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► To cite this version:

Olivier Besson. Adaptive detection of second-order subspace signals: Derivation and characterization of the maximal invariant statistic. *Signal Processing*, 2026, 245, pp.110605. <10.1016/j.sigpro.2026.110605>. <hal-05575215>

HAL Id: hal-05575215

<https://hal.science/hal-05575215v1>

Submitted on 1 Apr 2026

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ELSEVIER

Contents lists available at ScienceDirect

Signal Processing

journal homepage: www.elsevier.com/locate/sigpro

Short communication

Adaptive detection of second-order subspace signals: Derivation and characterization of the maximal invariant statistic

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ARTICLE INFO

Keywords:

Adaptive detection
Second-order subspace signals
Invariance

ABSTRACT

We address the problem of detecting a multidimensional subspace signal among colored noise with unknown covariance matrix using a second-order model, i.e. assuming that the presence of the signal of interest results in a rank- R modification of the noise covariance matrix. The problem is tackled through the prism of invariance. We derive the group of transformations that leave the problem invariant and subsequently derive the maximal invariant statistic (MIS) along with the induced maximal invariant (IMI). A statistical representation of the MIS is obtained which is quite different from that of a first-order model. Hints at possible test statistics based on the MIS are also provided.

1. Problem statement

Modeling or approximating a multidimensional signal of interest (SoI) by a linear combination of a few known basis functions is common practice in many engineering applications and is especially effective when the size of the observations is large since such a low-rank model achieves a significant reduction in the number of unknown signal parameters. In some applications the SoI may actually belong to a known subspace but a subspace model is also often used to take into account uncertainties about a SoI which would be rank-one: in this case the subspace model is aimed at including the nominal rank-one signal and capturing most of its variations around this nominal value. In other words the $N \times K_x$ matrix of observation \mathbf{X} can be written as $\mathbf{X} = \mathbf{H}\Theta + \mathbf{N}_x$ where the columns of the $N \times R$ matrix \mathbf{H} contain the known basis functions and a column of Θ provides the coordinates of the corresponding column of \mathbf{X} in the range space of \mathbf{H} . \mathbf{N}_x denotes an unknown noise whose statistics are generally unknown so that a set of K_y training samples gathered column-wise in a matrix $\mathbf{Y} = \mathbf{N}_y$ is assumed to be available with \mathbf{N}_x and \mathbf{N}_y sharing the same distribution. The set of all available data can be written as $[\mathbf{X} \ \mathbf{Y}] = \mathbf{H}\Theta[\mathbf{I}_{K_x} \ \mathbf{0}] + [\mathbf{N}_x \ \mathbf{N}_y]$. When Θ is assumed to be deterministic, this corresponds to a particular form of the so-called generalized multiple analysis of variance model [1–5]. Adaptive detection in such a framework has been extensively studied, especially when Θ is assumed to be deterministic, see e.g. the early work of Khatri [2] and the more recent work of Kelly [6] which has been very influential in the radar domain. In contrast, the case of a second-order model where Θ is assumed to be random is more scarcely addressed. We refer the reader to [7,8] and references therein for a very good review. In [7,8] an

exhaustive study of multidimensional subspace signals detection is carried out both for first and second-order models and a systematic derivation of the generalized likelihood ratio test (GLRT) or a two-step approximation of it is provided for many different cases, namely homogeneous or partially homogeneous noise environments, known or unknown subspace.

In this communication we tackle the problem of detecting second-order multidimensional subspace signals through the prism of invariance. Actually many composite hypotheses testing problems are invariant to some group of transformations and it is thus natural to look for test statistics that satisfy this invariance. All such invariant test statistics will depend on the so-called maximal invariant statistic (MIS) which achieves the greatest compression among invariant test statistics. Being able to derive the MIS is thus critical as it guarantees the most efficient use of the data. This approach has been very popular in the field of adaptive radar processing where a major contribution is due to Bose and Steinhart [9–11] who considered rank-one or uncertain rank-one waveforms. Raghavan also provided significant insights by deriving the MIS for rank-one signals and multiple observations [12], and for subspace signals and multiple observations [13]. A remarkable step was also taken by Ciunzo *et al.* who, in [14,15], provide a review of previous results and derive new results concerning first-order subspace signals in first-order subspace interference plus unstructured noise. Strangely enough derivation of the MIS for second-order subspace signals is not, to the best of our knowledge, available, perhaps because dealing with second-order subspace signals is considerably more complicated than dealing with first-order subspace signals and also because detection schemes based on a first-order model generally perform very well under a second-order

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Received 9 December 2025; Received in revised form 19 February 2026; Accepted 18 March 2026

Available online 20 March 2026

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model [7,8]. We fill this gap in the present communication which enables us to illustrate the similarities and the differences compared to the first-order model. While deriving the MIS is rather straightforward finding its distribution seems very complicated but at least a simple statistical representation can be obtained in terms of Gaussian and Wishart distributed random matrices.

2. Maximal invariant statistic

We assume that we have two initial sets of data $\mathbf{X}_i \in \mathbb{R}^{N \times K_x}$ and $\mathbf{Y}_i \in \mathbb{R}^{N \times K_y}$ with $K_y \geq N$ such that \mathbf{Y}_i contains noise only while \mathbf{X}_i possibly contains a random signal $\mathbf{H}_i \boldsymbol{\Theta}_i$ which belongs to a known subspace spanned by the columns of the $N \times R$ full-rank matrix \mathbf{H}_i . The problem considered here is that of deciding between the two following hypotheses

$$\begin{aligned} \mathbb{H}_0 : \mathbf{X}_i &\sim \mathcal{N}_{N, K_x}(\mathbf{0}, \boldsymbol{\Sigma}_i, \mathbf{I}_{K_x}), \mathbf{Y}_i \sim \mathcal{N}_{N, K_y}(\mathbf{0}, \boldsymbol{\Sigma}_i, \mathbf{I}_{K_y}) \\ \mathbb{H}_1 : \mathbf{X}_i &\sim \mathcal{N}_{N, K_x}(\mathbf{0}, \boldsymbol{\Sigma}_i + \mathbf{H}_i \boldsymbol{\Omega}_i \mathbf{H}_i^T, \mathbf{I}_{K_x}), \mathbf{Y}_i \sim \mathcal{N}_{N, K_y}(\mathbf{0}, \boldsymbol{\Sigma}_i, \mathbf{I}_{K_y}) \end{aligned} \quad (1)$$

where $\boldsymbol{\Omega}_i$ is the covariance matrix of $\boldsymbol{\Theta}_i$ and $\boldsymbol{\Sigma}_i$ stands for the noise covariance matrix. Let \mathbf{Q} be a unitary matrix whose first R columns are $\mathbf{H}_i(\mathbf{H}_i^T \mathbf{H}_i)^{-1/2}$ and let us consider the transformed data $\mathbf{X} = \mathbf{Q}^T \mathbf{X}_i$ and $\mathbf{Y} = \mathbf{Q}^T \mathbf{Y}_i$. Then (1) is equivalent to

$$\begin{aligned} \mathbb{H}_0 : \mathbf{X} &\sim \mathcal{N}_{N, K_x}(\mathbf{0}, \boldsymbol{\Sigma}, \mathbf{I}_{K_x}), \mathbf{Y} \sim \mathcal{N}_{N, K_y}(\mathbf{0}, \boldsymbol{\Sigma}, \mathbf{I}_{K_y}) \\ \mathbb{H}_1 : \mathbf{X} &\sim \mathcal{N}_{N, K_x}(\mathbf{0}, \boldsymbol{\Sigma} + \mathbf{H} \boldsymbol{\Omega} \mathbf{H}^T, \mathbf{I}_{K_x}), \mathbf{Y} \sim \mathcal{N}_{N, K_y}(\mathbf{0}, \boldsymbol{\Sigma}, \mathbf{I}_{K_y}) \end{aligned} \quad (2)$$

where $\boldsymbol{\Sigma} = \mathbf{Q}^T \boldsymbol{\Sigma}_i \mathbf{Q}$, $\boldsymbol{\Omega} = (\mathbf{H}_i^T \mathbf{H}_i)^{T/2} \boldsymbol{\Omega}_i (\mathbf{H}_i^T \mathbf{H}_i)^{1/2}$ and $\mathbf{H} = \begin{pmatrix} \mathbf{I}_R \\ \mathbf{0} \end{pmatrix}$. The previous transformation is classical and it confines the signal part to the first R rows of \mathbf{X} .

Herein, we look for invariant test statistics which thus depend on the MIS.

2.1. Derivation of the MIS and of the IMI

First we need to define the natural invariances of problem (2). Since \mathbf{X} and \mathbf{Y} are Gaussian and zero-mean under \mathbb{H}_0 we consider only linear transformations. \mathbf{X} and \mathbf{Y} being independent and their columns being independent, they can be post-multiplied by any orthogonal matrix without affecting this independence [16, Chapter 2]. Besides, the two data sets sharing the same (unknown) covariance matrix $\boldsymbol{\Sigma}$ under \mathbb{H}_0 they can be pre-multiplied by any non-singular matrix \mathbf{A} . Actually if $\mathbf{X} \rightarrow \mathbf{A}\mathbf{X}\mathbf{Q}_x$ and $\mathbf{Y} \rightarrow \mathbf{A}\mathbf{Y}\mathbf{Q}_y$, then $\boldsymbol{\Sigma} \rightarrow \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^T$. However, under \mathbb{H}_1 the covariance matrix of \mathbf{X} , namely $\boldsymbol{\Sigma} + \mathbf{H}\boldsymbol{\Omega}\mathbf{H}^T$, is transformed to $\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^T + \mathbf{A}\mathbf{H}\boldsymbol{\Omega}\mathbf{H}^T\mathbf{A}^T$ and should be of the form $\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^T + \mathbf{H}\boldsymbol{\Omega}'\mathbf{H}^T$. This implies that one must have $\mathbf{A}\mathbf{H} = \mathbf{H}\mathbf{T}$ for some matrix \mathbf{T} (and in this case $\boldsymbol{\Omega}' = \mathbf{T}\boldsymbol{\Omega}\mathbf{T}^T$). If we partition $\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}$, where \mathbf{A}_{11} is $R \times R$, it means that

$$\begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{I}_R \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{11} \\ \mathbf{A}_{21} \end{pmatrix} = \begin{pmatrix} \mathbf{T} \\ \mathbf{0} \end{pmatrix} \quad (3)$$

which implies that $\mathbf{A}_{21} = \mathbf{0}$. The group of transformations that leave the problem invariant is thus

$$\mathcal{G} \left(\begin{bmatrix} \mathbf{X}_1 & \mathbf{Y}_1 \\ \mathbf{X}_2 & \mathbf{Y}_2 \end{bmatrix} \right) = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{22} \end{pmatrix} \begin{bmatrix} \mathbf{X}_1 & \mathbf{Y}_1 \\ \mathbf{X}_2 & \mathbf{Y}_2 \end{bmatrix} \begin{pmatrix} \mathbf{Q}_x & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_y \end{pmatrix} \quad (4)$$

for arbitrary orthogonal matrices \mathbf{Q}_x and \mathbf{Q}_y . This is basically the same group of invariance as in the first-order subspace model where $\mathbf{X} \sim \mathcal{N}_{N, K_x}(\mathbf{H}\boldsymbol{\Theta}, \boldsymbol{\Sigma}, \mathbf{I}_{K_x})$. However the induced transformations on the parameters space is different and is given by $\bar{\mathcal{G}}(\boldsymbol{\Omega}, \boldsymbol{\Sigma}) = (\mathbf{A}_{11}\boldsymbol{\Omega}\mathbf{A}_{11}^T, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^T)$. In the proposition below we give the expressions of the MIS and the IMI.

Proposition 1. *The maximal invariant statistic and the induced maximal invariant are given by*

$$\text{MIS}(\mathbf{X}, \mathbf{Y}) = \{\mathbf{M}_1 = \mathbf{X}_{1,2}^T \mathbf{S}_{1,2}^{-1} \mathbf{X}_{1,2}, \mathbf{M}_2 = \mathbf{X}_2^T \mathbf{S}_{22}^{-1} \mathbf{X}_2\} \quad (5a)$$

$$\text{IMI}(\boldsymbol{\Omega}, \boldsymbol{\Sigma}) = \lambda(\boldsymbol{\Sigma}_{1,2}^{-1/2} \boldsymbol{\Omega} \boldsymbol{\Sigma}_{1,2}^{-T/2}) \quad (5b)$$

where $\mathbf{S} = \mathbf{Y}\mathbf{Y}^T = \begin{pmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{Y}_1 \mathbf{Y}_1^T & \mathbf{Y}_1 \mathbf{Y}_2^T \\ \mathbf{Y}_2 \mathbf{Y}_1^T & \mathbf{Y}_2 \mathbf{Y}_2^T \end{pmatrix}$, $\mathbf{S}_{1,2} = \mathbf{S}_{11} - \mathbf{S}_{12} \mathbf{S}_{22}^{-1} \mathbf{S}_{21}$, $\mathbf{X}_{1,2} = \mathbf{X}_1 - \mathbf{S}_{12} \mathbf{S}_{22}^{-1} \mathbf{X}_2$ and where $\lambda(\cdot)$ stands for the set of eigenvalues of the matrix between parentheses.

Proof. See the Appendix. \square

A few observations can be made. First, the MIS is the same as in the first-order model and hence any test statistic which is invariant under the first-order model remains invariant under the second-order model. Next, there is a significant difference regarding the induced maximal invariant between the first-order and the second-order model: in the former case the IMI is the $K_x \times K_x$ matrix $\boldsymbol{\Theta}^T \boldsymbol{\Sigma}_{1,2}^{-1} \boldsymbol{\Theta}$ with rank $\min(R, K_x)$ while in the latter case it consists of a set of R eigenvalues. The probability of detection of any invariant test will thus depend only on this set of eigenvalues.

2.2. Statistical characterization of the MIS

Let us turn to the distribution of $(\mathbf{M}_1, \mathbf{M}_2)$. First note that $\mathbf{S} = \mathbf{Y}\mathbf{Y}^T \sim \mathcal{W}_N(K_y, \boldsymbol{\Sigma})$ and it follows that $\mathbf{S}_{1,2}$ is independent of $(\mathbf{S}_{22}^{-1} \mathbf{S}_{21}, \mathbf{S}_{22})$ and that $\mathbf{S}_{1,2} \sim \mathcal{W}_R(K_y - N + R, \boldsymbol{\Sigma}_{1,2})$, $\mathbf{S}_{22} \sim \mathcal{W}_{N-R}(K_y, \boldsymbol{\Sigma}_{22})$ and $\mathbf{S}_{22}^{-1} \mathbf{S}_{21} | \mathbf{S}_{22} \sim \mathcal{N}_{N-R, R}(\boldsymbol{\Sigma}_{22}^{-1} \boldsymbol{\Sigma}_{21}, \mathbf{S}_{22}^{-1}, \boldsymbol{\Sigma}_{1,2})$ [16]. Since $\mathbf{X}_2 \sim \mathcal{N}_{N-R, K_x}(\mathbf{0}, \boldsymbol{\Sigma}_{22}, \mathbf{I}_{K_x})$, it ensues that $\mathbf{M}_2 = \mathbf{X}_2^T \mathbf{S}_{22}^{-1} \mathbf{X}_2$ follows a matrix-variate F distribution which does not depend on any unknown parameter [16], namely \mathbf{M}_2 is an ancillary variable.

Let us consider now the distribution of $\mathbf{M}_1 | \mathbf{M}_2$. We have

$$\begin{aligned} \mathbf{X}_1 | \mathbf{X}_2 &\sim \mathcal{N}_{R, K_x}(\boldsymbol{\Sigma}_{12} \boldsymbol{\Sigma}_{22}^{-1} \mathbf{X}_2, \boldsymbol{\Sigma}_{1,2} + \boldsymbol{\Omega}, \mathbf{I}_{K_x}) \\ \mathbf{Y}_1 | \mathbf{Y}_2 &\sim \mathcal{N}_{R, K_y}(\boldsymbol{\Sigma}_{12} \boldsymbol{\Sigma}_{22}^{-1} \mathbf{Y}_2, \boldsymbol{\Sigma}_{1,2}, \mathbf{I}_{K_x}) \end{aligned} \quad (6)$$

so that

$$\mathbf{Y}_1 \mathbf{Y}_2^T (\mathbf{Y}_2 \mathbf{Y}_2^T)^{-1} \mathbf{X}_2 | \mathbf{X}_2, \mathbf{Y}_2 \stackrel{d}{=} \boldsymbol{\Sigma}_{12} \boldsymbol{\Sigma}_{22}^{-1} \mathbf{X}_2 + \boldsymbol{\Sigma}_{1,2}^{1/2} \mathcal{N}_{R, Q}(\mathbf{0}, \mathbf{I}_R, \mathbf{I}_Q) \mathbf{L}_2^T \quad (7)$$

where \mathbf{L}_2 is a $K_x | Q$ matrix such that $\mathbf{L}_2 \mathbf{L}_2^T = \mathbf{M}_2$ where $Q = \min(N - R, K_x)$. It follows that

$$\begin{aligned} \mathbf{X}_{1,2} | \mathbf{M}_2 &\stackrel{d}{=} (\boldsymbol{\Sigma}_{1,2} + \boldsymbol{\Omega})^{1/2} \mathcal{N}_{R, K_x}(\mathbf{0}, \mathbf{I}_R, \mathbf{I}_{K_x}) + \boldsymbol{\Sigma}_{1,2}^{1/2} \mathcal{N}_{R, Q}(\mathbf{0}, \mathbf{I}_R, \mathbf{I}_Q) \mathbf{L}_2^T \\ &\stackrel{d}{=} \boldsymbol{\Sigma}_{1,2}^{1/2} [(\mathbf{I}_R + \boldsymbol{\Gamma})^{1/2} \mathbf{N}_a + \mathbf{N}_b \mathbf{L}_2^T] \end{aligned} \quad (8)$$

where $\boldsymbol{\Gamma} = \boldsymbol{\Sigma}_{1,2}^{-1/2} \boldsymbol{\Omega} \boldsymbol{\Sigma}_{1,2}^{-T/2}$, $\mathbf{N}_a \sim \mathcal{N}_{R, K_x}(\mathbf{0}, \mathbf{I}_R, \mathbf{I}_{K_x})$, $\mathbf{N}_b \sim \mathcal{N}_{R, Q}(\mathbf{0}, \mathbf{I}_R, \mathbf{I}_Q)$.

Additionally we have $\mathbf{S}_{1,2} \sim \mathcal{W}_R(K_y - N + R, \boldsymbol{\Sigma}_{1,2}) \stackrel{d}{=} \boldsymbol{\Sigma}_{1,2}^{1/2} \mathbf{W}_{1,2} \boldsymbol{\Sigma}_{1,2}^{T/2}$ with $\mathbf{W}_{1,2} \sim \mathcal{W}_r(K_y - N + R, \mathbf{I}_R)$ so that

$$\mathbf{M}_1 | \mathbf{M}_2 \stackrel{d}{=} [(\mathbf{I}_R + \boldsymbol{\Gamma})^{1/2} \mathbf{N}_a + \mathbf{N}_b \mathbf{L}_2^T]^T \mathbf{W}_{1,2}^{-1} [(\mathbf{I}_R + \boldsymbol{\Gamma})^{1/2} \mathbf{N}_a + \mathbf{N}_b \mathbf{L}_2^T] \quad (9)$$

We can derive another more convenient stochastic representation as follows. We have

$$\text{vec}(\boldsymbol{\Sigma}_{1,2}^{-1/2} \mathbf{X}_{1,2}) \stackrel{d}{=} [\mathbf{I}_{K_x} \otimes (\mathbf{I}_R + \boldsymbol{\Gamma})^{1/2}] \mathbf{n}_a + [\mathbf{L}_2 \otimes \mathbf{I}_R] \mathbf{n}_b \quad (10)$$

with covariance matrix

$$\mathbf{I}_{K_x} \otimes (\mathbf{I}_R + \boldsymbol{\Gamma}) + \mathbf{L}_2 \mathbf{L}_2^T \otimes \mathbf{I}_R = \mathbf{I}_{K_x} \otimes \boldsymbol{\Gamma} + (\mathbf{I}_{K_x} + \mathbf{L}_2 \mathbf{L}_2^T) \otimes \mathbf{I}_R \quad (11)$$

Hence we can use the alternative representation

$$\begin{aligned} \text{vec}(\boldsymbol{\Sigma}_{1,2}^{-1/2} \mathbf{X}_{1,2}) &\stackrel{d}{=} [\mathbf{I}_{K_x} \otimes \boldsymbol{\Gamma}^{1/2}] \mathbf{n}_1 + [(\mathbf{I}_{K_x} + \mathbf{M}_2)^{1/2} \otimes \mathbf{I}_R] \mathbf{n}_2 \\ &\Leftrightarrow \boldsymbol{\Sigma}_{1,2}^{-1/2} \mathbf{X}_{1,2} \stackrel{d}{=} \boldsymbol{\Gamma}^{1/2} \mathbf{N}_1 + \mathbf{N}_2 (\mathbf{I}_{K_x} + \mathbf{M}_2)^{T/2} \end{aligned} \quad (12)$$

with $\mathbf{N}_1 \sim \mathcal{N}_{R, K_x}(\mathbf{0}, \mathbf{I}_R, \mathbf{I}_{K_x})$, $\mathbf{N}_2 \sim \mathcal{N}_{R, K_x}(\mathbf{0}, \mathbf{I}_R, \mathbf{I}_{K_x})$. Therefore we obtain the following stochastic representation of $\mathbf{M}_1 | \mathbf{M}_2$:

$$\mathbf{M}_1 | \mathbf{M}_2 \stackrel{d}{=} [\boldsymbol{\Gamma}^{1/2} \mathbf{N}_1 + \mathbf{N}_2 (\mathbf{I}_{K_x} + \mathbf{M}_2)^{T/2}]^T \mathbf{W}_{1,2}^{-1} [\boldsymbol{\Gamma}^{1/2} \mathbf{N}_1 + \mathbf{N}_2 (\mathbf{I}_{K_x} + \mathbf{M}_2)^{T/2}] \quad (13)$$

with $\mathbf{W}_{1,2} \sim \mathcal{W}_R(K_y - N + R, \mathbf{I}_R)$. Finally, if we let $\mathbf{\Gamma} = \mathbf{U}\mathbf{A}\mathbf{U}^T$ be the eigenvalue decomposition of $\mathbf{\Gamma}$ then

$$\begin{aligned} \mathbf{M}_1|\mathbf{M}_2 &\stackrel{d}{=} [\mathbf{U}\mathbf{A}^{1/2}\mathbf{N}_1 + \mathbf{N}_2(\mathbf{I}_{K_x} + \mathbf{M}_2)^{T/2}]^T \mathbf{W}_{1,2}^{-1} [\mathbf{U}\mathbf{A}^{1/2}\mathbf{N}_1 \\ &\quad + \mathbf{N}_2(\mathbf{I}_{K_x} + \mathbf{M}_2)^{T/2}] \\ &\stackrel{d}{=} [\mathbf{A}^{1/2}\mathbf{N}_1 + \mathbf{U}^T \mathbf{N}_2(\mathbf{I}_{K_x} + \mathbf{M}_2)^{T/2}]^T (\mathbf{U}^T \mathbf{W}_{1,2} \mathbf{U})^{-1} [\mathbf{A}^{1/2}\mathbf{N}_1 \\ &\quad + \mathbf{U}^T \mathbf{N}_2(\mathbf{I}_{K_x} + \mathbf{M}_2)^{T/2}] \\ &\stackrel{d}{=} [\mathbf{A}^{1/2}\mathbf{N}_1 + \mathbf{N}_2(\mathbf{I}_{K_x} + \mathbf{M}_2)^{T/2}]^T \mathbf{W}_{1,2}^{-1} [\mathbf{A}^{1/2}\mathbf{N}_1 + \mathbf{N}_2(\mathbf{I}_{K_x} + \mathbf{M}_2)^{T/2}] \end{aligned} \quad (14)$$

where we used the fact that $\mathbf{U}^T \mathbf{N}_2 \stackrel{d}{=} \mathbf{N}_2$ and $\mathbf{U}^T \mathbf{W}_{1,2} \mathbf{U} \stackrel{d}{=} \mathbf{W}_{1,2}$ and kept the initial variables. Consequently the distribution of $\mathbf{M}_1|\mathbf{M}_2$ depends only on \mathbf{A} , i.e., on the set of eigenvalues of $\mathbf{\Gamma} = \mathbf{\Sigma}_{1,2}^{-1/2} \mathbf{\Omega} \mathbf{\Sigma}_{1,2}^{-T/2}$ which is nothing but $\text{IMI}(\mathbf{\Omega}, \mathbf{\Sigma})$ in (5b).

2.3. The matrix $(\mathbf{I}_{K_x} + \mathbf{M}_2)^{-1/2} \mathbf{M}_1 (\mathbf{I}_{K_x} + \mathbf{M}_2)^{-T/2}$

Let \mathbf{L} be a square-root of $\mathbf{I}_{K_x} + \mathbf{M}_2$, i.e., $\mathbf{L}\mathbf{L}^T = \mathbf{I}_{K_x} + \mathbf{M}_2$. Then

$$\begin{aligned} \mathbf{M}_1|\mathbf{M}_2 &\stackrel{d}{=} [\mathbf{A}^{1/2}\mathbf{N}_1 + \mathbf{N}_2\mathbf{L}^T]^T \mathbf{W}_{1,2}^{-1} [\mathbf{A}^{1/2}\mathbf{N}_1 + \mathbf{N}_2\mathbf{L}^T] \\ &\stackrel{d}{=} \mathbf{L} [\mathbf{A}^{1/2}\mathbf{N}_1\mathbf{L}^{-T} + \mathbf{N}_2]^T \mathbf{W}_{1,2}^{-1} [\mathbf{A}^{1/2}\mathbf{N}_1\mathbf{L}^{-T} + \mathbf{N}_2] \mathbf{L}^T \end{aligned} \quad (15)$$

and subsequently

$$\mathbf{F} = \mathbf{L}^{-1} \mathbf{M}_1 \mathbf{L}^{-T} |\mathbf{M}_2 \stackrel{d}{=} [\mathbf{A}^{1/2}\mathbf{N}_1\mathbf{L}^{-T} + \mathbf{N}_2]^T \mathbf{W}_{1,2}^{-1} [\mathbf{A}^{1/2}\mathbf{N}_1\mathbf{L}^{-T} + \mathbf{N}_2] \quad (16)$$

Under \mathbb{H}_0 we have $\mathbf{F} \stackrel{d}{=} \mathbf{N}_2^T \mathbf{W}_{1,2}^{-1} \mathbf{N}_2$ with $\mathbf{N}_2 \sim \mathcal{N}_{R, K_x}(\mathbf{0}, \mathbf{I}_R, \mathbf{I}_{K_x})$ and $\mathbf{W}_{1,2} \sim \mathcal{W}_R(K_y - N + R, \mathbf{I}_R)$ so that \mathbf{F} follows a central F distribution, similarly to the first-order model. Therefore any test statistic based on \mathbf{F} will ensure a constant false alarm rate. On the other hand, the distribution of \mathbf{F} under \mathbb{H}_1 is rather different from that of the first-order model where it obeys a non-central F distribution. This is no longer the case with a second-order model for which the distribution of \mathbf{F} under \mathbb{H}_1 seems very difficult to obtain due to the fact that $\mathbf{N} = \mathbf{A}^{1/2}\mathbf{N}_1\mathbf{L}^{-T} + \mathbf{N}_2$ does not follow a classical matrix Gaussian distribution. Indeed the covariance matrix of $\mathbf{n} = \text{vec}\{\mathbf{N}\}$, namely $\mathbb{E}\{\mathbf{n}\mathbf{n}^T\} = \mathbf{I}_{RK_x} + (\mathbf{I}_{K_x} + \mathbf{M}_2) \otimes \mathbf{A}$, does not have a Kronecker structure and therefore obtaining an explicit expression for the distribution of \mathbf{F} does not seem possible. Actually, if $K_x \leq R$, $\mathbf{F}|\mathbf{N} \sim \mathcal{W}_{K_x}(K_y - N, (\mathbf{N}^T \mathbf{N})^{-1})^{-1}$ which implies that

$$p(\mathbf{F}|\mathbf{N}) \propto |\mathbf{N}^T \mathbf{N}|^{\frac{1}{2}(K_y - N)} |\mathbf{F}|^{-\frac{1}{2}(K_y - N + K_x + 1)} \text{etr} \left\{ -\frac{1}{2} \mathbf{N}^T \mathbf{N} \mathbf{F}^{-1} \right\} \quad (17)$$

and

$$p(\mathbf{N}) \propto |\mathbf{I}_{RK_x} + (\mathbf{I}_{K_x} + \mathbf{M}_2) \otimes \mathbf{A}|^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} \mathbf{n}^T (\mathbf{I}_{RK_x} + (\mathbf{I}_{K_x} + \mathbf{M}_2) \otimes \mathbf{A})^{-1} \mathbf{n} \right\} \quad (18)$$

Performing the integration $\int p(\mathbf{F}|\mathbf{N})p(\mathbf{N})d\mathbf{N}$ does not seem feasible. To illustrate this a bit further, let us consider the special case of a single observation, i.e., $K_x = 1$. In this case the MIS becomes $(m_1 = \mathbf{x}_{1,2}^T \mathbf{S}_{1,2}^{-1} \mathbf{x}_{1,2}, m_2 = \mathbf{x}_2^T \mathbf{S}_{2,2}^{-1} \mathbf{x}_2)$ where $\mathbf{X} \rightarrow \mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$. Then we have

$$\mathbf{\Sigma}_{1,2}^{-1/2} \mathbf{x}_{1,2} | m_2 \stackrel{d}{=} \mathbf{\Gamma}^{1/2} \mathcal{N}_{R,1}(\mathbf{0}, \mathbf{I}_R) + (1 + m_2)^{1/2} \mathcal{N}_{R,1}(\mathbf{0}, \mathbf{I}_R) \quad (19)$$

$$\sim \mathcal{N}_{R,1}(\mathbf{0}, (1 + m_2)\mathbf{I}_R + \mathbf{\Gamma}) \stackrel{d}{=} [(1 + m_2)\mathbf{I}_R + \mathbf{\Gamma}]^{1/2} \mathbf{n}_{1,2} \quad (20)$$

where $\mathbf{n}_{1,2} \sim \mathcal{N}_{R,1}(\mathbf{0}, \mathbf{I}_R)$. Since $\mathbf{S}_{1,2} \sim \mathcal{W}_R(K_y - N + R, \mathbf{\Sigma}_{1,2}) \stackrel{d}{=} \mathbf{S}_{1,2}^{1/2} \mathbf{W}_{1,2} \mathbf{S}_{1,2}^{T/2}$ with $\mathbf{W}_{1,2} \sim \mathcal{W}_R(K_y - N + R, \mathbf{I}_R)$ we have

$$\mathbf{x}_{1,2}^T \mathbf{S}_{1,2}^{-1} \mathbf{x}_{1,2} | \mathbf{x}_{1,2} \stackrel{d}{=} \mathbf{x}_{1,2}^T \mathbf{\Sigma}_{1,2}^{-T/2} \mathbf{W}_{1,2}^{-1} \mathbf{\Sigma}_{1,2}^{-1/2} \mathbf{x}_{1,2} \stackrel{d}{=} [\chi_{K_y - N + 1}^2(0)]^{-1} [\mathbf{\Sigma}_{1,2}^{-1} \mathbf{x}_{1,2}]^T [\mathbf{\Sigma}_{1,2}^{-1} \mathbf{x}_{1,2}] \quad (21)$$

If we let $\mathbf{\Gamma} = \mathbf{U}\mathbf{A}\mathbf{U}^T$ be the eigenvalue decomposition of $\mathbf{\Gamma}$, then

$$[\mathbf{\Sigma}_{1,2}^{-1} \mathbf{x}_{1,2}]^T [\mathbf{\Sigma}_{1,2}^{-1} \mathbf{x}_{1,2}] | m_2 \stackrel{d}{=} \mathbf{n}_{1,2}^T \mathbf{U} [(1 + m_2)\mathbf{I}_R + \mathbf{A}] \mathbf{U}^T \mathbf{n}_{1,2} \stackrel{d}{=} \mathbf{n}^T [(1 + m_2)\mathbf{I}_R + \mathbf{A}] \mathbf{n} \quad (22)$$

with $\mathbf{n} \sim \mathcal{N}_{R,1}(\mathbf{0}, \mathbf{I}_R)$. Therefore $m_1|m_2$ is a quadratic form in complex normal variates, and

$$f = \frac{m_1}{1 + m_2} | m_2 \stackrel{d}{=} \frac{\sum_{r=1}^R (1 + \frac{\lambda_r}{1+m_2}) \chi_1^2(0)}{\chi_{K_y - N + 1}^2(0)} \quad (23)$$

Although its statistical representation is rather simple, such random variable does not admit a closed-form probability density function.

Anyway even having an expression of $p(\mathbf{F})$ does not guarantee that a test statistic can be obtained in a straightforward manner. In what follows we adopt a more pragmatic approach and look at test statistics which are simple functions of \mathbf{F} .

3. Some possible test statistics

In this section we briefly discuss invariant test statistics. Since the distribution of \mathbf{F} is parameter-free under \mathbb{H}_0 , we naturally look at test statistics which are a function of \mathbf{F} . First, it should be noted that the GLRT for the first-order model uses as test statistic

$$t_{\text{GLR}} = \frac{|\mathbf{I}_{K_x} + \mathbf{M}_1 + \mathbf{M}_2|}{|\mathbf{I}_{K_x} + \mathbf{M}_2|} = |\mathbf{I}_{K_x} + \mathbf{F}| \quad (24)$$

Note that the GLRT for the second-order model was derived in [7] and assessed in [8]. While much more computationally involved than its first-order counterpart it does not provide any significant improvement over (24) and hence we focus on t_{GLR} . Another possible test statistics could be $t_{\text{Khatri}} = \text{Tr}\{\mathbf{F}(\mathbf{I}_{K_x} + \mathbf{F})^{-1}\}$ which was proposed in [17]. A logical route is also to look for an estimate of $\text{Tr}\{\mathbf{A}\}$ since the latter is zero under \mathbb{H}_0 and positive under \mathbb{H}_1 . Towards this end, one can write

$$\begin{aligned} \mathbb{E}\{\mathbf{F}|\mathbf{M}_2\} &= \mathbb{E}\{\mathbf{L}^{-1} \mathbf{N}_1^T \mathbf{A}^{1/2} \mathbf{W}_{1,2}^{-1} \mathbf{A}^{1/2} \mathbf{N}_1 \mathbf{L}^{-T} |\mathbf{M}_2\} + \mathbb{E}\{\mathbf{N}_2^T \mathbf{W}_{1,2}^{-1} \mathbf{N}_2\} \\ &= \frac{1}{K_y - N - 1} [\mathbf{L}^{-1} \mathbb{E}\{\mathbf{N}_1^T \mathbf{A} \mathbf{N}_1\} \mathbf{L}^{-T} + \mathbf{R}_{\mathbf{I}_{K_x}}] \\ &= \frac{1}{K_y - N - 1} [\text{Tr}\{\mathbf{A}\} \mathbf{L}^{-1} \mathbf{L}^{-T} + \mathbf{R}_{\mathbf{I}_{K_x}}] \end{aligned} \quad (25)$$

where we used the fact that $\mathbb{E}\{\mathbf{N}_1^T \mathbf{A} \mathbf{N}_1\} = \text{Tr}\{\mathbf{A}\} \mathbf{I}_{K_x}$. As a consequence

$$\mathbb{E}\{\text{Tr}\{\mathbf{F}\}|\mathbf{M}_2\} = \frac{RK_x + \text{Tr}\{\mathbf{A}\} \text{Tr}\{(\mathbf{I}_{K_x} + \mathbf{M}_2)^{-1}\}}{K_y - N - 1} \quad (26)$$

and

$$t = \frac{(K_y - N - 1) \text{Tr}\{\mathbf{F}\} - RK_x}{\text{Tr}\{(\mathbf{I}_{K_x} + \mathbf{M}_2)^{-1}\}} \quad (27)$$

is an unbiased estimate of $\text{Tr}\{\mathbf{A}\}$ since $\mathbb{E}\{t\} = \mathbb{E}\{\mathbb{E}\{t|\mathbf{M}_2\}\} = \text{Tr}\{\mathbf{A}\}$. Since the latter is positive we may use as a test statistic

$$t_{\text{Tr}\{\mathbf{F}\}|\mathbf{M}_2} = \max(t, 0) \quad (28)$$

From (26), we note that $\mathbb{E}\{\text{Tr}\{\mathbf{F}\}\}$ will be an affine function of $\text{Tr}\{\mathbf{A}\}$ and thus $\text{Tr}\{\mathbf{F}\}$ can also serve a test statistic. Finally we also investigate the Frobenius norm $\|\mathbf{F}\|_F$ as a possible test statistic.

4. Numerical illustration

We provide an illustration of the probability of detection of the four detectors derived above. We have a scenario where $N = 32$ and the signal of interest is of the form $A \sin(2\pi n f_s + \varphi)$ where $f_s = 0.05$, A is Rayleigh distributed and φ is uniformly distributed. The columns of \mathbf{H}_i consist of sine and cosine functions and $R = 2$. The noise has covariance matrix $\mathbf{\Sigma}_i = \mathbf{\Sigma}_c + \sigma_n^2 \mathbf{I}_N$ with $\mathbf{\Sigma}_c(m, n) = \exp\{-2\pi^2 \sigma_f^2 (m - n)^2\}$ where $\sigma_f = 0.01$ and $\text{Tr}\{\mathbf{\Sigma}_c\} / \text{Tr}\{\sigma_n^2 \mathbf{I}_N\} = 20\text{dB}$. The signal to noise ratio is defined as $\text{SNR} = \text{Tr}\{\mathbf{\Omega}\} / \text{Tr}\{(\mathbf{H}^T \mathbf{\Sigma}^{-1} \mathbf{H})^{-1}\}$. The probability of false alarm is set to $P_{fa} = 10^{-3}$. In Fig. 1 we compare the probability of detection versus SNR. It can be observed that the trace-based detectors perform rather close to the GLR, especially t_{Khatri} . This is confirmed in Fig. 2 where the difference is small whatever K_x . In contrast, the detector based on $\|\mathbf{F}\|_F$ has a smaller P_d . Finally in Fig. 3 we consider the mismatched case where the signal has true frequency \bar{f}_s but the detectors use f_s in their hypotheses. Again, all trace-based detectors show a similar behavior in terms of robustness/selectivity.

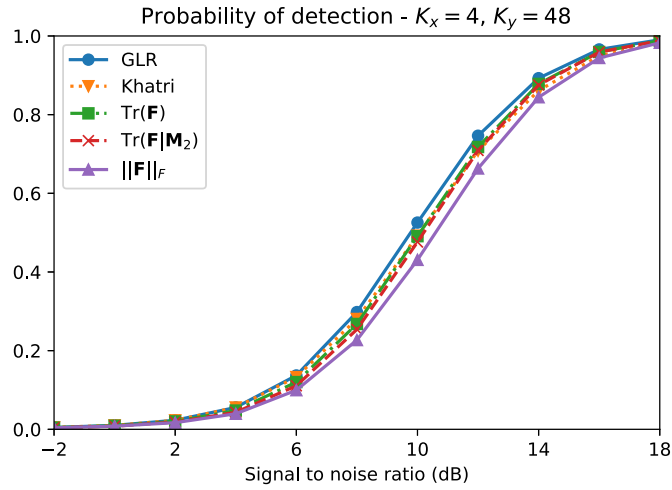


Fig. 1. Probability of detection versus SNR. $K_x = 4$, $K_y = 48$.

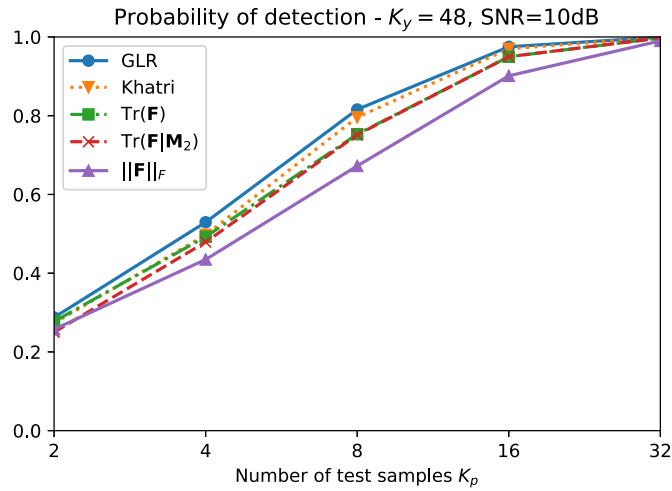


Fig. 2. Probability of detection versus K_x . $K_y = 48$ and SNR = 10dB.

5. Conclusion

In this short communication we derived the maximal invariant statistic and the induced maximal invariant for second-order subspace signals where the signal coordinates within its subspace are random and Gaussian distributed. While the MIS is the same as in the first-order model, we showed that its distribution is quite different under \mathbb{H}_1 . More precisely we derived a statistical representation of the MIS in terms of variables with known distributions. From there we proposed some simple detectors based on the trace of some matrix which were shown to perform closely to the GLRT. This paper thus completes previous results on first-order models and provides new insights into the differences/similarities between first-order and second-order subspace models.

CRedit authorship contribution statement

Olivier Besson: Writing - original draft, Software, Methodology, Conceptualization.

Data availability

No data was used for the research described in the article.

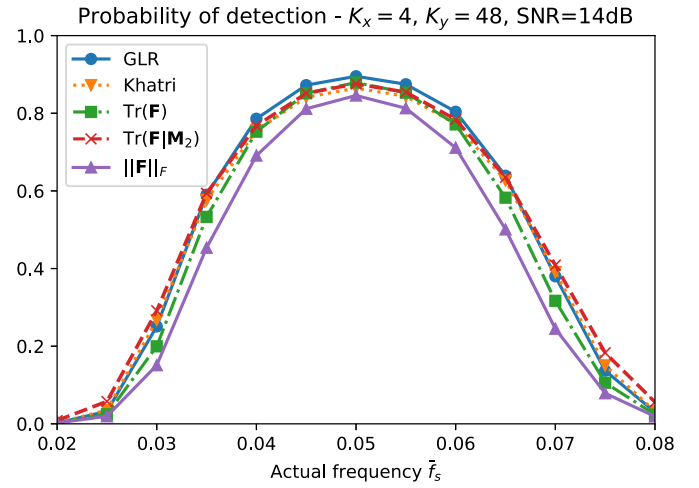


Fig. 3. Probability of detection versus \bar{f}_s . $K_x = 4$, $K_y = 48$ and SNR = 14dB.

Declaration of competing interest

Olivier BESSON reports was provided by ISAE-SUPAERO Higher Institute of Aerospace Engineering. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Proof of proposition

Since $g(\cdot)$ is the same as in the first-order model, the MIS is the same as in the first-order model and is given by two $K_x \times K_x$ matrices $\mathbf{M}_1 = \mathbf{X}_{1,2}^T \mathbf{S}_{1,2}^{-1} \mathbf{X}_{1,2}$ and $\mathbf{M}_2 = \mathbf{X}_2^T \mathbf{S}_{22}^{-1} \mathbf{X}_2$, see [13,18] for a proof.

Let us consider now the induced maximal invariant and prove (5b). Towards this end two steps are necessary. First we need to show that $\text{IMI}(\tilde{g}(\tilde{\Omega}, \tilde{\Sigma})) = \text{IMI}(\tilde{\Omega}, \tilde{\Sigma})$. Next we need to show that if $\text{IMI}(\tilde{\Omega}, \tilde{\Sigma}) = \text{IMI}(\tilde{\Omega}, \tilde{\Sigma})$ then $(\tilde{\Omega}, \tilde{\Sigma}) = \tilde{g}(\tilde{\Omega}, \tilde{\Sigma})$. As for the first part, since by $g(\cdot)$, $\tilde{\Sigma} \rightarrow \mathbf{A} \tilde{\Sigma} \mathbf{A}^T$ and \mathbf{A} is block-upper-triangular, we get $\tilde{\Sigma}_{1,2} \rightarrow \mathbf{A}_{11} \tilde{\Sigma}_{1,2} \mathbf{A}_{11}^T$. Consequently

$$\lambda(\tilde{\Sigma}_{1,2}^{-1/2} \tilde{\Omega} \tilde{\Sigma}_{1,2}^{-T/2}) \rightarrow \lambda(\tilde{\Sigma}_{1,2}^{-1/2} \mathbf{A}_{11}^{-1} \mathbf{A}_{11} \tilde{\Omega} \mathbf{A}_{11}^T \mathbf{A}_{11}^{-T} \tilde{\Sigma}_{1,2}^{-T/2}) = \lambda(\tilde{\Sigma}_{1,2}^{-1/2} \tilde{\Omega} \tilde{\Sigma}_{1,2}^{-T/2}) \quad (\text{A.1})$$

Suppose now that $(\tilde{\Omega}, \tilde{\Sigma})$ and $(\tilde{\Omega}, \tilde{\Sigma})$ are such that $(\tilde{\Omega}, \tilde{\Sigma}) = \tilde{g}(\tilde{\Omega}, \tilde{\Sigma})$. Whatever $\tilde{\Sigma}$ and $\tilde{\Sigma}$ there exists an upper-triangular matrix \mathbf{A} such that $\tilde{\Sigma} = \mathbf{A} \tilde{\Sigma} \mathbf{A}^T$: actually $\mathbf{A} = \text{chol}_{\text{up}}(\tilde{\Sigma}) \text{chol}_{\text{up}}(\tilde{\Sigma})^{-1}$ where $\text{chol}_{\text{up}}(\cdot)$ stands for the upper-triangular Cholesky factor. This implies that $\tilde{\Sigma}_{1,2} = \mathbf{A}_{11} \tilde{\Sigma}_{1,2} \mathbf{A}_{11}^T$. Since $\tilde{\Gamma} = \tilde{\Sigma}_{1,2}^{-1/2} \tilde{\Omega} \tilde{\Sigma}_{1,2}^{-T/2}$ and $\tilde{\Gamma} = \tilde{\Sigma}_{1,2}^{-1/2} \tilde{\Omega} \tilde{\Sigma}_{1,2}^{-T/2}$ share the same eigenvalues there exists a unitary matrix \mathbf{Q} such that $\tilde{\Gamma} = \mathbf{Q}^T \tilde{\Gamma} \mathbf{Q}$, that is to say

$$\tilde{\Sigma}_{1,2}^{-1/2} \tilde{\Omega} \tilde{\Sigma}_{1,2}^{-T/2} = \mathbf{Q}^T \tilde{\Sigma}_{1,2}^{-1/2} \tilde{\Omega} \tilde{\Sigma}_{1,2}^{-T/2} \mathbf{Q} \quad (\text{A.2})$$

Now $\text{IMI}(\tilde{\Omega}, \tilde{\Sigma})$ is independent of the square-root chosen for $\tilde{\Sigma}$ so we can choose $\tilde{\Sigma}_{1,2}^{-1/2} = \mathbf{A}_{11} \tilde{\Sigma}_{1,2}^{-1/2} \mathbf{Q}$ which leads to

$$\tilde{\Sigma}_{1,2}^{-1/2} \tilde{\Omega} \tilde{\Sigma}_{1,2}^{-T/2} = \mathbf{Q}^T \tilde{\Sigma}_{1,2}^{-1/2} \mathbf{A}_{11}^{-1} \tilde{\Omega} \mathbf{A}_{11}^{-T} \tilde{\Sigma}_{1,2}^{-T/2} \mathbf{Q} \quad (\text{A.3})$$

Gathering the two previous equations we obtain $\tilde{\Omega} = \mathbf{A}_{11}^{-1} \tilde{\Omega} \mathbf{A}_{11}^{-T}$ and therefore $\tilde{\Omega} = \mathbf{A}_{11} \tilde{\Omega} \mathbf{A}_{11}^T$ which shows that $(\tilde{\Omega}, \tilde{\Sigma}) = \tilde{g}(\tilde{\Omega}, \tilde{\Sigma})$ and completes the proof.

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