmented Lagrangian or invoke standard duality rules, e.g. that the dual to  $\inf\{\langle c, \chi \rangle \mid \mathcal{A} \chi = b\}_{\chi \in \mathcal{M}_+}$  is  $\sup\{\langle v, b \rangle \mid c - \mathcal{A}' v \in \mathcal{M}'_+\}_{v \in \mathcal{Y}}$  regardless of the nature of linear maps  $\mathcal{A}$ ,  $\langle \cdot, b \rangle$  and  $\langle c, \cdot \rangle$ , and convex cones  $\mathcal{M}_+$  and  $\mathcal{M}'_+$ . Hence, denoting  $\sigma = (\sigma_k)_{k=1}^N$  and  $\mathcal{E} = (\mathcal{E}_k)_{k=1}^N$ , and identifying

$$\mathcal{A}(\nu, \nu_T, \nu_R, \sigma) = \begin{bmatrix} \mathcal{D}(\nu, \nu_T, \nu_R) \\ \mathcal{E}(\nu, \nu_T, \nu_R, \sigma) \\ \nu^{(t,x)}, \nu_T^{(x)}, \nu_R^{(t)} \end{bmatrix} \quad (12a)$$

$$b = (\rho_{1/2}, (\rho_k)_{k=1}^N, \lambda_\Omega, \lambda_{\mathbf{X}}, \lambda_{\mathbf{T}}) \quad (12b)$$

$$c = (h, h_T, h_R, 0) \quad (12c)$$

the dual of problem (10) is straightforwardly formulated as:

$$\sup_{\substack{\varphi, \phi_1, \dots, \phi_N \\ w, w_T, w_R}} \int_\Omega \varphi d\rho_{1/2} + \sum_{k=1}^N \int_\Omega \phi_k d\rho_k + \int_\Omega w d\lambda + \int_{\mathbf{X}} w_T d\lambda + \int_{\mathbf{T}} w_R d\lambda \quad (13)$$

$$\begin{aligned}
\text{s.t. } \quad & \varphi, \phi_1, \dots, \phi_N \in \mathcal{C}^1(\Omega) w \in \mathcal{C}^0(\Omega), w_T \in \mathcal{C}^0(\mathbf{X}), w_R \in \mathcal{C}^0(\mathbf{T}) \\
& h + y \frac{\partial \varphi}{\partial t} + f \frac{\partial \varphi}{\partial x} + \sum_{k=1}^N \eta_k \frac{\partial \phi_k}{\partial t} + q_k \frac{\partial \phi_k}{\partial x} - w \geq 0 \\
& h_T - y \varphi(T, \cdot) - \sum_{k=1}^N \eta_k \phi_k(T, \cdot) - w_T \geq 0 \\
& h_R - f \varphi(\cdot, R) - \sum_{k=1}^N q_k \phi_k(\cdot, R) - w_R \geq 0 \\
& -\phi_k \geq 0, \quad k = 1, \dots, N
\end{aligned}$$

**Remark 12.** By definition of topological duality, the Lagrange multiplier  $\varphi$  exactly corresponds to test functions in the weak formulation (3), while the Lagrange multiplier  $\phi_k$  corresponds to  $-\psi$  in the entropy condition (4) for the entropy pair  $(\eta_k, q_k)$ .

**Remark 13.** Recalling the initial and boundary conditions, the criterion in  $(\varphi, \phi_1, \dots, \phi_N)$  can be made explicit as

$$\int_{\Omega} \varphi d\rho_{1/2} = \int_{\mathbf{X}} y_0(x) \varphi(0, x) dx + \int_{\mathbf{T}} f(y_L(t)) \varphi(t, L) dt \quad (14a)$$

$$\int_{\Omega} \phi_k d\rho_k = \int_{\mathbf{X}} \eta_k(y_0(x)) \phi_k(0, x) dx + \int_{\mathbf{T}} q_k(y_L(t)) \phi_k(t, L) dt \quad (14b)$$

**Theorem 14** (Strong Duality). *Problems (10) and (13) have the same optimal value.*

*Proof.* According to [20, Theorem 2.6], strong duality follows from uniform boundedness, in total variation norm, of all involved measure decision variables in the primal problem. Let then  $(\nu, \nu_T, \nu_R, \sigma)$  be feasible for (10). Since those are Borel measures, their total variation norm is their masses. Hence, we examine, using the marginal constraints in (10):

$$\int_{\mathbf{K}} 1 d\nu = \int_{\Omega} 1 d\nu^{(t,x)} = \int_{\Omega} dt dx = \lambda(\Omega).$$

Likewise, we get  $\int_{\mathbf{K}_T} 1 d\nu_T = \lambda(\mathbf{X})$  and  $\int_{\mathbf{K}_R} 1 d\nu_R = \lambda(\mathbf{T})$ . Eventually, it remains to compute, using (10) and (11b):

$$\int 1 d\sigma_k = \int_{\Omega} 1 d\rho_k - \int_{\mathbf{K}_T} \eta_k d\nu_T - \int_{\mathbf{K}_R} q_k d\nu_R \leq \int_{\Omega} 1 d\rho_k \stackrel{(14b)}{=} \|\eta_k \circ y_0\|_{\mathcal{L}^1(\mathbf{X})} + \|q_k \circ y_L\|_{\mathcal{L}^1(\mathbf{T})}$$

where the first equality was obtained by remembering that a partial derivative of a measure always maps constants to 0, and the inequality was obtained by remembering that, since  $\eta_k(y) = \frac{y^{2k}}{2k}$  and  $y \geq 0$  is ensured by the maximum principle,  $\eta_k$  and  $q_k$  are nonnegative. We conclude that any  $(\nu, \nu_T, \nu_R, \sigma)$  feasible in (10) satisfies the uniform bounds:

$$\|\nu\|_{TV} \leq \lambda(\Omega), \quad \|\nu_T\|_{TV} \leq \lambda(\mathbf{X}), \quad \|\nu_R\|_{TV} \leq \lambda(\mathbf{T}),$$

$$\|\sigma_k\|_{TV} \leq \|\eta_k \circ y_0\|_{\mathcal{L}^1(\mathbf{X})} + \|q_k \circ y_L\|_{\mathcal{L}^1(\mathbf{T})} < \infty$$

which proves strong duality as per [20, Theorem 2.6].  $\square$

## 4 PDE Solution from Dual Variables

In this section, we explain how an approximate solution of the conservation law can be reconstructed from a set of polynomial dual variables  $(\varphi, \psi, w, w_T, w_R)$  obtained by solving finite dimensional convex strengthenings of the dual problem on test functions. The reconstruction relies on strong duality and on the concentration of entropy measure-valued solutions on the graph of the weak solution of the PDE.

## 4.1 Finite-Dimensional Polynomial Recastings

Define the bulk slack function  $S(t, x, y)$  associated with the feasible set of dual variables posed on the whole domain  $\mathbf{K}$ , as well as the boundary slack functions  $S_T(x, y)$  and  $S_R(t, y)$ :

$$S := h - w + y \frac{\partial \varphi}{\partial t} + f \frac{\partial \varphi}{\partial x} + \eta \frac{\partial \psi}{\partial t} + q \frac{\partial \psi}{\partial x} \quad (15a)$$

$$S_T := h_T - w_T - (y \varphi(T, \cdot) + \eta \psi(T, \cdot)) \quad (15b)$$

$$S_R := h_R - w_R - (f \varphi(\cdot, R) + q \psi(\cdot, R)) \quad (15c)$$

Dual feasibility imposes  $S_T^* \geq 0$  on  $\mathbf{K}_T$ ,  $S_R^* \geq 0$  on  $\mathbf{K}_R$  and  $\psi^* \geq 0$  on  $\Omega$ , with respective Lagrange multipliers  $\nu, \nu_T, \nu_R$ . Suppose that  $h, h_T, h_R$  are polynomials, and restrict the search for  $\varphi, \psi, w, w_{\mathbf{X}}, w_{\mathbf{T}}$  to polynomial parametrizations. The Stone-Weierstrass theorem ensures that such restriction does not change the optimal value of the problem, hence strong duality is preserved. Moreover, with this parametrization,  $S(t, x, y)$ ,  $S_T(x, y)$  and  $S_R(t, y)$  are now polynomials too. Eventually, we also assume that  $\mathbf{K}, \mathbf{K}_T$  and  $\mathbf{K}_R$  are semi-algebraic sets, i.e. for instance  $\exists g_1, \dots, g_m \in \mathbb{R}[t, x, y]$ ;

$$\mathbf{K} = \{(t, x, y) \in \mathbb{R}^3 \mid g_1(t, x, y) \geq 0, \dots, g_m(t, x, y) \geq 0\}.$$

These assumptions allow us to leverage Putinar's Positivstellensatz to represent positivity constraints on  $S, S_T$  and  $S_R$ . Namely, under mild assumptions, Putinar's theorem states that if  $\forall (t, x, y) \in \mathbf{K}, S(t, x, y) > 0$ , then  $\exists N_0, \dots, N_m \in \mathbb{N}, p_0 \in \mathbb{R}[t, x, y]^{N_0}, \dots, p_m \in \mathbb{R}[t, x, y]^{N_m}$  such that

$$S(t, x, y) = \|p_0(t, x, y)\|_2^2 + \sum_{i=1}^m g_i(t, x, y) \|p_i(t, x, y)\|_2^2 \quad (16)$$

where  $\|p_0(t, x, y)\|_2^2 = p_{0,1}(t, x, y)^2 + \dots + p_{0,N_0}(t, x, y)^2$  denotes the squared Euclidean norm of  $p_0(t, x, y) \in \mathbb{R}^{N_0}$ , and can be represented by a finite-size positive semidefinite matrix. Consequently, by further restricting problem (13) to polynomials of uniformly bounded degree  $d$  and to bounded-size matrix representations of the inequality constraints, we obtain a convex finite-dimensional SDP strengthening of the original dual problem (13).

**Remark 15.** By “bounded-size matrix representation”, we mean that we restrict the parametrization of  $p_0, \dots, p_m$  such that  $\deg(\|p_0\|_2^2) \leq d$  and  $\deg(g_i \|p_i\|_2^2) \leq d, i = 1, \dots, m$ , to be consistent with the degree bound. This results in a uniform bound on the size of the corresponding matrix representations. In this way, one gets a hierarchy of nested finite-size SDP problems indexed by  $d \in \mathbb{N}$ , called a sum-of-squares (SoS) hierarchy, as  $\|p\|_2^2$  parametrizes polynomial sums of squares.

**Theorem 16** (Strong duality and convergence). *The above derived SDP strengthenings of (13) are in strong duality with the moment relaxations of (10) presented in [7]. Moreover, defining  $V^*$  as the common optimal value of problems (10) and (13), and  $V_d$  the common optimal value to their respective degree  $d$  finite dimensional representations, it holds*

$$V_d \nearrow_{d \rightarrow \infty} V^*. \quad (17)$$

*Proof.* Strong duality is proved identically as in Theorem 14, using [6, Theorem 4] instead of [20, Theorem 2.6]. Monotonicity is due to the fact that increasing the degree bound  $d$  enlarges the dual feasible set, resulting in a higher supremum value. Convergence follows from [7, Theorem 4].  $\square$

**Corollary 17** (Approximate Complementary Slackness). *Let  $(\varphi_d, \psi_d, w_d, w_{T,d}, w_{R,d}) \in \mathbb{R}_d[t, x]^3 \times \mathbb{R}_d[x] \times \mathbb{R}_d[t]$  be optimal for the degree  $d$  representation of (13), and  $S_d$  the corresponding slack*

function as per (15a), for  $d \geq \deg(h)$ . Let  $(t, x) \mapsto y^*(t, x)$  be the entropy weak solution to (1). Then,

$$0 \leq \int_{\Omega} S_d(t, x, y^*(t, x)) dt dx \leq V^* - V_d \xrightarrow{d \rightarrow \infty} 0, \quad (18a)$$

from which it follows directly that

$$S_d(t, x, y^*(t, x)) \xrightarrow{d \rightarrow \infty} 0 \quad \text{for } \lambda_{\Omega}\text{-almost all } (t, x). \quad (18b)$$

*Proof.* Let  $(\nu^*, \nu_T^*, \nu_R^*)$  be optimal in (10), by [7, Theorem 1] we know that  $\nu^*(dt, dx, dy) = \delta_{y^*(t, x)}(dy) dt dx$ , so that

$$\int_{\Omega} S_d(t, x, y^*(t, x)) dt dx = \int S_d d\nu^*.$$

Then, by definition of Lagrange duality, denoting by  $\mathcal{L}(\varphi, \phi, w, w_T, w_R)$  the optimization criterion of (13), it holds

$$\int S_d d\nu^* \leq \mathcal{J}(\nu^*, \nu_T^*, \nu_R^*) - \mathcal{L}(\varphi_d, \phi_d, w_d, w_{T,d}, w_{R,d}) = V^* - V_d \xrightarrow{d \rightarrow \infty} 0$$

by Theorem 16. We conclude by nonnegativity of  $S_d$ .  $\square$

## 4.2 Pointwise Solution Reconstruction

Despite the above results, at this point the reconstruction of solutions to the conservation law from dual optimizers is only heuristic. Mathematical proof of convergence of the extracted solution approximation would require subtle estimates and assumptions that are beyond the scope of this article. Hence, we only expose our rationale for the reconstruction method. In the framework of optimal control, a GMP formulation is also available, with strong duality guarantees, where the dual problem on functions approximates the value function from above via super-solutions  $v_d$  of the HJB equation. Then, under regularity and convexity assumptions, for a given time  $t$  and state  $x$ , a control feedback law is extracted as (see e.g. [21])

$$u_d(t, x) = \arg \min \{v_d(t, x, u) \mid u \in \mathbf{U}\}.$$

Likewise, here we propose to define the approximate solution

$$y_d(t, x) = \arg \min \{S_d(t, x, y) \mid y \in \mathbf{Y}\}. \quad (19)$$

**Remark 18.** While proving the convergence of  $y_d(t, x)$  towards the actual entropy weak solution  $y(t, x)$  is highly nontrivial, by design of the proposed approximation, Corollary 17 at least gives access to the pointwise estimate  $\lambda_{\Omega}$ -a.e.

$$S_d(t, x, y_d(t, x)) \leq S_d(t, x, y^*(t, x)) \leq V^* - V_d, \quad (20)$$

so that approximate complementary slackness is preserved.

**Theorem 19.** Assume that  $\mathbf{K}$  is convex and there exists  $\varepsilon > 0$  for which feasibility of  $(\varphi, \phi, w, w_T, w_R)$  for (13) enforces  $\varepsilon$ -strong convexity of the corresponding slack  $S(t, x, y)$  in  $y$ :

$$\forall (t, x, y) \in \mathbf{K}, \quad \frac{\partial^2 S}{\partial y^2}(t, x, y) \geq \varepsilon.$$

Then, (19) defines a unique  $(t, x) \mapsto y_d(t, x)$  and

$$y_d \xrightarrow{d \rightarrow \infty} y^* \quad \text{in } \mathcal{L}^2(\Omega). \quad (21)$$

*Proof.* Uniqueness of the minimizer is a direct consequence of  $\varepsilon$ -strong convexity. Moreover, again by  $\varepsilon$ -strong convexity, it holds for any  $(t, x, y) \in \mathbf{K}$ :

$$S_d(t, x, y) \geq S_d(t, x, y_d(t, x)) + \frac{\varepsilon}{2}|y - y_d(t, x)|^2.$$

In particular, we get for  $y = y^*(t, x)$ :

$$\begin{aligned} \|y^* - y_d\|_{\mathcal{L}^2(\Omega)}^2 &= \int_{\Omega} |y^*(t, x) - y_d(t, x)|^2 dt dx \\ &\leq \frac{2}{\varepsilon} \int_{\Omega} (S_d(t, x, y^*(t, x)) - S_d(t, x, y_d(t, x))) dt dx \\ &\leq \frac{2}{\varepsilon} \int_{\Omega} S_d(t, x, y^*(t, x)) dt dx \\ &\leq 2 \frac{V^* - V_d}{\varepsilon} \xrightarrow{d \rightarrow \infty} 0. \quad \square \end{aligned}$$

**Remark 20.** *The strong-convexity assumption in Theorem 19 efficiently enforces convergence but is difficult to establish. Actually, when the solution  $y^*$  is discontinuous, strong convexity of  $S$  is unlikely to hold. An option would be to add it to the constraints of the dual (13), and analyze the behavior of the primal solutions as  $\varepsilon$  goes to 0, in a manner similar to viscosity solutions. Another (complementary) possibility would consist in adjusting the primal cost  $h$  in order to enforce strong convexity. In any case, short of convincingly proving convergence in the general case, Theorem 19 illustrates the intuition for extracting solutions according to (19). This is further supported by the numerical results shown below.*

**Remark 21.** *Our approach to extracting solutions from moment-SoS optimal solutions may look similar to the one leveraging Christoffel-Darboux (CD) kernels in [14], in the sense that both resort to representing the solution as the arg min of a polynomial, hence covering a large variety of functions, including discontinuous shocks and piecewise-affine rarefaction waves. Still, they differ in the sense that extraction based on the CD kernel “only” exploits the moment matrix*

$$\mathbf{M}_d \simeq \int e(t, x, y) e(t, x, y)^\top \nu^*(dt, dx, dy)$$

where  $e(t, x, y) \in \mathbb{R}_d[t, x, y]^{\binom{d+3}{3}}$  is a basis of  $\mathbb{R}_d[t, x, y]$ . In the moment-SoS hierarchy, this moment matrix is dual to  $\|p_0(t, x, y)\|_2^2$  in the Putinar representation (16) of  $S_d(t, x, y)$ , while our approach accounts for the full slack polynomial  $S_d$ . Moreover, the spectral filtering required to represent singular measures in the CD kernel approach is not relevant here.

**Remark 22.** *In addition to proving convergence of  $y_d$  towards  $y^*$ , another important question would be to assess the rate at which the solutions can be expected to converge, which requires a detailed analysis of the regularity of optimal solutions to the dual problem (13). Such solutions are not even guaranteed to exist in the current formulation, see [22] for a complete description of the link between existence of regular solutions and convergence rates of the moment-SoS hierarchy.*

## 5 Numerical Results

### 5.1 Setup Description

We solve degree- $d = 10$  SoS strengthenings of (13) using the MATLAB Code available at [https://gitlab.com/slimj-group/mv-entropy-dual/-/blob/main/mv\\_entropy\\_dual.m](https://gitlab.com/slimj-group/mv-entropy-dual/-/blob/main/mv_entropy_dual.m). More precisely, we take  $\mathbf{T} = [0, 1] = \{t \in \mathbb{R} \mid t(1-t) \geq 0\}$ ,  $\mathbf{X} = [-0.5, 0.5] = \{x \in \mathbb{R} \mid (x+0.5)(0.5-x) \geq 0\}$ , and  $\mathbf{Y} = [0, 1]$ . We focus on two classical test cases for the Burgers equation (i.e.,  $f(y) = y^2/2$ ): a shock wave with  $y_0(x) = \mathbb{1}_{[-0.5, 0]}(x)$  and  $y_L(t) = 1$ , and a rarefaction wave

with  $y_0(x) = \mathbb{1}_{[0,0.5]}(x)$  and  $y_L(t) = 0$ . Finally, we run the SoS strengthening for three different choices of the cost

$$\mathcal{J}(\nu, \nu_T, \nu_R) = \int h \, d\nu + \int h_T \, d\nu_T + \int h_R \, d\nu_R,$$

namely  $h = h_T = h_R$ , with three alternatives:

$$h(t, x, y) = \sum_{\alpha=0}^{d/2} \sum_{\beta=0}^{d/2-\alpha} t^{2\alpha} x^{2\beta} y^{d-2\alpha-2\beta} \quad (\text{even})$$

$$h(t, x, y) = \sum_{\alpha=1}^d \frac{y^\alpha + (1-y)^\alpha}{\alpha} \quad (\text{log})$$

$$h(t, x, y) = \sum_{\alpha=0}^{d/2} \frac{y^{2\alpha+1} + (1-y)^{2\alpha+1}}{2\alpha+1} \quad (\text{atanh})$$

where (even) corresponds to the trace of the moment matrix, as in [7], (log) converges to  $-\log(y(1-y)) = \int_0^1 \frac{dz}{1-|y-z|}$ ; and (atanh) converges to  $\tanh^{-1}(y) + \tanh^{-1}(1-y) = \int_0^1 \frac{dz}{1-(y-z)^2}$  as  $d \rightarrow \infty$ . The last two criteria reflect a trade-off between strong convexity of  $h$  in  $y$  and the penalization of large values of  $y(t, x)$  close to the bounds.

While [7] advocates using multiple entropies in the degree- $d$  truncations, we observed that this strategy drastically increases the computational time while providing only limited improvement over a single well-chosen entropy. As a result, we performed the computations with  $N = 1$  and

$$\eta(y) = \sum_{\alpha=1}^{d/2} \frac{y^{2\alpha}}{2\alpha}, \quad q(y) = \sum_{\alpha=1}^{d/2} \frac{y^{2\alpha+1}}{2\alpha+1}. \quad (22)$$

## 5.2 Numerical Experiments

For each of the above test cases, we solved the corresponding degree- $d$  representations with both MOSEK and SeDuMi as SDP solvers within YALMIP. For comparison, we approximated the normalized distance  $\frac{1}{\lambda(\Omega)} \|y_d - y^*\|_{\mathcal{L}^1(\Omega)}$  by the empirical mean error

$$\hat{y} := \frac{1}{K} \sum_{n=1}^K |y_d(t_n, x_n) - y^*(t_n, x_n)|, \quad (23)$$

where the points  $(t_n, x_n)$  are sampled on a relatively coarse  $50 \times 50$  grid (i.e.,  $K = 2500$ ) in order to keep the post-processing time reasonable. The resulting approximate  $\mathcal{L}^1$  distances are reported in Tables 1 and 2 together with two graphs representing the shock and rarefaction waves reconstructed by our algorithm. In addition to the dual formulation, we also computed the primal relaxations with Christoffel-Darboux kernel extraction, as per [7], which we in turn compared to the analytic solution using (23) once again.

A shock-wave discontinuity propagates along  $x_s(t) = st$  at the Rankine-Hugoniot speed  $s = \frac{f(y_0(L)) - f(y_0(R))}{y_0(L) - y_0(R)}$ , so that  $y^*(t, x) = \mathbb{1}_{]-\infty, st]}(x)$ .

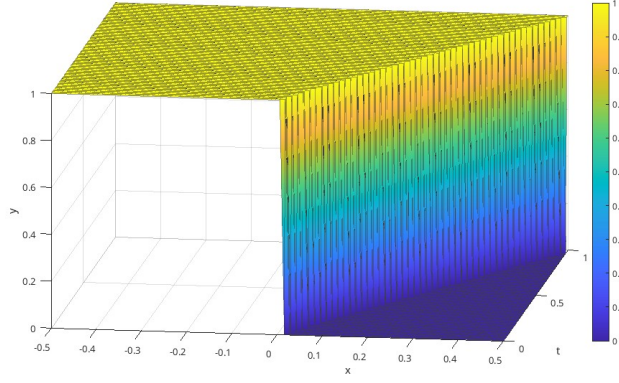


Figure 1: Shock-Wave Graph

Table 1: $L^1$ -distance from the exact shock-wave solution.			
	even	log	atanh
SeDuMi	0.000832986255	0.000832986255	0.000832986255
Mosek	0.000832986255	0.001249479383	0.001249479383

Comparatively, the primal only achieves  $\hat{y} = 0.0122186$  using Mosek with even entropy costs at equal  $d$ -relaxation degree.

As for the rarefaction wave ( $y_0(x) = \mathbb{1}_{[0,R]}(x)$ ,  $y_L(t) = 0$ ), the exact solution is  $y^*(t, x) = \frac{x}{t} \mathbb{1}_{[0,t]}(x) + \mathbb{1}_{]t,+\infty[}(x)$ .

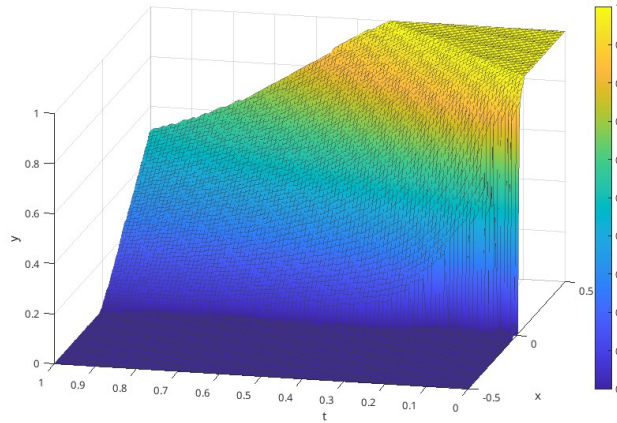


Figure 2: Rarefaction-Wave Graph

Table 2: $L^1$ -distance from the exact rarefaction-wave solution.			
	even	log	atanh
SeDuMi	0.014615751581	0.016534437954	0.016481358316
Mosek	0.015114338538	0.018346819141	0.017678366146

Comparatively, the primal only achieves  $\hat{y} = 0.028886$  using Mosek with even entropy costs at equal  $d$ -relaxation degree.

These experiments suggest that, at a fixed relaxation/strengthening degree  $d$ , our dual approach outperforms the primal relaxations from [7]. They also pave the way for future developments on convergence towards the solution, as well as more sophisticated problems such as control synthesis.

## 6 Conclusion

This paper developed a dual SoS framework for entropy measure-valued solutions of scalar nonlinear hyperbolic conservation laws on compact space-time domains. Starting from the generalized moment formulation, we emphasized the intrinsic Lagrangian dual over test functions, entropy multipliers, and slack variables, and showed how this viewpoint yields both admissibility certificates and a practical reconstruction principle based on slack minimization.

On the computational side, the dual problem was approximated by polynomial restrictions on compact semialgebraic sets and enforced through Putinar-type SoS certificates, leading to semidefinite programs implemented in YALMIP. The numerical experiments on Burgers-type shock and rarefaction configurations show that the dual formulation is bounded and can recover the expected entropy solution with good accuracy. In particular, the reconstructed profiles reproduce both discontinuous and continuous wave patterns, and the reported discrete  $L^1$ -errors quantify the influence of both the entropy-based cost integrand and the SDP solver on the final approximation.

These results indicate that the dual hierarchy is not only a theoretical certificate of admissibility, but also an effective tool for solution recovery. They also suggest that entropy selection, cost design, and solver robustness play a decisive role in reconstruction quality. Future work will focus on refined multi-entropy formulations, more systematic comparisons with primal moment relaxations and Christoffel–Darboux recovery, stronger numerical reconstruction procedures based on relaxed contact sets, with convergence guarantees and rates, and eventually extensions to parameter-dependent conservation laws, as well as control and stabilization of PDEs.

## Acknowledgements

This work was carried out within the EUROfusion Consortium and funded by the European Union through the Euratom Research and Training Programme (Grant Agreement No. 101052200 – EUROfusion). The views expressed herein are those of the authors only and do not necessarily reflect those of the European Union, the European Commission, or EUROfusion, which cannot be held responsible for them.

The work of Matteo Tacchi-Bénard is supported by the French National Research Agency (ANR) under the PIVOINE project grant ANR-25-CE48-1599-01. The work of Slim Jmal is supported by the program "Initiatives de Recherche à Grenoble Alpes" through the grant SOS-Fusion.

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