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BREAKING EXPONENTIAL COMPLEXITY IN GAMES OF ORDERED PREFERENCE: A TRACTABLE REFORMULATION*

DONG HO LEE[†], JINGQI LI[‡], LASSE PETERS[§], GEORGIOS BAKIRTZIS[¶], AND DAVID FRIDOVICH-KEIL^{†‡}

Abstract. Games of ordered preference (GOOPs) model multi-player equilibrium problems in which each player maintains a distinct hierarchy of strictly prioritized objectives. Existing approaches solve GOOPs by deriving and enforcing the necessary optimality conditions that characterize lexicographically constrained Nash equilibria through a single-level reformulation. However, the number of primal and dual variables in the resulting Karush-Kuhn-Tucker (KKT) system grows exponentially with the number of preference levels, leading to severe scalability challenges. We derive a compact reformulation of these necessary conditions that preserves the essential primal stationarity structure across hierarchy levels, yielding a “reduced” KKT system whose size grows polynomially with both the number of players and the number of preference levels. The reduced system constitutes a relaxation of the complete KKT system, yet it remains a valid necessary condition for local GOOP equilibria. For GOOPs with quadratic objectives and linear constraints, we prove that the primal solution sets of the reduced and complete KKT systems coincide. More generally, for GOOPs with arbitrary (but smooth) nonlinear objectives and constraints, the reduced KKT conditions recover all local GOOP equilibria but may admit spurious non-equilibrium solutions. We introduce a second-order sufficient condition to certify when a candidate point corresponds to a local GOOP equilibrium. We also develop a primal-dual interior-point method for computing a local GOOP equilibrium with local quadratic convergence. The resulting framework enables scalable and efficient computation of GOOP equilibria beyond the tractable range of existing exponentially complex formulations.

Key words. Lexicographic preferences, hierarchical games, mathematical programming

MSC codes. 49K99, 91A65, 90C99

1. Introduction. Making decisions under strictly ordered, potentially conflicting, objectives is a central problem in optimization. In many applications—ranging from autonomous driving [33] and power systems [2] to supply chains [32]—higher-priority objectives must be satisfied before lower-priority ones are considered. Lexicographic optimization [16, 2, 4] provides a natural framework for such problems by imposing a strict priority ordering among multiple objectives for a single decision-maker. In contrast, classical multi-objective optimization [15, 13, 9, 24] studies competing objectives for a single decision-maker through Pareto-optimality [25, 17] and methods such as weighted sums [12, 13], ϵ -constraint formulations [14], and goal programming [3, 17]. In these approaches, improving one objective may degrade another, and no intrinsic priority structure is imposed.

We consider the multi-agent extension of the lexicographic setting, namely *games*

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of *ordered preference* (GOOPs) [19, 6], in which each player solves a lexicographically constrained, coupled optimization problem. GOOPs inherit aspects of noncooperative game theory [1, 10, 7], but differ fundamentally from standard generalized Nash games [8, 18, 20]: each player faces a hierarchy of ordered objectives rather than a single objective. This hierarchical structure makes the computation of equilibrium solutions substantially more challenging, since a GOOP equilibrium must encode both coupling across players and lexicographic optimality within each player’s problem.

A natural route for modeling and solving GOOPs is through multilevel optimization [5, 22, 29, 31], where lower-level (high-priority) optimality conditions define upper-level (low-priority) feasibility. The existing solution strategy [19] adopts this perspective and recursively replaces lower-level problems with their KKT conditions, thereby obtaining a single-level mathematical program with complementarity constraints (MPCC) [30, 23, 27]. However, the MPCC reformulation introduces lower-level dual variables that become primal variables at the upper levels. The expanding set of primal variables causes the number of variables and conditions to grow exponentially with the number of preference levels, and severely limits scalability. By contrast, lexicographic optimization enforces higher-priority optimality by treating objectives sequentially and constraining each subproblem so as not to degrade higher-priority, objective values. However, the principle in lexicographic optimization does not extend readily to noncooperative equilibrium settings, where each player’s feasible set and objectives depend on the decisions of the others.

In this paper, we show that the exponential growth of the existing GOOP reformulation is not intrinsic to the equilibrium conditions, but rather a consequence of recursively flattening the problem hierarchy. We derive a *reduced* KKT system that preserves the nested primal stationarity structure while avoiding redundant dual variable propagation. The proposed system grows polynomially in the number of players and preference levels. Our contributions are threefold. We first show that the reduced system is a relaxation of the complete KKT system. Second, for quadratic GOOPs with linear constraints, we prove primal solution equivalence between the reduced and complete KKT systems. For GOOPs with arbitrary (but smooth) nonlinear objectives and constraints, we show that the reduced KKT system may potentially admit additional solutions. Accordingly, we establish a second-order sufficient condition that certifies local GOOP equilibria. Finally, we develop a primal-dual interior-point method for the reduced KKT system and prove its local quadratic convergence.

2. Games of Ordered Preference. In this section, we formalize games of ordered preference (GOOPs), define the local generalized Nash solution concept, and derive corresponding first-order necessary conditions.

2.1. Local Generalized Nash Equilibrium for GOOP. We consider a non-cooperative GOOP involving N players. Let $[N] := \{1, 2, \dots, N\}$. For a player $i \in [N]$, we denote their decision variables as a real-valued vector $z^i \in \mathbb{R}^{n^i}$. We define the collection of all players’ variables as $z := \{z^1, \dots, z^N\}$, and the collection of all players’ variables except those of player i as $z^{-i} := \{z^j\}_{j \neq i}$. When convenient, we identify z and z^{-i} as vectors, i.e., $z = [(z^1)^\top, \dots, (z^N)^\top]^\top \in \mathbb{R}^n$, where $n := \sum_{i=1}^N n^i$, and $z^{-i} = [(z^j)^\top]_{j \neq i}^\top \in \mathbb{R}^{n-n^i}$. Each player has a hierarchy of strictly prioritized objectives, which we refer to as *preferences*. We denote the number of preference levels for player i as $K^i \in \mathbb{N}$. Unless stated otherwise, players may have different numbers of preference levels, i.e., $K^i \neq K^j$ for $i \neq j$. We denote the primal variable of player i at preference level k as $z_k^i \in \mathbb{R}^{n^i}$. Player i ’s decision variable is common to all preference

levels; throughout, we may suppress the level index for the top level ($k = 1$) and define $z^i := z_1^i$ to denote player i 's top-level decision variable. Each preference level k is associated with a real-valued objective function $J_k^i(z_k^i, z^{-i})$, where z^{-i} denotes the collection of the other players' top-level variables. Using $\arg \min$ to denote the argument of *local* minimizers, we state player i 's problem in the GOOP framework:

$$(2.1a) \quad \min_{z_1^i} J_1^i(z_1^i, z^{-i})$$

$$(2.1b) \quad \text{s.t.} \quad z_1^i \in \arg \min_{z_2^i} J_2^i(z_2^i, z^{-i})$$

⋮

$$(2.1c) \quad \text{s.t.} \quad z_{K^i-1}^i \in \arg \min_{z_{K^i}^i} J_{K^i}^i(z_{K^i}^i, z^{-i})$$

$$(2.1d) \quad \text{s.t.} \quad z_{K^i}^i \in \mathcal{Z}_{K^i}^i(z^{-i}).$$

The innermost feasible set (at level K^i) is given by

$$(2.2) \quad \mathcal{Z}_{K^i}^i(z^{-i}) := \{z^i \mid h^i(z^i, z^{-i}) = \mathbf{0}, g^i(z^i, z^{-i}) \geq \mathbf{0}\},$$

where $\mathbf{0}$ denotes a vector (or matrix) of all zeros of appropriate dimension, $h^i(z^i, z^{-i}) \in \mathbb{R}^{m_h^i}$ and $g^i(z^i, z^{-i}) \in \mathbb{R}^{m_g^i}$ are equality and inequality constraints, respectively.

In this paper, we consider the following assumption:

Assumption 2.1. For each player $i \in [N]$ and each level $k \in [K^i]$, we assume the objective function $J_k^i(z_k^i, z^{-i})$ is sufficiently differentiable with respect to z^i , but not necessarily convex in z^i . The innermost feasible set $\mathcal{Z}_{K^i}^i(z^{-i})$ is compact, and an MPCC-tailored constraint qualification, such as MPCC-linear independence constraint qualification (LICQ) [30], holds.

Under the above assumption, we formally define the *local generalized Nash equilibrium* for GOOPs. This definition extends the local generalized Nash equilibrium concept in [10, 28] to the lexicographic setting.

DEFINITION 2.2 (Local Generalized Nash Equilibrium for GOOPs). *A vector z^* is a local generalized Nash equilibrium for GOOP (2.1), if for each player i , the strategy z^{i*} is feasible for player i 's nested problem and there exists an $\epsilon > 0$ such that*

$$(2.3) \quad J_1^i(z^{i*}, z^{-i*}) \leq J_1^i(z^i, z^{-i*}), \quad \forall z^i \in \mathcal{Z}_1^i(z^{-i*}) \text{ s.t. } \|z^i - z^{i*}\|_2 \leq \epsilon, \quad \forall i \in [N],$$

where $\mathcal{Z}_1^i(z^{-i*})$ denotes the top-level feasible set that encodes the lexicographic preferences of player i . Equivalently, no player can unilaterally locally deviate from z^{i*} and achieve a better top-level objective given z^{-i*} while remaining consistent with their hierarchy of preferences.

In the rest of this work, we refer to a local generalized Nash equilibrium of a GOOP simply as a local GOOP equilibrium. In the next subsection, building on [19], we review the complete KKT conditions for a local GOOP equilibrium and show that its size grows exponentially with hierarchy depth. This exponential complexity motivates the construction of the reduced KKT system, presented in Section 3, whose size grows polynomially in the number of players and preference levels.

2.2. Complete Necessary Conditions for GOOP Equilibria. The reformulation in [19] replaces each lower-level problem in (2.1) with its *complete* KKT

optimality conditions; we refer to this reformulation as the complete KKT system. Construction proceeds by backward induction for each player, starting from the innermost level (highest priority) and recursively deriving the KKT systems for outer levels. For clarity, we denote quantities of the complete KKT system with an overbar to distinguish them from terms in the reduced KKT system introduced in Section 3.

Innermost level $k = K^i$ for player i . Consider the bottom-level problem in (2.1c) and (2.1d). Let $\bar{\lambda}_{K^i}^i$ and $\bar{\gamma}_{K^i}^i$ represent the dual variables for the equality and inequality constraints, respectively. To simplify the notation for player i 's Lagrangian, we collect the dual variables $\bar{\lambda}_{K^i}^i$ and $\bar{\gamma}_{K^i}^i$ as $\bar{\eta}_{K^i}^i$, i.e.,

$$(2.4) \quad \bar{\eta}_{K^i}^i := [(\bar{\lambda}_{K^i}^i)^\top, (\bar{\gamma}_{K^i}^i)^\top]^\top \in \mathbb{R}^{m_\varepsilon^i + m_z^i}.$$

The bottom-level Lagrangian is defined in terms of the primal variables \bar{z}^i , the dual variables $\bar{\eta}_{K^i}^i$, and the other players' primal variables \bar{z}^{-i} :

$$(2.5) \quad \bar{\mathcal{L}}_{K^i}^i(\bar{z}^i, \bar{z}^{-i}, \bar{\eta}_{K^i}^i) = J_{K^i}^i(\bar{z}^i, \bar{z}^{-i}) - \bar{\lambda}_{K^i}^{i\top} h^i(\bar{z}^i, \bar{z}^{-i}) - \bar{\gamma}_{K^i}^{i\top} g^i(\bar{z}^i, \bar{z}^{-i}).$$

Let \odot denote the elementwise product. The bottom-level KKT system is given by:

$$(2.6) \quad \bar{\mathcal{F}}_{K^i}^i(\bar{z}^i, \bar{z}^{-i}, \bar{\eta}_{K^i}^i) = \begin{bmatrix} \nabla_{\bar{z}^i} \bar{\mathcal{L}}_{K^i}^i(\bar{z}^i, \bar{z}^{-i}, \bar{\eta}_{K^i}^i) \\ h^i(\bar{z}^i, \bar{z}^{-i}) \\ g^i(\bar{z}^i, \bar{z}^{-i}) \odot \bar{\gamma}_{K^i}^i \end{bmatrix} = \mathbf{0},$$

$$(2.7) \quad \bar{\mathcal{G}}_{K^i}^i(\bar{z}^i, \bar{z}^{-i}, \bar{\gamma}_{K^i}^i) = \begin{bmatrix} g^i(\bar{z}^i, \bar{z}^{-i}) \\ \bar{\gamma}_{K^i}^i \end{bmatrix} \geq \mathbf{0}.$$

Upper level $k \leq K^i - 1$. At level $k = K^i - 1$, the subproblem inherits the equality constraints (2.6) and inequality constraints (2.7) from the lower level K^i . Under this reformulation, dual variables $\bar{\eta}_{K^i}^i$ introduced at level K^i reappear in the level- $(K^i - 1)$ problem as additional *primal* variables; we refer to these as *induced primals*. Consequently, the Lagrangian stationarity conditions are imposed not only with respect to the original primal variables \bar{z}^i , but also with respect to the induced primals $\bar{\eta}_{K^i}^i$. This procedure is then applied recursively at each upper level. In what follows, we formalize this construction of the complete KKT system for any level k .

We first introduce shorthand notation for level-indexed variables used in the complete KKT system. At each level k , the complete KKT system introduces dual variables comprising: (i) $\bar{\psi}_k^i$, the multipliers associated with lower-level Lagrangian stationarity; (ii) $\bar{\phi}_k^i$, the multipliers associated with complementarity constraints; (iii) $\bar{\lambda}_k^i$, the multipliers for innermost-level equality constraints h^i ; and (iv) $\bar{\gamma}_k^i$, the multipliers for innermost-level inequality constraints g^i and lower-level dual feasibility conditions. We collect these dual variables into $\bar{\eta}_k^i := [(\bar{\psi}_k^i)^\top, (\bar{\phi}_k^i)^\top, (\bar{\lambda}_k^i)^\top, (\bar{\gamma}_k^i)^\top]^\top$, and define $\bar{\eta}_{k:K^i}^i := [(\bar{\eta}_k^i)^\top, \dots, (\bar{\eta}_{K^i}^i)^\top]^\top$ and similarly, $\bar{\gamma}_{k:K^i}^i := [(\bar{\gamma}_k^i)^\top, \dots, (\bar{\gamma}_{K^i}^i)^\top]^\top$. At level k , $\bar{\eta}_{k+1:K^i}^i$ functions as induced primal variables. The Lagrangian function at level k is given as

$$(2.8) \quad \bar{\mathcal{L}}_k^i(\bar{z}^i, \bar{z}^{-i}, \bar{\eta}_{k:K^i}^i) = J_k^i(\bar{z}^i, \bar{z}^{-i}) - \bar{\lambda}_k^{i\top} h^i(\bar{z}^i, \bar{z}^{-i}) - \bar{\gamma}_{k,1}^{i\top} g^i(\bar{z}^i, \bar{z}^{-i}) \\ - \bar{\psi}_k^{i\top} \begin{bmatrix} \nabla_{\bar{z}^i} \bar{\mathcal{L}}_{k+1}^i \\ \nabla_{\bar{\eta}_{k+2:K^i}^i} \bar{\mathcal{L}}_{k+1}^i \\ \vdots \\ \nabla_{\bar{z}^i} \bar{\mathcal{L}}_{K^i}^i \end{bmatrix} - \bar{\phi}_k^{i\top} \begin{bmatrix} g^i(\bar{z}^i, \bar{z}^{-i}) \odot \bar{\gamma}_{k+1,1}^i \\ \bar{\gamma}_{k+2:K^i}^i \odot \bar{\gamma}_{k+1,2}^i \\ \vdots \\ g^i(\bar{z}^i, \bar{z}^{-i}) \odot \bar{\gamma}_{K^i}^i \end{bmatrix} - \bar{\gamma}_{k,\cdot}^{i\top} \begin{bmatrix} \bar{\gamma}_{k+1}^i \\ \vdots \\ \bar{\gamma}_{K^i}^i \end{bmatrix}.$$

Using (2.8), the complete KKT system at level k is

$$(2.9) \quad \bar{\mathcal{F}}_k^i(\bar{z}^i, \bar{z}^{-i}, \bar{\eta}_{k:K^i}^i) = \begin{bmatrix} \nabla_{\bar{z}^i} \bar{\mathcal{L}}_k^i(\bar{z}^i, \bar{z}^{-i}, \bar{\eta}_{k:K^i}^i) \\ \nabla_{\bar{\eta}_{k+1:K^i}^i} \bar{\mathcal{L}}_k^i(\bar{z}^i, \bar{z}^{-i}, \bar{\eta}_{k:K^i}^i) \\ \bar{\mathcal{G}}_{k+1}^i(\bar{z}^i, \bar{z}^{-i}, \bar{\gamma}_{k+1:K^i}^i) \odot \bar{\gamma}_k^i \\ \bar{\mathcal{F}}_{k+1}^i(\bar{z}^i, \bar{z}^{-i}, \bar{\eta}_{k+1:K^i}^i) \end{bmatrix} = \mathbf{0},$$

$$(2.10) \quad \bar{\mathcal{G}}_k^i(\bar{z}^i, \bar{z}^{-i}, \bar{\gamma}_{k:K^i}^i) = \begin{bmatrix} g^i(\bar{z}^i, \bar{z}^{-i}) \\ \bar{\gamma}_{k:K^i}^i \end{bmatrix} \geq \mathbf{0}.$$

Observe that at any level $k \leq K^i - 1$, the subproblem inherits all equality (2.9) and inequality (2.10) constraints from the lower level $k + 1$. By this nesting property, the complete KKT system for player i is achieved at the top level, $(\bar{\mathcal{F}}_1^i, \bar{\mathcal{G}}_1^i)$, which we denote by $(\bar{\mathcal{F}}^i, \bar{\mathcal{G}}^i)$ for convenience. Aggregating $(\bar{\mathcal{F}}^i, \bar{\mathcal{G}}^i)$ over players yields the complete KKT system $(\bar{\mathcal{F}}, \bar{\mathcal{G}})$, which provides necessary conditions for a local GOOP equilibrium under Assumption 2.1. We present the complete KKT system next.

THEOREM 2.3 (Complete necessary conditions, cf. [19]). *Suppose that \bar{z}^{i*} is a local solution to player i 's GOOP problem (2.1) under Assumption 2.1. Then, for each level $k \in [K^i]$, there exist (i) induced primal variables from lower levels $\bar{\eta}_{k+1:K^i}^i$, and (ii) dual variables $\bar{\eta}_k^i$ such that the conditions in (2.9) and (2.10) are satisfied at $(\bar{z}^{i*}, \bar{z}^{-i*}, \bar{\eta}_{k:K^i}^i)$.*

We now show that the number of variables and conditions in the complete KKT system scales exponentially in the number of preference levels.

PROPOSITION 2.4 (Exponential growth of the complete KKT system). *Consider player i 's GOOP problem in (2.1). The complete KKT system $(\bar{\mathcal{F}}^i, \bar{\mathcal{G}}^i)$ has*

$$\begin{aligned} N_{\bar{z}^i, \bar{\eta}^i} &= 2^{K^i-1} (n^i + m_{\mathcal{E}}^i + K^i m_{\mathcal{I}}^i) && \text{variables,} \\ N_{\bar{\mathcal{F}}^i} &= 2^{K^i-1} (n^i + m_{\mathcal{E}}^i + K^i m_{\mathcal{I}}^i) && \text{equations in } \bar{\mathcal{F}}^i(\cdot), \text{ and} \\ N_{\bar{\mathcal{G}}^i} &= 2^{K^i} m_{\mathcal{I}}^i && \text{inequalities in } \bar{\mathcal{G}}^i(\cdot). \end{aligned}$$

Thus, the number of variables and conditions in the complete KKT system grows exponentially with the number of preference levels.

Proof. The proof can be found in the Appendix. \square

The preceding result shows that the complete KKT system $(\bar{\mathcal{F}}, \bar{\mathcal{G}})$ grows linearly in the number of players but exponentially in the number of preference levels in each player's hierarchy depth. Section 3 derives a reduced KKT system whose size grows polynomially in *both* the number of players *and* the number of preference levels.

3. A Reduced Set of Necessary Conditions for GOOP Equilibria. We derive a new formulation of the KKT system for the GOOP problem (2.1), which avoids the exponential scaling in Theorem 2.3. Unless otherwise noted, we adopt the notation of Subsection 2.2, omitting overbars to denote quantities for the reduced KKT system. We construct the reduced KKT system recursively from the innermost level.

Innermost level $k = K^i$ for player i . For the bottom-level (2.1c) and (2.1d), we define the dual variables $\eta_{K^i}^i := [(\lambda_{K^i}^i)^\top, (\gamma_{K^i}^i)^\top]^\top \in \mathbb{R}^{m_{\mathcal{E}}^i + m_{\mathcal{I}}^i}$. The corresponding

Lagrangian $\mathcal{L}_{K^i}^i$ and KKT system $(\mathcal{F}_{K^i}^i, \mathcal{G}_{K^i}^i)$ are

$$(3.1) \quad \mathcal{L}_{K^i}^i(z^i, z^{-i}, \eta_{K^i}^i) = J_{K^i}^i(z^i, z^{-i}) - \lambda_{K^i}^{i\top} h^i(z^i, z^{-i}) - \gamma_{K^i}^{i\top} g^i(z^i, z^{-i}),$$

$$(3.2) \quad \pi_{K^i}^i(z^i, z^{-i}, \eta_{K^i}^i) = \nabla_{z^i} \mathcal{L}_{K^i}^i(z^i, z^{-i}, \eta_{K^i}^i),$$

$$(3.3) \quad \mathcal{F}_{K^i}^i(z^i, z^{-i}, \eta_{K^i}^i) = \begin{bmatrix} \nabla_{z^i} \mathcal{L}_{K^i}^i(z^i, z^{-i}, \eta_{K^i}^i) \\ h^i(z^i, z^{-i}) \\ g^i(z^i, z^{-i}) \odot \gamma_{K^i}^i \end{bmatrix} = \mathbf{0},$$

$$(3.4) \quad \mathcal{G}_{K^i}^i(z^i, z^{-i}, \gamma_{K^i}^i) = \begin{bmatrix} g^i(z^i, z^{-i}) \\ \gamma_{K^i}^i \end{bmatrix} \geq \mathbf{0}.$$

At the innermost level, the reduced KKT system (3.3) and (3.4) coincides with the complete KKT system (2.6) and (2.7).

Upper level $k \leq K^i - 1$. Let the function $\pi_{k+1}^i(z^i, z^{-i}, \eta_{k+1:K^i}^i)$ collect the stationarity conditions of levels $k+1, \dots, K^i$. We associate the dual variable ψ_k^i with the function π_{k+1}^i and the dual variable ϕ_k^i only with the complementarity constraints involving the innermost inequality constraint g^i . Unlike the complete KKT system, we impose stationarity of the Lagrangian function only with respect to the primal variable z^i . Defining the dual variables $\eta_k^i := [(\psi_k^i)^\top, (\phi_k^i)^\top, (\lambda_k^i)^\top, (\gamma_k^i)^\top]^\top$, we write the corresponding Lagrangian \mathcal{L}_k^i and KKT system $(\mathcal{F}_k^i, \mathcal{G}_k^i)$ as

$$(3.5) \quad \mathcal{L}_k^i(z^i, z^{-i}, \eta_{k:K^i}^i) = J_k^i(z^i, z^{-i}) - \lambda_k^{i\top} h^i(z^i, z^{-i}) - \gamma_k^{i\top} g^i(z^i, z^{-i}) \\ - \psi_k^{i\top} \pi_{k+1}^i(z^i, z^{-i}, \eta_{k+1:K^i}^i) - \sum_{\ell=1}^{K^i-k} \phi_{k,\ell}^{i\top} [g^i(z^i, z^{-i}) \odot \gamma_{K^i-\ell+1}^i],$$

$$(3.6) \quad \pi_k^i(z^i, z^{-i}, \eta_{k:K^i}^i) = \begin{bmatrix} \nabla_{z^i} \mathcal{L}_k^i(z^i, z^{-i}, \eta_{k:K^i}^i) \\ \pi_{k+1}^i(z^i, z^{-i}, \eta_{k+1:K^i}^i) \end{bmatrix},$$

$$(3.7) \quad \mathcal{F}_k^i(z^i, z^{-i}, \eta_{k:K^i}^i) = \begin{bmatrix} \nabla_{z^i} \mathcal{L}_k^i(z^i, z^{-i}, \eta_{k:K^i}^i) \\ g^i(z^i, z^{-i}) \odot \gamma_k^i \\ \mathcal{F}_{k+1}^i(z^i, z^{-i}, \eta_{k+1:K^i}^i) \end{bmatrix} = \mathbf{0},$$

$$(3.8) \quad \mathcal{G}_k^i(z^i, z^{-i}, \gamma_{k:K^i}^i) = \begin{bmatrix} g^i(z^i, z^{-i}) \\ \gamma_{k:K^i}^i \end{bmatrix} \geq \mathbf{0}.$$

After reordering, the reduced KKT system for player i $(\mathcal{F}^i, \mathcal{G}^i)$ can be written as

$$(3.9) \quad \mathcal{F}^i(z^i, z^{-i}, \eta_{1:K^i}^i) = \begin{bmatrix} \nabla_{z^i} \mathcal{L}_1^i(z^i, z^{-i}, \eta_{1:K^i}^i) \\ \vdots \\ \nabla_{z^i} \mathcal{L}_{K^i}^i(z^i, z^{-i}, \eta_{K^i}^i) \\ h^i(z^i, z^{-i}) \\ g^i(z^i, z^{-i}) \odot \gamma_1^i \\ \vdots \\ g^i(z^i, z^{-i}) \odot \gamma_{K^i}^i \end{bmatrix} = \mathbf{0},$$

$$(3.10) \quad \mathcal{G}^i(z^i, z^{-i}, \gamma_{1:K^i}^i) = \begin{bmatrix} g^i(z^i, z^{-i}) \\ \gamma_{1:K^i}^i \end{bmatrix} \geq \mathbf{0}.$$

Aggregating $(\mathcal{F}^i, \mathcal{G}^i)$ over players yields the reduced KKT system $(\mathcal{F}, \mathcal{G})$. We characterize the relationship between the solution sets of the complete KKT system $(\bar{\mathcal{F}}, \bar{\mathcal{G}})$ and reduced KKT system $(\mathcal{F}, \mathcal{G})$ in the following result.

THEOREM 3.1 (Reduced KKT system is a relaxation of the complete KKT system). *Suppose Assumption 2.1 holds. Let $(\bar{z}^*, \bar{\eta}_{1:K^1}^{1*}, \dots, \bar{\eta}_{1:K^N}^{N*})$ be a solution to the complete KKT system $(\bar{\mathcal{F}}, \bar{\mathcal{G}})$ in (2.9) and (2.10). Then there exists reduced-system dual variables $(\eta_{1:K^i}^{i*})_{i=1}^N$ such that $(\bar{z}^*, \eta_{1:K^1}^{1*}, \dots, \eta_{1:K^N}^{N*})$ satisfies the reduced KKT system $(\mathcal{F}, \mathcal{G})$ in (3.7) and (3.8). Consequently, every primal solution \bar{z}^* of the complete KKT system is a solution of the reduced KKT system and the reduced system is a relaxed set of necessary conditions that must hold at any local GOOP equilibrium.*

Proof. The proof can be found in the Appendix. \square

In Proposition 3.2, we prove that the number of variables and conditions in the reduced KKT system $(\mathcal{F}, \mathcal{G})$ scales polynomially with the number of levels, in contrast to the exponential scaling of the complete KKT system.

PROPOSITION 3.2 (Polynomial growth of reduced KKT system). *Consider player i 's GOOP problem in (2.1). The corresponding reduced KKT system $(\mathcal{F}^i, \mathcal{G}^i)$ has*

$$\begin{aligned} N_{z^i, \eta^i} &= \left(1 + \frac{K^i(K^i - 1)}{2}\right) n^i + K^i m_{\mathcal{E}}^i + \frac{K^i(K^i + 1)}{2} m_{\mathcal{I}}^i && \text{variables,} \\ N_{\mathcal{F}^i} &= K^i n^i + m_{\mathcal{E}}^i + K^i m_{\mathcal{I}}^i && \text{equations in } \mathcal{F}^i(\cdot), \text{ and} \\ N_{\mathcal{G}^i} &= (K^i + 1) m_{\mathcal{I}}^i && \text{inequalities in } \mathcal{G}^i(\cdot). \end{aligned}$$

Thus, the number of variables grows quadratically, and the number of equations and inequalities grows linearly with the number of preference levels for the reduced system.

Proof. The proof can be found in the Appendix. \square

The reduction in complexity naturally raises the question of whether the reduced KKT system preserves the set of primal solutions. Theorem 3.1 establishes one direction: every primal solution of the complete KKT system satisfies the reduced KKT system via reconstruction of reduced-system dual variables. We now study the converse, namely whether a reduced-system solution can be lifted to a complete-system solution. We prove this converse for quadratic GOOPs. For general GOOPs with non-quadratic objectives and nonlinear constraints, establishing the same equivalence is substantially more challenging, because newly arising higher-order terms break the term-by-term correspondence between the two systems.

3.1. Equivalence Between Complete and Reduced KKT Systems in Quadratic GOOPs. We now establish primal solution equivalence between the complete and reduced KKT systems for quadratic GOOPs. We first treat the equality-constrained case and then extend the result to the inequality-constrained case.

3.1.1. Equality-Constrained Quadratic GOOPs. At any level k , assume that each player i 's objective is given by:

$$(3.11) \quad J_k^i(z) := \frac{1}{2} z^\top Q_k^i z + q_k^{i\top} z,$$

where $z := [(z^i)^\top, (z^{-i})^\top]^\top \in \mathbb{R}^n$, $Q_k^i \in \mathbb{R}^{n \times n}$, and $q_k^i \in \mathbb{R}^n$. We partition the matrix Q_k^i and vector q_k^i into blocks $Q_k^{i,j,k} \in \mathbb{R}^{n^j \times n^k}$ and $q_k^{i,j} \in \mathbb{R}^{n^j}$ for all $i, j, k \in [N]$:

$$(3.12) \quad Q_k^i := \begin{bmatrix} Q_k^{i,1,1} & \dots & Q_k^{i,1,N} \\ \vdots & \ddots & \vdots \\ Q_k^{i,N,1} & \dots & Q_k^{i,N,N} \end{bmatrix}, \quad q_k^i := \begin{bmatrix} q_k^{i,1} \\ \vdots \\ q_k^{i,N} \end{bmatrix}.$$

We impose the following regularity assumption:

$$(3.13) \quad Q_k^{i,j,j} \succeq \mathbf{0}, \quad \forall i, j \in [N],$$

and define the following matrices, which combine components in (3.12):

$$(3.14) \quad Q_k := \begin{bmatrix} Q_k^{1,1,1} & Q_k^{1,1,2} & \dots & Q_k^{1,1,N} \\ Q_k^{2,2,1} & Q_k^{2,2,2} & \dots & Q_k^{2,2,N} \\ \vdots & \vdots & \ddots & \vdots \\ Q_k^{N,N,1} & Q_k^{N,N,2} & \dots & Q_k^{N,N,N} \end{bmatrix}, \quad \hat{Q}_k := \begin{bmatrix} Q_k^{1,1,1} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & Q_k^{N,N,N} \end{bmatrix}, \quad q_k := \begin{bmatrix} q_k^1 \\ \vdots \\ q_k^N \end{bmatrix}.$$

The matrix Q_k consists of i^{th} -block rows of Q_k^i . By (3.13), the block-diagonal matrix \hat{Q}_k is positive semidefinite, and, thus, player i 's objective is partially convex in z^i , i.e., $\nabla_{z^i}^2 J_k^i$ is a positive semidefinite matrix for each player $i \in [N]$ at each level $k \in [K]$.

The linear equality constraints imposed on each player i take the form of:

$$(3.15) \quad H^i z = h^i, \quad H^i := [H_1^i \quad \dots \quad H_N^i],$$

for matrix $H^i \in \mathbb{R}^{m_\varepsilon^i \times n}$, vector $h^i \in \mathbb{R}^{m_\varepsilon^i}$, and submatrices $H_j^i \in \mathbb{R}^{m_\varepsilon^i \times n^j}$. Define

$$(3.16) \quad H := \begin{bmatrix} H_1^1 & \dots & H_N^1 \\ \vdots & \ddots & \vdots \\ H_1^N & \dots & H_N^N \end{bmatrix}, \quad \hat{H} := \begin{bmatrix} H_1^1 & & \\ & \ddots & \\ & & H_N^N \end{bmatrix}, \quad h := \begin{bmatrix} h^1 \\ \vdots \\ h^N \end{bmatrix}.$$

The matrix H is the block-row concatenation of $\{H^i\}_{i=1}^N$, and the matrix \hat{H} is a block-diagonal matrix that retains only the diagonal blocks H_i^i from H .

Assumption 3.3 (Full row rank of \hat{H}). We assume that the block-diagonal matrix \hat{H} has full row rank. Equivalently, each diagonal block H_i^i has full row rank.

Different numbers of preference levels. Players need not share a common number of preference levels. For analytical convenience, we regard each player as having levels $k \in \{1, \dots, K\}$ where $K := \max\{K^i : i \in [N]\}$. If player i has no level k (i.e., $K^i < k \leq K$), we set the player's objective to $Q_k^i = \mathbf{0}$, $q_k^i = \mathbf{0}$, which retains the regularity assumption (3.13) and positive semidefiniteness of \hat{Q}_k .

Using the objective and constraint terms defined above, we develop the recursive structure of the complete and reduced KKT systems. The key idea in the recursion is that, at each level k , the complete KKT system matrix \bar{M}_k (resp., the reduced KKT system matrix M_k) is *nested*: it contains, as a principal block, the level- $(k+1)$ KKT system matrix \bar{M}_{k+1} (resp., M_{k+1}). This enforces the inter-level optimality chain where the upper-level's decision space is constrained by lower-level optimality. Furthermore, we will repeatedly use the following matrix for the recursive structure:

$$(3.17) \quad \bar{R}_k := \left[\begin{array}{cc|c} \hat{Q}_k & \mathbf{0} & \bar{R}_{k+1} \\ \mathbf{0} & \mathbf{0} & \\ \hline \bar{R}_{k+1} & & \mathbf{0} \end{array} \right], \quad \forall k \in [K-1], \quad \bar{R}_K := \begin{bmatrix} \hat{Q}_K & \hat{H}^\top \\ \hat{H} & \mathbf{0} \end{bmatrix}.$$

By construction, \bar{R}_k is symmetric for all k . To align dimensions between the complete and reduced systems, we use the matrix

$$(3.18) \quad R_{k+1} := \left[\begin{array}{c} \bar{R}_{k+1, [1:n, :]} \\ \mathbf{0} \end{array} \right],$$

which takes the first n rows of the matrix \bar{R}_{k+1} and appends zeros below (see (3.20)). These n rows correspond to the stationarity condition with respect to the primal variable z in (3.7). Moreover, we augment playerwise lower-level dual variables (e.g., $\bar{\lambda}_K = [(\bar{\lambda}_K^1)^\top, \dots, (\bar{\lambda}_K^N)^\top]^\top$) and use the notation $\bar{\eta}_{K:k+1} = [(\bar{\eta}_K)^\top, \dots, (\bar{\eta}_{k+1})^\top]^\top$. We present the resulting complete and reduced KKT systems next.

Recursive structure of the complete KKT system. The conditions (2.9) for a quadratic GOOP at any level k admit the following linear representation:

$$(3.19) \quad \underbrace{\left[\begin{array}{cc|c} Q_k & \mathbf{0} & \bar{R}_{k+1} \\ \mathbf{0} & \mathbf{0} & \\ \hline \bar{M}_{k+1} & & \mathbf{0} \end{array} \right]}_{\bar{M}_k} \underbrace{\left[\begin{array}{c} \bar{z} \\ \bar{\eta}_{K:k+1} \\ \bar{\eta}_k \end{array} \right]}_{\bar{v}_k} = \underbrace{\left[\begin{array}{c} q_k \\ \mathbf{0} \\ q_{k+1} \end{array} \right]}_{\bar{p}_k}, \quad \forall k \in [K-1].$$

Recursive structure of the reduced KKT system. The conditions (3.7) for a quadratic GOOP at any level k can be expressed as:

$$(3.20) \quad \underbrace{\left[\begin{array}{cc|c} Q_k & \mathbf{0} & R_{k+1} \\ \mathbf{0} & \mathbf{0} & \\ \hline M_{k+1} & & \mathbf{0} \end{array} \right]}_{M_k} \underbrace{\left[\begin{array}{c} z \\ \eta_{K:k+1} \\ \eta_k \end{array} \right]}_{v_k} = \underbrace{\left[\begin{array}{c} q_k \\ \mathbf{0} \\ q_{k+1} \end{array} \right]}_{p_k}, \quad \forall k \in [K-1].$$

For the base level $k = K$, the complete and reduced KKT systems coincide:

$$(3.21) \quad \underbrace{\left[\begin{array}{cc} Q_K & \hat{H}^\top \\ H & \mathbf{0} \end{array} \right]}_{M_K = \bar{M}_K} \underbrace{\left[\begin{array}{c} z \\ \lambda_K \end{array} \right]}_{v_K = \bar{v}_K} = \underbrace{\left[\begin{array}{c} q_K \\ h \end{array} \right]}_{p_K = \bar{p}_K}.$$

For any $k \in [K]$, the top-left blocks of \bar{M}_k in (3.19) and M_k in (3.20) contain the same Q_k term on the leading diagonal; thus, the two matrices share the first n columns. Furthermore, because the right-hand-side vector is identical, $p_k = \bar{p}_k$, it follows that $\bar{M}_k \bar{v}_k = M_k v_k$. This fact is the basis for the next result, where we establish equivalence of the primal solution sets via a column-space relationship between \bar{M}_k and M_k .

THEOREM 3.4 (Primal solution equivalence in quadratic GOOPs). *Consider a GOOP problem (2.1) with partially convex quadratic objectives (3.11) and linear equality constraints (3.15) under Assumption 3.3. Then the following hold:*

- (a) $\text{Col}(R_k) \subseteq \text{Col}(\bar{R}_k)$, $\forall k \in [K]$.
- (b) Denote the first n columns shared by both matrices \bar{M}_k and M_k as C_k , and consider the partition: $\bar{M}_k = [C_k \mid \bar{M}_k^c]$ and $M_k = [C_k \mid M_k^c]$. Then we have $\text{Col}(M_k^c) \subseteq \text{Col}(\bar{M}_k^c)$, $\forall k \in [K-1]$.
- (c) The primal solution sets of the complete KKT system (3.19) and reduced KKT system (3.20) coincide for all $k \in [K]$.

Proof. The proof can be found in the Appendix. \square

Theorem 3.4 formalizes the structural relationships that ensure coincidence of the primal solution sets (even though the corresponding dual variables may differ).

3.1.2. Inequality-Constrained Quadratic GOOP. We now extend the preceding result in Subsection 3.1.1 to quadratic GOOPs with linear inequality constraints. Consider a GOOP problem (2.1) with quadratic objectives (3.11) and linear equality constraints (3.15). Assume that each player i also has inequality constraints:

$$(3.22) \quad G^i z \geq g^i, \quad G^i := [G_1^i \quad \cdots \quad G_N^i],$$

for matrix $G^i \in \mathbb{R}^{m_x^i \times n}$, vector $g^i \in \mathbb{R}^{m_x^i}$ and submatrices $G_j^i \in \mathbb{R}^{m_x^i \times n^j}$. Define the active set at a solution z^* as

$$(3.23) \quad \mathcal{A}^i(z^*) := \{j \mid (G^i z^* - g^i)_j = 0\}.$$

Let matrix $G_{\mathcal{A}^i}^i$ be the submatrix of G^i with rows indexed by $\mathcal{A}^i(z^*)$, and likewise index the entries of the vector g^i corresponding to active inequality constraints as $g_{\mathcal{A}^i}^i$. Define the vector $\tilde{h}^i := [h^i; g_{\mathcal{A}^i}^i]$. We present the primal solution equivalence result for the inequality-constrained case in Theorem 3.5.

THEOREM 3.5 (Primal solution equivalence in inequality-constrained quadratic GOOPs). *Consider a GOOP problem (2.1) with partially convex quadratic objectives (3.11), linear equality constraints (3.15), and inequality constraints (3.22) for each player. Assume that at every solution z^* , the following hold for player i :*

1. Strict complementarity. *At the innermost level K^i , every active inequality constraint has a positive multiplier, i.e., $(G^i z^* - g^i)_j = 0$ and $\gamma_{K^i, j}^{i*} > 0$, $\forall j \in \mathcal{A}^i(z^*)$.*

2. Regularity. *The matrix*

$$(3.24) \quad \tilde{H} := \begin{bmatrix} \begin{bmatrix} H_1^1 \\ G_{1, \mathcal{A}^1}^1 \end{bmatrix} & & \\ & \ddots & \\ & & \begin{bmatrix} H_N^N \\ G_{N, \mathcal{A}^N}^N \end{bmatrix} \end{bmatrix}$$

has full row rank. Equivalently, each diagonal block $\begin{bmatrix} H_i^i \\ G_{i, \mathcal{A}^i}^i \end{bmatrix}$ has full row rank.

Then, the set of primal solutions of the reduced KKT system $(\mathcal{F}, \mathcal{G})$ coincides with that of the complete KKT system $(\tilde{\mathcal{F}}, \tilde{\mathcal{G}})$.

Proof. The proof can be found in the Appendix. □

The strict complementarity assumption in Theorem 3.5 provides the nondegeneracy needed at the innermost level to reconstruct the relevant reduced-system dual variables and initiate the recursive argument via Theorem 3.4 across higher levels.

Remark 3.6. The assumption of strict complementarity at the innermost level guarantees the local reduction to an equality-constrained subproblem. Characterizing primal equivalence under weaker conditions (even in quadratic GOOPs) remains a challenge as the active set may no longer be uniquely identified by the strictly positive multipliers in these degenerate cases.

3.2. From Quadratic GOOPs to Nonquadratic GOOPs. The preceding analysis in Subsections 3.1.1 and 3.1.2 raises the question of whether the reduced and complete KKT systems continue to share the same primal solution set for general smooth GOOPs with nonquadratic objectives and nonlinear constraints. Empirically, we observe such primal equivalence in Section 6, but a general proof is nontrivial.

The difficulty is structural: when players' objectives are nonquadratic, higher-order derivatives of inner-level objectives introduce additional terms that destroy the block-wise correspondence between the two KKT systems. Consequently, lower-level dual variables—which generally take different values in the complete and reduced systems—generate *different coefficients* in the upper-level nonlinear stationarity conditions. This misalignment makes it challenging to reconstruct dual variables for the complete KKT system from a reduced KKT-system primal solution.

The coefficient mismatch *first* appears at level $k = K^i - 2$ when player i is subject only to linear equality constraints (3.15). The multipliers for the complete system are $\bar{\eta}_{K^i-2}^i := [(\bar{\psi}_{K^i-2}^i)^\top, (\bar{\lambda}_{K^i-2}^i)^\top]^\top$. The complete-system Lagrangian stationarity is

$$(3.25) \quad \begin{aligned} \nabla_{\bar{z}^i} \bar{\mathcal{L}}_{K^i-2}^i(\bar{z}, \bar{\eta}_{K^i-1:K^i}^i, \bar{\eta}_{K^i-2}^i) &= \nabla_{\bar{z}^i} J_{K^i-2}^i(\bar{z}) - \left(\nabla_{\bar{z}^i}^2 \bar{\mathcal{L}}_{K^i-1}^i(\bar{z}, \bar{\eta}_{K^i-1:K^i}^i) \right)^\top \bar{\psi}_{K^i-2,1}^i \\ &\quad - \left(\nabla_{\bar{z}^i}^2 \bar{\mathcal{L}}_{K^i}^i(\bar{z}, \bar{\eta}_{K^i}^i) \right)^\top \bar{\psi}_{K^i-2,2}^i - H_i^{i\top} \bar{\lambda}_{K^i-2}^i. \end{aligned}$$

The stationarity equation for the reduced system is obtained analogously. Next, expand the Hessian of the lower level ($K^i - 1$) Lagrangian appearing in (3.25):

$$(3.26) \quad \nabla_{\bar{z}^i}^2 \bar{\mathcal{L}}_{K^i-1}^i(\bar{z}, \bar{\eta}_{K^i-1:K^i}^i) = \nabla_{\bar{z}^i}^2 J_{K^i-1}^i(\bar{z}) - \left(\nabla_{\bar{z}^i}^3 J_{K^i}^i(\bar{z}) \right)^\top \bar{\psi}_{K^i-1}^i.$$

The final term in (3.26) arises from the third derivative of the innermost objective and is weighted by the lower-level dual variable $\bar{\psi}_{K^i-1}^i$. Notice that $\bar{\psi}_{K^i-1}^i$ need not coincide with its reduced KKT system counterpart $\psi_{K^i-1}^i$. As a result, the misaligned dual variables generate different coefficients in the upper-level stationarity condition (3.25), i.e., $\nabla_{\bar{z}^i}^2 \bar{\mathcal{L}}_{K^i-1}^i(\bar{z}, \bar{\eta}_{K^i-1:K^i}^i) \neq \nabla_{\bar{z}^i}^2 \mathcal{L}_{K^i-1}^i(\bar{z}, \eta_{K^i-1:K^i}^i)$. In this way, the complete and reduced systems progressively accumulate mismatched coefficients in their stationarity conditions. The proof of Theorem 3.4 relied upon consistent coefficients to align the dual variables between the two KKT systems. Consequently, establishing primal solution equivalence in this more general regime requires a different mechanism to align—or otherwise relate—the relevant dual variables across the two systems.

Even in the absence of a primal-equivalence guarantee in general settings, we can still certify the local optimality of candidate solutions for general GOOPs. Accordingly, we shift our focus to establishing second-order sufficient conditions (SOSC) that characterize when a candidate solution corresponds to a local equilibrium.

4. Second-order Sufficient Conditions. In this section, we describe sufficient conditions for local optimality in a general GOOP problem. As illustrated in Figure 1, a solution to the reduced KKT system may be *spurious*: it satisfies the reduced system, yet may fail to satisfy the complete KKT system and therefore cannot correspond to a GOOP equilibrium. We present Theorem 4.1, which provides a set of conditions that each subproblem (for every player and level) must satisfy in order for a candidate solution to the reduced KKT system to be a true local GOOP equilibrium.

We proceed along the lines established by [11], in the context of identifying weak local minimizers. Using the reduced KKT system (3.7) and (3.8), we define the linearized feasible cone and the critical cone of the k^{th} -level subproblem. First, we define player i 's k^{th} -level active set at $(z^*, \eta_{k+1:K^i}^{i*})$

$$(4.1) \quad \mathcal{A}_k^i(z^*, \eta_{k+1:K^i}^{i*}) := \{j : \mathcal{G}_{k,j}^i(z^*, \eta_{k+1:K^i}^{i*}) = 0\},$$

where \mathcal{G}_k^i refers to player i 's reduced KKT system (inequalities) in (3.8). The set of

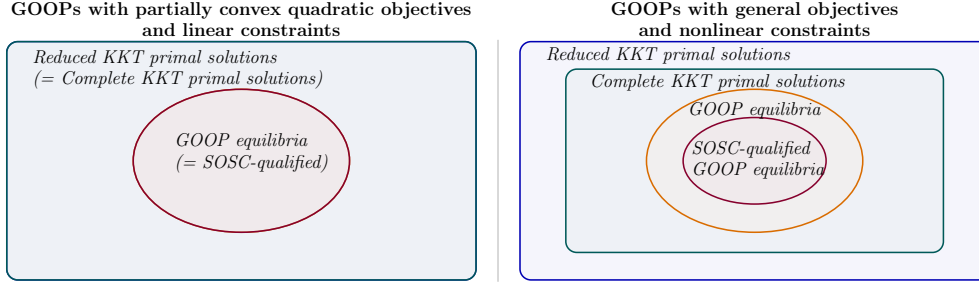


Fig. 1: **(Left)** For quadratic objectives and linear constraints, under the assumptions of Subsection 3.1, the reduced and complete KKT systems share the same primal solution set, and every GOOP equilibrium is SOSC-qualified (hence the corresponding sets coincide). **(Right)** For general smooth GOOPs, the reduced KKT primal solution set is a relaxation (superset) of the complete KKT primal solution set, which contains the GOOP equilibria; Theorem 4.1 certifies a subset of such equilibria.

linearized feasible directions is

$$(4.2) \quad \mathcal{L}_F(z^*, \eta_{k+1:K^i}^{i*}) = \left\{ d \mid \begin{cases} \nabla_{(z^i, \eta_{k+1:K^i}^i)} \mathcal{F}_k^i(z^*, \eta_{k+1:K^i}^{i*})^\top d = \mathbf{0}, \\ \nabla_{(z^i, \eta_{k+1:K^i}^i)} \mathcal{G}_{k,j}^i(z^*, \eta_{k+1:K^i}^{i*})^\top d \geq \mathbf{0}, \quad \forall j \in \mathcal{A}_k^i \end{cases} \right\}.$$

The critical cone is then

$$(4.3) \quad \mathcal{C}_k(z^*, \eta_{k+1:K^i}^{i*}) = \left\{ d \in \mathcal{L}_F \mid \nabla_{(z^i, \eta_{k+1:K^i}^i)} \mathcal{G}_{k,j}^i(z^*, \eta_{k+1:K^i}^{i*})^\top d = \mathbf{0}, \quad \forall j \in \mathcal{A}_k^{i+} \right\},$$

where $\mathcal{A}_k^{i+} := \{j \in \mathcal{A}_k^i(z^*, \eta_{k+1:K^i}^{i*}) : \gamma_{k,j}^{i*} > 0\}$, and the feasible neighborhood set is

$$(4.4) \quad \mathcal{P}_k(\epsilon, \delta) := \left\{ p \mid \begin{cases} \|p - d\| \leq \epsilon \text{ for some critical direction } d \in \mathcal{C}_k(z^*, \eta_{k+1:K^i}^{i*}), \\ (z^{i*}, \eta_{k+1:K^i}^{i*}) + \tilde{\delta} p \text{ satisfies (3.7) and (3.8) for some } \tilde{\delta} \text{ such that} \\ 0 < \tilde{\delta} < \delta, \text{ and } \|p\| = 1. \end{cases} \right\}.$$

We now present a second-order sufficient condition for the k^{th} -level subproblem of player i using the reduced KKT system.

THEOREM 4.1 (Second-order sufficient condition for local GOOP equilibrium).

Let $(z^{i*}, z^{-i*}, \eta_{1:K^i}^{i*})$ be a solution to the reduced KKT system in (3.7) and (3.8) under Assumption 2.1. Suppose that, for each level $k \in [K^i - 1]$, the k^{th} -level Lagrangian function \mathcal{L}_k^i is stationary with respect to $\eta_{k+1:K^i}^i$, i.e., $\nabla_{\eta_{k+1:K^i}^i} \mathcal{L}_k^i(z^{i*}, z^{-i*}, \eta_{k+1:K^i}^{i*}) = 0$. Suppose that there exists $\epsilon' > 0$, $\delta' > 0$ such that for every $p \in \mathcal{P}_k(\epsilon', \delta')$, we have

$$(4.5) \quad p^\top \nabla_{(z^i, \eta_{k+1:K^i}^i)}^2 \mathcal{L}_k^i((z^{i*}, \eta_{k+1:K^i}^{i*}) + \alpha \tilde{\delta} p, \eta_{k+1:K^i}^{i*}) p \geq 0$$

for all $\alpha \in (0, 1)$. Then, z^{i*} is a weak (possibly non-isolated) local minimizer for player i 's k^{th} -level subproblem. Furthermore, if (4.5) holds at every level $k \in [K^i - 1]$ for every player $i \in [N]$, then the solution of the reduced KKT system $(z^*, \eta_{1:K^1}^{1*}, \dots, \eta_{1:K^N}^{N*})$ is a weak local solution of the GOOP problem and satisfies Definition 2.2.

Proof. By [11, Theorem 2.1], condition (4.5) implies that $(z^{i*}, \eta_{k+1:K^i}^{i*})$ is a weak local minimizer of player i 's k^{th} -level subproblem. Applying this argument recursively for $k = K^i - 1, \dots, 1$ shows that player i 's candidate solution is weakly locally minimizing at every level of its hierarchy. Repeating this argument for all players $i \in [N]$ yields that $(z^*, \eta_{1:K^1}^{1*}, \dots, \eta_{1:K^N}^{N*})$ is a weak local solution of the GOOP problem. \square

Remark 4.2 (Implications of a strict local minimizer). If the inequality in (4.5) holds strictly at level k , the candidate point corresponds to a *strict* local minimizer at that level. Thus, in a neighborhood of the candidate point, the feasible set for the upper levels $k-1, \dots, 1$ collapses to the unique solution of the lower level. As a result, the verification of sufficient conditions at these upper levels becomes unnecessary.

Remark 4.3 (Limitations of Theorem 4.1). Per Remark 4.2, the second-order condition established in Theorem 4.1 checks whether a candidate solution is a weak local minimizer at each level for each player. Unfortunately, the condition is not easily checked, in general. This is true even in the case of single-level optimization as discussed in [11]. Establishing easily verified conditions for this case is an important direction for future work.

In the next section, we develop a primal-dual interior-point method and analyze its convergence properties. This common solver framework enables a controlled comparison in Section 6, where we quantify the computational benefits of solving the reduced system relative to the complete formulation.

5. Primal-Dual Interior Point Method for Reduced KKT System and Convergence Analysis. In this section, we present a numerical approach for solving the reduced KKT conditions. We base our approach on the primal-dual interior point (PDIP) method [26], which we view as a homotopy method [21]. To this end, we (i) introduce nonnegative slack variables $\{s_k^i\}_{k=1, i=1}^{K^i, N}$, which transform the inequality constraints into equality constraints for all players $i \in [N]$ and levels $k \in [K^i]$, and (ii) introduce a scalar homotopy parameter $\rho > 0$ to perturb the complementarity slackness condition for each player i at each level k :

$$(5.1) \quad g^i(z^i, z^{-i}) - s_k^i = \mathbf{0}, \quad s_k^i \odot \gamma_k^i = \rho \mathbf{1}, \quad s_k^i \geq \mathbf{0}, \quad \gamma_k^i \geq \mathbf{0},$$

where $\mathbf{1}$ denotes a vector of all ones. Player i 's perturbed reduced KKT system is

$$(5.2) \quad \mathcal{K}_\rho^i(z^i, z^{-i}, \eta_{1:K^i}^i, s_{1:K^i}^i) = \begin{bmatrix} \nabla_{z^i} \mathcal{L}_1^i(z^i, z^{-i}, \eta_{1:K^i}^i) \\ \vdots \\ \nabla_{z^i} \mathcal{L}_{K^i}^i(z^i, z^{-i}, \eta_{K^i}^i) \\ h^i(z^i, z^{-i}) \\ g^i(z^i, z^{-i}) - s_1^i \\ \vdots \\ g^i(z^i, z^{-i}) - s_{K^i}^i \\ s_{1:K^i}^i \odot \gamma_{1:K^i}^i - \rho \mathbf{1} \end{bmatrix} = \mathbf{0}.$$

Aggregating (5.2) over all players and defining $\eta := [(\eta_{1:K^1}^1)^\top, \dots, (\eta_{1:K^N}^N)^\top]^\top$, $s := [(s_{1:K^1}^1)^\top, \dots, (s_{1:K^N}^N)^\top]^\top$, and $y := [z^\top, \eta^\top, s^\top]^\top$, we obtain the ρ -perturbed reduced KKT system: $\mathcal{K}_\rho(y) = \left[\mathcal{K}_\rho^i(z^i, z^{-i}, \eta_{1:K^i}^i, s_{1:K^i}^i) \right]_{i=1}^N = \mathbf{0}$. We present our method in Algorithm 5.1. The algorithm gradually decreases homotopy parameter ρ

Algorithm 5.1 Primal-Dual Interior Point Method for the Reduced KKT System

Require: Initial homotopy parameter ρ , tolerance ϵ , maximum outer iterations ℓ_{\max} , line search parameter $\beta \in (0, 1)$, contraction rate $\sigma \in (0, 1)$, initial solution $y_\rho^{(0)} := [z_\rho^{(0)}, \eta_\rho^{(0)}, s_\rho^{(0)}]$ with $s_\rho^{(0)} > \mathbf{0}$ and $\gamma_\rho^{(0)} > \mathbf{0}$.

- 1: **for** $\ell = 0, 1, 2, \dots, \ell_{\max} - 1$ **do**
- 2: **while** $\|\mathcal{K}_\rho(y_\rho^{(\ell)})\|_2 > \epsilon$ **do**
- 3: Compute the Newton update direction $\Delta y_\rho = -(\nabla \mathcal{K}_\rho(y_\rho^{(\ell)}))^+ \mathcal{K}_\rho(y_\rho^{(\ell)})$
- 4: Initialize step size $\alpha \leftarrow 1$
- 5: **while** $\|\mathcal{K}_\rho(y_\rho^{(\ell)} + \alpha \Delta y_\rho)\|_2 > \|\mathcal{K}_\rho(y_\rho^{(\ell)})\|_2$ or $\hat{y} := y_\rho^{(\ell)} + \alpha \Delta y_\rho$ has a nonpositive element in its subvector $[\hat{s}_\rho, \hat{\gamma}_\rho]$ **do**
- 6: $\alpha \leftarrow \beta \cdot \alpha$
- 7: **end while**
- 8: **If** $\alpha < \epsilon$, **then declare** “line-search failure” **and break.**
- 9: $y_\rho^{(\ell)} \leftarrow y_\rho^{(\ell)} + \alpha \Delta y_\rho$.
- 10: **end while**
- 11: $y_\rho^{(\ell+1)} \leftarrow y_\rho^{(\ell)}$ and $\rho \leftarrow \sigma \cdot \rho$
- 12: **end for**
- 13: **return** $y_\rho^{(\ell)}$

to zero. For each ρ , at iteration ℓ , we compute $y_\rho^{(\ell)}$ that drives the homotopy residual to zero, i.e., $\|\mathcal{K}_\rho(y_\rho^{(\ell)})\|_2 \rightarrow 0$. Since $\nabla \mathcal{K}_\rho(y_\rho^{(\ell)})$ is not a square matrix in general, we compute the Newton update direction Δy_ρ using the pseudoinverse $(\nabla \mathcal{K}_\rho(y_\rho^{(\ell)}))^+$,

$$(5.3) \quad \Delta y_\rho := (\nabla \mathcal{K}_\rho(y_\rho^{(\ell)}))^+ (-\mathcal{K}_\rho(y_\rho^{(\ell)})).$$

We then choose a step size $\alpha^{(\ell)} \in (0, 1]$ via backtracking line search and update

$$(5.4) \quad y_\rho^{(\ell+1)} = y_\rho^{(\ell)} + \alpha^{(\ell)} \Delta y_\rho.$$

During line search, we ensure that the slack variables $s_\rho^{(\ell)}$ and dual variables $\gamma_\rho^{(\ell)}$ remain nonnegative after the update. Using the KKT residual $\|\mathcal{K}_\rho(y_\rho^{(\ell)})\|_2$ as the merit function, we repeat this procedure until convergence, after which ρ is reduced and the process continues. Next, we characterize regularity conditions under which our method is provably convergent in Theorem 5.1.

THEOREM 5.1 (Local quadratic convergence). *Let $y_\rho^{(0)}$ be an initial point with $\gamma_\rho^{(0)} \odot s_\rho^{(0)} \geq \epsilon \mathbf{1}$, for some $\epsilon \in (0, \rho)$. Define $\mathcal{S}_{y_\rho} := \{y_\rho : \|\mathcal{K}_\rho(y_\rho)\|_2 \leq \|\mathcal{K}_\rho(y_\rho^{(0)})\|_2, \gamma_\rho \odot s_\rho \geq \epsilon \mathbf{1}\}$ and $L := \|\mathcal{K}_\rho(y_\rho^{(0)})\|_2$. Suppose that $\mathcal{K}_\rho(y_\rho)$ is in the column space of $\nabla \mathcal{K}_\rho(y_\rho)$ for all $y_\rho \in \mathcal{S}_{y_\rho}$ and that there exists an optimal solution $y_\rho^* \in \mathcal{S}_{y_\rho}$ such that $\mathcal{K}_\rho(y_\rho^*) = 0$. Furthermore, suppose that there exist constants $C, D > 0$ such that the pseudoinverse of $\nabla \mathcal{K}_\rho(y_\rho)$ is upper bounded by D , $\|(\nabla \mathcal{K}_\rho(y_\rho))^+\|_2 \leq D, \forall y_\rho \in \mathcal{S}_{y_\rho}$, and the Jacobian $\nabla \mathcal{K}_\rho(y_\rho)$ is C -Lipschitz continuous:*

$$(5.5) \quad \|\nabla \mathcal{K}_\rho(y_\rho) - \nabla \mathcal{K}_\rho(\tilde{y}_\rho)\|_2 \leq C \|y_\rho - \tilde{y}_\rho\|_2, \forall y_\rho, \tilde{y}_\rho \in \mathcal{S}_{y_\rho}.$$

Define $\hat{\alpha} := \min\{1, \frac{1}{CD^2L}, \frac{\rho - \epsilon}{D^2L^2}\}$, and $\Delta y_\rho = -(\nabla \mathcal{K}_\rho(y_\rho))^+ \mathcal{K}_\rho(y_\rho)$. Then, for all $y_\rho \in \mathcal{S}_{y_\rho}$, there exists a stepsize $\alpha \in (0, 1]$ such that $y_\rho + \alpha \Delta y_\rho \in \mathcal{S}_{y_\rho}$ and

$$(5.6) \quad \|\mathcal{K}_\rho(y_\rho + \alpha \Delta y_\rho)\|_2 \leq \left(1 - \frac{\hat{\alpha}}{2}\right) \|\mathcal{K}_\rho(y_\rho)\|_2.$$

Moreover, when $\|\mathcal{K}_\rho(y_\rho)\|_2 \leq \min\left\{\frac{2}{CD^2}, \frac{\sqrt{\rho-\epsilon}}{D}\right\}$, we have $y_\rho + \Delta y_\rho \in \mathcal{S}_{y_\rho}$ and quadratic convergence holds:

$$(5.7) \quad \frac{D^2C}{2} \|\mathcal{K}_\rho(y_\rho + \Delta y_\rho)\|_2 \leq \left(\frac{D^2C}{2} \|\mathcal{K}_\rho(y_\rho)\|_2\right)^2.$$

Proof. The proof is provided in the Appendix. \square

Given a fixed $\rho > 0$, \mathcal{K}_ρ and \mathcal{K}_0 differ only in the complementarity conditions, and consequently the proximity between y_ρ^* and y_0^* is directly controlled by ρ . We characterize the solution error between the converged solution y_ρ^* and a ground truth solution y_0^* to the reduced KKT system in the following result.

THEOREM 5.2 (Central path). *Let y_0^* satisfy $\mathcal{K}_0(y_0^*) = 0$ under Assumption 2.1, with $\|\nabla\mathcal{K}_0(y)^+\|_2 \leq D_0 < \infty$ and $\nabla\mathcal{K}_0(y)$ having constant row rank in a neighborhood of y_0^* . Let $N_c = \sum_{i=1}^N m_{\mathcal{I}}^i$ be the total number of complementarity pairs. Then, the solution y_ρ^* to $\mathcal{K}_\rho(y_\rho^*) = 0$ satisfies*

$$(5.8) \quad \|y_\rho^* - y_0^*\|_2 \leq D_0 \sqrt{N_c} \rho.$$

Proof. The proof is provided in the Appendix. \square

Specifically, Theorem 5.2 shows that $\|y_\rho^* - y_0^*\|_2 = \mathcal{O}(\rho)$, meaning that the error between the converged PDIP iterate and the solution of the reduced KKT system at $\rho = 0$ decomposes into two independent contributions:

$$(5.9) \quad \|y_\rho^{(\ell)} - y_0^*\|_2 \leq \|y_\rho^{(\ell)} - y_\rho^*\|_2 + D_0 \sqrt{N_c} \rho$$

where the first term is the algorithmic error, controlled by the PDIP convergence tolerance ϵ and decreasing quadratically by Theorem 5.1, and the second term is the perturbation error, arising from the ρ -relaxation of the complementarity condition $s \odot \gamma = \rho \mathbf{1}$ and decreasing linearly with ρ . Therefore, driving both ϵ and ρ to zero is sufficient to recover a ground truth solution to the reduced KKT conditions.

6. Numerical Study. In this section, we report a numerical study¹ to quantify the advantages of the reduced KKT system formulation compared with the complete formulation.

6.1. Complexity Study. We compare reduced and complete KKT systems on quadratic and nonquadratic GOOPs with $N = 4$ players. Each player i has $n^i = 10$ primal variables, $m_{\mathcal{E}}^i = 3$ linear equality constraints, and $m_{\mathcal{I}} = 2$ linear inequality constraints. We vary the number of preference levels $K \in \{2, \dots, 6\}$ and, for each K , generate 100 random instances with initial values (for primal variables) perturbed as $\mathcal{N}(0, 1)$ around a feasible primal solution $z_\rho^{(0)}$. In the quadratic setting, each Q_k^i in (3.11) is sampled as rank-2 positive semidefinite from Gaussian matrices, and q_k^i is chosen from the column space of Q_k^i to ensure boundedness; constraints satisfy (3.24). To generate nonquadratic problem instances, we replace the outermost objective with $(\mathbf{1}^\top z)^4$. We adapt Algorithm 5.1 to the complete KKT system by approximating complementarity as in (5.1), and run it on both the reduced and complete formulations using a geometric schedule $\rho \in \{1, 10^{-1}, \dots, 10^{-10}\}$.

Table 1 reports: (i) total wall-clock solve time of Algorithm 5.1 summed over all values of ρ per instance, reported as mean \pm standard deviation over 100 random instances after trimming the top and bottom 2.5% of runtimes; (ii) the number

¹Code is available at <https://github.com/CLearRoboticsLab/Reduced-GOOP>.

Table 1: Comparison of reduced and complete KKT systems across preference levels. R denotes the reduced KKT system, and C denotes the complete KKT system.

Level (K)	System size		Variable size		Solve time (s) for Quad. GOOP		Solve time (s) for Nonquad. GOOP	
	R	C	R	C	R	C	R	C
2	132	168	128	136	0.05 ± 0.06	0.07 ± 0.06	0.08 ± 0.12	0.19 ± 0.21
3	188	368	244	304	0.17 ± 0.16	0.34 ± 0.12	0.53 ± 0.44	1.11 ± 0.35
4	244	800	408	672	0.32 ± 0.32	2.32 ± 2.20	0.54 ± 0.56	5.09 ± 2.82
5	300	1728	620	1472	0.60 ± 0.75	Failed	0.96 ± 0.88	Failed
6	356	3712	880	3200	1.19 ± 0.80	Failed	1.79 ± 3.01	Failed

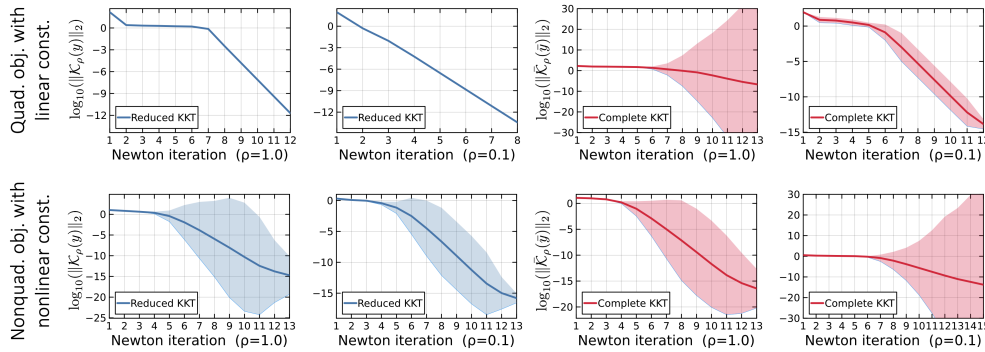


Fig. 2: Convergence of Algorithm 5.1 for reduced and complete KKT systems under varying ρ . The curve and the shaded area denote the mean and the variance of $\log_{10}(\|\mathcal{K}_\rho(y)\|_2)$, respectively.

of variables in each KKT system, N_{z^i, η^i} (reduced) and $N_{\bar{z}^i, \bar{\eta}^i}$ (complete); and (iii) the number of equality and inequality constraints appearing in each KKT system, $(\mathcal{F}^i, \mathcal{G}^i)$ (reduced) and $(\bar{\mathcal{F}}^i, \bar{\mathcal{G}}^i)$ (complete). The complete system exhibits exponential growth in system size with respect to K , reflected by the rapid increase in the number of variables and constraints. This exponential growth translates directly into greater computational burden: across all tested preference levels, the complete formulation requires substantially longer solve times, and the performance gap widens as K increases. The same qualitative behavior is observed in both the quadratic and non-quadratic settings. We do not report nonquadratic GOOP results for the complete KKT system formulation when considering larger preference hierarchies ($K > 4$), since the symbolic compilation became intractable due to the large size of complete KKT systems.

6.2. Convergence Study. We empirically demonstrate linear and local quadratic convergence of Algorithm 5.1 on two problem classes. Class (i): $N = 3$ players with $n^i = 10$ variables and $K = 4$ levels, using rank-2 quadratic objectives $J_k^i(z) = z^\top Q_k^i z + q_k^{i\top} z$ and linear constraints whose normals lie in the column space of Q_k^i . Class (ii): $N = 2$ players with $n^i = 4$ variables and $K = 3$ levels, with a top-level L_2 objective $J_1^i(z) = \|z\|_2^2$ ensuring a unique equilibrium, inner objectives

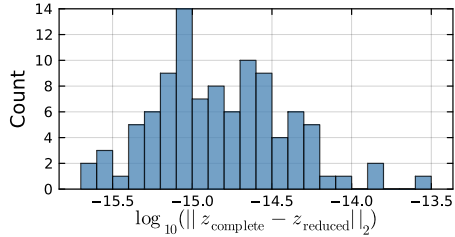


Fig. 3: Monte Carlo comparison over 100 randomly generated GOOP instances with nonquadratic objectives and nonlinear constraints, showing close agreement between primal solutions z_{complete} and z_{reduced} , obtained from the complete and reduced KKT systems, respectively.

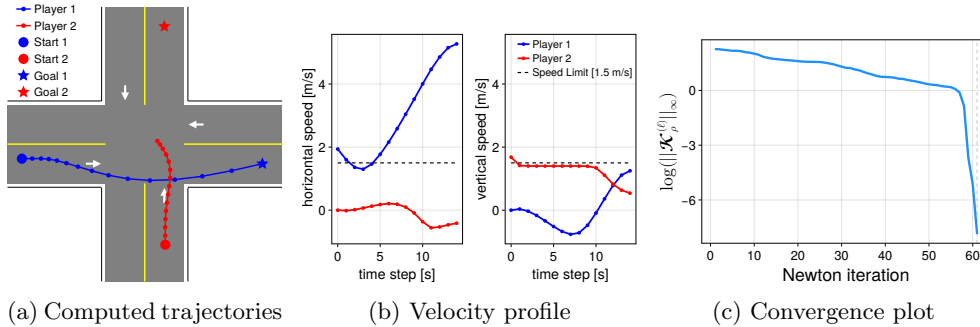


Fig. 4: Two-player intersection scenario solved using the reduced KKT system. We use the four-preference setup as in [19]. Player 1 (blue) prioritizes reaching its goal and accelerates beyond the speed limit. Player 2 (red) prioritizes obeying the speed limit and proceeds toward its goal while remaining within the limit. The convergence plot shows local quadratic convergence of the KKT residual to 10^{-8} . The homotopy parameter ρ is reduced following a geometric schedule $\{1, 2^{-1}, \dots, 2^{-10}\}$; the reported results correspond to $\rho = 2^{-10}$.

$J_k^i(z) = e^{v_{i,k}^\top z} + e^{-v_{i,k}^\top z}$ for random unit vectors $v_{i,k}$ (obtained by normalizing Gaussian samples), and nonlinear coupling constraints. In both settings, the convergence curves in Figure 2, showing the mean and variance of $\log_{10}(\|\mathcal{K}_\rho(y)\|_2)$ over 10 runs with initializations perturbed as $\mathcal{N}(0, 0.5)$ around a feasible solution z_0 , empirically support Theorem 5.1. Figure 3 further presents a Monte Carlo study over 100 instances of class (ii), indicating that the reduced KKT system recovers the same primal solution as the complete system, although theoretical guarantees remain challenging due to the complexity of the nonlinear KKT solution space.

6.3. Application to a practical scenario. Finally, Figure 4 demonstrates the applicability of the reduced GOOP formulation and Algorithm 5.1 on a practical multi-vehicle intersection planning problem, originally formulated in [19], illustrating the potential of games of ordered preference in practical settings.

7. Conclusion. In this paper, we studied games of ordered preference and introduced a compact reduced KKT system that avoids the exponential complexity of standard single-level reformulations while preserving the essential primal stationarity structure across preference levels. For quadratic GOOPs with linear constraints, we established primal solution equivalence between the reduced and complete KKT systems. For general smooth nonlinear GOOPs, we showed that the reduced system is

a relaxation of the complete system and supplemented this result with second-order sufficient conditions for local optimality. We developed a primal-dual interior-point method and proved its local quadratic convergence. Numerical study shows that the reduced KKT system formulation yields faster solve times than the complete KKT formulation, with benefits that grow in the number of preference levels. Future work will address primal solution equivalence beyond the quadratic setting, and broaden the framework to richer classes of hierarchical games and multi-agent decision problems.

Appendix A. Supplementary results.

Proof of Proposition 2.4. For player i at level k , we define $\bar{d}_k^i := \dim(\bar{\eta}_k^i)$, $\bar{p}_k^i := \dim(\bar{\eta}_{k+1:K^i}^i)$, $\bar{v}_k^i := \dim(\bar{z}^i, \bar{\eta}_{k:K^i}^i)$, $\bar{f}_k^i := \dim(\bar{\mathcal{F}}_k^i)$, $\bar{g}_k^i := \dim(\bar{\mathcal{G}}_k^i)$, respectively denoting the numbers of dual variables, induced primals, total variables, equations, and inequality constraints. For the innermost level, $\bar{d}_{K^i}^i = m_{\mathcal{E}}^i + m_{\mathcal{T}}^i$, $\bar{p}_{K^i}^i = 0$, $\bar{v}_{K^i}^i = n^i + m_{\mathcal{E}}^i + m_{\mathcal{T}}^i$, $\bar{f}_{K^i}^i = n^i + m_{\mathcal{E}}^i + m_{\mathcal{T}}^i$, and $\bar{g}_{K^i}^i = 2m_{\mathcal{T}}^i$. At any $k \in [K^i - 1]$, $\bar{d}_k^i = \bar{f}_{k+1}^i + \bar{g}_{k+1}^i$, $\bar{p}_k^i = \sum_{j=k+1}^{K^i} \bar{d}_j^i$, $\bar{v}_k^i = n^i + \bar{p}_k^i + \bar{d}_k^i$, $\bar{f}_k^i = n^i + \bar{p}_k^i + \bar{d}_k^i$, and $\bar{g}_k^i = 2^{K^i-k+1}m_{\mathcal{T}}^i$. Observe that $\bar{d}_k^i = \bar{f}_k^i - n^i - \bar{p}_k^i$ and $\bar{p}_{k-1}^i = \bar{d}_k^i + \bar{p}_k^i$ implies $\bar{p}_{k-1}^i = \bar{f}_k^i - n^i$. Substituting this into $\bar{f}_{k-1}^i = n^i + \bar{p}_{k-1}^i + \bar{f}_k^i + \bar{g}_k^i$ yields $\bar{f}_k^i = 2\bar{f}_{k-1}^i + \bar{g}_k^i$. Then, applying the previous identity repeatedly and substituting $\bar{f}_{K^i}^i$ and \bar{g}_{k+j}^i ,

$$\begin{aligned} \bar{f}_k^i &= 2\bar{f}_{k+1}^i + \bar{g}_{k+1}^i = 2(2\bar{f}_{k+2}^i + \bar{g}_{k+2}^i) + \bar{g}_{k+1}^i = \dots = 2^{K^i-k} \bar{f}_{K^i}^i + \sum_{j=1}^{K^i-k} 2^{j-1} \bar{g}_{k+j}^i \\ &= 2^{K^i-k} (n^i + m_{\mathcal{E}}^i + (K^i - k + 1)m_{\mathcal{T}}^i). \quad \square \end{aligned}$$

Proof of Theorem 3.1. We show that, for each player i , every solution of the complete KKT system induces a solution of the reduced KKT system at the same primal point. The argument proceeds by backward induction on the preference level.

Fix a player i , and let $(\bar{z}^i, \bar{\eta}_{1:K^i}^i)$ satisfy the complete KKT system. Throughout the proof, we identify the primal variables in the two systems by setting $z^i := \bar{z}^i$. We then construct reduced-system dual variables $\eta_{1:K^i}^i$ from the complete-system dual variables $\bar{\eta}_{1:K^i}^i$ so that the reduced KKT conditions hold at the same primal point.

Base case $k = K^i - 1$. Consider the complete KKT system at level $K^i - 1$. Let $\bar{\gamma}_{K^i-1,1}^i$ be the dual variable for the inequality constraint $g^i(\bar{z}^i, \bar{z}^{-i}) \geq \mathbf{0}$ and $\bar{\gamma}_{K^i-1,2}^i$ the dual variable for $\bar{\gamma}_{K^i}^i \geq \mathbf{0}$ constraint. Relative to the complete system $\bar{\mathcal{F}}_{K^i-1}^i$, the reduced system $\mathcal{F}_{K^i-1}^i$ omits (i) $\nabla_{\bar{\eta}_{K^i}^i} \bar{\mathcal{L}}_{K^i}^i$ from the stationarity constraints and (ii) the complementarity conditions $\bar{\gamma}_{K^i}^i \odot \bar{\gamma}_{K^i-1,2}^i$. Accordingly, we set the reduced multipliers as $\psi_{K^i-1}^i = \bar{\psi}_{K^i-1}^i$, $\phi_{K^i-1}^i = \bar{\phi}_{K^i-1}^i$, $\lambda_{K^i-1}^i = \bar{\lambda}_{K^i-1}^i$, $\gamma_{K^i-1}^i = \bar{\gamma}_{K^i-1,1}^i$, and discard the remaining dual variable $\bar{\gamma}_{K^i-1,2}^i$ (associated with the omitted complementarity conditions). Next, we obtain the Lagrangian of the reduced KKT system at level $K^i - 1$ from the complete-system Lagrangian:

$$\begin{aligned} \text{(A.1)} \quad \bar{\mathcal{L}}_{K^i-1}^i(\bar{z}, \bar{\eta}_{K^i-1:K^i}^i) &= J_{K^i-1}^i(\bar{z}) - \bar{\lambda}_{K^i-1}^{i\top} h^i(\bar{z}) - \bar{\gamma}_{K^i-1,1}^{i\top} g^i(\bar{z}) \\ &\quad - \bar{\psi}_{K^i-1}^{i\top} \nabla_{\bar{z}^i} \bar{\mathcal{L}}_{K^i}^i - \bar{\phi}_{K^i-1}^{i\top} (g^i(\bar{z}) \odot \bar{\gamma}_{K^i}^i) - \cancel{\bar{\gamma}_{K^i-1,2}^{i\top} \bar{\gamma}_{K^i}^i} \\ &= \mathcal{L}_{K^i-1}^i(\bar{z}, \eta_{K^i-1:K^i}^i). \end{aligned}$$

Hence the reduced system $\mathcal{F}_{K^i-1}^i$ is satisfied by the complete-system solution. Likewise, the reduced inequality system $\mathcal{G}_{K^i-1}^i$ is obtained from $\bar{\mathcal{G}}_{K^i-1}^i$ by removing the nonnegativity conditions associated with the discarded multipliers, i.e., $\bar{\gamma}_{K^i-1,2}^i \geq \mathbf{0}$.

Inductive step. Let $k \in \{2, \dots, K^i - 1\}$, and suppose that reduced-system dual variables have already been constructed at levels k, \dots, K^i so that the reduced KKT conditions hold on those levels. We now construct the reduced dual variables at level $k-1$. Consider the complete KKT system at level $k-1$. Let $\bar{\gamma}_{k-1,1}^i$ be the dual variable for the inequality constraint $g^i(\bar{z}^i, \bar{z}^{-i}) \geq \mathbf{0}$ and $\bar{\gamma}_{k-1,2}^i$ the dual variable for $\bar{\gamma}_{k:K^i}^i \geq \mathbf{0}$ constraint. In passing from the complete system $\bar{\mathcal{F}}_{k-1}^i$ to the reduced system \mathcal{F}_{k-1}^i , we omit: (i) the stationarity conditions with respect to lower-level dual variables, $\nabla_{\bar{\eta}_{k:K^i}^i} \bar{\mathcal{L}}_{k-1}^i$, (ii) the complementarity conditions involving lower-level inequality multipliers, $\bar{\gamma}_{k:K^i}^i \odot \bar{\gamma}_{k-1,2}^i$. Next, we set the reduced dual variables at level $k-1$ by retaining only a subset of components of $\bar{\eta}_{k-1}^i$: (i) the components of ψ_{k-1}^i are taken from $\bar{\psi}_{k-1}^i$ for the terms associated with $\nabla_{z^i} \bar{\mathcal{L}}_{k:K^i}^i$, (ii) the components of ϕ_{k-1}^i are taken from $\bar{\phi}_{k-1}^i$ for the terms associated with $g^i(z^i, z^{-i}) \odot \bar{\gamma}_{k,1}^i, \dots, g^i(z^i, z^{-i}) \odot \bar{\gamma}_{K^i}^i$ and (iii) the current-level feasibility multipliers are preserved: $\lambda_{k-1}^i = \bar{\lambda}_{k-1}^i$, $\gamma_{k-1}^i = \bar{\gamma}_{k-1,1}^i$. The remaining complete-system dual variables at level $k-1$, including $\bar{\gamma}_{k-1,2}^i$, are discarded. We illustrate this construction below:

$$\begin{aligned}
 \text{(A.2)} \quad \bar{\mathcal{L}}_{k-1}^i(\bar{z}^i, \bar{z}^{-i}, \bar{\eta}_{k-1:K^i}^i) &= J_{k-1}^i(\bar{z}^i, \bar{z}^{-i}) - \bar{\lambda}_{k-1}^{i\top} h^i(\bar{z}^i, \bar{z}^{-i}) - \bar{\gamma}_{k-1,1}^{i\top} g^i(\bar{z}^i, \bar{z}^{-i}) \\
 &- \bar{\psi}_{k-1}^{i\top} \begin{bmatrix} \nabla_{\bar{z}^i} \bar{\mathcal{L}}_k^i \\ \nabla_{\bar{\eta}_{k+1:K^i}^i} \bar{\mathcal{L}}_k^i \\ \nabla_{\bar{z}^i} \bar{\mathcal{L}}_{k+1}^i \\ \nabla_{\bar{\eta}_{k+2:K^i}^i} \bar{\mathcal{L}}_{k+1}^i \\ \vdots \\ \nabla_{\bar{z}^i} \bar{\mathcal{L}}_{K^i}^i \end{bmatrix} - \bar{\phi}_{k-1}^{i\top} \begin{bmatrix} g^i(\bar{z}^i, \bar{z}^{-i}) \odot \bar{\gamma}_{k,1}^i \\ \bar{\gamma}_{k+1:K^i}^i \odot \bar{\gamma}_{k,2}^i \\ g^i(\bar{z}^i, \bar{z}^{-i}) \odot \bar{\gamma}_{k+1,1}^i \\ \bar{\gamma}_{k+2:K^i}^i \odot \bar{\gamma}_{k+1,2}^i \\ \vdots \\ g^i(\bar{z}^i, \bar{z}^{-i}) \odot \bar{\gamma}_{K^i}^i \end{bmatrix} - \bar{\gamma}_{k-1,2}^{i\top} \begin{bmatrix} \bar{\gamma}_k^i \\ \vdots \\ \bar{\gamma}_{K^i}^i \end{bmatrix} \\
 &= \mathcal{L}_{k-1}^i(z^i, z^{-i}, \eta_{k-1:K^i}^i).
 \end{aligned}$$

With this construction, the reduced system \mathcal{F}_{k-1}^i is satisfied by the complete-system solution $(\bar{z}, \bar{\eta}_{k:K^i}^i)$. The reduced system \mathcal{G}_{k-1}^i retains only the nonnegativity conditions of $\bar{\gamma}_{k-1,1}^i$ and inherits the conditions $\bar{\gamma}_{k,1}^i \geq \mathbf{0}, \dots, \bar{\gamma}_{K^i,1}^i \geq \mathbf{0}$ from level k , which hold by the induction hypothesis. Hence \mathcal{G}_{k-1}^i is also satisfied. Since the construction is valid for every player $i \in [N]$, we conclude that every solution of the complete KKT system induces a solution of the reduced KKT system at the same primal solution. \square

Proof of Proposition 3.2. Fix a player $i \in [N]$. The total number of variables in the reduced KKT system $(\mathcal{F}^i, \mathcal{G}^i)$ consists of (i) primal variables z^i and (ii) dual variables $\eta_{1:K^i}^i$. For any $k \leq K^i$, the dimension of the dual vector η_k^i is $(K^i - k)n^i + (K^i - k + 1)m_{\mathcal{I}}^i + m_{\mathcal{E}}^i$. The sum of all variables' dimensions is then

$$\begin{aligned}
 \text{(A.3)} \quad N_{z^i, \eta^i} &= n^i + \sum_{k=1}^{K^i} (K^i - k)n^i + (K^i - k + 1)m_{\mathcal{I}}^i + m_{\mathcal{E}}^i \\
 &= \left(1 + \frac{K^i(K^i - 1)}{2}\right)n^i + K^i m_{\mathcal{E}}^i + \frac{K^i(K^i + 1)}{2} m_{\mathcal{I}}^i. \quad \square
 \end{aligned}$$

Proof of Theorem 3.4. The proof has three parts: (a) we establish the column-space inclusion relation between the reduced and complete linear systems, (b) we propagate this relation across levels using the recursive block structure of the two KKT matrices, and (c) we use the resulting column-space inclusions to show that the primal solution sets of the two systems coincide.

(a) We first prove the equivalent statements: $\text{Col}(R_k) \subseteq \text{Col}(\bar{R}_k) \iff \mathcal{N}(\bar{R}_k^\top) \subseteq \mathcal{N}(R_k^\top)$. Let $y = [u^\top, v^\top, w^\top, z^\top]^\top \in \mathcal{N}(\bar{R}_k^\top)$ where $u \in \mathbb{R}^n$. Then,

$$(A.4) \quad \bar{R}_k^\top y = \left[\begin{array}{cc|c} \hat{Q}_k & \mathbf{0} & \bar{R}_{k+1} \\ \mathbf{0} & \mathbf{0} & \\ \hline \bar{R}_{k+1} & \mathbf{0} & \mathbf{0} \\ & \mathbf{0} & \mathbf{0} \end{array} \right] \begin{bmatrix} u \\ v \\ w \\ x \end{bmatrix} = \left[\begin{array}{c} [\hat{Q}_k u] + \bar{R}_{k+1} \begin{bmatrix} w \\ x \end{bmatrix} \\ \bar{R}_{k+1} \begin{bmatrix} u \\ v \end{bmatrix} \end{array} \right] = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}.$$

Premultiplying the first block row in (A.4) by $[u^\top, v^\top]$ and using the symmetry of \bar{R}_{k+1} :

$$(A.5) \quad u^\top \hat{Q}_k u + \begin{bmatrix} u \\ v \end{bmatrix}^\top \bar{R}_{k+1} \begin{bmatrix} w \\ x \end{bmatrix} = u^\top \hat{Q}_k u + \left(\bar{R}_{k+1} \begin{bmatrix} u \\ v \end{bmatrix} \right)^\top \begin{bmatrix} w \\ x \end{bmatrix} = u^\top \hat{Q}_k u = \mathbf{0}.$$

Since \hat{Q}_k is positive semi-definite, it follows that $\hat{Q}_k u = \mathbf{0}$. Now, consider

$$R_k^\top y = \left[\begin{array}{c|ccc} \hat{Q}_k & & & \\ \hline \bar{R}_{k+1,[:,1:n]}^\top & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right] \begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix} = \begin{bmatrix} \hat{Q}_k u \\ \bar{R}_{k+1,[:,1:n]}^\top u \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

where the second row follows from $\bar{R}_{k+1} \begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{0}$ in (A.4) and the symmetry of \bar{R}_{k+1} .

Hence, $y \in \mathcal{N}(R_k^\top)$, which implies $\mathcal{N}(\bar{R}_k^\top) \subseteq \mathcal{N}(R_k^\top)$.

(b) We prove this result by induction on k .

Base case $k = K - 1$. We partition the matrix M_{K-1} as follows.

$$(A.6) \quad M_{K-1} = \left[\begin{array}{cc|cc} Q_{K-1} & \mathbf{0} & \hat{Q}_K & \hat{H}^\top \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ Q_K & \hat{H}^\top & \mathbf{0} & \mathbf{0} \\ H & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right] \Rightarrow M_{K-1}^c = \left[\begin{array}{c|cc} \mathbf{0} & & R_K \\ \mathbf{0} & & \\ \hline \hat{H}^\top & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right].$$

$C_{K-1} \qquad M_{K-1}^c$

Similarly, we partition the matrix \bar{M}_{K-1} as

$$(A.7) \quad \bar{M}_{K-1} = \left[\begin{array}{cc|cc} Q_{K-1} & \mathbf{0} & \hat{Q}_K & \hat{H}^\top \\ \mathbf{0} & \mathbf{0} & \hat{H} & \mathbf{0} \\ Q_K & \hat{H}^\top & \mathbf{0} & \mathbf{0} \\ H & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right] \Rightarrow \bar{M}_{K-1}^c = \left[\begin{array}{c|cc} \mathbf{0} & & \bar{R}_K \\ \mathbf{0} & & \\ \hline \hat{H}^\top & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right].$$

$C_{K-1} \qquad \bar{M}_{K-1}^c$

By (a), $\text{Col}(R_K) \subseteq \text{Col}(\bar{R}_K)$. This implies that $\text{Col}(M_{K-1}^c) \subseteq \text{Col}(\bar{M}_{K-1}^c)$.

Induction step. Assume $\text{Col}(M_k^c) \subseteq \text{Col}(\bar{M}_k^c)$ for any $k \in [K-1]$. Consider the level $k-1$. Partitioning the matrices M_{k-1} and \bar{M}_{k-1} are:

$$(A.8) \quad M_{k-1}^c = \begin{bmatrix} \mathbf{0} & R_k \\ M_k^c & \mathbf{0} \end{bmatrix}, \quad \bar{M}_{k-1}^c = \begin{bmatrix} \mathbf{0} & \bar{R}_k \\ \bar{M}_k^c & \mathbf{0} \end{bmatrix}.$$

Define the subspaces: $A = \text{Col}\left(\begin{bmatrix} \mathbf{0} \\ M_k^c \end{bmatrix}\right)$, $\bar{A} = \text{Col}\left(\begin{bmatrix} \mathbf{0} \\ \bar{M}_k^c \end{bmatrix}\right)$, $B = \text{Col}\left(\begin{bmatrix} R_k \\ \mathbf{0} \end{bmatrix}\right)$, and $\bar{B} = \text{Col}\left(\begin{bmatrix} \bar{R}_k \\ \mathbf{0} \end{bmatrix}\right)$. By the induction hypothesis, $\text{Col}(M_k^c) \subseteq \text{Col}(\bar{M}_k^c) \Rightarrow A \subseteq \bar{A}$.

By (a), $B \subseteq \bar{B}$. Since A and B are orthogonal subspaces that span $\text{Col}(M_{k-1}^c)$ and, similarly, \bar{A} and \bar{B} are orthogonal subspaces that span $\text{Col}(\bar{M}_{k-1}^c)$, we conclude $\text{Col}(M_{k-1}^c) \subseteq \text{Col}(\bar{M}_{k-1}^c)$.

(c) We now use the results from parts (a) and (b) to show primal solution equivalence. Write $\bar{v}_k = \begin{bmatrix} \bar{z} \\ \bar{\eta}_{k:K} \end{bmatrix}$ and $v_k = \begin{bmatrix} z \\ \eta_{k:K} \end{bmatrix}$ and define the *primal* solution sets

$$(A.9) \quad \bar{\mathcal{X}}(\bar{p}_k) = \left\{ \bar{z} \mid \exists \bar{\eta}_k : \bar{M}_k \bar{v}_k = \bar{p}_k \right\}, \quad \mathcal{X}(\bar{p}_k) = \left\{ z \mid \exists \eta_k : M_k v_k = \bar{p}_k \right\}.$$

(i) $\mathcal{X}(\bar{p}_k) \subseteq \bar{\mathcal{X}}(\bar{p}_k)$. Partition the matrices \bar{M}_k, M_k and let $z \in \mathcal{X}(\bar{p}_k)$ so that

$$(A.10) \quad M_k \begin{bmatrix} z \\ \eta_k \end{bmatrix} = [c_k \mid M_k^c] \begin{bmatrix} z \\ \eta_k \end{bmatrix} = c_k z + M_k^c \eta_k.$$

Set $\bar{z} = z$. Choose a vector $\bar{\eta}_k$ such that $M_k^c \eta_k = \bar{M}_k^c \bar{\eta}_k$, which exists by (b). Then,

$$(A.11) \quad \bar{M}_k \begin{bmatrix} \bar{z} \\ \bar{\eta}_k \end{bmatrix} = [c_k \mid \bar{M}_k^c] \begin{bmatrix} \bar{z} \\ \bar{\eta}_k \end{bmatrix} = \underbrace{c_k \bar{z}}_{c_k z} + \underbrace{\bar{M}_k^c \bar{\eta}_k}_{M_k^c \eta_k}.$$

This implies that $z \in \bar{\mathcal{X}}(\bar{p}_k)$ and thus $\mathcal{X}(\bar{p}_k) \subseteq \bar{\mathcal{X}}(\bar{p}_k)$.

(ii) $\bar{\mathcal{X}}(\bar{p}_k) \subseteq \mathcal{X}(\bar{p}_k)$. We proceed via induction on k .

Base case $k = K - 1$. Let $\bar{z} \in \bar{\mathcal{X}}(\bar{p}_{K-1})$. Recalling the right-hand-side vector \bar{p}_{K-1} in (3.19), we write

$$(A.12) \quad \bar{M}_{K-1} \bar{v}_{K-1} = \begin{bmatrix} Q_{K-1} & \mathbf{0} & \hat{Q}_K & \hat{H}^\top \\ \mathbf{0} & \mathbf{0} & \hat{H} & \mathbf{0} \\ Q_K & \hat{H}^\top & \mathbf{0} & \mathbf{0} \\ H & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \bar{z} \\ \bar{\lambda}_K \\ \bar{\psi}_{K-1} \\ \bar{\lambda}_{K-1} \end{bmatrix} = \begin{bmatrix} -q_{K-1} \\ \mathbf{0} \\ -q_K \\ b \end{bmatrix} = \bar{p}_{K-1}.$$

Set $v_{K-1} = \bar{v}_{K-1}$ (so $z = \bar{z}$). Then,

$$(A.13) \quad M_{K-1} \bar{v}_{K-1} = \begin{bmatrix} Q_{K-1} & \mathbf{0} & \hat{Q}_K & \hat{H}^\top \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ Q_K & \hat{H}^\top & \mathbf{0} & \mathbf{0} \\ H & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \bar{z} \\ \bar{\lambda}_K \\ \bar{\psi}_{K-1} \\ \bar{\lambda}_{K-1} \end{bmatrix} = \begin{bmatrix} -q_{K-1} \\ \mathbf{0} \\ -q_K \\ b \end{bmatrix} = \bar{p}_{K-1}$$

so $\bar{z} \in \mathcal{X}(\bar{p}_{K-1})$.

Induction step. Assume the following holds for $k \in [K - 1]$

$$(A.14) \quad \bar{M}_k \bar{v}_k = \bar{p}_k \Rightarrow M_k \bar{v}_k = \bar{p}_k.$$

From (3.19), the first n rows are

$$(A.15) \quad Q_{k-1} \bar{z} + \bar{R}_{k,[1:n,:]} \bar{\eta}_{k-1} = q_{k-1}.$$

Set $v_{k-1} = \bar{v}_{k-1}$ (so $z = \bar{z}$) Now consider $M_{k-1} \bar{v}_{k-1}$. From (3.20) using (3.18) and (A.15),

$$(A.16) \quad \underbrace{\begin{bmatrix} Q_{k-1} & \mathbf{0} & R_k \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ M_k & & \mathbf{0} \end{bmatrix}}_{M_{k-1}} \underbrace{\begin{bmatrix} \bar{z} \\ \bar{\eta}_{K:k} \\ \bar{\eta}_{k-1} \end{bmatrix}}_{\bar{v}_{k-1}} = \underbrace{\begin{bmatrix} q_{k-1} \\ \mathbf{0} \\ q_k \end{bmatrix}}_{\bar{p}_{k-1}},$$

where the last row follows from the induction hypothesis and $v_k = [z^\top, (\eta_{K:k})^\top]^\top$. Thus, $\bar{z} \in \mathcal{X}(\bar{p}_k)$. Combining (i) and (ii) gives $\bar{\mathcal{X}}(\bar{p}_k) = \mathcal{X}(\bar{p}_k)$ for all $k \in [K-1]$. \square

Proof of Theorem 3.5. The proof proceeds in three steps. We first fix the active inequality constraints at the candidate primal solution z^* and show that, at z^* , the corresponding inequality-constrained problem can be reduced to an equality-constrained problem on the active set. We then apply the equality-constrained primal equivalence result from Theorem 3.4. Finally, we reconstruct complete-system inequality multipliers from the equality-constrained multipliers using strict complementarity.

Step 1. Local active-set reduction. Fix a candidate local solution z^* . Because the active set is fixed at z^* , the solution set of the inequality-constrained problem (2.1) for player i is identical to the solution set of the equality-constrained problem obtained by replacing $G^i z \geq g^i$ with $G_{\mathcal{A}^i}^i z = g_{\mathcal{A}^i}^i$ and discarding the constraints that are inactive (slack) at the optimum. Let $\mathcal{P}_{\mathcal{I}}$ denote inequality-constrained optimization problem (2.1) and $\mathcal{P}_{\mathcal{E}}$ denote the corresponding equality-constrained problem.

Step 2. The equality-constrained problem $\mathcal{P}_{\mathcal{E}}$ shares the same primal solution set as its reduced KKT system. This follows directly from Theorem 3.4 applied to $\mathcal{P}_{\mathcal{E}}$, given that \hat{H} satisfies the required full row rank assumption.

Step 3. The reduced KKT system for the equality-constrained problem (\mathcal{E}) and the reduced KKT system for the inequality-constrained problem (\mathcal{I}) share the same set of primal solutions. Consider the stationarity conditions for level k for both systems.

Reduced KKT System for \mathcal{E} : Let $\mu^\mathcal{E} := (z^\mathcal{E}, \lambda^\mathcal{E}, \psi^\mathcal{E})$. For $i \in [N]$ and $k \in [K^i]$:

$$(A.17) \quad \sum_{j=1}^N Q_k^{i,i,j} z^{j,\mathcal{E}} - \sum_{\ell=1}^{K^i-k} \left[Q_{K^i-\ell+1}^{i,i,i} \psi_{k,\ell}^{i,\mathcal{E}} \right] - H_i^{i\top} \lambda_{k,1}^{i,\mathcal{E}} - G_{i,\mathcal{A}^i}^{i\top} \lambda_{k,2}^{i,\mathcal{E}} = -q_k^i,$$

subject to $H^i z^\mathcal{E} = h^i$ and $G_{\mathcal{A}^i}^i z^\mathcal{E} = g_{\mathcal{A}^i}^i$.

Reduced KKT System for \mathcal{I} : Let $\mu^\mathcal{I} := (z^\mathcal{I}, \lambda^\mathcal{I}, \psi^\mathcal{I}, \phi^\mathcal{I}, \gamma^\mathcal{I})$. For $i \in [N]$ and $k \in [K^i]$:

$$(A.18) \quad \sum_{j=1}^N Q_k^{i,i,j} z^{j,\mathcal{I}} - \sum_{\ell=1}^{K^i-k} \left[Q_{K^i-\ell+1}^{i,i,i} \psi_{k,\ell}^{i,\mathcal{I}} + G_{i,\mathcal{A}^i}^{i\top} \left(\phi_{k,\ell}^{i,\mathcal{I}} \odot \gamma_{K^i-\ell+1}^{i,\mathcal{I}} \right) \right] \\ - H_i^{i\top} \lambda_k^{i,\mathcal{I}} - G_{i,\mathcal{A}^i}^{i\top} \gamma_k^{i,\mathcal{I}} = -q_k^i,$$

subject to $H^i z^\mathcal{I} = h^i$, $G_{\mathcal{A}^i}^i z^\mathcal{I} \geq g_{\mathcal{A}^i}^i$, $\gamma_{1:K^i}^{i,\mathcal{I}} \geq \mathbf{0}$, and complementarity conditions.

3.1 If $\mu^\mathcal{I}$ satisfies (\mathcal{I}), then there exists $\mu^\mathcal{E}$ satisfying (\mathcal{E}) such that $z^\mathcal{E} = z^\mathcal{I}$. Assume $\mu^\mathcal{I} := (z^\mathcal{I}, \lambda^\mathcal{I}, \psi^\mathcal{I}, \phi^\mathcal{I}, \gamma^\mathcal{I})$ satisfies (\mathcal{I}). Define (\mathcal{E}) variables for $i \in [N]$: $z^{i,\mathcal{E}} := z^{i,\mathcal{I}}$, $\lambda_{k,1}^{i,\mathcal{E}} := \lambda_k^{i,\mathcal{I}}$, $\psi^{i,\mathcal{E}} := \psi^{i,\mathcal{I}}$, $\lambda_{k,2}^{i,\mathcal{E}} := \gamma_k^{i,\mathcal{I}} + \sum_{\ell=1}^{K^i-k} \phi_{k,\ell}^{i,\mathcal{I}} \odot \gamma_{K^i-\ell+1}^{i,\mathcal{I}}$. Substituting these definitions into (A.18) yields stationarity in (A.17). Next, verify feasibility. We have $H^i z^\mathcal{I} = h^i$. By the construction of the active set \mathcal{A}^i , any solution to (\mathcal{I}) must satisfy $G_{\mathcal{A}^i}^i z^\mathcal{I} = g_{\mathcal{A}^i}^i$. Thus $z^\mathcal{E}$ satisfies the equality constraints of (\mathcal{E}).

3.2 If $\mu^\mathcal{E}$ satisfies (\mathcal{E}), then there exists $\mu^\mathcal{I}$ satisfying (\mathcal{I}) such that $z^\mathcal{I} = z^\mathcal{E}$. Assume $\mu^\mathcal{E} := (z^\mathcal{E}, \psi^\mathcal{E}, \lambda^\mathcal{E})$ satisfies (\mathcal{E}). We construct a solution for (\mathcal{I}) by induction on the level index k , proceeding backwards from K^i to 1. First, set shared variables for $i \in [N]$: $z^{i,\mathcal{I}} := z^{i,\mathcal{E}}$, $\lambda_{k,1}^{i,\mathcal{I}} := \lambda_{k,1}^{i,\mathcal{E}}$, $\psi^{i,\mathcal{I}} := \psi^{i,\mathcal{E}}$. Since $H^i z^\mathcal{E} = h^i$ and $G_{\mathcal{A}^i}^i z^\mathcal{E} = g_{\mathcal{A}^i}^i$, $z^\mathcal{I}$ satisfies primal feasibility with respect to the innermost constraints.

3.2.1 Base Case ($k = K^i$). Fix a player i . Define $\gamma_{K^i}^{i,\mathcal{I}} = \lambda_{K^i}^{i,\mathcal{E}}$, where $\lambda_{K^i}^{i,\mathcal{E}}$ denotes the multipliers associated with the equality constraints obtained from the

active inequalities at level K^i . By strict complementarity at the candidate solution, these multipliers are strictly positive. Hence $\gamma_{K^i}^{i,\mathcal{I}} > \mathbf{0}$.

3.2.2 Induction Step Assume that, for all levels $m^i > k$, we have determined $\gamma_{m^i}^{i,\mathcal{I}} \geq \mathbf{0}$ and $\phi_{m^i}^{i,\mathcal{I}}$. We now determine $\gamma_k^{i,\mathcal{I}} \geq \mathbf{0}$ and $\phi_k^{i,\mathcal{I}}$ for (A.18). From (A.18) and (A.17), matching the terms requires $\lambda_{k,2}^{i,\mathcal{E}} = \gamma_k^{i,\mathcal{I}} + \sum_{\ell=1}^{K^i-k} \phi_{k,\ell}^{i,\mathcal{I}} \odot \gamma_{K^i-\ell+1}^{i,\mathcal{I}}$. To satisfy this, it suffices to choose one lower-level term and set the remaining $\phi_{k,\ell}^{i,\mathcal{I}}$ equal to zero. We use the term corresponding to $\ell = 1$, which involves $\gamma_{K^i}^{i,\mathcal{I}}$, already known to be strictly positive from the base case. Set $\phi_{k,\ell}^{i,\mathcal{I}} = \mathbf{0}$ for all $\ell > 1$. Then, we have $\lambda_{k,2}^{i,\mathcal{E}} = \gamma_k^{i,\mathcal{I}} + \phi_{k,1}^{i,\mathcal{I}} \odot \gamma_{K^i}^{i,\mathcal{I}}$. For each component $r \in \{1, \dots, m_{\mathcal{I}}^i\}$:

- If $(\lambda_{k,2}^{i,\mathcal{E}})_r \geq 0$: Set $(\gamma_k^{i,\mathcal{I}})_r := (\lambda_{k,2}^{i,\mathcal{E}})_r$ and $(\phi_{k,1}^{i,\mathcal{I}})_r := 0$.
- If $(\lambda_{k,2}^{i,\mathcal{E}})_r < 0$: Set $(\gamma_k^{i,\mathcal{I}})_r := 0$. We must solve $(\lambda_{k,2}^{i,\mathcal{E}})_r = (\phi_{k,1}^{i,\mathcal{I}})_r \cdot (\gamma_{K^i}^{i,\mathcal{I}})_r$. Since $(\gamma_{K^i}^{i,\mathcal{I}})_r > 0$ (from Base Case 3.2.1), we can uniquely define: $(\phi_{k,1}^{i,\mathcal{I}})_r := \frac{(\lambda_{k,2}^{i,\mathcal{E}})_r}{(\gamma_{K^i}^{i,\mathcal{I}})_r}$.

This construction ensures $\gamma_k^{i,\mathcal{I}} \geq \mathbf{0}$, satisfying nonnegativity. The complementarity condition $(G_{\mathcal{A}^i}^i z^{i,\mathcal{E}} - g_{\mathcal{A}^i}) \odot \gamma_k^{i,\mathcal{I}} = \mathbf{0}$ holds naturally, because primal feasibility gives $G_{\mathcal{A}^i}^i z^{i,\mathcal{E}} = g_{\mathcal{A}^i}$. Repeat steps 3.2.1 and 3.2.2 for all players $i \in [N]$. Thus, we obtain $\mu^{\mathcal{I}}$ satisfying (\mathcal{I}) such that $z^{\mathcal{I}} = z^{\mathcal{E}}$. Combining Steps 1, 2, and 3 proves the theorem. \square

Proof of Theorem 5.1. Let $y_\rho \in \mathcal{S}_{y_\rho}$. We begin the proof by characterizing how the pseudoinverse $(\nabla \mathcal{K}_\rho(y_\rho))^+$ affects the Newton update. Since $\mathcal{K}_\rho(y_\rho)$ is in the column space of $\nabla \mathcal{K}_\rho(y_\rho)$ for all $y_\rho \in \mathcal{S}_{y_\rho}$, the variable $\Delta y_\rho = (\nabla \mathcal{K}_\rho(y_\rho))^+ (-\mathcal{K}_\rho(y_\rho))$ satisfies the equation $\nabla \mathcal{K}_\rho(y_\rho) \Delta y_\rho = -\mathcal{K}_\rho(y_\rho)$, where the complementarity rows of this equation give for each i^{th} element of the vectors of $\{s_\rho, \gamma_\rho, \Delta s_\rho, \Delta \gamma_\rho\}$:

$$(A.19) \quad s_{\rho,i} \Delta \gamma_{\rho,i} + \gamma_{\rho,i} \Delta s_{\rho,i} = \rho - \gamma_{\rho,i} s_{\rho,i}.$$

We then proceed to prove the linear convergence property in (5.6). It suffices to show the existence of a stepsize $\alpha \in (0, 1]$ satisfying (5.6) by showing that $\hat{\alpha}$ is a stepsize rendering $\|\mathcal{K}_\rho(y_\rho + \hat{\alpha} \Delta y_\rho)\|_2 \leq (1 - \frac{\hat{\alpha}}{2}) \|\mathcal{K}_\rho(y_\rho)\|_2$. Following the fundamental theorem of calculus again, we have, for all stepsize $\alpha \in (0, 1]$,

$$(A.20) \quad \begin{aligned} \|\mathcal{K}_\rho(y_\rho + \alpha \Delta y_\rho)\|_2 &= \left\| \mathcal{K}_\rho(y_\rho) + \int_0^1 \nabla \mathcal{K}_\rho(y_\rho + \tau \alpha \Delta y_\rho) \alpha \Delta y_\rho d\tau \right\|_2 \\ &\leq \|\mathcal{K}_\rho(y_\rho) + \alpha \nabla \mathcal{K}_\rho(y_\rho) \Delta y_\rho\|_2 + \left\| \int_0^1 (\nabla \mathcal{K}_\rho(y_\rho + \tau \alpha \Delta y_\rho) - \nabla \mathcal{K}_\rho(y_\rho)) \alpha \Delta y_\rho d\tau \right\|_2 \\ &\leq \|\mathcal{K}_\rho(y_\rho) - \alpha \mathcal{K}_\rho(y_\rho)\|_2 + \|\alpha \Delta y_\rho\|_2 \cdot \int_0^1 C \|\alpha \tau \Delta y_\rho\|_2 d\tau \\ &\leq (1 - \alpha) \|\mathcal{K}_\rho(y_\rho)\|_2 + \frac{1}{2} \alpha^2 C D^2 \|\mathcal{K}_\rho(y_\rho)\|_2^2. \end{aligned}$$

Let $y_\rho \in \mathcal{S}_{y_\rho}$. Since $\hat{\alpha} \leq \frac{1}{C D^2 L}$ and $\|\mathcal{K}_\rho(y_\rho)\|_2 \leq L$, we have $\frac{C \hat{\alpha} D^2}{2} \|\mathcal{K}_\rho(y_\rho)\|_2 \leq \frac{1}{2}$, and

$$(A.21) \quad \|\mathcal{K}_\rho(y_\rho + \hat{\alpha} \Delta y_\rho)\|_2 \leq (1 - \hat{\alpha}) \|\mathcal{K}_\rho(y_\rho)\|_2 + \frac{1}{2} \hat{\alpha} \|\mathcal{K}_\rho(y_\rho)\|_2 \leq \left(1 - \frac{1}{2} \hat{\alpha}\right) \|\mathcal{K}_\rho(y_\rho)\|_2 \leq L.$$

which establishes (5.6). To complete the proof of $y_\rho + \hat{\alpha} \Delta y_\rho \in \mathcal{S}_{y_\rho}$, we need to show $(s_\rho + \hat{\alpha} \Delta s_\rho) \odot (\gamma_\rho + \hat{\alpha} \Delta \gamma_\rho) \geq \epsilon \mathbf{1}$. For every i^{th} element of the vectors $\{s_\rho, \gamma_\rho, \Delta s_\rho, \Delta \gamma_\rho\}$, using the complementarity rows in (A.19) and substituting the Newton's update,

$$(A.22) \quad (\gamma_{\rho,i} + \hat{\alpha} \Delta \gamma_{\rho,i})(s_{\rho,i} + \hat{\alpha} \Delta s_{\rho,i}) = (1 - \hat{\alpha}) \gamma_{\rho,i} s_{\rho,i} + \hat{\alpha} \rho + \hat{\alpha}^2 \Delta \gamma_{\rho,i} \Delta s_{\rho,i}.$$

From $\Delta y_\rho = -(\nabla \mathcal{K}_\rho(y_\rho))^+ \mathcal{K}_\rho(y_\rho)$, we have $\|\Delta y_\rho\|_2 \leq D \|\mathcal{K}_\rho(y_\rho)\|_2 \leq DL$, and $\|\Delta \gamma_{\rho,i} \Delta s_{\rho,i}\|_2 \leq \|\Delta y_\rho\|_2^2 \leq D^2 L^2$. Since $\gamma_{\rho,i} s_{\rho,i} \geq \epsilon$ and $\hat{\alpha} \leq \frac{\rho - \epsilon}{D^2 L^2}$, we have

$$(A.23) \quad (\gamma_{\rho,i} + \hat{\alpha} \Delta \gamma_{\rho,i})(s_{\rho,i} + \hat{\alpha} \Delta s_{\rho,i}) \geq \epsilon + \hat{\alpha} ((\rho - \epsilon) - \hat{\alpha} D^2 L^2) \geq \epsilon$$

which completes the proof of $y_\rho + \hat{\alpha} \Delta y_\rho \in \mathcal{S}_{y_\rho}$.

In what follows, we show the quadratic convergence of our PDIP method when $\|\mathcal{K}_\rho(y_\rho)\|_2 \leq \min\{\frac{2}{CD^2}, \frac{\sqrt{\rho - \epsilon}}{D}\}$. From (A.20), we have

$$(A.24) \quad \mathcal{K}_\rho(y_\rho + \Delta y_\rho) \leq \frac{1}{2} CD^2 \|\mathcal{K}_\rho(y_\rho)\|_2^2 \leq \frac{1}{2} CD^2 \frac{2}{CD^2} \|\mathcal{K}_\rho(y_\rho)\|_2 = \|\mathcal{K}_\rho(y_\rho)\|_2 \leq L.$$

Moreover, since $\|\Delta s_{\rho,i} \Delta \gamma_{\rho,i}\|_2 \leq D^2 \|\mathcal{K}_\rho(y_\rho)\|_2^2$, we have

$$(A.25) \quad (\gamma_{\rho,i} + \Delta \gamma_{\rho,i})(s_{\rho,i} + \Delta s_{\rho,i}) \geq \epsilon + (\rho - \epsilon) - D^2 \|\mathcal{K}_\rho(y_\rho)\|_2^2 \geq \epsilon$$

and therefore $y_\rho + \Delta y_\rho \in \mathcal{S}_{y_\rho}$.

Observing $\frac{CD^2}{2} \|\mathcal{K}_\rho(y_\rho + \Delta y_\rho)\|_2 \leq \left(\frac{CD^2}{2} \|\mathcal{K}_\rho(y_\rho)\|_2\right)^2 \leq 1$, we have $\|\mathcal{K}_\rho(y_\rho + \Delta y_\rho)\|_2 \leq \min\{\frac{2}{CD^2}, \frac{\sqrt{\rho - \epsilon}}{D}\}$. By induction, this condition holds at all iterates. \square

Proof of Theorem 5.2. Let $\mathbf{1}_\rho$ be defined as a binary vector, where we have ones for the entries corresponding to the complementarity slackness condition.

Since $\mathcal{K}_\rho(y_\rho) = \mathcal{K}_0(y_\rho) - \rho \cdot \mathbf{1}_\rho$, evaluating at y_ρ^* , we have $\mathcal{K}_0(y_\rho^*) = \rho \mathbf{1}_\rho$. By the fundamental theorem of calculus, we have

$$(A.26) \quad \mathcal{K}_0(y_\rho^*) - \mathcal{K}_0(y_0^*) = \underbrace{\int_0^1 \nabla_y \mathcal{K}_0(y_0^* + t(y_\rho^* - y_0^*)) dt}_{\bar{J}} \cdot (y_\rho^* - y_0^*).$$

Since $\nabla_y \mathcal{K}_0$ has constant row rank in a neighborhood of y_0^* , for ρ sufficiently small y_ρ^* lies in this neighborhood and the right hand side of (A.26) has the same constant row rank, with $\|\bar{J}^+\|_2 \leq D_0$ by continuity of the pseudoinverse under constant rank. Multiplying both sides of (A.26) by \bar{J}^+ , we have

$$(A.27) \quad \|y_\rho^* - y_0^*\|_2 \leq \|\bar{J}^+\|_2 \|\rho \mathbf{1}_\rho\|_2 \leq D_0 \sqrt{N_c} \rho \quad \square$$

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