



HAL
open science

On the solution of Scalar Non local Traffic Equation with Random Flux Functions

Francesca Angrisani

► **To cite this version:**

Francesca Angrisani. On the solution of Scalar Non local Traffic Equation with Random Flux Functions. 2026. <hal-05576519>

HAL Id: hal-05576519

<https://hal.science/hal-05576519v1>

Preprint submitted on 1 Apr 2026

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons CC BY 4.0 - Attribution - International License

On the solution of Scalar Non local Traffic Equation with Random Flux Functions

Francesca Angrisani *
Laboratoire Jacques Louis Lions,
Sorbonne Universite,
4 place Jussieu,
75005 Paris (France).

Abstract

We study scalar conservation laws with nonlocal fluxes depending on random parameters, motivated by applications to macroscopic traffic flow models. First, we consider a nonlocal conservation law in which randomness affects the velocity function, the convolution kernel, and the initial datum. We introduce a suitable notion of random solution and prove existence and uniqueness results, together with stability and moment estimates. The analysis relies on the well-posedness theory for deterministic nonlocal equations and on spectral representations of the random inputs via Karhunen–Loève expansions.

In the second part of the paper, we investigate a stochastic nonlocal traffic model driven by Brownian noise. We establish existence and uniqueness of strong solutions for suitable finite-dimensional approximations and pass to the limit to obtain well-posedness results for the full stochastic problem. Several qualitative properties of solutions are also discussed.

Keywords. Nonlocal conservation laws; random fluxes; stochastic partial differential equations; traffic flow models; well-posedness.

AMS Subject Classifications. 35L65, 35R60, 60H15, 90B20.

1 Introduction

In the literature, the chaotic dynamics of vehicles has been the subject of several studies starting from the middle of the previous century. There are various possible approaches for traffic modeling. Those based on differential equations are mainly categorized as microscopic or macroscopic. Microscopic traffic models use ordinary differential equations (ODEs) to describe the dynamics of each individual vehicle in the traffic flow, while macroscopic models use partial differential equations (PDEs) to describe the evolution of macroscopic quantities, such as the density $u = u(t, x)$ and the average velocity $v = v(t, x)$ of cars, as in fluid mechanics.

*E-mail: francesca.angrisani@sorbonne-universite.fr.

The first macroscopic model was introduced by Lighthill–Whitham [1] and independently by Richards [2]. It is based on a single equation in conservation form describing the evolution of traffic density on a unidirectional road. The key assumption is that the velocity v is not an unknown, but it depends in a non-increasing way on the density u . Hence, the resulting model can be described by the conservation law

$$\partial_t u + \partial_x f(u) = 0,$$

where $f(u) = uv(u)$ is the flux. Subsequently, various extensions and generalizations have been proposed; see [3, 4, 5, 6] and the references therein.

More recently, another class of models has attracted some interest in the traffic community, namely the so-called nonlocal models. They have the peculiarity that the velocity function depends in a functional way on the traffic density, trying to mimic the phenomenon that each driver chooses her/his speed according to the distribution of the other vehicles in a neighborhood; see for example [11, 12, 7, 10, 9].

All the previous models are deterministic, in the sense that, once the initial conditions are known, the whole dynamics is exactly determined. However, traffic evolution is affected by several sources of uncertainty, which can be represented by random variables with specified probability distributions. For example, they can describe disturbances in the car velocity and/or in the vehicle distribution, due to inaccuracies in the data or to random behaviors of drivers. The first papers in which stochastic terms are introduced in traffic models are [15, 14, 13].

More recently, hyperbolic conservation laws with noise have been studied also in [17, 18, 19, 20].

In the first part of this paper we consider the stochastic version of the nonlocal equation

$$\partial_t u + \partial_x (uv(u * \eta)) = 0, \tag{1}$$

where η is a kernel function and the convolution $u * \eta$ represents an averaged density. The well-posedness of (1), coupled with the initial condition

$$u(0, x) = u_o(x), \tag{2}$$

is proved in various papers; see [9, 7, 12, 11, 24].

Inspired by [14], we focus on the following Cauchy problem:

$$\begin{cases} \partial_t u + \partial_x (uv(\omega, u * \eta(\omega))) = 0, \\ u(0, x) = u_o(\omega, x), \end{cases} \tag{3}$$

with stochastic initial condition, velocity, and kernel. We introduce the notion of random solution for (3), which requires measurability with respect to the random variable ω . We prove the existence of a unique solution to (3), together with some additional properties. A crucial idea in the proof of our result is the decomposition of the functions v , η and u_o using Karhunen–Loève spectral expansions. This permits us to construct a sequence of approximate solutions which converges to the solution to the Cauchy problem (3). A similar approach is used in [14]. The uniqueness of the solution is a straightforward consequence of the well-posedness of the deterministic model (1).

In [16] the authors consider a similar problem for a nonlocal traffic flow model with a specific random structure, in which the random variable appears only in the velocity function, differently from our case. Here we also allow the dependence on the random variable in the kernel function η and in the initial condition u_o .

In the second part of the paper we consider the stochastic Cauchy problem

$$\begin{cases} \partial_t u + \partial_x(u v(\omega, u * \eta(\omega))) \, dW = 0, \\ u(0, x) = u_o(\omega, x), \end{cases} \quad (4)$$

where W is a one-dimensional Brownian motion defined on a probability space and adapted to some filtration.

We prove the existence and uniqueness of a weak solution for (4), inspired by ...

2 The Nonlocal Problem with Random Terms

Here we consider a conservation law with both nonlocal and random terms. More precisely, we consider the following scalar equation:

$$\partial_t u + \partial_x(u v(\omega, u * \eta(\omega))) = 0, \quad t > 0, x \in \mathbb{R}, \omega \in \Omega, \quad (5)$$

coupled with the initial condition

$$u(0, x) = u_o(\omega, x), \quad x \in \mathbb{R}, \omega \in \Omega. \quad (6)$$

The element $\omega \in \Omega$ is a possible outcome of a random process. In particular, the triplet $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, i.e. Ω is a set (sample space), \mathcal{F} is a σ -algebra on Ω , and \mathbb{P} is a probability measure.

Throughout the paper, for $\omega \in \Omega$, $t \geq 0$ and $x \in \mathbb{R}$, we write

$$(\eta(\omega, \cdot) * u(\omega, t, \cdot))(x) = \int_{\mathbb{R}} \eta(\omega, x - y) u(\omega, t, y) \, dy$$

for the convolution in the space variable.

For the velocity function v in (5) we assume the following hypothesis.

(V) $v : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies:

- (a) for \mathbb{P} -a.e. $\omega \in \Omega$, $v(\omega, \cdot) \in \mathbf{W}^{3,\infty}(\mathbb{R}; \mathbb{R})$;
- (b) the map

$$\begin{array}{ccc} \Omega & \longrightarrow & \mathbf{W}^{3,\infty}(\mathbb{R}; \mathbb{R}) \\ \omega & \longmapsto & v(\omega, \cdot) \end{array}$$

is measurable, where the topological space $\mathbf{W}^{3,\infty}(\mathbb{R}; \mathbb{R})$ is endowed with the Borel σ -algebra.

The convolution kernel η satisfies the assumption:

(η) $\eta : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies:

- (a) for \mathbb{P} -a.e. $\omega \in \Omega$, $\eta(\omega, \cdot) \in \mathbf{W}^{2,\infty}(\mathbb{R}; \mathbb{R})$;

(b) the map

$$\begin{aligned} \Omega &\longrightarrow \mathbf{W}^{2,\infty}(\mathbb{R}; \mathbb{R}) \\ \omega &\longmapsto \eta(\omega, \cdot) \end{aligned}$$

is measurable, where the topological space $\mathbf{W}^{2,\infty}(\mathbb{R}; \mathbb{R})$ is endowed with the Borel σ -algebra.

For the initial condition u_o in (6) we assume the following hypothesis.

(IC) $u_o : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies:

- (a) for \mathbb{P} -a.e. $\omega \in \Omega$, $u_o(\omega, \cdot) \in \mathbf{L}^1(\mathbb{R}; \mathbb{R})$;
- (b) for \mathbb{P} -a.e. $\omega \in \Omega$, $\text{Tot.Var.}(u_o(\omega, \cdot)) < \infty$;
- (c) the map

$$\begin{aligned} \Omega &\longrightarrow \mathbf{L}^1(\mathbb{R}; \mathbb{R}) \\ \omega &\longmapsto u_o(\omega, \cdot) \end{aligned}$$

is measurable, where the topological space $\mathbf{L}^1(\mathbb{R}; \mathbb{R})$ is endowed with the Borel σ -algebra.

Let us introduce the definition of solution for the Cauchy problem (5)–(6).

Definition 2.1 *Given $T > 0$, a map $u : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is said to be a solution to (5)–(6) on the time interval $[0, T]$ if the following conditions are satisfied.*

1. *The map $\omega \mapsto u(\omega, \cdot, \cdot)$ belongs to $\mathbf{C}^0([0, T]; \mathbf{L}^1(\mathbb{R}; \mathbb{R}))$ for \mathbb{P} -a.e. ω and*

$$u : \Omega \rightarrow \mathbf{C}^0([0, T]; \mathbf{L}^1(\mathbb{R}; \mathbb{R}))$$

is a random field, i.e., a measurable function when $\mathbf{C}^0([0, T]; \mathbf{L}^1(\mathbb{R}; \mathbb{R}))$ is endowed with the Borel σ -algebra.

2. *For \mathbb{P} -a.e. $\omega \in \Omega$ and for a.e. $x \in \mathbb{R}$,*

$$u(\omega, 0, x) = u_o(\omega, x).$$

3. *For \mathbb{P} -a.e. $\omega \in \Omega$ and for every test function $\varphi \in \mathbf{C}_c^1((0, T) \times \mathbb{R}; \mathbb{R})$, it holds*

$$\begin{aligned} 0 &= \int_0^T \int_{\mathbb{R}} u(\omega, t, x) \partial_t \varphi(t, x) \, dx \, dt \\ &\quad + \int_0^T \int_{\mathbb{R}} u(\omega, t, x) v(\omega, (\eta(\omega, \cdot) * u(\omega, t, \cdot))(x)) \partial_x \varphi(t, x) \, dx \, dt. \end{aligned} \tag{7}$$

The following well-posedness result holds.

Theorem 1 *Assume hypotheses (V), (η), and (IC). Then, for every $T > 0$, there exists a solution $u : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ to the Cauchy problem (5)–(6) in the sense of Definition 2.1.*

Moreover, for every $k \geq 1$, for every $0 \leq t \leq T$, and for \mathbb{P} -a.e. $\omega \in \Omega$,

$$\|u\|_{\mathbf{L}^k(\Omega; \mathbf{C}^0([0, T]; \mathbf{L}^1(\mathbb{R}; \mathbb{R})))} \leq \|u_o\|_{\mathbf{L}^k(\Omega; \mathbf{L}^1(\mathbb{R}; \mathbb{R}))}, \tag{8}$$

$$\|u(\omega, t)\|_{\mathbf{L}^1(\mathbb{R}; \mathbb{R})} \leq \|u_o(\omega)\|_{\mathbf{L}^1(\mathbb{R}; \mathbb{R})}, \quad (9)$$

$$\text{Tot.Var.}(u(\omega, t, \cdot)) \leq C_1 \text{Tot.Var.}(u_o(\omega, \cdot)) + \exp(C_2 t), \quad (10)$$

and, with \bar{R} as above,

$$\sup_{0 \leq t \leq T} \|u(\omega, t, \cdot)\|_{\mathbf{L}^1(\mathbb{R}; \mathbb{R})} \leq C \bar{R} \quad \mathbb{P}\text{-a.e. } \omega \in \Omega. \quad (11)$$

Proof. We divide the proof into several steps.

Step 1: construction of the candidate solution. We construct the candidate solution for the Cauchy problem (5)–(6), by considering $\omega \in \Omega$ as a fixed parameter.

By (V), (η), and (IC), there exists $\Omega_1 \subseteq \Omega$ such that $\mathbb{P}(\Omega \setminus \Omega_1) = 0$ and, for every $\omega \in \Omega_1$,

$$\begin{aligned} v(\omega, \cdot) &\in \mathbf{C}^3(\mathbb{R}; \mathbb{R}), \\ \eta(\omega, \cdot) &\in (\mathbf{L}^1 \cap \mathbf{W}^{2, \infty})(\mathbb{R}; \mathbb{R}), \quad \text{Tot.Var.}(\eta(\omega, \cdot)) < +\infty, \\ u_o(\omega, \cdot) &\in \mathbf{L}^1(\mathbb{R}; \mathbb{R}), \quad \text{Tot.Var.}(u_o(\omega, \cdot)) < +\infty. \end{aligned}$$

Let us fix $\omega \in \Omega_1$. Then Theorem 3 implies that there exists an ω -dependent semigroup

$$\mathcal{S}^\omega : \mathbb{R}_+ \times \mathcal{D} \longrightarrow \mathcal{D},$$

where

$$\mathcal{D} = \{u \in \mathbf{L}^1(\mathbb{R}; \mathbb{R}) : \text{Tot.Var.}(u) < +\infty\},$$

such that the orbit $t \mapsto \mathcal{S}_t^\omega \bar{u}$ is the solution to the deterministic problem (32) with velocity $v_d = v(\omega, \cdot)$, with kernel $\eta_d = \eta(\omega, \cdot)$, and with initial condition (33) given by $u_{o,d} = \bar{u}$. In this way, we construct a map $u : \Omega_1 \times [0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}$, defined by

$$u(\omega, t, x) = \mathcal{S}_t^\omega(u_o(\omega, \cdot))(x).$$

We extend it to a function

$$u : \Omega \times [0, +\infty) \times \mathbb{R} \longrightarrow \mathbb{R} \quad (12)$$

by setting $u(\omega, t, x) = 0$ for every $\omega \in \Omega \setminus \Omega_1$, $t \geq 0$, and $x \in \mathbb{R}$. Note that, by construction, u satisfies conditions 2 and 3 of Definition 2.1.

Step 2: measurability of the candidate solution.

Consider the function

$$\begin{aligned} H_1 : \quad \mathbf{W}^{3, \infty}(\mathbb{R}; \mathbb{R}) \times \mathbf{W}^{2, \infty}(\mathbb{R}; \mathbb{R}) \times \mathcal{D} &\longrightarrow \mathbf{C}^0([0, T]; \mathbf{L}^1(\mathbb{R}; \mathbb{R})) \\ (v, \eta, \bar{u}) &\longmapsto \mathcal{S}^{v, \eta}(\bar{u}), \end{aligned} \quad (13)$$

where $\mathcal{S}^{v, \eta}$ is the semigroup generated by v and η ; see Theorem 3. By Theorem 3, the map H_1 is locally Lipschitz continuous; hence it is also measurable.

Consider now the map

$$\begin{aligned} H_2 : \quad \Omega &\longrightarrow \mathbf{W}^{3, \infty}(\mathbb{R}; \mathbb{R}) \times \mathbf{W}^{2, \infty}(\mathbb{R}; \mathbb{R}) \times \mathcal{D} \\ \omega &\longmapsto (v(\omega, \cdot), \eta(\omega, \cdot), u_o(\omega, \cdot)), \end{aligned} \quad (14)$$

which is measurable thanks to (V), (η), and (IC).

We have that

$$H_1 \circ H_2(\omega) = u(\omega, \cdot, \cdot). \quad (15)$$

This permits us to conclude that the map $\omega \mapsto u(\omega, \cdot, \cdot)$ is measurable. \square

Let us define

$$U = [-1, 1]^{\mathbb{N}}.$$

Since (40), (39) and (??) hold, for every $J \in \mathbb{N} \setminus \{0\}$ we define the truncations:

$$u_o^J(y, x) = \bar{u}_o^0(x) + \sum_{j=1}^J y_j \bar{u}_o^j(x), \quad y \in U, \quad (16)$$

$$v^J(z, r) = \psi^0(r) + \sum_{j=1}^J z_j \psi^j(r), \quad z \in U, \quad (17)$$

$$\eta^J(h, x) = \zeta^0(x) + \sum_{j=1}^J h_j \zeta^j(x), \quad h \in U. \quad (18)$$

Thanks to [Theorem 3](#), for every $y \in U$, $z \in U$, $h \in U$, we can consider the solution of the deterministic Cauchy problem

$$\begin{cases} \partial_t u + \partial_x(u v^J(z, u * \eta^J(h))) = 0, \\ u(0, x) = u_o^J(y, x). \end{cases} \quad (19)$$

We denote this solution by $u^J = u^J(y, z, h, t, x)$.

The functions (16), (17), (18) converge respectively to

$$\bar{u}_o(y, x) = \bar{u}_o^0(x) + \sum_{j=1}^{+\infty} y_j \bar{u}_o^j(x), \quad y \in U, \quad (20)$$

$$\bar{v}(z, r) = \psi^0(r) + \sum_{j=1}^{+\infty} z_j \psi^j(r), \quad z \in U, \quad (21)$$

$$\bar{\eta}(h, x) = \zeta^0(x) + \sum_{j=1}^{+\infty} h_j \zeta^j(x), \quad h \in U, \quad (22)$$

in the spaces \mathbf{L}^1 , $\mathbf{W}^{3,\infty}$, and $\mathbf{W}^{2,\infty}$, respectively, uniformly with respect to y , z , h .

For every $y \in U$, $z \in U$, $h \in U$, we define $\tilde{u} = \tilde{u}(y, z, h, t, x)$ as the solution of

$$\begin{cases} \partial_t u + \partial_x(u \bar{v}(z, u * \bar{\eta}(h))) = 0, \\ u(0, x) = \bar{u}_o(y, x). \end{cases} \quad (23)$$

By (S2), (S4), and (S5) we deduce that, for every $y \in U$, $z \in U$, $h \in U$, and $t > 0$,

$$\lim_{J \rightarrow +\infty} \|u^J(y, z, h, t, \cdot) - \tilde{u}(y, z, h, t, \cdot)\|_{\mathbf{L}^1(\mathbb{R}; \mathbb{R})} = 0. \quad (24)$$

Let us now consider the stochastic Cauchy problem (3) and let us take a sequence $\{U_j\}_{j \in \mathbb{N}}$ such that $U_j \subseteq [-1, 1]^3$ for all $j \in \mathbb{N}$, U_k is a proper subset of $[-1, 1]^3$ only for a finite number of indices k , and

$$U = \prod_{j=1}^{\infty} U_j.$$

Let \mathcal{G} be the corresponding product σ -algebra. Let us set $s := (y, z, h)$ and consider the probability measure

$$d\sigma(s) := \bigotimes_{j \geq 1} \frac{ds_j}{2},$$

i.e., the product of the normalized Lebesgue measures on the components.

By assumption the random coordinates s_j are independent and identically distributed, uniform on U_j , and for $U = \prod_{j=1}^{\infty} U_j$ we have

$$\sigma(U) = \prod_{j=1}^{\infty} \mathbb{P}\{\omega : s_j(\omega) \in U_j\}.$$

We take $y_j = Y_j(\omega)$, $z_j = Z_j(\omega)$, $h_j = H_j(\omega)$, and define $S_j(\omega) = (Y_j(\omega), Z_j(\omega), H_j(\omega))$ for every $j \in \mathbb{N}$. Let $S(\omega) := (S_j(\omega))_{j \geq 1} \in U$ and define

$$\hat{u}^J(\omega, t, x) := u^J(S(\omega), t, x). \quad (25)$$

Note that the map $\omega \mapsto \hat{u}^J(\omega, \cdot, \cdot)$ is measurable, as it is a composition of measurable maps. Moreover $\{\hat{u}^J\}_{J \in \mathbb{N}}$ is a Cauchy sequence in $\mathbf{C}^0([0, T]; \mathbf{L}^1(\mathbb{R}; \mathbb{R}))$ (which is separable, hence we can apply Lemma 1), and therefore the limit \hat{u} is measurable from (Ω, \mathcal{G}) into

$$(\mathbf{C}^0([0, T]; \mathbf{L}^1(\mathbb{R}; \mathbb{R})), \mathcal{B}(\mathbf{C}^0([0, T]; \mathbf{L}^1(\mathbb{R}; \mathbb{R}))).$$

Step 3: \hat{u} is the unique solution of the Cauchy problem (3). By construction, the functions $u^J(S(\omega), t, x)$ satisfy condition (7) for $S(\omega) \in U$, \mathbb{P} -a.s.

Passing to the limit as $J \rightarrow \infty$ in $\mathbf{C}^0([0, T]; \mathbf{L}^1(\mathbb{R}; \mathbb{R}))$, thanks to the stability and the uniqueness of entropy solutions, we obtain the claim.

Step 4: the estimates (8), (9), (10), (11).

- Let us prove (8). We have

$$\begin{aligned} \|u\|_{\mathbf{L}^k(\Omega; \mathbf{C}^0([0, T]; \mathbf{L}^1(\mathbb{R}; \mathbb{R})))}^k &= \int_{\Omega} \|u(\omega)\|_{\mathbf{C}^0([0, T]; \mathbf{L}^1)}^k d\omega \\ &= \int_{\Omega} \left(\sup_{t \in [0, T]} \|u(\omega, t)\|_{\mathbf{L}^1} \right)^k d\omega \\ &= \int_{\Omega} \left(\sup_{t \in [0, T]} \int_{\mathbb{R}} |u(\omega, t, x)| dx \right)^k d\omega \\ &\leq \int_{\Omega} \left(\sup_{t \in [0, T]} \int_{\mathbb{R}} |u_o(\omega, x)| dx \right)^k d\omega \end{aligned}$$

$$\begin{aligned}
&= \int_{\Omega} \left(\int_{\mathbb{R}} |u_o(\omega, x)| \, dx \right)^k \, d\omega \\
&= \|u_o\|_{\mathbf{L}^k(\Omega; \mathbf{L}^1(\mathbb{R}; \mathbb{R}))}^k.
\end{aligned}$$

- Estimate (9) follows directly from 35, observing that if $u_o = 0$ then the unique solution of (32) is $u \equiv 0$.
- Thanks to (S6) there exist two constants $C_1, C_2 > 0$ such that (10) holds true.
- Let us prove (11). We have

$$\begin{aligned}
\sup_{0 \leq t \leq T} \|u(\omega, t, \cdot)\|_{\mathbf{L}^1(\mathbb{R}; \mathbb{R})} &= \sup_{0 \leq t \leq T} \|\mathcal{S}_t u_o(\omega, \cdot)\|_{\mathbf{L}^1(\mathbb{R}; \mathbb{R})} \\
&\leq C \|u_o(\omega, \cdot)\|_{\mathbf{L}^1(\mathbb{R}; \mathbb{R})} \\
&\leq C \bar{R}, \quad \mathbb{P}\text{-a.e. } \omega \in \Omega,
\end{aligned}$$

where 35 has been used. □

3 The Nonlocal Traffic SPDE

Here we consider the following SPDE Cauchy problem:

$$\begin{cases} \partial_t u + \partial_x (u v(\omega, u * \eta(\omega))) \, dW_t = 0, \\ u(0, x) = u_o(\omega, x), \end{cases} \quad (26)$$

where $W = (W_t)_{t \geq 0}$ is a one-dimensional Brownian motion defined on a probability space and adapted to some filtration.

Let us consider a filtration $\{F_t\}_{t \in [0, T]}$ that is complete and right-continuous. We call

$$\mathcal{S} := (\Omega, \mathcal{F}, \{F_t\}_{t \geq 0}, \mathbb{P})$$

a *stochastic basis*.

The “weak” (distributional) solutions in the PDE sense in the Sobolev spaces for (5), defined by (7), are pathwise (probabilistically) strong.

This is because weak/strong probabilistic solutions have a different meaning from weak/strong solutions in the PDE literature. In probability theory, the so-called martingale solutions are considered “weak” solutions.

If we fix a priori a stochastic basis \mathcal{S} and a Brownian motion W on it, we speak of a strong (or pathwise) solution. If (\mathcal{S}, W) is part of the unknown, the relevant notion is that of a martingale solution.

Definition 3.1 *The triple (\mathcal{S}, u, W) is a weak (or martingale) \mathbf{H}^m -solution of the SPDE (26) with initial distribution Λ if the following conditions hold:*

1. $\mathcal{S} = (\Omega, \mathcal{F}, \{F_t\}_{t \geq 0}, \mathbb{P})$ is a stochastic basis;
2. W is a standard Wiener (Brownian motion) process on \mathcal{S} ;
3. $u : \Omega \times [0, T] \rightarrow \mathbf{H}^1(\mathbb{R})$ is adapted, with

$$u \in \mathbf{L}_0^p(\Omega; \mathbf{C}^0([0, T]; \mathbf{H}^1(\mathbb{R}))) \quad \text{and} \quad u \in \mathbf{L}^2([0, T]; \mathbf{H}^m(\mathbb{R})) \quad \mathbb{P}\text{-a.s.},$$

for some $p_0 > 4$. Moreover,

$$\begin{cases} u \in \mathbf{L}^2(\Omega; \mathbf{L}^2([0, T]; \mathbf{L}^2(\mathbb{R}))) & \text{if } m = 1, \\ u \in_{\text{sb}} \mathbf{L}^2([0, T]; \mathbf{H}^{m+1}(\mathbb{R})) \cap \mathbf{L}^\infty([0, T]; \mathbf{H}^m(\mathbb{R})) & \text{if } m \geq 2, \end{cases} \quad (27)$$

where \in_{sb} means stochastically bounded,¹

4. the law of the initial data $u_0 := u(0)$ on $\mathbf{H}^m(\mathbb{R})$ is Λ , i.e., $(u(0))_* \mathbb{P} = \Lambda$, that is, $\Lambda(A) = \mathbb{P}((u(0))^{-1}(A))$ for measurable sets A ;
5. for all $t \in [0, T]$ and all $\varphi \in \mathbf{C}_c^1(\mathbb{R})$, the following equation holds \mathbb{P} -almost surely:

$$\int_0^T \int_{\mathbb{R}} \left[u(\omega, t, x) \frac{\partial \varphi}{\partial t}(t, x) + u(\omega, t, x) v(\eta * u(\omega, t, \cdot))(x) \frac{\partial \varphi}{\partial x}(t, x) \right] dx dt dW_t = 0. \quad (28)$$

Definition 3.2 Fix a stochastic basis \mathcal{S} , and a Wiener process W defined on \mathcal{S} . Fix $m \in \mathbb{N}$ and $p_0 > 4$, and consider a random variable $u_0 \in \mathbf{L}_0^p(\Omega; \mathbf{H}^1(\mathbb{R}))$. A process u , defined relative to \mathcal{S} , is a strong \mathbf{H}^m solution to (26) if (\mathcal{S}, u, W) is a weak \mathbf{H}^m -solution to (26) with initial law $\Lambda := (u_0)_* \mathbb{P}$.

3.1 Existence and uniqueness of a strong solution

Theorem 2 Let us consider the problem

$$\begin{cases} \partial_t u + \partial_x(u v^J(z, u * \eta^J(h))) dW_t = 0, \\ u(0, x) = u_0^J(y, x). \end{cases} \quad (29)$$

For every $J \in \mathbb{N}$ there exists a unique $\mathbf{L}^1 \cap \mathbf{L}^\infty$ -valued adapted process u^J that is a strong solution to problem (29).

Proof. First of all let us notice that (29) is an SPDE system with coefficients that are locally Lipschitz continuous on the finite-dimensional space S_J spanned by the truncation.

By a standard well-posedness theorem for SDEs (see, e.g., [21, Thm. IX.2.1]), this immediately implies the existence and uniqueness of a continuous (strong) solution of (29) on $[0, T]$.

It remains to prove that system (29) can be seen as an SPDE in S_J . Since, for every J , $u^J \in \mathbf{L}^2([0, T]; \mathbb{R})$, using the Karhunen–Loève expansion we obtain

$$u^J(\omega, x, t) = \mathbb{E}(u^J) + \sum_{j=1}^J \alpha_j(\omega, t) \gamma_j^R(x), \quad (30)$$

¹A sequence $\{v_n\}$ of B -valued random variables is stochastically bounded in B if $\mathbb{P}(\|v_n\|_B > R) \rightarrow 0$ as $R \rightarrow \infty$, uniformly in n .

where, for a fixed $0 < R < +\infty$, $\{\gamma_j^R\}_{j=1}^J$ is an orthonormal basis of $\mathbf{L}^2((-R, R); \mathbb{R})$.

Since the $\{\gamma_j^R\}_j$ are eigenfunctions, we have:

•

$$\partial_t u^J(\omega, x, t) = \sum_{j=1}^J \frac{\partial \alpha_j}{\partial t}(\omega, t) \gamma_j^R(x);$$

• for the convolution,

$$\begin{aligned} (u^J * \eta^J(h, \cdot))(x) &= \int_{B_\varepsilon(x)} u^J(\omega, k, t) \eta^J(h, k - x) dk \\ &= \sum_{j=1}^J \alpha_j(\omega, t) \int_{B_\varepsilon(x)} \gamma_j^R(k) \eta^J(h, k - x) dk; \end{aligned}$$

• for the derivative of the flux, we write

$$\begin{aligned} \partial_x (u^J v^J(z, u^J * \eta^J(h, \cdot))) &= \partial_x u^J(\omega, x, t) v^J(z, (u^J * \eta^J(h, \cdot))(x)) \\ &\quad + u^J(\omega, x, t) \partial_x v^J(z, (u^J * \eta^J(h, \cdot))(x)). \end{aligned}$$

Using the expansions above and the regularity of $\psi^0, \psi^j, \zeta^0, \zeta^j$, one obtains expressions of the form

$$\begin{aligned} \partial_x u^J(\omega, x, t) v^J(z, u^J * \eta^J(h, \cdot)) \\ = \sum_{j=1}^J C_j \alpha_j(\omega, t) \gamma_j^R(x) \Psi_1(\alpha(\omega, t), z, h, x), \end{aligned}$$

and

$$\begin{aligned} u^J(\omega, x, t) \partial_x v^J(z, u^J * \eta^J(h, \cdot)) \\ = \sum_{j=1}^J \alpha_j(\omega, t) \gamma_j^R(x) \Psi_2(\alpha(\omega, t), z, h, x), \end{aligned}$$

for suitable smooth functions Ψ_1, Ψ_2 .

Thanks to the regularity assumptions on the data of the problem and to the fact that the eigenfunctions $\{\gamma_j^R\}_j$ can be chosen in \mathbf{C}^∞ , every term of the SPDE in (29) is locally Lipschitz in the coefficients $\{\alpha_j\}_{j=1}^J$.

□

Corollary 1 *There exists a unique \mathbf{L}^1 -valued adapted process u that is a strong solution to problem (26).*

Proof. Thanks to Theorem 2 there exists, for every J , a unique sequence of coefficients $\{\alpha_j\}_{j=1}^J$ and a unique solution $u^J(\omega, t, x) \in \mathbf{L}^1 \cap \mathbf{L}^2 \cap \mathbf{L}^\infty$ of problem (29), with

$$u^J(\omega, x, t) = \mathbb{E}(u^J) + \sum_{j=1}^J \alpha_j(\omega, t) \gamma_j^R(x). \quad (31)$$

The map $\omega \mapsto u^J(\alpha(\omega, t), x)$ is measurable, as it is a composition of measurable maps, and $\{u^J\}_{J \in \mathbb{N}}$ is a Cauchy sequence in $\mathbf{C}^0([0, T]; \mathbf{L}^1(\mathbb{R}; \mathbb{R}))$ (which is separable, hence we can apply Lemma 1). Therefore the limit \hat{u} is measurable from (Ω, \mathcal{G}') into $(\mathbf{C}^0([0, T]; \mathbf{L}^1(\mathbb{R}; \mathbb{R})), \mathcal{B}(\mathbf{C}^0([0, T]; \mathbf{L}^1(\mathbb{R}; \mathbb{R})))$, where \mathcal{G}' is a suitable σ -algebra (it can be constructed as in Theorem 1).

Passing to the limit as $J \rightarrow \infty$ in $\mathbf{C}^0([0, T]; \mathbf{L}^1(\mathbb{R}; \mathbb{R}))$, thanks to the stability and to the uniqueness of a strong solution of (26), we obtain the thesis.

□

4 Preliminaries

This section contains some preliminary results about the theory developed in the paper.

4.1 The deterministic nonlocal case

Here we consider the deterministic nonlocal equation

$$\partial_t u + \partial_x (u v_d(u * \eta_d)) = 0, \quad t > 0, x \in \mathbb{R}, \quad (32)$$

coupled with the initial condition

$$u(0, x) = u_{o,d}(x), \quad x \in \mathbb{R}. \quad (33)$$

For the velocity function v_d and for the convolution kernel η_d in (32) we introduce the following assumptions.

(V_d) $v_d \in \mathbf{W}^{3,\infty}(\mathbb{R}; \mathbb{R})$.

(η_d) $\eta_d \in \mathbf{W}^{2,\infty}(\mathbb{R}; \mathbb{R})$.

Let us introduce the definition of solution for the Cauchy problem (32)–(33).

Definition 4.1 *Given $T > 0$, a function $u \in \mathbf{C}^0([0, T]; \mathbf{L}^1(\mathbb{R}; \mathbb{R}))$ is a solution of the nonlocal deterministic Cauchy problem (32)–(33) on the time interval $[0, T]$ if*

1. $u(0, x) = u_{o,d}(x)$ for a.e. $x \in \mathbb{R}$;
2. for all test functions $\varphi \in \mathbf{C}_c^1((0, T) \times \mathbb{R}; \mathbb{R})$, it holds

$$\int_0^T \int_{\mathbb{R}} [u(t, x) \partial_t \varphi(t, x) + u(t, x) v_d((\eta_d * u(t, \cdot))(x)) \partial_x \varphi(t, x)] dx dt = 0. \quad (34)$$

The following well posedness result holds; see [24] and [8] for the proof.

Theorem 3 *Assume (V_d) and (η_d). Then, there exist a domain*

$$\mathcal{D} = \{u \in \mathbf{L}^1(\mathbb{R}; \mathbb{R}) : \text{Tot. Var.}(u) < +\infty\},$$

and a unique semigroup $\mathcal{S}: \mathbb{R}_+ \times \mathcal{D} \rightarrow \mathcal{D}$ with the following properties:

(S1) For every $u_o \in \mathcal{D}$ and $T > 0$, the orbit $t \mapsto \mathcal{S}_t u_o$ is the unique global solution to (32) with initial datum (33) on $[0, T]$ in the sense of Definition 4.1.

(S2) \mathcal{S} is locally Lipschitz continuous with respect to the initial datum: for all $T > 0$ and $\hat{u}_o \in \mathcal{D}$, there exists a positive constant C , depending on η_d , v_d , T , $\|\hat{u}_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})}$, and $\text{Tot.Var.}(\hat{u}_o)$, such that, for every $t \in [0, T]$ and $\check{u}_o \in \mathcal{D}$,

$$\|\mathcal{S}_t \check{u}_o - \mathcal{S}_t \hat{u}_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})} \leq C \|\check{u}_o - \hat{u}_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})}. \quad (35)$$

(S3) \mathcal{S} is locally Lipschitz continuous in time: for all $T > 0$ and $u_o \in \mathcal{D}$, there exists a positive constant $C > 0$, depending on η_d , v_d , T , $\|u_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})}$, and $\text{Tot.Var.}(u_o)$, such that, for every $t_1, t_2 \in [0, T]$,

$$\|\mathcal{S}_{t_1} u_o - \mathcal{S}_{t_2} u_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})} \leq C |t_1 - t_2|. \quad (36)$$

(S4) Let $v_1, v_2 \in \mathbf{W}^{3,\infty}(\mathbb{R};\mathbb{R})$ and call $\mathcal{S}^1, \mathcal{S}^2$ the corresponding semigroups. For all $T > 0$, $u_o \in \mathcal{D}$, there exists a positive constant $C > 0$, depending on η_d , v_1 , T , $\|u_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})}$, and $\text{Tot.Var.}(u_o)$, such that, for every $t \in [0, T]$,

$$\|\mathcal{S}_t^1 u_o - \mathcal{S}_t^2 u_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})} \leq C \|v_1' - v_2'\|_{\mathbf{W}^{2,\infty}(\mathbb{R};\mathbb{R})}. \quad (37)$$

(S5) Let $\eta_1, \eta_2 \in \mathbf{W}^{2,\infty}(\mathbb{R};\mathbb{R})$ and call $\mathcal{S}^1, \mathcal{S}^2$ the corresponding semigroups. For all $T > 0$, $u_o \in \mathcal{D}$, there exists a positive constant $C > 0$, depending on η_1 , η_2 , v_d , T , $\|u_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})}$, and $\text{Tot.Var.}(u_o)$, such that, for every $t \in [0, T]$,

$$\|\mathcal{S}_t^1 u_o - \mathcal{S}_t^2 u_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})} \leq C \|\eta_1 - \eta_2\|_{\mathbf{W}^{2,\infty}(\mathbb{R};\mathbb{R})}. \quad (38)$$

(S6) For all $t \in \mathbb{R}_+$ and $u_o \in \mathcal{D}$, there exists a positive constant $C > 0$, depending on η_d , v_d , T , and on $\|u_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})}$, such that, for every $t \in [0, T]$,

$$\text{Tot.Var.}(\mathcal{S}_t u_o) \leq \sqrt{n} [\text{Tot.Var.}(u_o) + Ct \exp(Ct)].$$

(S7) If $u_o \geq 0$, then, for all $t \in \mathbb{R}_+$,

$$u(t, x) \geq 0$$

for a.e. $x \in \mathbb{R}$.

4.2 Some technical tools

Other tools which are useful to prove our main result are the following lemmas.

Lemma 1 ([15, Lemma 2.1]) Let E be a separable Banach space and let $X : \Omega \rightarrow E$ be an E -valued random variable on (Ω, \mathcal{F}) . Then the mapping

$$\Omega \ni \omega \longmapsto \|X(\omega)\|_E \in \mathbb{R}$$

is measurable.

Lemma 2 Let us assume that the hypotheses (H1), (H2), (H3) hold true. Then:

1. the velocity function v admits the expansion

$$v(\omega, r) = \psi^0(r) + \sum_{j=1}^{+\infty} V^j(\omega) \psi^j(r), \quad (39)$$

where $\psi^j \in W^{3,\infty}(\mathbb{R}; \mathbb{R})$ for every $j \in \mathbb{N}$ and $V^j(\omega) \in [-1, 1]$ for every $j \in \mathbb{N} \setminus \{0\}$;

2. the kernel function η admits the expansion

$$\eta(\omega, x) = \zeta^0(x) + \sum_{j=1}^{+\infty} A^j(\omega) \zeta^j(x), \quad (40)$$

where $\zeta^j \in W^{2,\infty}(\mathbb{R}; \mathbb{R})$ for every $j \in \mathbb{N}$ and $A^j(\omega) \in [-1, 1]$ for every $j \in \mathbb{N} \setminus \{0\}$;

3.

$$u_o \in L^0(\Omega, \mathcal{F}, \mathbb{P}; L^1(\mathbb{R})). \quad (41)$$

Proof. Since $v, \eta \in L^\infty(\Omega; C^1(\mathbb{R}; \mathbb{R})) \cap L^2(\Omega; C^1(\mathbb{R}; \mathbb{R}))$, it is possible to consider a Karhunen–Loève expansion for v and η (see [Section A](#)) and to obtain (39) and (40).

Property (41) follows from the fact that u_o is a measurable map from Ω to $L^2(\mathbb{R})$ (and hence to $L^1(\mathbb{R})$ under our assumptions). \square

Now we are ready to prove our main theorem.

5 Some properties

Theorem 4 *If u satisfies (3), then the following properties hold true:*

1. for a.e. $t \in [0, T]$,

$$\mathbb{E} \left(\int_{\mathbb{R}} u(\omega, x, t) \, dx \right) = \mathbb{E} \left(\int_{\mathbb{R}} u_o(\omega, x) \, dx \right); \quad (42)$$

2. for a.e. $t \in [0, T]$, there exists a constant $C > 0$, independent of t , such that

$$\mathbb{E} \left(\left(\int_{\mathbb{R}} u(\omega, x, t) \, dx - \mathbb{E} \left(\int_{\mathbb{R}} u(\omega, x, t) \, dx \right) \right)^k \right) \leq C \mathbb{E} \left(\int_{\mathbb{R}} |u_o(\omega, x)|^k \, dx \right). \quad (43)$$

Proof.

1. Thanks to [24] and to our main theorem 1, we can deduce an explicit representation formula for the solution of (3), namely

$$u(\omega, t, x) = u_o(\omega, X(\omega, t, 0, x)) \exp \left(- \int_0^t (\partial_r v)(u(\omega, s, \cdot)) * \eta(\omega, \cdot)(x) \, ds \right), \quad (44)$$

where the function $t \mapsto X(\omega, t, 0, x)$ is the solution of the Cauchy problem

$$\begin{cases} \frac{dX}{dt}(t) = v(t, X(t)), & t \in (0, T], \\ X(0) = x_0. \end{cases} \quad (45)$$

Thanks to property $(\Sigma 3)$ and Lemma 4.6 in [24], together with the explicit formula above, we obtain

$$\|u(\omega, t)\|_{L^1(\mathbb{R}; \mathbb{R})} = \|u_o(\omega)\|_{L^1(\mathbb{R}; \mathbb{R})}. \quad (46)$$

Integrating with respect to Ω , we conclude the proof of property (1).

2. Let us now prove property (2). Thanks to (1), we have, for a.e. $t \in [0, T]$,

$$\begin{aligned} & \mathbb{E} \left(\left(\int_{\mathbb{R}} u(\omega, x, t) \, dx - \mathbb{E} \left(\int_{\mathbb{R}} u(\omega, x, t) \, dx \right) \right)^k \right) \\ &= \mathbb{E} \left(\left(\int_{\mathbb{R}} u(\omega, x, t) \, dx - \mathbb{E} \left(\int_{\mathbb{R}} u_o(\omega, x) \, dx \right) \right)^k \right) \\ &\leq \int_{\Omega} \left| \int_{\mathbb{R}} (u(\omega, x, t) - \mathbb{E}(u_o(\cdot, x))) \, dx \right|^k \, d\omega \\ &\leq \int_{\mathbb{R}} \sup_{t \in [0, T]} \|u(\cdot, x, t) - \mathbb{E}(u_o(\cdot, x))\|_{L^1(\Omega; \mathbb{R})}^k \, dx \\ &\leq \int_{\mathbb{R}} \sup_{t \in [0, T]} \|u(\cdot, x, t) - \mathbb{E}(u_o(\cdot, x))\|_{L^k(\Omega; \mathbb{R})}^k \, dx \\ &\leq C \int_{\mathbb{R}} \int_{\Omega} |u_o(\omega, x)|^k \, d\omega \, dx = C \mathbb{E} \left(\int_{\mathbb{R}} |u_o(\omega, x)|^k \, dx \right), \end{aligned}$$

for a suitable constant $C > 0$ depending only on the a priori estimates established in [Theorem 1](#).

□

Theorem 5 *Let us suppose $u \in H^1(\mathbb{R})$ and that u is the unique solution of (3). Then the following equality holds true:*

$$\mathbb{E}(\|u(\omega, t)\|_{H^1(\mathbb{R})}) = \mathbb{E}(\|u_o(\omega)\|_{H^1(\mathbb{R})}) \quad (47)$$

for a.e. $t \in [0, T]$.

Proof. By an explicit computation starting from formula (44), we obtain

$$\|u(\omega, \cdot, t)\|_{L^2(\mathbb{R})} = \|u_o(\omega, \cdot)\|_{L^2(\mathbb{R})}.$$

Hence, for a.e. $t \in [0, T]$,

$$\mathbb{E}(\|u(\omega, t)\|_{L^2(\mathbb{R})}) = \mathbb{E}(\|u_o(\omega)\|_{L^2(\mathbb{R})}). \quad (48)$$

By a Morrey-type embedding, we also have $u \in C^1$ in space under our assumptions. Differentiating (44) with respect to x and computing the $L^2(\mathbb{R})$ -norm of $\partial_x u$ we obtain

$$\|\partial_x u(\omega, \cdot, t)\|_{L^2(\mathbb{R})} = \|\partial_x u_o(\omega, \cdot)\|_{L^2(\mathbb{R})}.$$

Therefore, for a.e. $t \in [0, T]$,

$$\mathbb{E}(\|\partial_x u(\omega, t)\|_{L^2(\mathbb{R})}) = \mathbb{E}(\|\partial_x u_o(\omega, \cdot)\|_{L^2(\mathbb{R})}), \quad (49)$$

and the claim follows combining the L^2 and gradient parts. \square

5.1 Martingale solutions

First of all, we recall the following classical result.

Theorem 6 (Skorokhod–Jakubowski a.s. representations) *Let $(\mathcal{X}, \tau, \mathcal{B}_\tau)$ be a quasi-Polish space,² and denote by $\Sigma_f \subset \mathcal{B}_\tau$ the σ -algebra generated by a sequence $\{f_i\}$ of continuous functions that separate points. Then:*

1. every τ -compact subset of \mathcal{X} is metrizable;
2. every Borel subset of a σ -compact set in \mathcal{X} belongs to Σ_f ;
3. every probability measure supported by a σ -compact set in \mathcal{X} has a unique Radon extension to the Borel σ -algebra $\mathcal{B}_\tau = \mathcal{B}(\mathcal{X})$.

Moreover, if $\{\mu_J\}$ is a tight sequence of probability measures on (\mathcal{X}, Σ_f) , then there exist a subsequence $\{J_k\}_k$, a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, and Borel measurable \mathcal{X} -valued random variables \tilde{v}_k, \tilde{v} such that μ_{J_k} is the law of \tilde{v}_k and $\tilde{v}_k \rightarrow \tilde{v}$ $\tilde{\mathbb{P}}$ -a.s. in \mathcal{X} . Besides, the law μ of \tilde{v} is a Radon measure on \mathcal{B}_τ .

Proof. See [22, pp. 169–173]. \square

Theorem 7 (Tightness of probability laws) *Let us suppose that $\mathbb{E}(\|u_o\|_{H^1(\mathbb{R})}^p) < +\infty$ for some $p > 2$ and let $\theta' \in (0, \frac{2-p}{4p})$.*

Then, for each $J \in \mathbb{N}$, let u_J be a solution to (29) with $\mathbb{E}(\|u_o\|_{H^1(\mathbb{R})}^p) < +\infty$. Then the laws of $\{u_J\}$ are tight on

$$L^2([0, T]; H^1(\mathbb{R})) \times C([0, T]; H_w^1(\mathbb{R})) \times C([0, T]) \times H^1(\mathbb{R}).$$

Proof. Lemma 5.3 and Lemma 5.5 in [17] remain valid in our framework, since we have all the needed a priori estimates at disposal. \square

We need the following variant of the Aubin–Lions lemma [?, Thm. 2.1] to establish tightness on $L^2([0, T]; H^1(\mathbb{R}))$.

²A topological space with a countable basis and completely quasi-metrizable.

Lemma 3 Let $B_0 \subseteq B \subseteq B_1$ be Banach spaces, with B_0 and B_1 reflexive, and with compact embedding of B_0 into B . Fix $p \in (1, \infty)$ and $\alpha \in (0, 1)$.

Let

$$\mathcal{Y} = L^p([0, T]; B_0) \cap W^{\alpha, p}([0, T]; B_1),$$

endowed with the natural norm. Then the embedding of \mathcal{Y} into $L^p([0, T]; B)$ is compact.

Lemma 4 For each $J \in \mathbb{N}$, let u_J be a solution to (29) with $\mathbb{E}(\|u_o\|_{H^1(\mathbb{R})}^2) < +\infty$.

There exists a constant $C = C(T, \mathbb{E}(\|u_o\|_{H^1(\mathbb{R})}^2))$, independent of J and ε , such that

$$\mathbb{E}\left(\|u_J\|_{L^\infty([0, T]; H^1(\mathbb{R}))}^2\right) + \varepsilon \mathbb{E}\left(\int_0^T \|\partial_x u_J\|_{L^2(\mathbb{R})}^2 dt\right) \leq C.$$

Proof. Since $\mathbb{E}(\|u_o\|_{H^1(\mathbb{R})}^2) < +\infty$ and thanks to Theorem 5, $\sup_{t \in [0, T]} \|u_J(t)\|_{H^1(\mathbb{R})}^2 = \sup_{t \in [0, T]} \|u_{J, o}\|_{H^1(\mathbb{R})}^2$, there exists $C_1 > 0$ such that

$$\mathbb{E}\left(\|u_J\|_{L^\infty([0, T]; H^1(\mathbb{R}))}^2\right) < C_1.$$

Let us compute

$$\mathbb{E}\left(\int_0^T \|\partial_x u_J\|_{L^2(\mathbb{R})}^2 dt\right).$$

We have

$$\mathbb{E}\left(\int_0^T \|\partial_x u_J\|_{L^2(\mathbb{R})}^2 dt\right) = \mathbb{E}\left(\int_0^T \sum_{j=1}^J C_j^2 \alpha_j^2(\omega, t) \|\partial_x \psi_j\|_{L^2(\mathbb{R})}^2 dt\right). \quad (50)$$

It remains to show that, for every j , there exist $\varepsilon > 0$ and $C_2 > 0$ such that

$$\varepsilon \mathbb{E}\left(\int_0^T \alpha_j^2(\omega, t) dt\right) < C_2.$$

Lemma 5 (Temporal L^2 continuity) For each $J \in \mathbb{N}$, let u_J be a solution to (29) with $\mathbb{E}(\|u_o\|_{H^1(\mathbb{R})}^p) < +\infty$ for some $p > 2$. For any $\theta \in [0, \frac{p-2}{4p})$, there exists a constant

$$C = C(T, p, \theta, \mathbb{E}(\|u_o\|_{H^1(\mathbb{R})}^{4p})),$$

independent of J and ε , such that

$$\mathbb{E}(\|u_J\|_{C^\theta([0, T]; L^2(\mathbb{R}))}^{2p}) \leq C.$$

Proof. The thesis follows from the explicit representation

$$u_J(\omega, t, x) = u_{J, o}(\omega, X_J(\omega, 0, x)) \exp\left(-\int_0^t \partial_r v_J(u_J(\omega, s, \cdot)) * \eta_J(\omega, \cdot)(x) ds\right), \quad (51)$$

combined with the H^1 -estimates and standard interpolation arguments in time.

□

We denote by $\mu_J^{u,s}, \mu_J^{u,w}, \mu_J^W$ and μ_0^J the laws of u_J (in its strong topology), u_J (in its weak topology), W and $u_{J,o}$ on $L^2([0, T]; H^1(\mathbb{R}))$, $C([0, T]; H_w^1(\mathbb{R}))$, $C([0, T])$ and $H^1(\mathbb{R})$ respectively. (The subscripts “s” and “w” refer to the “strong” and “weak” topologies used in the subscripted path spaces and laws defined on them.) Note carefully that we have used two copies of u_J in the separate spaces $L^2([0, T]; H^1(\mathbb{R}))$ and $C([0, T]; H_w^1(\mathbb{R}))$ that do not inject continuously into one another. The aim of this manoeuvre is to ensure convergence in two separate topologies of the Skorokhod–Jakubowski representations of u_J .

Theorem 8 *Let us consider*

$$\mathcal{X} = L^2([0, T]; H^1(\mathbb{R})) \times C([0, T]; H_w^1(\mathbb{R})) \times C([0, T]) \times H^1(\mathbb{R}),$$

and the functions $u^J(\alpha_j(\omega, t), x)$ as in Theorem 2.

There exist a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and \mathcal{X} -valued random variables $(\tilde{u}_s^J, \tilde{u}_w^J, \tilde{W}^J, \tilde{u}_0^J)$ defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, such that along a subsequence (not relabelled),

$$u_s^J \sim \tilde{u}_s^J, \quad u_w^J \sim \tilde{u}_w^J, \quad W \sim \tilde{W}^J, \quad u_o \sim \tilde{u}_0^J,$$

and $\tilde{\mathbb{P}}$ -almost surely,

$$(\tilde{u}_s^J, \tilde{u}_w^J, \tilde{W}^J, \tilde{u}_0^J) \rightarrow (u_s, u_w, W, u_0) \quad \text{in } \mathcal{X}. \quad (52)$$

Proof. The thesis follows from Theorem 6 applied to the tight family of laws constructed above. \square

Lemma 6 (Identification of doubled variables) *For the sequence of variables defined above, we have $\tilde{u}_{J,s} = \tilde{u}_{J,w}$ $\tilde{\mathbb{P}} \times dt \times dx$ -a.e. Moreover, $\tilde{u}_s = \tilde{u}_w$ $\tilde{\mathbb{P}} \times dt \times dx$ -a.e. (so from now on it is sufficient to write $\tilde{u}_J := \tilde{u}_{J,s} = \tilde{u}_{J,w}$ and $\tilde{u} = \tilde{u}_s = \tilde{u}_w$).*

Proof. See Lemma 6.4 in [17]. \square

Let us fix $t \in [0, T]$ and let \mathcal{X} denote one of the spaces $L^2([0, T]; H^1(\mathbb{R}))$, $C([0, T]; H_w^1(\mathbb{R}))$ or $C([0, T])$.

Let $f \mapsto f|_{[0,t]} : \mathcal{X} \rightarrow \mathcal{X}_{[0,t]}$ denote the restriction operator to $[0, t]$.

We define $\{\tilde{\mathcal{F}}_t\}_{t \geq 0}$ to be the $\tilde{\mathbb{P}}$ -augmented canonical filtration of $(\tilde{u}, \tilde{W}, \tilde{u}_0)$, i.e.,

$$\tilde{\mathcal{F}}_t = \sigma\left(\tilde{u}|_{[0,t]}, \tilde{W}|_{[0,t]}, \tilde{u}_0\right) \vee \{N \in \tilde{\mathcal{F}} : \tilde{\mathbb{P}}(N) = 0\}, \quad (53)$$

where, for a collection E of subsets of $\tilde{\Omega}$, $\sigma(E)$ denotes the smallest σ -algebra containing E .

We denote by $\tilde{\mathcal{S}}$ the corresponding stochastic basis, that is,

$$\tilde{\mathcal{S}} := (\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_{t \geq 0}, \tilde{\mathbb{P}}). \quad (54)$$

Similarly, based on $\tilde{u}^J, \tilde{W}^J, \tilde{u}_0^J$ we define

$$(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t^J\}_{t \geq 0}, \tilde{\mathbb{P}}).$$

Then \tilde{u}, \tilde{W} and \tilde{u}^J, \tilde{W}^J are adapted relative to the stochastic bases $\tilde{\mathcal{S}}, \tilde{\mathcal{S}}^J$ respectively. Besides, by the equality of laws, each \tilde{W}^J is a Brownian motion on $\tilde{\mathcal{S}}^J$.

Lemma 7 (Brownian motion) *The process \tilde{W} is a Brownian motion.*

Proof. Since in our context W is a martingale relative to $\mathcal{S} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, the proof is identical to the proof of Lemma 6.6 in [17]. \square

Lemma 8 (Convergence) *Let u^J, \tilde{u}^J and \tilde{u} be defined as above and set $\tilde{p}^J = \partial_t \tilde{u}^J, \tilde{p} = \partial_t \tilde{u}, \tilde{q}^J := \partial_x(\tilde{u}^J v^J(\tilde{u}^J * \eta^J))$ and $\tilde{q} := \partial_x(\tilde{u} v(\tilde{u} * \eta))$. Then $p^J \sim \tilde{p}^J, q^J \sim \tilde{q}^J$, and the following convergences hold \mathbb{P} -almost surely:*

$$\tilde{u}^J \rightarrow \tilde{u} \quad \text{in } C([0, T]; H^1(\mathbb{R})), \quad (55)$$

$$\tilde{p}^J \rightarrow \tilde{p} \quad \text{in } L^2([0, T] \times \mathbb{R}), \quad (56)$$

$$\tilde{q}^J \rightarrow \tilde{q} \quad \text{in } L^2([0, T] \times \mathbb{R}). \quad (57)$$

Proof. Thanks to Theorem 8 we know that $\tilde{u}^J \sim u^J$ in $L^2([0, T]; H^1(\mathbb{R}))$. Combining Theorem 8, Lemma 5 and Lemma 6 we obtain (55) and (56).

To prove (57), consider

$$\partial_x(\tilde{u}^J v^J(\tilde{u}^J * \eta^J)).$$

We know that:

1. $\tilde{u}^J \rightarrow \tilde{u}$ in $C([0, T]; H^1(\mathbb{R}))$;
2. $v^J \rightarrow v$ in $W^{3, \infty}(\mathbb{R})$, uniformly with respect to $z(\omega)$;
3. $\eta^J \rightarrow \eta$ in $W^{2, \infty}(\mathbb{R})$, uniformly with respect to $h(\omega)$.

Thus,

$$\begin{aligned} & \|\tilde{u}^J v^J(\tilde{u}^J * \eta^J) - \tilde{u} v(\tilde{u} * \eta)\|_{C^0} \\ & \leq \|\tilde{u}^J v^J(\tilde{u}^J * \eta^J) - \tilde{u} v^J(\tilde{u}^J * \eta^J)\|_{C^0} + \|\tilde{u} v^J(\tilde{u}^J * \eta^J) - \tilde{u} v(\tilde{u}^J * \eta^J)\|_{C^0} \\ & \quad + \|\tilde{u} v(\tilde{u}^J * \eta^J) - \tilde{u} v(\tilde{u} * \eta)\|_{C^0} \\ & \leq \|v^J\|_{W^{3, \infty}} \|\tilde{u}^J(\tilde{u}^J * \eta^J) - \tilde{u}(\tilde{u}^J * \eta^J)\|_{C^0} \\ & \quad + \|\tilde{u}\|_{L^\infty} \|v^J(\tilde{u}^J * \eta^J) - v(\tilde{u}^J * \eta^J)\|_{W^{3, \infty}} \\ & \quad + \|v\|_{W^{3, \infty}} \|\tilde{u}(\tilde{u}^J * \eta^J) - \tilde{u}(\tilde{u} * \eta)\|_{C^0} \\ & \quad + \|\tilde{u}\|_{C^0} \|v^J(\tilde{u}^J * \eta^J) - v^J(\tilde{u} * \eta)\|_{W^{3, \infty}}. \end{aligned}$$

Since $W^{2, \infty}([0, T]; \mathbb{R}) \hookrightarrow C^0([0, T]; \mathbb{R})$, one has $\tilde{u}^J * \eta^J \rightarrow \tilde{u} * \eta$ in C^0 , and (57) follows. \square

Theorem 9 (Weak H^1 solution) *Suppose $p > 2$, and that $\mathbb{E}(\|u_o\|_{H^1(\mathbb{R})}^p) < \infty$. Let $\tilde{u}, \tilde{W}, \tilde{u}_0$ be the Skorokhod–Jakubowski a.s. representations from Theorem 8 and let $\tilde{\mathcal{S}}$ be the corresponding stochastic basis defined in (54). Then $(\tilde{\mathcal{S}}, \tilde{u}, \tilde{W})$ is a weak H^1 -solution of (26) with initial law $\tilde{\Lambda} := (\tilde{u}_0)_* \tilde{\mathbb{P}}$:*

(c') $\tilde{u} : \tilde{\Omega} \times [0, T] \rightarrow H^1(\mathbb{R})$ satisfies $\tilde{u}(\omega, \cdot) \in C([0, T]; H_w^1(\mathbb{R}))$ for $\tilde{\mathbb{P}}$ -almost every $\omega \in \tilde{\Omega}$.

Moreover,

$$\tilde{u} \in L^p(\tilde{\Omega}; L^\infty([0, T]; H^1(\mathbb{R}))) \cap L^2(\tilde{\Omega} \times [0, T]; H^2(\mathbb{R})).$$

Proof. We continue to use the notations from Lemma 8.

By the equality of joint laws on \mathcal{X} , see Theorem 8, we also have

$$(u^J, u^J, W, u_0) \sim (\tilde{u}^J, \tilde{u}^J, \tilde{W}^J, \tilde{u}_0^J) \quad \text{on } \mathcal{X}. \quad (58)$$

For each fixed $n \in \mathbb{N}$ and $\varphi \in C_c^1(\mathbb{R})$, consider the functional $F_{\varphi, n} : \mathcal{X} \rightarrow \mathbb{R}$ defined by

$$F_{\varphi, n}(u^J, W^J, u_0^J) = I_1^n + I_2^n,$$

where

$$I_1^n = \int_0^T \int_{\mathbb{R}} u^J(t, x) \partial_t \varphi(t, x) \, dx \, dt, \quad (59)$$

and

$$I_2^n = \int_0^T \int_{\mathbb{R}} u^J(t, x) v(\eta * u^J(t, \cdot)(x)) \partial_x \varphi(t, x) \, dx \, dt \, dW_t^J. \quad (60)$$

The mapping $F_{\varphi, n}$ is a Baire function of class 1 (since the stochastic integral can be approximated by temporally mollified integrals, see e.g. Bensoussan).

By the equivalence of joint laws (58), we have (see [23, p. 105])

$$\tilde{\mathbb{P}}\left(\{F_{\varphi, n}[(\tilde{u}^J, \tilde{W}^J, \tilde{u}_0^J)] = 0\}\right) = 1.$$

Passing to the limit as $J \rightarrow \infty$ and using Lemma 8, we obtain that the limit process \tilde{u} satisfies the weak formulation of (26) on the stochastic basis $\tilde{\mathcal{S}}$. The claimed integrability and regularity properties follow from the uniform estimates established above and standard lower semicontinuity arguments. \square

\square

A The Karhunen–Loève expansion

Let us consider

$$F \in \mathbf{L}^\infty(\Omega; \mathbf{C}^1(\mathbb{R}; \mathbb{R})) \cap \mathbf{L}^2(\Omega; \mathbf{C}^1(\mathbb{R}; \mathbb{R})).$$

For $R > 0$, we define the linear operator

$$C_F^R : \mathbf{L}^2((-R, R); \mathbb{R}) \longrightarrow \mathbf{L}^2((-R, R); \mathbb{R})$$

by

$$C_F^R[\Phi](u) := \int_{|h| \leq R} \text{Cov}[F](u, h) \Phi(h) \, dh, \quad u \in (-R, R), \quad (61)$$

where $\text{Cov}[F]$ denotes the covariance function associated with F .

Choosing

$$R > \text{ess sup}_{\omega \in \Omega} \|u_\omega(\omega, \cdot)\|_{L^\infty(\mathbb{R})}$$

we describe all possible states, \mathbb{P} -a.s., for the class of initial data under consideration.

It is known (see [14]) that, for the covariance function of a bounded random flux, the following Lipschitz-type estimate holds:

$$|\text{Cov}[F](u, h) - \text{Cov}[F](u', h')| \leq 2B(F) C(u, h) (|u - u'| + |h - h'|), \quad (62)$$

where $B(F)$ is a suitable constant depending on F , and $C(u, h)$ is a suitable function depending on u and h . Therefore (see [14] for explicit computations), C_F^R is a linear, compact and self-adjoint operator on $L^2(-R, R)$. Thus, by the spectral theorem, it admits a sequence of eigenfunctions

$$(\Phi_j^R)_{j \geq 1}$$

which forms a complete orthonormal basis of $L^2(-R, R)$.

Any bounded random flux $F(\omega, u)$ therefore admits, for every fixed $0 < R < +\infty$, a Karhunen–Loève expansion of the form

$$F(\omega, u) = \mathbb{E}(F(\cdot, u)) + \sum_{j \geq 1} Y_j^R(\omega) \Psi_j^R(u), \quad |u| \leq R, \quad (63)$$

which converges in $L^2(\Omega; L^2(-R, R))$. Here $(Y_j^R(\omega))_{j \geq 1}$ is a suitable sequence of (uncorrelated, in fact independent) real-valued random variables (see [14] for their explicit expression).

In the context of our article it is useful to relate the Karhunen–Loève expansion to a parametric, deterministic representation of the random flux.

Rescaling the eigenfunctions Ψ_j^R and the realizations $Y_j^R(\omega)$, expansion (63) can be rewritten as

$$F(\omega, u) = \mathbb{E}(F(\cdot, u)) + \sum_{j \geq 1} \bar{Y}_j^R(\omega) \bar{\Psi}_j^R(u), \quad |u| \leq R, \quad (64)$$

where \bar{Y}_j^R and $\bar{\Psi}_j^R$ denote suitable rescalings of Y_j^R and Ψ_j^R , respectively. By a slight abuse of notation, we denote by $F(y, u)$ the same random flux $F(\omega, u)$ expressed in terms of the parameters

$$y_j = \bar{Y}_j^R(\omega), \quad |u| \leq R.$$

Thus, thanks to (64), we can consider the parametric, deterministic representation

$$F(y, u) = \mathbb{E}(F(\cdot, u)) + \sum_{j \geq 1} y_j \bar{\Psi}_j^R(u), \quad y \in U = [-1, 1]^{\mathbb{N}}, \quad |u| \leq R. \quad (65)$$

Acknowledgements

MG was partly supported by the European Union—NextGenerationEU (National Sustainable Mobility Center CN00000023, Italian Ministry of University

and Research Decree n. 1033—17/06/2022, Spoke 8).

The first author is supported by MathInParis project by Fondation Sciences mathématiques de Paris (FSMP), funding from the European Union’s Horizon 2020 research and innovation programme, under the Marie Skłodowska-Curie grant agreement No 101034255.

She is also supported by Sorbonne Université, being affiliated at Laboratoire Jacques Louis Lions (LJLL).

Both the author during the writing of this paper were members of the Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

In particular the first author was member of ”INdAM -GNAMPA Project 2024”, codice CUP E53C23001670001, ”Non-smooth optimal control problems” and the second author...

The second author is still a GNAMPA member.

The second author are also members of: PRIN 2022, Progr-2022238YY5-PE1, ”Optimal control problems: analysis, approximations and applications”

References

- [1] Lighthill, M. J., Whitham, G. B. On kinematic waves. II. A theory of traffic flow on long crowded roads. *Proc. R. Soc. Lond., Ser. A*, 229: 317–345, 1955, doi:10.1098/rspa.1955.0089.
- [2] Richards, Paul I. Shock waves on the highway. *Operations Res.*, 4: 42–51, 1956.
- [3] Aw, A., Rascle, M. Resurrection of ’second order’ models of traffic flow. *SIAM Journal on Applied Mathematics*, 60(3): 916–938, 2000, doi:10.1137/S0036139997332099.
- [4] Colombo, Rinaldo M. Phase transitions in a traffic flow model. *PAMM, Proc. Appl. Math. Mech.*, 3: 20–23, 2003, doi:10.1002/pamm.200310306.
- [5] Colombo, Rinaldo M., Marcellini, Francesca, Rascle, Michel. A 2-phase traffic model based on a speed bound. *SIAM J. Appl. Math.*, 70(7): 2652–2666, 2010, doi:10.1137/090752468.
- [6] Goatin, Paola, Rossi, Elena. A multilane macroscopic traffic flow model for simple networks. *SIAM J. Appl. Math.*, 79(5): 1967–1989, 2019, doi:10.1137/19M1254386.
- [7] Blandin, Sebastien, Goatin, Paola. Well-posedness of a conservation law with non-local flux arising in traffic flow modeling. *Numer. Math.*, 132(2): 217–241, 2016, doi:10.1007/s00211-015-0717-6.
- [8] Keimer, Alexander, Pflug, Lukas. Existence, uniqueness and regularity results on nonlocal balance laws. *J. Differ. Equations*, 263(7): 4023–4069, 2017, doi:10.1016/j.jde.2017.05.015.
- [9] Keimer, Alexander, Pflug, Lukas, Spinola, Michele. Nonlocal scalar conservation laws on bounded domains and applications in traffic flow. *SIAM J. Math. Anal.*, 50(6): 6271–6306, 2018, doi:10.1137/18M119817X.

- [10] Friedrich, Jan, Göttlich, Simone, Keimer, Alexander, Pflug, Lukas. Conservation laws with nonlocality in density and velocity and their applicability in traffic flow modelling. Preprint, arXiv:2302.12797 [math.AP], 2023, arXiv:2302.12797.
- [11] Aggarwal, Aekta, Holden, Helge, Vaidya, Ganesh. Systems of nonlocal balance laws for dense multilane vehicular traffic. Preprint, arXiv:2312.16928 [math.NA], 2023, arXiv:2312.16928.
- [12] Aggarwal, Aekta, Holden, Helge, Vaidya, Ganesh. Well-posedness and error estimates for coupled systems of nonlocal conservation laws. Preprint, arXiv:2308.01411 [math.NA], 2023, arXiv:2308.01411.
- [13] Šukys, Jonas, Mishra, Siddhartha, Schwab, Christoph. Multi-level Monte Carlo finite difference and finite volume methods for stochastic linear hyperbolic systems. In *Monte Carlo and quasi-Monte Carlo methods 2012*, pp. 649–666. Springer, 2013, doi:10.1007/978-3-642-41095-6_34.
- [14] Mishra, Siddhartha, Risebro, Nils Henrik, Schwab, Christoph, Tokareva, Svetlana. Numerical solution of scalar conservation laws with random flux functions. *SIAM/ASA J. Uncertain. Quantif.*, 4: 552–591, 2016, doi:10.1137/120896967.
- [15] Mishra, Siddhartha, Schwab, Christoph. Sparse tensor multi-level Monte Carlo finite volume methods for hyperbolic conservation laws with random initial data. *Math. Comput.*, 81: 1979–2018, 2012.
- [16] Böhme, Timo, Göttlich, Simone, Neuenkirch, Andreas. A nonlocal traffic flow model with stochastic velocity. Preprint, arXiv:2407.02962 [math.NA], 2024, arXiv:2407.02962.
- [17] Holden, Helge, Karlsen, Kenneth H., Pang, Peter H. C. Global well-posedness of the viscous Camassa–Holm equation with gradient noise. *Discrete Contin. Dyn. Syst.*, 43(2): 568–618, 2023, doi:10.3934/dcds.2022163.
- [18] Holden, Helge, Karlsen, Kenneth H., Pang, Peter H. C. The Hunter–Saxton equation with noise. *J. Differ. Equations*, 270: 725–786, 2021, doi:10.1016/j.jde.2020.07.031.
- [19] Gerster, Stephan, Herty, Michael, Yu, Hui. Hypocoercivity of stochastic Galerkin formulations for stabilization of kinetic equations. *Commun. Math. Sci.*, 19(3): 787–806, 2021, doi:10.4310/CMS.2021.v19.n3.a10.
- [20] Herty, Michael, Kolb, Adrian, Müller, Siegfried. Multiresolution-analysis for stochastic hyperbolic conservation laws. Preprint, arXiv:2203.11534 [math.NA], 2023. arXiv:2203.11534.
- [21] Revuz, D., Yor, M. *Continuous Martingales and Brownian Motion*. Springer, Berlin, 3rd edition, 1999.
- [22] Jakubowski, A. The almost sure Skorokhod representation for subsequences in nonmetric spaces. *Theory Probab. Appl.*, 42: 167–174, 1997.
- [23] Situ, R. *Theory of Stochastic Differential Equations with Jumps and Applications*. Springer, New York, 2005.

- [24] Colombo, Rinaldo M., Garavello, Mauro, Nocita, Claudia. General stability estimates in nonlocal traffic models for several populations. *NoDEA, Non-linear Differ. Equ. Appl.*, 32(2): 29, 2025, doi:10.1007/s00030-025-01034-w.