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THE SHI VARIETY CORRESPONDING TO AN AFFINE WEYL GROUP

NATHAN CHAPELIER-LAGET

ABSTRACT. Let W be an irreducible Weyl group and W_a its affine Weyl group. In this article we show that there exists a bijection between W_a and the integral points of an affine variety, denoted \widehat{X}_{W_a} , which we call the Shi variety of W_a . In order to do so, we use Jian-Yi Shi's characterization of alcoves in affine Weyl groups. We then study this variety further. We introduce a new representation of W_a , called the Φ^+ -representation, and we highlight combinatorial and geometrical properties of the irreducible components of \widehat{X}_{W_a} via this representation. We also show how the components are related to a fundamental parallelepiped $P_{\mathcal{H}}$.

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1. INTRODUCTION

One of the reasons that Coxeter groups are so well explored is due to their appearance as a fundamental structure in several mathematical theories, such as group theory, representation of algebras, combinatorics, geometry, mathematical physics etc.

The cell theory of a Coxeter group W , as defined by Kazhdan and Lusztig in [9], is related to many disparate and deep topics in mathematics, including the representation theory of W and its Hecke algebra, the representation theory of algebraic groups, finite groups of Lie type, the geometry of unipotent conjugacy classes in simple complex algebraic groups, Lie algebras, primitive ideals in universal enveloping algebras, singularities of Schubert varieties etc.

One of the main challenge of this theory is to give a precise description of the cells of W and to explore their properties. Describing the cells is in general a very difficult question since there is no general method. Many questions on this subject are still open.

When the Coxeter group is an affine Weyl group, this theory yielded a lot of research works in recent decades. It was first studied by G. Lusztig in [10] where he described all the cells of the affine Weyl groups of rank ≤ 2 .

In 1983, Shi described all the cells of $W(\tilde{A}_n)$ (see [12]). To do so he introduced the notion of sign types, which leads to the notion of signed regions (the notion of signed region is discussed in Section 1.4). These notions stem from a characterization of the elements of $W(\tilde{A}_n)$ in terms of certain inequalities. A few years later he generalized in [13, 14] this work to any other affine Weyl group (see Theorem 2.1). We refer to this as *Shi's characterization*.

1.1. Goals. This article serves as foundation of Shi varieties. In this paper we revisit, extend and explore further Shi's characterization. We then introduce and investigate an affine variety \hat{X}_{W_a} associated to an affine Weyl group W_a , which we call the *the Shi variety* of W_a . Our main result, besides introducing the Shi varieties, is stated in Theorem 4.3. We reformulate it as follows:

Theorem 1.1. *There is a bijection between W_a and the integral points of \hat{X}_{W_a} , that is*

$$W_a \simeq \hat{X}_{W_a}(\mathbb{Z}).$$

Before delving into the notation, the definitions, and more technical aspects, let us explain why the Shi variety deserves our attention.

On the one hand, the set of inequalities in Theorem 2.1 is significant. We need at least as many inequalities as irreducible subroot systems of rank 2. For example, in type A_n , there are $(n-1)^2$ such root systems. These inequalities only provide “local” information on the Shi coefficients over each position $\alpha \in \Phi^+$. Achieving a “global” understanding of all possible vectors that satisfy all the inequalities simultaneously is quite challenging. For example, given a root α , it might be tedious, first to find all the irreducible subroot systems of rank 2 containing α , and second to check what coefficients can be chosen over this position. Moreover, the construction of a Shi vector $(k_\alpha)_{\alpha \in \Phi^+}$ follows an inductive approach with respect to the height of the roots. Therefore, we need to first satisfy all the inequalities for the roots of height less than m in order to build a Shi coefficient over a position of height $m+1$. This final phase is often the most strenuous, as it can be quite time-consuming.

On the other hand, the Shi variety is governed by two factors: a set of equations (see Section 4.3), and the elements in the fundamental parallelepiped $P_{\mathcal{H}}$ (see Section 2.3), which turn out to be in bijection with a family of vectors: the admitted vectors, but also with the conjugacy class of the full cycle $(1\ 2\ \cdots\ n+1)$ [5, Thm. 1.3]. More precisely, the variety \hat{X}_{W_a} is given by means of an ideal $J_{W_a} = \prod_{\lambda \text{ admitted}} J_{W_a}[\lambda]$ where the ideals $J_{W_a}[\lambda]$ are generated by the same linear polynomials up to a collection of affine terms (the coefficients of λ), see Examples 4.2 and 4.4. Therefore, in order to obtain \hat{X}_{W_a} it is sufficient to understand these polynomials. Furthermore, the number of polynomials generating $J_{W_a}[\lambda]$ is $|\Phi^+| - |\Delta|$. In type A_n this number is $n(n+1)/2 - n = n(n-1)/2$. In other words, the complexity of constructing a Shi vector, in terms of the inequalities is $(n-1)^2$, and in terms of the equations is $n(n-1)/2$. With the second approach, the time complexity is reduced by a factor of $(n-1)(n-2)/2$.

1.2. Notation and setup. For further developments we need to recall some basics about affine Weyl groups (all this material is explained in detail in Section 2).

Let V be a Euclidean space, $\Phi \subset V$ be an irreducible crystallographic root system, and W be the corresponding Weyl group. Associated to Φ there exists an infinite hyperplane arrangement, denoted \mathcal{H} , known as the affine Coxeter arrangement. This hyperplane arrangement cuts V into simplices of the same volume which are called alcoves. The set of alcoves is denoted by \mathcal{A} .

The associated affine Weyl group W_a acts regularly on \mathcal{A} , implying then a one-to-one correspondence between W_a and \mathcal{A} (see [7] Ch. 4 or Section 2 for more details). Let A_w be the alcove corresponding to $w \in W_a$ via this bijection. It turns out that A_w gives rise to a collection of integers $k(w, \alpha)$ for $\alpha \in \Phi^+$, and those integers, together with their relations between them, will be at the core of this paper (see Section 2.2 for the definition of $k(w, \alpha)$).

1.3. The plan of this work. The present paper has several goals that we describe now. First we recall in Section 2 some standard definitions and terminologies related to affine Weyl groups.

- The first goal is to introduce the necessary tools in order to see elements of W_a as isometries in $\bigoplus_{\theta \in \Phi^+} \mathbb{R}e_\theta$. This leads us to define the Φ^+ -representation in Section 3. This new object has an important role in the understanding of the natural action of W_a on the variety \widehat{X}_{W_a} . To do so, we first generalize a few formulas of [13] about the coefficients $k(w, \alpha)$. This is done in Section 3 and mainly in Proposition 3.2.

- The second goal, achieved in Section 4, is to build the variety \widehat{X}_{W_a} , whose integral points $\widehat{X}_{W_a}(\mathbb{Z})$ are in bijection with W_a , and to study the set of its irreducible components. The above statement is one of the main result of this article (see Theorem 4.3). To establish this, we first obtain a set of equations that cuts out an affine variety denoted X_{W_a} , and where the Φ^+ -tuples of integers of W_a are solutions of these equations. However, this variety is too large and we only have an injection of W_a into $X_{W_a}(\mathbb{Z})$. We then see how to shrink X_{W_a} in order to get a one-to-one correspondence between W_a and the integers points of a subvariety of X_{W_a} . This is where we introduce the notion of admissible and admitted vectors (see Definitions 4.1 and 4.3). These definitions yield a decomposition of X_{W_a} as follows (see Theorem 4.3):

$$X_{W_a} := \bigsqcup_{\lambda \text{ admissible}} X_{W_a}[\lambda],$$

whereas the subvariety \widehat{X}_{W_a} inherits the decomposition:

$$\widehat{X}_{W_a} := \bigsqcup_{\lambda \text{ admitted}} X_{W_a}[\lambda],$$

where the $X_{W_a}[\lambda]$ are the irreducible components of these two varieties. The difference between being admissible or admitted leads us to recover a polytope $P_{\mathcal{H}}$, which has already been well studied (see [2] Ch. VI, § 1.10 or [7] 4.9 or [8] pages 131-134), and which plays a crucial role thereafter. Indeed, thanks to it, we give among others the number of irreducible components of \widehat{X}_{W_a} (see Theorem 4.3).

1.4. Some Consequences of the Shi variety. Besides revisiting Shi's characterization and extending it in terms of equations, the Shi variety encompasses several notions, both algebraic and combinatorial.

- As a first consequence, that we don't explain in this paper, the Shi regions (regions defined in full generality in the Tits cone) are easy to construe and describe in terms of \widehat{X}_{W_a} since they are naturally in bijection with the signed regions introduced in [14] and since the signed regions lie naturally in the Shi variety.

- The second consequence of the Shi variety, established in [5], is the relation between a specific conjugacy class of $W(A_n)$ and the set of irreducible components of $\widehat{X}_{W(\tilde{A}_n)}$, that we denote by

$H^0(\widehat{X}_{W(\widetilde{A}_n)})$. More precisely we show that there is a natural bijection between the conjugacy class of $(1\ 2\ \cdots\ n+1)$ and the aforementioned set (see [5] Theorem 1.3). This result gives rise to several questions related to conjugacy class theory, such as: What can we say about other conjugacy classes in terms of a variant of the Shi variety (in any type) ?

- The parameterization of the irreducible components of \widehat{X}_{W_a} by the admitted vectors (see Theorem 4.3) gave rise to several interesting questions in type A . The set of admitted vector is partially ordered. Since this set is in bijection with the conjugacy class of $(1\ 2\ \cdots\ n+1)$, we obtain a partial order on this conjugacy class. In a joint work with A. Abram and C. Reutenauer, we study in [1] this poset with its different instances. We show in particular that this poset is a ranked poset of height $\binom{n}{3}$, showing us at the same time that it is cubic in n , which is rather unusual as we were told by R. Stanley. We then show that these two posets are isomorphic. From this isomorphism we derive several consequences, such as: it is a semidistributive lattice. We also connect these posets with Eulerian numbers, triangulations of an n -gon together with their mutations and Young's lattice. In particular we show that Young's lattice is isomorphic to $\lim_{n \rightarrow \infty} H^0(\widehat{X}_{W(\widetilde{A}_n)})$ (see [1] Theorem 32). Finally, we introduce in [1] a topological object, that we call *the line diagram*, and which allows us to describe precisely the bijection between our posets. The line diagrams are connected with some knot theory transformations: the Reidemeister moves. We show that performing Reidemeister moves on these diagrams is equivalent to move in the poset.

- Using [1, 5] we established another connection between the two aforementioned posets and the poset of the right weak order of $W(A_n)$: the Hasse diagram of the poset of irreducible components of $\widehat{X}_{W(\widetilde{A}_n)}$ has the same number of edges as the poset of the right weak order of $W(A_n)$, even though these two posets are not isomorphic. In the light of the work done in [1], it was natural to investigate the combinatorial properties of this poset in full generality. In [4] we study the poset of irreducible components of \widehat{X}_{W_a} for any affine Weyl group and we show that it also has a structure of semidistributive lattice.

- Darij Grinberg brought to our attention the problem 2 proposed by David Speyer on the USAMO 2010. It turns out that the properties of the poset, studied in [1], give a direct answer to this problem.

- The poset of irreducible components of \widehat{X}_{W_a} seems to be related to another poset introduced by D. Speyer et al in [6]. In this article the authors study the looping case of Mozes's game of numbers, which concerns the orbits in the reflection representation of affine Weyl groups situated on the boundary of the Tits cone. Using an operation called "firing a vertex" on the set of vertices of a Dynkin diagram, they obtain a poset which is isomorphic to the dual of triangulation of the unit hypercube by Weyl chambers. Their hypercube is what we call in this article the fundamental parallelepiped $P_{\mathcal{H}}$. It would be interesting to connect these two constructions.

- In [3] we use the Shi variety and the cohomology in degree 1 of W with coefficients in the root lattice: $H^1(W, \mathbb{Z}\Phi)$, in order to describe the orientations of alcoves. We provide a criteria via modular equations that enables us to say whether two alcoves have the same orientation or not in terms of the Shi coefficients. This criteria is a first step toward the direction of expressing the extended affine Weyl group \widehat{W}_a in terms of the Shi variety.

2. AFFINE WEYL GROUPS AND SHI PARAMETERIZATION

Let V be a Euclidean space with inner product $(-, -)$. We denote $\|x\| = \sqrt{(x, x)}$ for $x \in V$. Let Φ be an irreducible crystallographic root system in V . We assume here that Φ is essential, that is, taking the lattice $\mathbb{Z}\Phi$, one has $\mathbb{Z}\Phi \otimes_{\mathbb{Z}} \mathbb{R} = V$. From now on, when we will say ‘‘root system’’ it will always mean irreducible essential crystallographic root system.

Let W be the *Weyl group* associated to $\mathbb{Z}\Phi$, that is the maximal (for inclusion) reflection subgroup of $O(V)$ admitting $\mathbb{Z}\Phi$ as a W -equivariant lattice.

Let $\alpha \in \Phi$. We write

$$\begin{aligned} s_{\alpha} &: V \longrightarrow V \\ x &\longmapsto x - 2\frac{(\alpha, x)}{(\alpha, \alpha)}\alpha. \end{aligned}$$

It is known that the Coxeter group associated to Φ , i.e the subgroup of $O(V)$ generated by the reflections s_{α} , is actually the Weyl group W . Each Weyl group has a structure of Coxeter group with set of Coxeter generators $S := \{s_{\alpha_1}, \dots, s_{\alpha_n}\}$.

Because of the classification of irreducible crystallographic root systems, we know that there are at most two possible root lengths in Φ . We call short root the shorter ones. We require here that $\|\alpha\| = 1$ for any short root $\alpha \in \Phi^+$.

Let $\alpha \in \Phi$ such that $\alpha = a_1\alpha_1 + \dots + a_n\alpha_n$ with $a_i \in \mathbb{Z}$. The height of α (with respect to Δ) is defined by the number $h(\alpha) = a_1 + \dots + a_n$. Height provides a preorder on Φ^+ defined as $\alpha \leq \beta$ if and only if $h(\alpha) \leq h(\beta)$. We also have $h(\alpha + \beta) = h(\alpha) + h(\beta)$ and $h(-\alpha) = -h(\alpha)$ for all $\alpha, \beta, \alpha + \beta \in \Phi$. We denote by $-\alpha_0$ the *highest short root* of Φ .

2.1. Affine Weyl groups. From now on we will identify $\mathbb{Z}\Phi$ and the group of its associated translations. For $\alpha \in \Phi$ we write $\alpha^{\vee} := \frac{2\alpha}{(\alpha, \alpha)}$. Let $k \in \mathbb{Z}$. Define the affine reflection as follows

$$\begin{aligned} s_{\alpha, k} &: V \longrightarrow V \\ x &\longmapsto x - (2\frac{(\alpha, x)}{(\alpha, \alpha)} - k)\alpha. \end{aligned}$$

We consider the subgroups W_a and W_a^{\vee} of $\text{Aff}(V)$ defined as follows

$$W_a = \langle s_{\alpha, k} \mid \alpha \in \Phi, k \in \mathbb{Z} \rangle \text{ and } W_a^{\vee} = \langle s_{\alpha^{\vee}, k} \mid \alpha \in \Phi, k \in \mathbb{Z} \rangle.$$

It is known that $W_a \simeq \mathbb{Z}\Phi \rtimes W$ (see [7], Ch 4). The group W_a is called the *affine Weyl group* associated to Φ . The Coxeter group structure on W induces a Coxeter group structure on W_a . The set $S_a := \{s_{\alpha_1}, \dots, s_{\alpha_n}\} \cup \{s_{-\alpha_0, 1}\}$ is a set of Coxeter generators of W_a . For short we will write $S_a = \{s_0, s_1, \dots, s_n\}$ where $s_0 := s_{-\alpha_0, 1}$ and $s_i = s_{\alpha_i}$ for $i = 1, \dots, n$. Throughout this article, affine Weyl group means irreducible affine Weyl group.

Let us make the following comment. In the literature about Weyl groups and affine Weyl groups it is more common to define the affine Weyl groups (associated to the root system Φ) with the reflections associated to the hyperplanes $\{x \in V \mid (x, \alpha) = k\}$. Therefore what we would call the affine Weyl group associated to Φ would commonly be described as the affine Weyl group associated to Φ^{\vee} , that is W_a^{\vee} . Our choice to call W_a the affine Weyl group associated to Φ is because in [13] the author gave the definition of W_a via the hyperplanes $\{x \in V \mid (x, \alpha^{\vee}) = k\}$, that is via the reflections $s_{\alpha, k}$.

2.2. Shi parameterization. For any $\alpha \in \Phi$, any $k \in \mathbb{Z}$ and any $m \in \mathbb{R}$, we define the hyperplanes

$$\begin{aligned} H_{\alpha, k} &= \{x \in V \mid s_{\alpha, k}(x) = x\} \\ &= \{x \in V \mid (x, \alpha^{\vee}) = k\}, \end{aligned}$$

the half spaces

$$H_{\alpha, k}^+ = \{x \in V \mid k < (x, \alpha^{\vee})\}$$

and

$$H_{\alpha, k}^- = \{x \in V \mid (x, \alpha^{\vee}) < k\}$$

and the strips

$$\begin{aligned} H_{\alpha,k}^m &= \{x \in V \mid k < (x, \alpha^\vee) < k + m\} \\ &= H_{\alpha,k}^+ \cap H_{\alpha,k+m}^- \end{aligned}$$

We denote by \mathcal{H} the set of all the hyperplanes $H_{\alpha,k}$ with $\alpha \in \Phi^+$, $k \in \mathbb{Z}$, and by \mathcal{H}^\vee the set of all the hyperplanes $H_{\alpha^\vee,k}$ with $\alpha \in \Phi^+$, $k \in \mathbb{Z}$. We have the relation $H_{-\alpha,k} = H_{\alpha,-k}$. Therefore we need only to consider the hyperplanes $H_{\alpha,k}$ with $\alpha \in \Phi^+$ and $k \in \mathbb{Z}$. This last remark is quite convenient, because when we look at the collection of all the hyperplanes in V , we can think of them as being indexed either by elements of Φ or by elements of Φ^+ .

Furthermore, if we have two hyperplanes $H_{\alpha,k}$ and $H_{\alpha,k'}$ the geometric space strictly contained between them is

$$(1) \quad H_{\alpha, \min(k,k')}^{|k-k'|}$$

The lattice $\mathbb{Z}\Phi$ acts naturally on the strips and this action is given by

$$(2) \quad \tau_x H_{\alpha,k}^m = H_{\alpha, k+(\alpha^\vee, x)}^m$$

Moreover W also acts on the strips as isometries of \mathbb{R}^n and it follows that W_a acts also on the set of strips.

The connected components of

$$(3) \quad V \setminus \bigcup_{\substack{\alpha \in \Phi^+ \\ k \in \mathbb{Z}}} H_{\alpha,k}$$

are called *alcoves*. We denote \mathcal{A} the set of all the alcoves and A_e the alcove defined as $A_e = \bigcap_{\alpha \in \Phi^+} H_{\alpha,0}^1$. Since W_a acts on the strips it also acts on \mathcal{A} . It turns out that this action on \mathcal{A} is regular (see [7] chapter 4). Thus, there is a bijective correspondence between the elements of W_a and all the alcoves. This bijection is defined by $w \mapsto A_w$ where $A_w := wA_e$. We call A_w the corresponding alcove associated to $w \in W_a$. Any alcove of V can be written as an intersection of particular strips, that is there exists a Φ^+ -tuple of integers $(k(w, \alpha))_{\alpha \in \Phi^+}$ such that

$$(4) \quad A_w = \bigcap_{\alpha \in \Phi^+} H_{\alpha, k(w, \alpha)}^1$$

For any $w \in W_a$ and any $\alpha \in \Phi$ we use the convention

$$(5) \quad k(w, -\alpha) = -k(w, \alpha).$$

In the setting of affine Weyl groups the length of any element $w \in W_a$ is easy to compute via the coefficients $k(w, \alpha)$. Indeed, thanks to Proposition 4.3 in [13] we have

$$(6) \quad \ell(w) = \sum_{\alpha \in \Phi^+} |k(w, \alpha)|.$$

In [13] Shi gave a characterization of the possible Φ^+ -tuples $(k_\alpha)_{\alpha \in \Phi^+}$ over \mathbb{Z} such that these tuples are the tuples of elements in W_a . In 1999 the same author gave an easier statement of this characterization. We give below these two characterizations.

Theorem 2.1 ([13], Theorem 5.2). *Let $A = \bigcap_{\alpha \in \Phi^+} H_{\alpha, k_\alpha}^1$ with $k_\alpha \in \mathbb{Z}$. Then A is an alcove, if and only if, for all $\alpha, \beta \in \Phi^+$ satisfying $\alpha + \beta \in \Phi^+$, we have the following inequality*

$$(7) \quad \|\alpha\|^2 k_\alpha + \|\beta\|^2 k_\beta + 1 \leq \|\alpha + \beta\|^2 (k_{\alpha+\beta} + 1) \leq \|\alpha\|^2 k_\alpha + \|\beta\|^2 k_\beta + \|\alpha\|^2 + \|\beta\|^2 + \|\alpha + \beta\|^2 - 1.$$

Theorem 2.2 ([15], Theorem 1.1). *Let $A = \bigcap_{\alpha \in \Phi^+} H_{\alpha, k_\alpha}^1$ with $k_\alpha \in \mathbb{Z}$. Then A is an alcove, if and only if, for all $\alpha, \beta \in \Phi^+$ satisfying $\gamma := (\alpha^\vee + \beta^\vee)^\vee \in \Phi^+$ the following inequality holds*

$$(8) \quad k_\alpha + k_\beta \leq k_\gamma \leq k_\alpha + k_\beta + 1.$$

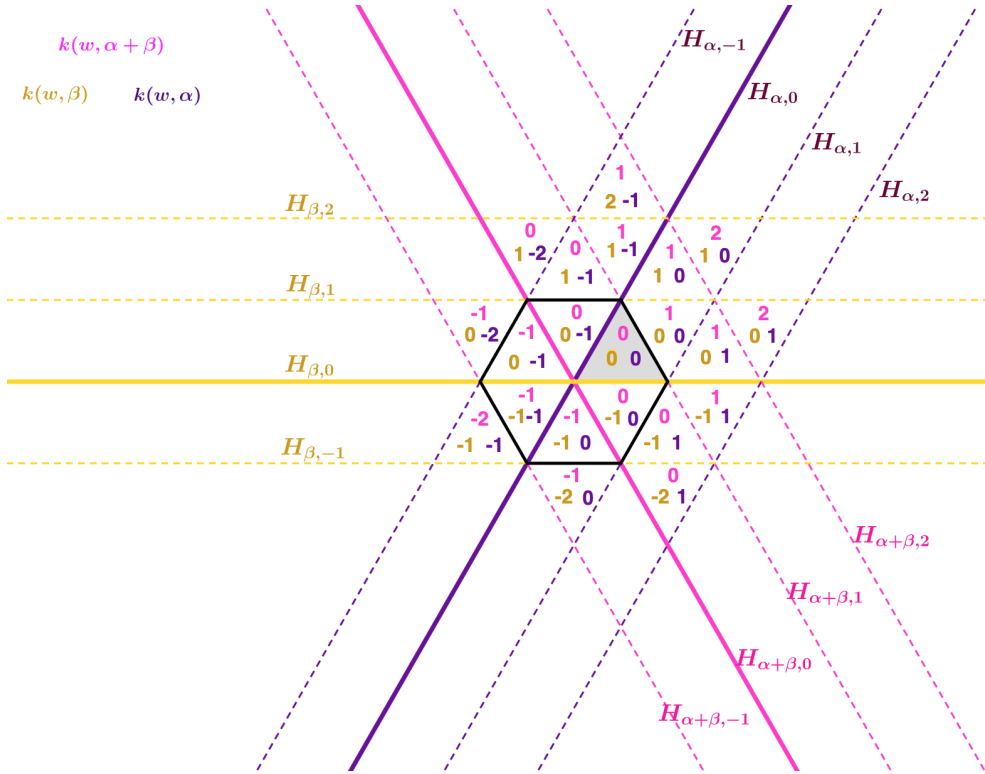


FIGURE 1. Shi parameterization of $W(\tilde{A}_2)$.

Example 2.1. For $W_a = \tilde{A}_2$, the positive root system of A_2 is given by 3 roots, say $\Phi^+ = \{\alpha, \beta, \alpha + \beta\}$.

Remark 2.1.

- (1) Notice that it's a priori not easy for a Φ^+ -tuple of integers to satisfy the inequalities (8).
- (2) We also want to warn the reader that compared to the conventions of Shi in [13] and [14], we swap left and right multiplication. The left multiplication is defined as follows: the alcove corresponding to $w'w$ is obtained by acting by the isometry w' on the alcove A_w . The right multiplication is defined as follows: the alcove corresponding to ws is obtained by crossing the wall of A_w associated to the generator s . See for example Figure 2.

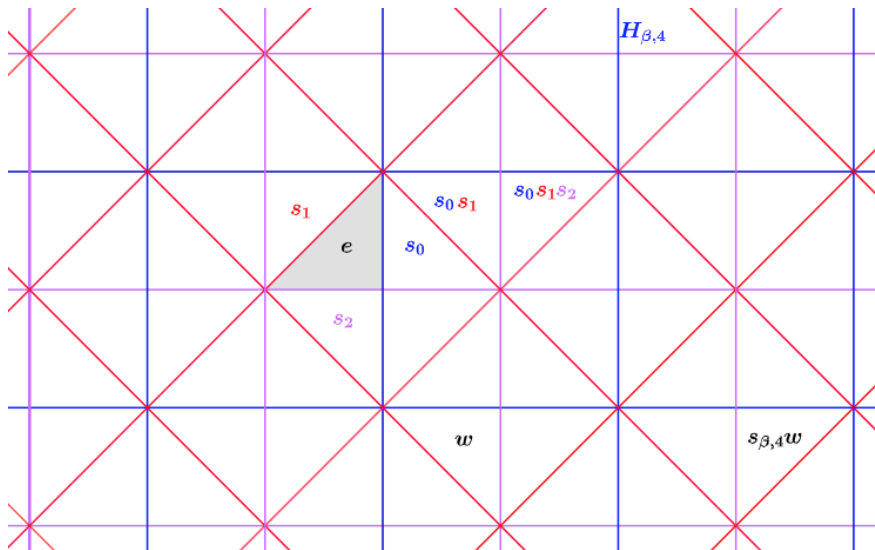


FIGURE 2. Folding of the generators of $W(\tilde{B}_2)$.

2.3. Fundamental parallelepiped $P_{\mathcal{H}}$. We set one for all $\Psi := \Phi^{\vee}$ and $\varepsilon_i := \alpha_i^{\vee}$ for any $i = 1, \dots, n$. Let $\mathbb{Z}\Psi$ be the coroot lattice and let us write $\mathbb{Z}\Psi = \mathbb{Z}\varepsilon_1 \oplus \dots \oplus \mathbb{Z}\varepsilon_n$. We define its dual lattice $\mathbb{Z}\Psi^*$ as

$$(9) \quad \mathbb{Z}\Psi^* := \{x \in V \mid (x, y) \in \mathbb{Z} \ \forall y \in \mathbb{Z}\Psi\}.$$

The lattice $\mathbb{Z}\Psi^*$ is called the *weight lattice*. This lattice can be decomposed as

$$(10) \quad \mathbb{Z}\Psi^* = \mathbb{Z}\omega_1 \oplus \dots \oplus \mathbb{Z}\omega_n$$

where ω_i is such that $(\varepsilon, \omega_j) = \delta_{ij}$. The elements ω_i are called the *fundamental weights* (with respect to Δ). Notice that it is also possible to define the weight lattice by

$$(11) \quad \mathbb{Z}\Psi^* = \{x \in V \mid (x, y) \in \mathbb{Z} \ \forall y \in \Psi\}.$$

It is clear that $\mathbb{Z}\Phi$ is a normal subgroup of $\mathbb{Z}\Psi^*$. Thus we can define their corresponding quotient group. The *index of connection* of Φ is by definition the cardinality of this quotient group. We denote it as

$$(12) \quad f_{\Phi} := \left| \mathbb{Z}\Psi^* / \mathbb{Z}\Phi \right|.$$

We define $P_{\mathcal{H}} := \bigcap_{\alpha \in \Delta} H_{\alpha,0}^1$ and $P_{\mathcal{H}^{\vee}} := \bigcap_{\alpha \in \Delta} H_{\alpha^{\vee},0}^1$. The fundamental weights ω_i are some of the vertices of $P_{\mathcal{H}}$ and we have $P_{\mathcal{H}} = \left\{ \sum_{i=1}^n c_i \omega_i \mid c_i \in [0, 1] \right\}$.

It is known (see [2] ch. VI, § 1, exercice 7) that the index of connection of Φ is the determinant of the Cartan matrix associated to Φ . Moreover, the Cartan matrix associated to Φ^{\vee} is the transpose of the Cartan matrix associated to Φ . It follows that the indexes of connection of Φ and Φ^{\vee} are the same. Finally, we define the two sets

$$\text{Alc}(P_{\mathcal{H}}) := \{w \in W_a \mid A_w \subset P_{\mathcal{H}}\} \quad \text{and} \quad \text{Alc}(P_{\mathcal{H}^{\vee}}) := \{w \in W_a^{\vee} \mid A_w \subset P_{\mathcal{H}^{\vee}}\},$$

for which it is known (see [8], Section 11-6, Lemma C) that

$$(13) \quad |\text{Alc}(P_{\mathcal{H}^{\vee}})| = \frac{|W(\Delta)|}{f_{\Phi}}.$$

Example 2.2. Let us take $W_a = W(\tilde{B}_2)$ with simple system $\{\alpha_1, \alpha_2\}$. A short computation shows that $\omega_1 = \frac{1}{2}(2\alpha_1 + \alpha_2)$ and $\omega_2 = \alpha_1 + \alpha_2$.

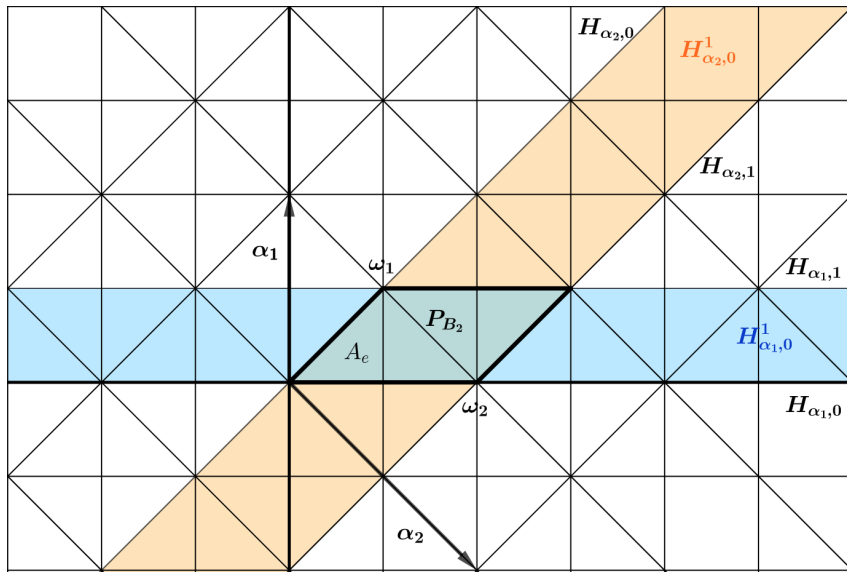


FIGURE 3. Fundamental parallelepiped P_{B_2} .

3. Φ^+ -REPRESENTATION

The idea of this section is to see the elements of W_a as affine isometries of a bigger space. Since any element $w \in W_a$ is characterized by a Φ^+ -tuple of integers $(k(w, \alpha))_{\alpha \in \Phi^+}$ and since W_a acts geometrically on itself it is natural to seek to see this action in $\mathbb{R}^{|\Phi^+|}$. For a Euclidean space E , we identify E with the space of its translations, and we denote by $\text{Isom}(E)$ the space of its affine isometries, that is the elements of $E \rtimes GL(E)$ such that the Euclidean scalar product is preserved. In other words $\text{Isom}(E) = E \rtimes O(E)$.

Let us write $\Phi^+ = \{\beta_1, \beta_2, \dots, \beta_m\}$ with $m = |\Phi^+|$. We send Φ^+ to the canonical basis of \mathbb{R}^m via the map ι . More specifically, since an element $w \in W_a$ is identified with the Φ^+ -tuple $(k(w, \gamma))_{\gamma \in \Phi^+}$ we set $\iota(w) := (k(w, \gamma))_{\gamma \in \Phi^+}$. The natural action of $s_{\alpha, k} \in \text{Isom}(\mathbb{R}^{|\Delta|})$ corresponds to acting on W_a by left multiplication, that is if $x \in A_w$ then $s_{\alpha, k}(x) \in A_{w'}$ where $w' = s_{\alpha, k}w$. We will denote the left multiplication by w as L_w . The main goal of this section is to prove the following theorem:

Theorem 3.1. *There exists an injective morphism $F : W_a \rightarrow \text{Isom}(\mathbb{R}^m)$ such that for any $w \in W_a$ the following diagram commutes. This morphism is called the Φ^+ -representation of W_a .*

$$(14) \quad \begin{array}{ccc} W_a & \xrightarrow{L_w} & W_a \\ \downarrow \iota & & \downarrow \iota \\ \mathbb{R}^m & \xrightarrow{F(w)} & \mathbb{R}^m. \end{array}$$

We recall first of all a few results of [13] that we will use later. The following lemmas show how the coefficients $k(w, \alpha)$ behave when we act on w by the available operations in W_a . Proposition 4.2 of [13] has a typo, which was noticed by M. Dyer and which is fixed in Proposition 3.1.

Lemma 3.1 ([13], Lemma 3.1). *Let w be an element of $W \subset W_a$ and $\alpha \in \Phi^+$. Then*

$$(15) \quad k(w, \alpha) = \begin{cases} 0 & \text{if } w^{-1}(\alpha) \in \Phi^+ \\ -1 & \text{if } w^{-1}(\alpha) \in \Phi^- \end{cases}$$

Lemma 3.2 ([13], Lemma 3.2). *Let w be an element of W and $x \in \mathbb{Z}\Phi$. Then for all $\alpha \in \Phi$ one has the following formula*

$$(16) \quad k(\tau_x w, \alpha) = k(w, \alpha) + (x, \alpha^\vee).$$

Corollary 3.1 ([13], Formula (3.3.1)). *Let $\alpha \in \Phi$. As defined in Section 2 we recall that α_0 is the negative of the highest short root of Φ and $s_0 := s_{-\alpha_0, 1}$. Then for all $0 \leq i \leq n$ we have*

$$(17) \quad k(s_i, \alpha) = \begin{cases} 1 & \text{if } \alpha = -\alpha_i \\ 0 & \text{if } \alpha \neq \pm\alpha_i \\ -1 & \text{if } \alpha = \alpha_i. \end{cases}$$

Proposition 3.1 ([13], Proposition 4.2). *Let $w \in W_a$ and $s \in S_a$. Then for all $\alpha \in \Phi^+$ one has the following formula*

$$(18) \quad k(sw, \alpha) = k(w, \bar{s}(\alpha)) + k(s, \alpha).$$

Proposition 3.2. *Let $w \in W_a$, $\alpha, \beta \in \Phi^+$ and $p \in \mathbb{Z}$. Then we have*

$$(1) \quad k(s_\alpha w, \beta) = \begin{cases} k(w, s_\alpha(\beta)) & \text{if } s_\alpha(\beta) \in \Phi^+ \\ k(w, s_\alpha(\beta)) - 1 & \text{if } s_\alpha(\beta) \in \Phi^- \end{cases}$$

$$(2) \quad k(s_{\alpha, p} w, \beta) = \begin{cases} k(w, s_\alpha(\beta)) - p(\alpha, s_\alpha(\beta)^\vee) & \text{if } s_\alpha(\beta) \in \Phi^+ \\ k(w, s_\alpha(\beta)) - 1 - p(\alpha, s_\alpha(\beta)^\vee) & \text{if } s_\alpha(\beta) \in \Phi^- \end{cases}$$

Proof. (1) Let $s_\alpha = s_1 s_2 \dots s_n$ be a reduced expression of s_α . Since s_α is a reflection one has $s_1 s_2 \dots s_n = s_n \dots s_2 s_1$. Using Proposition 3.1 enough times we get the following identity

$$k(s_\alpha w, \beta) = k(w, s_n \dots s_1(\beta)) + \sum_{i=1}^n k(s_i, s_{i-1} \dots s_1(\beta)).$$

When $i = 1$, the empty product on the righthand side is understood as the identity element. Further, by using again Proposition 3.1 enough times we get $k(s_\alpha, \beta) = \sum_{i=1}^n k(s_i, s_{i-1} \dots s_1(\beta))$. Thus we have

$$\begin{aligned} k(s_\alpha w, \beta) &= k(w, s_n \dots s_1(\beta)) + \sum_{i=1}^n k(s_i, s_{i-1} \dots s_1(\beta)) \\ &= k(w, s_\alpha(\beta)) + k(s_\alpha, \beta). \end{aligned}$$

Since $s_\alpha \in W$ and $\beta \in \Phi^+$ we have by Lemma 3.1 that $k(s_\alpha, \beta) = 0$ if $s_\alpha(\beta) \in \Phi^+$ and $k(s_\alpha, \beta) = -1$ if $s_\alpha(\beta) \in \Phi^-$. The result follows.

(2) This is just a consequence of 1) and Lemma 3.2. Let us write $w = \tau_x \bar{w}$. Notice first of all that $s_\alpha(\beta)^\vee = s_\alpha(\beta^\vee)$ and

$$k(s_\alpha w, \beta) = k(s_\alpha \tau_x \bar{w}, \beta) = k(\tau_{s_\alpha(x)} s_\alpha \bar{w}, \beta) = k(s_\alpha \bar{w}, \beta) + (s_\alpha(x), \beta^\vee).$$

Thus it follows that

$$\begin{aligned} k(s_{\alpha,p} w, \beta) &= k(\tau_{p\alpha} s_\alpha \tau_x \bar{w}, \beta) \\ &= k(\tau_{p\alpha + s_\alpha(x)} s_\alpha \bar{w}, \beta) \\ &= k(s_\alpha \bar{w}, \beta) + (p\alpha + s_\alpha(x), \beta^\vee) \\ &= k(s_\alpha \bar{w}, \beta) + p(\alpha, \beta^\vee) + (s_\alpha(x), \beta^\vee) \\ &= k(s_\alpha \bar{w}, \beta) - p(\alpha, s_\alpha(\beta^\vee)) + (s_\alpha(x), \beta^\vee) \\ &= k(s_\alpha w, \beta) - p(\alpha, s_\alpha(\beta^\vee)) \\ &= \begin{cases} k(w, s_\alpha(\beta)) - p(\alpha, s_\alpha(\beta^\vee)) & \text{if } s_\alpha(\beta) \in \Phi^+ \\ k(w, s_\alpha(\beta)) - 1 - p(\alpha, s_\alpha(\beta^\vee)) & \text{if } s_\alpha(\beta) \in \Phi^- \end{cases} \\ &= \begin{cases} k(w, s_\alpha(\beta)) - p(\alpha, s_\alpha(\beta)^\vee) & \text{if } s_\alpha(\beta) \in \Phi^+ \\ k(w, s_\alpha(\beta)) - 1 - p(\alpha, s_\alpha(\beta)^\vee) & \text{if } s_\alpha(\beta) \in \Phi^- \end{cases}. \end{aligned}$$

□

Definition 3.1. Let $\alpha \in \Phi^+$ and $p \in \mathbb{Z}$.

(1) We define $L_\alpha \in GL_m(\mathbb{R})$ via the matrix $(\ell_{i,j}(\alpha))_{i,j \in \llbracket 1, m \rrbracket}$ where

$$\ell_{\beta_j, \beta_i}(\alpha) := \ell_{j,i}(\alpha) = \begin{cases} 1 & \text{if } s_\alpha(\beta_i) = \beta_j \\ 0 & \text{if } s_\alpha(\beta_i) \neq \pm \beta_j \\ -1 & \text{if } s_\alpha(\beta_i) = -\beta_j. \end{cases}$$

(2) We define the vector $v_{p,\alpha} \in \mathbb{R}^m$ as $v_{p,\alpha} = (v_{p,\alpha}(\gamma))_{\gamma \in \Phi^+}$ where

$$v_{p,\alpha}(\gamma) := \begin{cases} -p(\alpha, s_\alpha(\gamma)^\vee) & \text{if } s_\alpha(\gamma) \in \Phi^+ \\ -1 - p(\alpha, s_\alpha(\gamma)^\vee) & \text{if } s_\alpha(\gamma) \in \Phi^- \end{cases}.$$

For $\gamma \in \Phi^+$ we set the convention $v_{p,\alpha}(-\gamma) := -v_{p,\alpha}(\gamma)$.

(3) We define $F(s_{\alpha,p})$ as the affine map such that for all $x \in \mathbb{R}^m$ one has

$$F(s_{\alpha,p})(x) := L_\alpha(x) + v_{p,\alpha}.$$

For short we will write $L_\alpha(w)$ instead of $L_\alpha(\iota(w))$. We denote $L_\alpha(w)[\theta]$ the coefficient in position θ of the vector $L_\alpha(w)$. This coefficient is called the θ -coefficient of $L_\alpha(w)$. Similarly, the coefficient in position θ of $F(s_{\alpha,p})(w)$ is denoted $F(s_{\alpha,p})(w)[\theta]$ and we have $F(s_{\alpha,p})(w)[\theta] = L_\alpha(w)[\theta] + v_{p,\alpha}(\theta)$. We call it the θ -coefficient of $F(s_{\alpha,p})(w)$.

Proposition 3.3. *Let $\alpha, \theta \in \Phi^+$ and $p \in \mathbb{Z}$. We have*

- (1) $L_\alpha^2 = id$ and $F(s_{\alpha,p}) \in Isom(\mathbb{R}^m)$.
- (2) $L_\alpha(w)[\theta] = k(w, s_\alpha(\theta))$.
- (3) *The following diagram commutes*

$$\begin{array}{ccc} W_\alpha & \xrightarrow{L_{s_{\alpha,p}}} & W_\alpha \\ \iota \downarrow & & \downarrow \iota \\ \mathbb{R}^m & \xrightarrow{F(s_{\alpha,p})} & \mathbb{R}^m. \end{array}$$

Proof. (1) Since s_α is an involution, the way we defined L_α shows that $\ell_{i,j}(\alpha) = \ell_{j,i}(\alpha)$, that is $L_\alpha = L_\alpha^t$. Moreover each line and each column has only one $\ell_{i,j}(\alpha) \neq 0$. Then it follows that $L_\alpha^2 = id$ and for instance we have $L_\alpha \in GL(\mathbb{R}^m)$. In order to show that $F(s_{\alpha,p}) \in Isom(\mathbb{R}^m)$ we just have to show that $L_\alpha \in O(\mathbb{R}^m)$. But $L_\alpha^t = L_\alpha = L_\alpha^{-1}$ which means that $L_\alpha \in O(\mathbb{R}^m)$.

(2) From the way we defined L_α the θ -coefficient of $L_\alpha(w)$ has the value

$$\sum_{\gamma \in \Phi^+} \ell_{\theta,\gamma}(\alpha) k(w, \gamma).$$

Furthermore, we know that $\ell_{\theta,\gamma}(\alpha) \neq 0$ if and only if $\gamma = \pm s_\alpha(\theta)$. Therefore, if $\gamma = s_\alpha(\theta)$ we have $s_\alpha(\theta) \in \Phi^+$ and it follows $\ell_{\theta,\gamma}(\alpha) = 1$, which implies that the θ -coefficient of $L_\alpha(w)$ is $\ell_{\theta,\gamma}(\alpha) k(w, \gamma) = k(w, s_\alpha(\theta))$. If now we have $s_\alpha(\theta) = -\gamma$ then $s_\alpha(\theta) \in \Phi^-$, which implies that $\ell_{\theta,\gamma}(\alpha) = -1$. Thus, the θ -coefficient of $L_\alpha(w)$ is $\ell_{\theta,\gamma}(\alpha) k(w, \gamma) = (-1) k(w, -s_\alpha(\theta)) = k(w, s_\alpha(\theta))$. Hence, in all cases we have the θ -coefficient of $L_\alpha(w)$ equal to $k(w, s_\alpha(\theta))$.

(3) Let $\theta \in \Phi^+$. Let us show that the θ -coefficient of the vector $F(s_{\alpha,p})(w)$ is the same as the θ -coefficient of the vector $\iota(s_{\alpha,p}w)$. By point 2) before we know that $L_\alpha(w)[\theta] = k(w, s_\alpha(\theta))$. Then we have

$$F(s_{\alpha,p})(w)[\theta] = L_\alpha(w)[\theta] + v_{p,\alpha}(\theta) = k(w, s_\alpha(\theta)) + v_{p,\alpha}(\theta).$$

With respect to $\iota(s_{\alpha,p}w)$, via the way we defined ι we know that the θ -coefficient of $\iota(s_{\alpha,p}w)$ is $k(s_{\alpha,p}w, \theta)$. However, because of Proposition 3.2 we know that

$$\begin{aligned} k(s_{\alpha,p}w, \theta) &= \begin{cases} k(w, s_\alpha(\theta)) - p(\alpha, s_\alpha(\theta)^\vee) & \text{if } s_\alpha(\theta) \in \Phi^+ \\ k(w, s_\alpha(\theta)) - 1 - p(\alpha, s_\alpha(\theta)^\vee) & \text{if } s_\alpha(\theta) \in \Phi^- \end{cases} \\ &= k(w, s_\alpha(\theta)) + \begin{cases} -p(\alpha, s_\alpha(\theta)^\vee) & \text{if } s_\alpha(\theta) \in \Phi^+ \\ -1 - p(\alpha, s_\alpha(\theta)^\vee) & \text{if } s_\alpha(\theta) \in \Phi^- \end{cases} \\ &= k(w, s_\alpha(\theta)) + v_{p,\alpha}(\theta). \end{aligned}$$

□

We are now ready in proving Theorem 3.1.

Proof of Theorem 3.1. Let $w \in W_\alpha$. Since $W_\alpha = \langle s_{\alpha,p}, \alpha \in \Phi^+, p \in \mathbb{Z} \rangle$ we can write $w = s_{\alpha_1, p_1} \dots s_{\alpha_q, p_q}$ with $\alpha_i \in \Phi^+$ and $p_i \in \mathbb{Z}$. We now want to extend to W_α the definition of F by $F(w) = F(s_{\alpha_1, p_1}) \dots F(s_{\alpha_q, p_q})$.

However we need to check that this definition doesn't depend on the choice of the expression of w . Let w_0 be the element of maximal length of W (the uniqueness of this element is well known, see for example [7] Section 1.8). This element is of length $|\Phi^+| = m$. Let $s_{i_1} s_{i_2} \dots s_{i_m}$ be a reduced expression of w_0 where $s_{i_j} \in S$. For all $k = 1, \dots, m$ and for all $\alpha \in \Phi^+$, Lemma 3.1 implies that $k(s_{i_1} s_{i_2} \dots s_{i_k}, \alpha)$ is either 0 or -1. Moreover, it is clear that in order to express $\iota(s_{i_1} s_{i_2} \dots s_{i_{k+1}})$ in terms of $\iota(s_{i_1} s_{i_2} \dots s_{i_k})$ we just need to change only one 0 of $\iota(s_{i_1} s_{i_2} \dots s_{i_k})$ into a -1.

Let us order the coordinates of \mathbb{R}^m such that

$$\begin{aligned}\iota(s_{i_1}) &= (-1, 0, \dots, 0) \\ \iota(s_{i_1}s_{i_2}) &= (-1, -1, 0, \dots, 0) \\ &\vdots \\ \iota(s_{i_1}s_{i_2}\dots s_{i_{m-1}}) &= (-1, -1, \dots, -1, 0) \\ \iota(s_{i_1}s_{i_2}\dots s_{i_m}) &= (-1, -1, \dots, -1, -1).\end{aligned}$$

We see then that the set $\{\iota(s_{i_1}), \iota(s_{i_1}s_{i_2}), \dots, \iota(s_{i_1}s_{i_2}\dots s_{i_m})\}$ is a basis of \mathbb{R}^m . Adding $\iota(e)$ to this basis we get an affine basis of \mathbb{R}^m that we denote by \mathcal{B} .

Let us assume now that we have two affine maps $\varphi_1(w), \varphi_2(w) \in \text{Aff}(\mathbb{R}^m)$ making the diagram (14) commute. In particular for all $x \in W$ we have

$$\varphi_1(w)(\iota(x)) = (\iota \circ L_w)(x) = \varphi_2(w)(\iota(x)).$$

Thus, it follows that

$$\varphi_1(w)(u) = \varphi_2(w)(u) \text{ for all } u \in \mathcal{B}.$$

Hence $\varphi_1(w) = \varphi_2(w)$ and then there is at most one affine map making the diagram (14) commute. Using Proposition 3.3 3) we see that $F(s_{\alpha_1, p_1}) \dots F(s_{\alpha_q, p_q})$ is such an affine map. Thus $F(w)$ is well defined and makes the diagram (14) commute.

Using Proposition 3.3 1) it is clear that $F(w) \in \text{Isom}(\mathbb{R}^m)$, and by definition of F it is also clear that the next diagram commutes for all $w_1, w_2 \in W_a$

$$\begin{array}{ccccc} & & L_{w_2 w_1} & & \\ & \curvearrowright & & \curvearrowleft & \\ W_a & \xrightarrow{L_{w_1}} & W_a & \xrightarrow{L_{w_2}} & W_a \\ \downarrow \iota & & \downarrow \iota & & \downarrow \iota \\ \mathbb{R}^m & \xrightarrow{F(w_1)} & \mathbb{R}^m & \xrightarrow{F(w_2)} & \mathbb{R}^m \\ & \curvearrowleft & & \curvearrowright & \\ & & F(w_2 w_1) & & \end{array}$$

Thus F is a morphism. This morphism is injective because $F(w_1) = F(w_2)$ implies for instance that $F(w_1)(\iota(e)) = F(w_2)(\iota(e))$ which means that $\iota \circ L_{w_1}(e) = \iota \circ L_{w_2}(e)$, that is $\iota(w_1) = \iota(w_2)$. Since $\iota(w_1) = (k(w_1, \alpha))_{\alpha \in \Phi^+}$ and $\iota(w_2) = (k(w_2, \alpha))_{\alpha \in \Phi^+}$ it follows that $w_1 = w_2$. \square

4. STRUCTURE OF AFFINE VARIETY ON W_a

Let W_a be an affine Weyl group, Φ its crystallographic root system with $\Phi^+ = \{\beta_1, \dots, \beta_m\}$, and $\Delta = \{\alpha_1, \dots, \alpha_n\}$ its simple system. The aim of this section is to understand, from a new geometric point of view, affine Coxeter groups based on their Φ^+ -tuple of integers obtained in [13]. Shi gave in [13] a characterization of elements in W_a via a collection of inequalities. He also gave an easier characterization of the Φ^+ -tuples in Theorem 8. The goal here is to transform these inequalities into equations. From now on we will denote $A[X_\Delta] := A[X_{\alpha_1}, \dots, X_{\alpha_n}]$ and $A[X_{\Phi^+}] := A[X_{\beta_1}, \dots, X_{\beta_m}]$ where A is a commutative ring. In particular we have $A[X_{\alpha_1}, \dots, X_{\alpha_n}] \subset A[X_{\beta_1}, \dots, X_{\beta_m}]$. For $w \in W_a$ and $Q \in A[X_\Delta]$ we denote by

$$Q(w) := Q(k(w, \alpha_1), \dots, k(w, \alpha_n)).$$

4.1. Decomposition of the coefficients $k(w, \alpha)$. In this section we give the main tool in order to construct the Shi variety.

Theorem 4.1. *Let $w \in W_a$. We have*

- (1) *For all $\theta \in \Phi^+ - \Delta$ there exists a linear polynomial $P_\theta \in \mathbb{Z}[X_\Delta]$ with positive coefficients and $\lambda_\theta(w) \in \llbracket 0, h(\theta^\vee) - 1 \rrbracket$ such that*

$$(19) \quad k(w, \theta) = P_\theta(w) + \lambda_\theta(w).$$

- (2) *$P_\theta = P_\alpha + P_\beta$ for all $\alpha, \beta \in \Phi^+$ such that $\theta^\vee = \alpha^\vee + \beta^\vee$.*

Proof. If $\theta \in \Delta$ then we set $P_\theta = X_\theta$ and $\lambda_\theta(w) = 0$. Assume now that $\theta \in \Phi^+ \setminus \Delta$. We proceed by induction on the height h of coroots. We have $\theta^\vee \in (\Phi^\vee)^+$ and then there exist $\alpha, \beta \in \Phi^+$ such that $\theta^\vee = \alpha^\vee + \beta^\vee$. Notice that it is always possible to write an element of $(\Phi^\vee)^+ \setminus \Delta^\vee$ as a sum of two others elements of $(\Phi^\vee)^+$. Thus we have $\theta = (\alpha^\vee + \beta^\vee)^\vee \in \Phi^+$ and because of Theorem 2.2 it follows $k(w, \alpha) + k(w, \beta) \leq k(w, \theta) \leq k(w, \alpha) + k(w, \beta) + 1$. Hence $k(w, \theta) = k(w, \alpha) + k(w, \beta) + \gamma_\theta(w)$ where $\gamma_\theta(w) \in \llbracket 0, 1 \rrbracket$.

If $h(\theta^\vee) = 2$ one has necessarily $\alpha^\vee, \beta^\vee \in \Delta^\vee$. Therefore by setting $P_\theta := X_\alpha + X_\beta$ and $\lambda_\theta(w) := \gamma_\theta(w)$ we have shown the base case.

Assume now that $h(\theta^\vee) = d + 1$ and assume that the formula (19) is true for all coroots of height less than $d + 1$. We also assume that for all $\gamma \in \Phi^+$ satisfying $h(\gamma^\vee) < d + 1$ the polynomial P_γ is linear and its coefficients are positive. Since $\alpha^\vee + \beta^\vee = \theta^\vee$, we have $h(\alpha^\vee) < h(\theta^\vee)$ and $h(\beta^\vee) < h(\theta^\vee)$. It follows that $k(w, \alpha) = P_\alpha(w) + \lambda_\alpha(w)$ and $k(w, \beta) = P_\beta(w) + \lambda_\beta(w)$ with $P_\alpha, P_\beta \in \mathbb{Z}[X_\Delta]$ both linear, $\lambda_\alpha(w) \in \llbracket 0, h(\alpha^\vee) - 1 \rrbracket$ and $\lambda_\beta(w) \in \llbracket 0, h(\beta^\vee) - 1 \rrbracket$. It follows that:

$$k(w, \theta) = P_\alpha(w) + \lambda_\alpha(w) + P_\beta(w) + \lambda_\beta(w) + \gamma_\theta(w).$$

By setting $P_\theta := P_\alpha + P_\beta$ and $\lambda_\theta(w) = \lambda_\alpha(w) + \lambda_\beta(w) + \gamma_\theta(w)$ we have P_θ linear, its coefficients are positive since those of P_α and P_β are positive, and finally since $h(\theta^\vee) = h(\alpha^\vee) + h(\beta^\vee)$ it results that $\lambda_\theta(w) \in \llbracket 0, h(\theta^\vee) - 1 \rrbracket$. This ends the induction and the proof of points (1) and (2). \square

Example 4.1. Let $\Delta(B_n) := \{\alpha_1, \dots, \alpha_n\}$ with its Dynkin diagram:



It is well-known (see [2]) that its positive root system decomposes as

$$\Phi^+(B_n) = I_1 \sqcup I_2 \sqcup I_3,$$

where $I_1 := \{\sum_{t=i}^{j-1} \alpha_t \mid 1 \leq i < j \leq n\}$, $I_2 := \{\sum_{t=i}^n \alpha_t \mid 1 \leq i \leq n\}$ and $I_3 := \{\sum_{t=i}^{j-1} \alpha_t + 2 \sum_{t=j}^n \alpha_t \mid 1 \leq i < j \leq n\}$.

It turns out that the short roots are those in I_2 and the long ones are those in $I_1 \sqcup I_3$. Moreover, the only simple root that is short is α_n . Let $\alpha \in \Phi^+(B_n)$ and $w \in W_a$. After computation we get:

- (i) If $\alpha = \sum_{t=i}^{j-1} \alpha_t \in I_1$ then there exists $\lambda_\alpha(w) \in \llbracket 0, j - i - 1 \rrbracket$ such that $k(w, \alpha) = \sum_{t=i}^{j-1} k(w, \alpha_t) + \lambda_\alpha(w)$. Here $P_\alpha = \sum_{k=i}^{j-1} X_{\alpha_k}$ and it is easy to see that $j - i = h(\alpha) = h(\alpha^\vee)$.
- (ii) If $\alpha = \sum_{t=i}^n \alpha_t \in I_2$ then there exists $\lambda_\alpha(w) \in \llbracket 0, 2(n - i) \rrbracket$ such that $k(w, \alpha) = 2 \sum_{t=i}^{n-1} k(w, \alpha_t) + k(w, \alpha_n) + \lambda_\alpha(w)$. Here $P_\alpha := 2 \sum_{k=i}^{n-1} X_{\alpha_k} + X_{\alpha_n}$ and it is easy to see that $h(\alpha^\vee) = 2(n - i) + 1$.

(iii) If $\alpha = \sum_{t=i}^{j-1} \alpha_t + 2 \sum_{t=j}^n \alpha_t \in I_3$ then there exists $\lambda_\alpha(w) \in \llbracket 0, 2n - (i + j) \rrbracket$ such that

$$k(w, \alpha) = \begin{cases} \sum_{t=i}^{j-1} k(w, \alpha_t) + 2 \sum_{t=j}^{n-1} k(w, \alpha_t) + k(w, \alpha_n) + \gamma_\alpha(w) & \text{if } j < n \\ \sum_{t=i}^n k(w, \alpha_t) + \gamma_\alpha(w) & \text{if } j = n. \end{cases}$$

Here we have

$$P_\alpha = \begin{cases} \sum_{k=i}^{j-1} X_{\alpha_k} + 2 \sum_{k=j}^{n-1} X_{\alpha_k} + X_{\alpha_n} & \text{if } j < n \\ \sum_{k=i}^n X_{\alpha_k} & \text{if } j = n, \end{cases}$$

and it is easy to see that $h(\alpha^\vee) = 2n - (i + j) + 1$.

4.2. Definition of the Shi variety. Before giving the main definitions of this section we want to discuss some terminologies and notation, particularly to clarify what we mean by ‘‘affine variety’’ in this article. For some people an affine variety V over a field k is just the set of solutions of polynomial equations with coefficients in k , which is sometimes called an algebraic set. For others it is the ringed space (V, \mathcal{O}_V) where \mathcal{O}_V is the sheaf of regular functions over V , or again for others it is a ringed space (X, \mathcal{O}_X) isomorphic to (V, \mathcal{O}_V) . In this article the term *affine variety* is employed as solutions of polynomial equations.

Let k be a field and \mathcal{I} be an ideal of $k[X_1, \dots, X_p]$. Let $V(\mathcal{I})$ be the set given by

$$V(\mathcal{I}) = \{x \in k^p \mid P(x) = 0 \forall P \in \mathcal{I}\}.$$

Definition 4.1. We say that $v = (v_\alpha)_{\alpha \in \Phi^+} \in \mathbb{N}^m$ is an *admissible vector* (or just *admissible*) if it satisfies the boundary conditions, that is if for all $\alpha \in \Phi^+$ one has $v_\alpha \in I_\alpha := \llbracket 0, h(\alpha^\vee) - 1 \rrbracket$. Notice that if α is a simple root then $I_\alpha = \{0\}$. For short we will write λ instead of $(\lambda_\alpha)_{\alpha \in \Phi^+}$.

Definition 4.2. Let $\theta \in \Phi^+$. For any root $\theta \in \Delta$ we set $P_\theta = X_\theta$ and $\lambda_\theta = 0$. We denote by $P_\theta[\lambda_\theta]$ the polynomial in $\mathbb{Z}[X_{\Phi^+}]$ given by

$$(20) \quad P_\theta[\lambda_\theta] := X_\theta - P_\theta - \lambda_\theta.$$

Let λ be an admissible vector. We define the ideals $J_{W_a}[\lambda]$ and J_{W_a} of $\mathbb{Z}[X_{\Phi^+}]$ as follows

$$(21) \quad J_{W_a}[\lambda] := \sum_{\alpha \in \Phi^+} \langle P_\alpha[\lambda_\alpha] \rangle = \langle P_\alpha[\lambda_\alpha], \alpha \in \Phi^+ \rangle,$$

and

$$(22) \quad J_{W_a} := \prod_{\lambda \text{ admissible}} J_{W_a}[\lambda].$$

We define then the affine variety X_{W_a} by

$$(23) \quad X_{W_a} := V(J_{W_a})$$

together with

$$(24) \quad X_{W_a}[\lambda] := V(J_{W_a}[\lambda]) \text{ and } X_{W_a}[0] := V(J_{W_a}[0_{\mathbb{R}^m}]).$$

Remark 4.1. We recall that our goal is to have a bijection between W_a and the integral points of a variety. The set X_{W_a} seems to be a good candidate because the way we defined it shows that $W_a \hookrightarrow X_{W_a}(\mathbb{Z})$. However, it turns out that X_{W_a} is too large. We explain this later on.

Example 4.2. Let us take $W_a = W(\tilde{B}_2)$ with simple system $\Delta = \{\alpha, \beta\}$ and α the short root of Δ . The positive root system is $\Phi^+ = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta\}$. Because of Example 4.1 we know that there exists $\lambda_{\alpha+\beta}(w) \in \llbracket 0, 2 \rrbracket$ such that $k(w, \alpha + \beta) = k(w, \alpha) + 2k(w, \beta) + \lambda_{\alpha+\beta}(w)$, and there exists $\lambda_{2\alpha+\beta}(w) \in \llbracket 0, 1 \rrbracket$ such that $k(w, 2\alpha + \beta) = k(w, \alpha) + k(w, \beta) + \lambda_{\alpha+2\beta}(w)$. Hence we have 6 choices of $\lambda = (\lambda_\theta)_{\theta \in \Phi^+}$.

$$\begin{array}{ccc} & \beta & \\ \alpha + \beta & \alpha & = (\alpha, \beta, \alpha + \beta, 2\alpha + \beta) \\ & 2\alpha + \beta & \end{array}$$

FIGURE 4. Reading direction of a Φ^+ -tuple in $W(\tilde{B}_2)$.

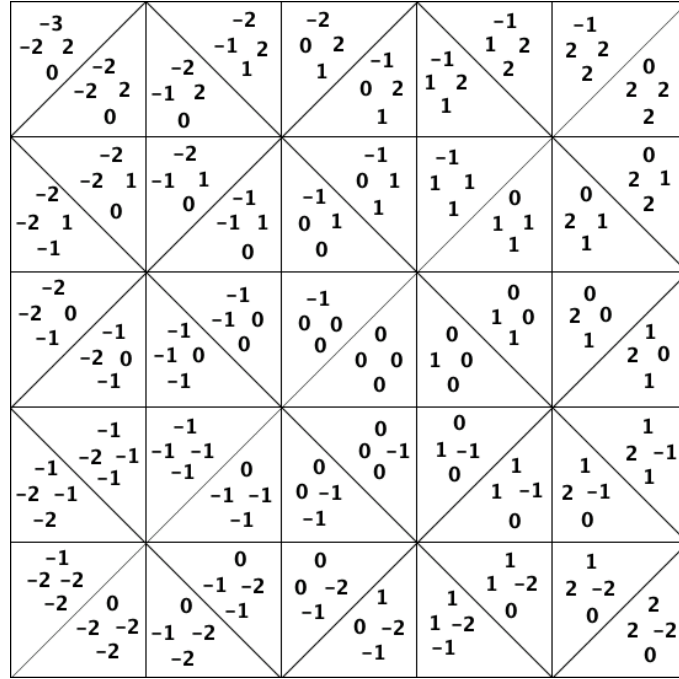


FIGURE 5. Alcoves in $W(\tilde{B}_2)$.

Let us now explain what is $X_{\tilde{B}_2}$. Using the reading direction of Figure 4, we have 6 admissible vectors λ_i ($i = 1, \dots, 6$), respectively given by

$$\{(0, 0, 0, 0), (0, 0, 1, 0), (0, 0, 1, 1), (0, 0, 2, 1), (0, 0, 2, 0), (0, 0, 0, 1), \}.$$

We have then

$$\begin{aligned} J_{\tilde{B}_2}[\lambda_1] &= \langle X_{\alpha+\beta} - 2X_\alpha - X_\beta, X_{2\alpha+\beta} - X_\alpha - X_\beta \rangle \\ J_{\tilde{B}_2}[\lambda_2] &= \langle X_{\alpha+\beta} - 2X_\alpha - X_\beta - 1, X_{2\alpha+\beta} - X_\alpha - X_\beta \rangle \\ J_{\tilde{B}_2}[\lambda_3] &= \langle X_{\alpha+\beta} - 2X_\alpha - X_\beta - 1, X_{2\alpha+\beta} - X_\alpha - X_\beta - 1 \rangle \\ J_{\tilde{B}_2}[\lambda_4] &= \langle X_{\alpha+\beta} - 2X_\alpha - X_\beta - 2, X_{2\alpha+\beta} - X_\alpha - X_\beta - 1 \rangle \\ J_{\tilde{B}_2}[\lambda_5] &= \langle X_{\alpha+\beta} - 2X_\alpha - X_\beta - 2, X_{2\alpha+\beta} - X_\alpha - X_\beta \rangle \\ J_{\tilde{B}_2}[\lambda_6] &= \langle X_{\alpha+\beta} - 2X_\alpha - X_\beta, X_{2\alpha+\beta} - X_\alpha - X_\beta - 1 \rangle \end{aligned}$$

and

$$J_{\tilde{B}_2} = J_{\tilde{B}_2}[\lambda_1] \cdot J_{\tilde{B}_2}[\lambda_2] \cdot J_{\tilde{B}_2}[\lambda_3] \cdot J_{\tilde{B}_2}[\lambda_4] \cdot J_{\tilde{B}_2}[\lambda_5] \cdot J_{\tilde{B}_2}[\lambda_6].$$

Therefore

$$X_{\tilde{B}_2} = V(J_{\tilde{B}_2}).$$

4.3. Admissible, admitted vectors and components of \widehat{X}_{W_a} .

Definition 4.3. We will denote by $S[W_a]$ the system of all the inequalities coming from Theorem 2.1 or equivalently Theorem 2.2. Let λ be an admissible vector. We say that λ is *admitted* if it satisfies the system $S[W_a]$.

Proposition 4.1. *Let $w \in W_a$. Let us write $k(w, \theta) = P_\theta(w) + \lambda_\theta(w)$ for all $\theta \in \Phi^+$. Then $(k(w, \theta))_{\theta \in \Phi^+}$ satisfies $S[W_a]$ if and only if $(\lambda_\theta(w))_{\theta \in \Phi^+}$ satisfies $S[W_a]$.*

Proof. Let $\theta, \alpha, \beta \in \Phi^+$ such that $\theta^\vee = \alpha^\vee + \beta^\vee$. Hence by Theorem 2.2 one has $k(w, \alpha) + k(w, \beta) \leq k(w, \theta) \leq k(w, \alpha) + k(w, \beta) + 1$. Moreover by Theorem 4.1 we know that $k(w, \alpha) = P_\alpha(w) + \lambda_\alpha(w)$ and $k(w, \beta) = P_\beta(w) + \lambda_\beta(w)$. We also know by Theorem 4.1 that $P_\theta = P_\alpha + P_\beta$. Therefore we get $P_\theta(w) = P_\alpha(w) + P_\beta(w)$ and the previous inequalities become

$$P_\alpha(w) + \lambda_\alpha(w) + P_\beta(w) + \lambda_\beta(w) \leq P_\theta(w) + \lambda_\theta(w) \leq P_\alpha(w) + \lambda_\alpha(w) + P_\beta(w) + \lambda_\beta(w) + 1.$$

These are the same as

$$\lambda_\alpha(w) + \lambda_\beta(w) \leq \lambda_\theta(w) \leq \lambda_\alpha(w) + \lambda_\beta(w) + 1.$$

Thus $(k(w, \alpha))_{\alpha \in \Phi^+} \in S[W_a]$ if and only if $(\lambda_\alpha)_{\alpha \in \Phi^+} \in S[W_a]$. \square

The following proposition gives the irreducible components of X_{W_a} together with their shape. We shall use this decomposition later.

Proposition 4.2. X_{W_a} decomposes as $X_{W_a} = \bigsqcup_{\lambda \text{ admissible}} X_{W_a}[\lambda]$. The irreducible components of X_{W_a} are the $X_{W_a}[\lambda]$ for λ admissible and we have $X_{W_a}[\lambda] = X_{W_a}[0] + \lambda$. Moreover, each irreducible component is of dimension $|\Delta| = n$.

Proof. From the way we defined X_{W_a} it's clear that $X_{W_a} = \bigsqcup_{\lambda \text{ admissible}} X_{W_a}[\lambda]$. Moreover, being an element of $X_{W_a}[\lambda]$, for λ admissible, is the same as being a solution of the system

$$\{P_\alpha + \lambda_\alpha - X_\alpha = 0 \mid \alpha \in \Phi^+\}.$$

However the solutions of this system are exactly the solutions of the system

$$\{P_\alpha - X_\alpha = 0 \mid \alpha \in \Phi^+\} + (\lambda_\alpha)_{\alpha \in \Phi^+},$$

that is the solutions of the system $X_{W_a}[0] + \lambda$. Since $X_{W_a}[0] + \lambda$ is an affine space it is clear that it is an irreducible component of X_{W_a} . As $X_{W_a}[\lambda] := V(J_{W_a}[\lambda])$ we see that this is just the intersection of the hyperplanes associated to the polynomials $P_\alpha[\lambda_\alpha]$ for all $\alpha \in \Phi^+$. Since there are $|\Phi^+| - |\Delta|$ such polynomials and since they are linearly independent as linear forms on $\mathbb{R}^{|\Phi^+|}$, it follows that the dimension of $X_{W_a}[\lambda]$ is $|\Delta|$. \square

Notation 4.1. Let $Y \subset \mathbb{R}^m$. We denote by $Y(\mathbb{Z})$ the set of integral points of Y .

Theorem 4.2. *Let γ be an admissible vector. The following points are equivalent*

- (i) γ is admitted.
- (ii) There is $x \in X_{W_a}[\gamma]$ such that there exists $w \in W_a$ satisfying $x = (k(w, \alpha))_{\alpha \in \Phi^+}$.
- (iii) All the integral points of $X_{W_a}[\gamma]$ arise as in (ii).

Proof. From Theorem 2.2 we know that a Φ^+ -tuple of integers $(k(w, \alpha))_{\alpha \in \Phi^+}$ associated to an element $w \in W_a$ is entirely characterized by the system of inequalities $S[W_a]$. We also know that a Φ^+ -tuple $(k_\alpha)_{\alpha \in \Phi^+} \in \mathbb{Z}^m$ satisfying the system $S[W_a]$ provides an element $w \in W_a$ such that $(k(w, \alpha))_{\alpha \in \Phi^+} = (k_\alpha)_{\alpha \in \Phi^+}$.

Let $x = (x_\alpha)_{\alpha \in \Phi^+} \in X_{W_a}(\mathbb{Z})[\gamma]$. We can write x_α as $x_\alpha = P_\alpha(x) + \gamma_\alpha$. Then, thanks to Proposition 4.2 if γ is admitted we have x_α that satisfies $S[W_a]$. However, satisfying $S[W_a]$ implies being a Φ^+ -tuple of an element of W_a . Thus, there exists $w \in W_a$ such that $x = (k(w, \alpha))_{\alpha \in \Phi^+}$. The direction $i) \Rightarrow ii)$ follows.

Conversely, if $x \in X_{W_a}[\gamma]$ such that $x = (k(w, \alpha))_{\alpha \in \Phi^+}$ for some $w \in W_a$, then we have again $k(w, \alpha) = P_\alpha(w) + \gamma_\alpha$ for all $\alpha \in \Phi^+$. Since $w \in W_a$, $(k(w, \alpha))_{\alpha \in \Phi^+}$ must satisfy $S[W_a]$, but once again because of Proposition 4.2 this is equivalent to say that $\gamma = (\gamma_\alpha)_{\alpha \in \Phi^+}$ is also a solution of $S[W_a]$, that is γ is admitted. The direction $ii) \Rightarrow i)$ follows.

If we have now a point $x' = (x'_\alpha)_{\alpha \in \Phi^+} \in X_{W_a}(\mathbb{Z})[\gamma]$ with $x' \neq x$ then x' decomposes as $x'_\alpha = P_\alpha(x') + \gamma_\alpha$ for all $\alpha \in \Phi^+$. However, since x is coming from an element of W_a we know that γ is admitted, which means that it is a solution of $S[W_a]$. By using Proposition 4.2 it follows that $(x'_\alpha)_{\alpha \in \Phi^+}$ satisfies $S[W_a]$. Thus, x' is coming from a point of W_a , that is there exists $w' \in W_a$ such that $x' = (k(w', \alpha))_{\alpha \in \Phi^+}$. Thus $ii) \Rightarrow iii)$. The direction $iii) \Rightarrow ii)$ is clear. \square

Theorem 4.3. *We have*

- (1) *The map $\iota : W_a \rightarrow X_{W_a}(\mathbb{Z})$ defined by $w \mapsto (k(w, \alpha))_{\alpha \in \Phi^+}$ induces by corestriction a bijective map from W_a to the integral points of a subvariety of X_{W_a} , denoted \widehat{X}_{W_a} , which we call the Shi variety of W_a . This subvariety is nothing but $\widehat{X}_{W_a} := \bigsqcup_{\lambda \text{ admitted}} X_{W_a}[\lambda]$.*

In other words, one has the following diagram:

$$\begin{array}{ccc} W_a & \xrightarrow{\iota} & X_{W_a}(\mathbb{Z}) \\ & \searrow \tilde{\iota} & \uparrow \circlearrowleft \\ & & \widehat{X}_{W_a}(\mathbb{Z}). \end{array}$$

- (2) *Let $\tilde{\alpha} = \sum_{i=1}^n c_i \alpha_i$ be the highest root in Φ^+ . The number of irreducible components of \widehat{X}_{W_a} is $n! \prod_{i=1}^n c_i$.*

Proof. (1) From the way we defined X_{W_a} we know that each element of W_a , seen as Φ^+ -tuple, belongs to $X_{W_a}(\mathbb{Z})$. That is ι is well defined. Moreover, we also know because of Proposition 5.1 in [13] that ι is injective. Via Theorem 4.2, one has $\iota(w) \in X_{W_a}[\lambda]$ if and only if λ is admitted. Therefore, by deleting all the components $X_{W_a}[\gamma]$ with γ admissible but non-admitted, ι becomes bijective.

(2) The number of irreducible components of \widehat{X}_{W_a} is equal to the number of admitted vectors. Moreover, by Theorem 4.2 we know that for each admitted vector $\gamma = (\gamma_\alpha)_{\alpha \in \Phi^+}$ there exists a unique $w \in W_a$ such that $\gamma_\alpha = k(w, \alpha)$ for all $\alpha \in \Phi^+$. Since $\gamma_\alpha = 0$ for all $\alpha \in \Delta$, we have a bijection between the admitted vectors and the alcoves that belong to $P_{\mathcal{H}} = \bigcap_{\alpha \in \Delta} H_{\alpha,0}^1$.

Thanks to Section 2.3 we know that $|\text{Alc}(P_{\mathcal{H}^\vee})| = \frac{|W(\Delta)|}{f_\Phi}$. Therefore, by duality it follows that $|\text{Alc}(P_{\mathcal{H}})| = \frac{|W(\Delta^\vee)|}{f_{\Phi^\vee}}$. However it is clear that $|W(\Phi^\vee)| = |W(\Phi)|$, and since $f_\Phi = f_{\Phi^\vee}$ (see Section 2.3) it follows that $|\text{Alc}(P_{\mathcal{H}})| = \frac{|W(\Delta)|}{f_\Phi}$. However the number $|W(\Delta)|$ is known and is equal to $f_\Phi n! \prod_{i=1}^n c_i$ (see for example [2], Ch. VI, § 2, prop. 7). The result follows. \square

Notation 4.2. For $\iota(w) \in \widehat{X}_{W_a}[\lambda]$ we denote $\lambda(w) := \lambda$. For instance $X_{W_a}[\lambda(s)]$ is the irreducible component having the Φ^+ -tuple $(k(s, \alpha))_{\alpha \in \Phi^+}$ with $s \in S$.

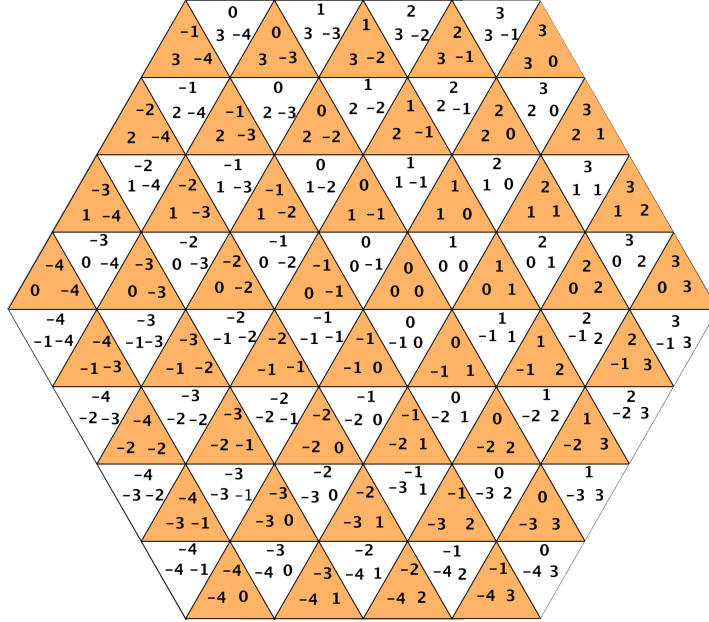
Definition 4.4. We define $H^0(\widehat{X}_{W_a})$ to be the set of irreducible components of \widehat{X}_{W_a} . For $\lambda = (\lambda_\alpha)_{\alpha \in \Phi^+}$ an admitted vector we denote by w_λ the associated element of $\text{Alc}(P_{\mathcal{H}})$, that is w_λ is such that $k(w_\lambda, \alpha) = \lambda_\alpha$ for all $\alpha \in \Phi^+$. We give in the next table the number of irreducible components of any affine Weyl group.

TABLE 1. Number of irreducible components of the Shi variety in any type.

Type	Coefficients of $\tilde{\alpha}$	Index of connection	$ W $	$ H^0(\widehat{X}_{W_a}) $
A_n	1,1,...,1	$n + 1$	$(n + 1)!$	$n!$
B_n	1,2,2,...,2	2	$2^n n!$	$2^{n-1} n!$
C_n	2,2,...,2,1	2	$2^n n!$	$2^{n-1} n!$
D_n	1,2,...,2,1,1	4	$2^{n-1} n!$	$2^{n-3} n!$
E_6	1,2,2,3,2,1	3	$2^7 3^4 5$	$2^7 3^3 5$
E_7	2,2,3,4,3,2,1	2	$2^{10} 3^4 5 \cdot 7$	$2^9 3^4 5 \cdot 7$
E_8	2,3,4,6,5,4,3,2	1	$2^{14} 3^5 5^2 7$	$2^{14} 3^5 5^2 7$
F_4	2,3,4,2	1	$2^7 3^2$	$2^7 3^2$
G_2	3,2	1	$2^2 3$	$2^2 3$

Example 4.3. Let us take the group $W(\tilde{A}_2)$. The variety $\widehat{X}_{W(\tilde{A}_2)}$ has two components which are drawn in Figure 6 either in orange or in white. Thus, we have the decomposition as follows:

$$\widehat{X}_{W(\tilde{A}_2)} = X_{W(\tilde{A}_2)}[(0, 0, 0)] \sqcup X_{W(\tilde{A}_2)}[(0, 0, 1)].$$

FIGURE 6. Irreducible components of $\widehat{X}_{W(\tilde{A}_2)}$.

Example 4.4. We continue Example 4.2. It is easy to see that $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are admitted but λ_5 and λ_6 are not. Therefore, the Shi variety associated to $W(\tilde{B}_2)$ is given by

$$\begin{aligned} \widehat{X}_{\tilde{B}_2} &= V\left(J_{\tilde{B}_2}[\lambda_1] \cdot J_{\tilde{B}_2}[\lambda_2] \cdot J_{\tilde{B}_2}[\lambda_3] \cdot J_{\tilde{B}_2}[\lambda_4]\right) \\ &= X_{\tilde{B}_2}[\lambda_1] \sqcup X_{\tilde{B}_2}[\lambda_2] \sqcup X_{\tilde{B}_2}[\lambda_3] \sqcup X_{\tilde{B}_2}[\lambda_4] \end{aligned}$$

The first component corresponds with the pink, the second one with the yellow, the third one with the blue and the last one with the white. Moreover we can see that all the information is contained in the parallelogram where there are the 4 alcoves associated to the 4 admitted vectors. We also see that any reflection sends a component to another one. We will see in Proposition 4.3 that it is not a coincidence. It is also possible to see that the lattice leaves stable the components. For example if we pick up the alcove $A_w = (0, 0, 0, 0)$ and if we take $x = e_1 + e_2$ then $A_{\tau_x w} = (0, 2, 2, 2)$ and they are both of the same color pink.

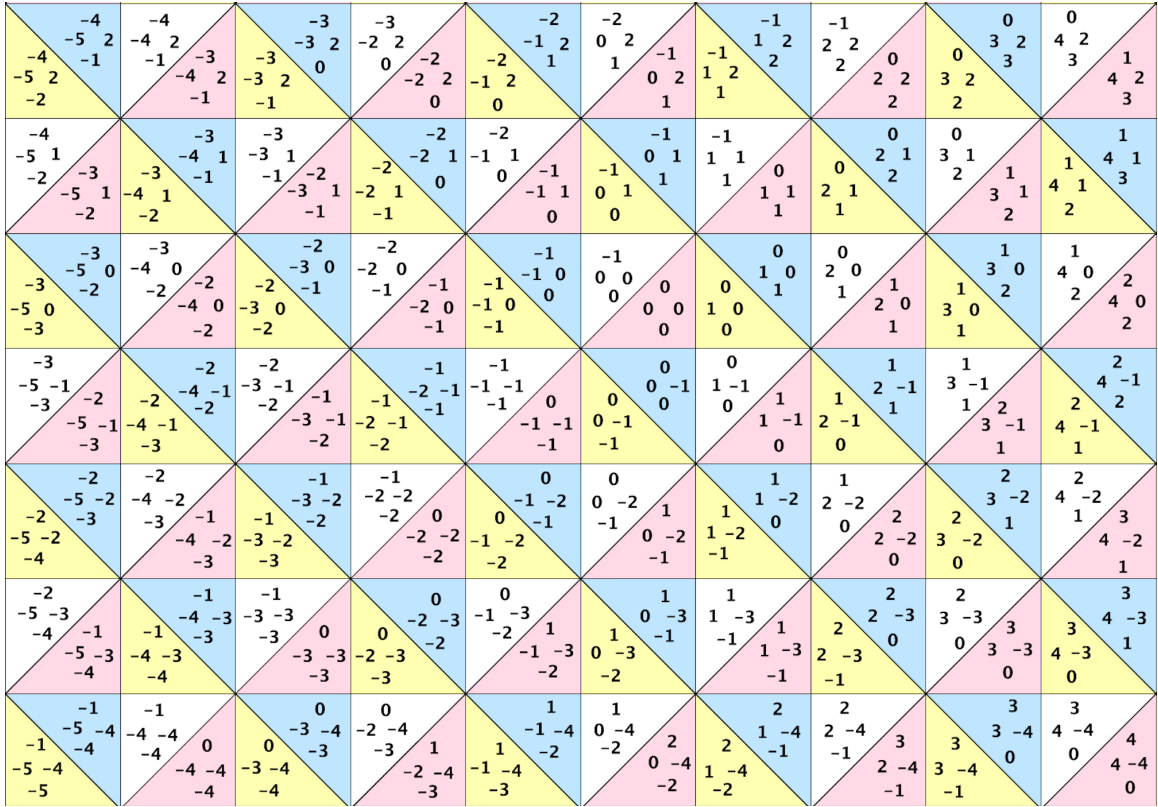


FIGURE 7. Irreducible components of $\widehat{X}_{W(\widetilde{B}_2)}$.

Example 4.5. Let us take the group $W(\widetilde{G}_2)$. We can see all the irreducible components in the polytope P_{G_2} . Here for example we have the 12 irreducible components of $W(\widetilde{G}_2)$. If we would color any element of the finite part $W(G_2)$ according to the components in which they are, we would see that all the colors appear in $W(G_2)$. The reason is that $f_{G_2} = 1$.

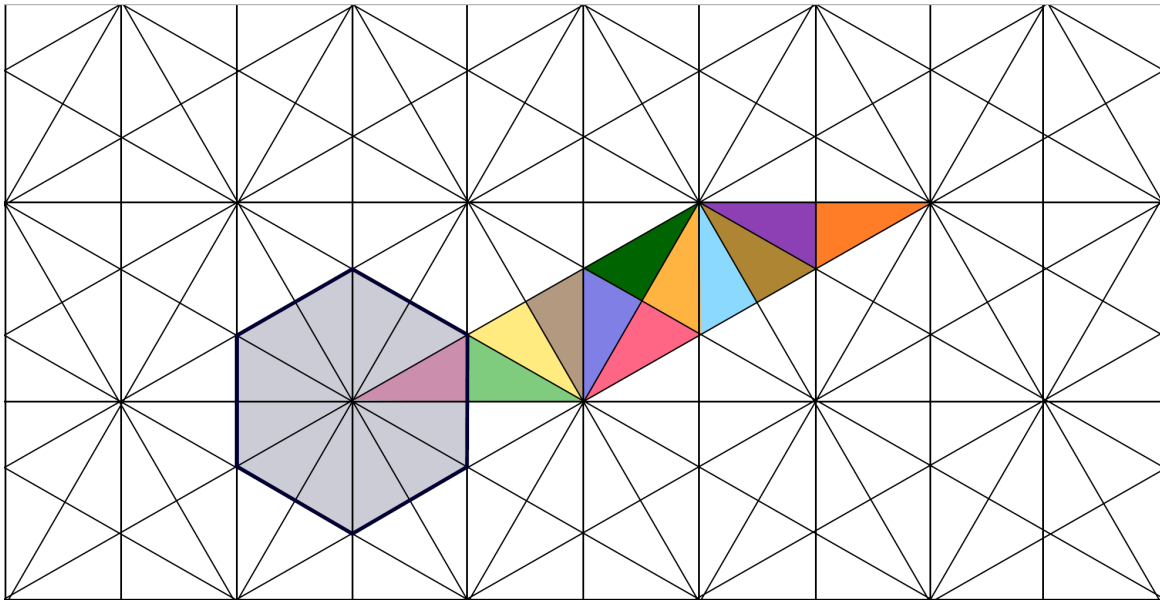


FIGURE 8. Alcoves of P_{G_2} seen as set of representatives of irreducible components of $\widehat{X}_{W(\widetilde{G}_2)}$.

4.4. Action on the components. In this section we investigate the geometrical action associated to the Φ^+ -representation that we define in Proposition 4.3. We show in Proposition 4.4 that this action is related to the index of connection of Φ . The last result of this section shows where the Coxeter generators of W lie in the Shi variety.

Lemma 4.1. *Let θ be in Φ^+ . Then $\theta^\vee = P_\theta(\alpha_1^\vee, \dots, \alpha_n^\vee)$.*

Proof. Let us write $\theta^\vee = c_1\alpha_1^\vee + \dots + c_n\alpha_n^\vee$ with $c_i \in \mathbb{N}$. Then there exists $i \in \llbracket 1, n \rrbracket$ such that $\mu^\vee := c_1\alpha_1^\vee + \dots + (c_i - 1)\alpha_i^\vee + \dots + c_n\alpha_n^\vee \in (\Phi^\vee)^+$. Thus, by using Theorem 4.1 we have $P_\theta = P_\mu + P_{\alpha_i}$. Proceeding by induction the same way on μ^\vee we get that $P_\theta = c_1P_{\alpha_1} + \dots + c_nP_{\alpha_n}$. However $P_{\alpha_i} = X_{\alpha_i}$ for all $i \in \llbracket 1, n \rrbracket$. It follows that $P_\theta = c_1X_{\alpha_1} + \dots + c_nX_{\alpha_n}$. \square

Lemma 4.2. *Let $w = \tau_x \bar{w}$ be an element of W_a such that $x \in \mathbb{Z}\Phi$ and $\bar{w} \in W$. Let λ be an admitted vector. Assume that $(k(w, \alpha))_{\alpha \in \Phi^+} \in X_{W_a}[\lambda]$. Then $(k(\bar{w}, \alpha))_{\alpha \in \Phi^+} \in X_{W_a}[\lambda]$. In particular $X_{W_a}[\lambda]$ is stable under the action of $\mathbb{Z}\Phi$.*

Proof. We just have to show that for any $\alpha \in \Phi^+$ one has $\lambda_\alpha(w) = \lambda_\alpha(\bar{w})$. We know that $k(w, \alpha) = P_\alpha(w) + \lambda_\alpha(w)$. For all $\varepsilon \in \Delta$, thanks to Lemma 3.2, we have

$$k(w, \varepsilon) = k(\bar{w}, \varepsilon) + (x, \varepsilon^\vee) = P_\alpha(\bar{w}) + \lambda_\alpha(\bar{w}) + (x, \varepsilon^\vee).$$

However, we also have that

$$k(w, \alpha) = P_\alpha(w) + \lambda_\alpha(w) = P_\alpha(\{k(\bar{w}, \varepsilon) + (x, \varepsilon^\vee)\}_{\varepsilon \in \Delta}) + \lambda_\alpha(w).$$

Since P_α is a linear polynomial we can split up the sum inside the variables and we get

$$\begin{aligned} k(w, \alpha) &= P_\alpha(\{k(\bar{w}, \varepsilon)\}_{\varepsilon \in \Delta}) + P_\alpha(\{(x, \varepsilon^\vee)\}_{\varepsilon \in \Delta}) + \lambda_\alpha(w) \\ &= k(\bar{w}, \alpha) - \lambda_\alpha(\bar{w}) + P_\alpha(\{(x, \varepsilon^\vee)\}_{\varepsilon \in \Delta}) + \lambda_\alpha(w) \\ &= k(\bar{w}, \alpha) - \lambda_\alpha(\bar{w}) + (x, P_\alpha(\{\varepsilon^\vee\}_{\varepsilon \in \Delta})) + \lambda_\alpha(w) \\ &= k(\bar{w}, \alpha) - \lambda_\alpha(\bar{w}) + (x, \alpha^\vee) + \lambda_\alpha(w). \end{aligned}$$

It follows that $k(\bar{w}, \alpha) + (x, \alpha^\vee) = k(\bar{w}, \alpha) - \lambda_\alpha(\bar{w}) + (x, \alpha^\vee) + \lambda_\alpha(w)$, whence $\lambda_\alpha(w) = \lambda_\alpha(\bar{w})$.

For the second statement let us take $g \in W_a$ such that $(k(g, \alpha))_{\alpha \in \Phi^+} \in X_{W_a}[\lambda]$ and let us take $y \in \mathbb{Z}\Phi$. What we have done just before implies that $(k(\bar{\tau}_y g, \alpha))_{\alpha \in \Phi^+}$ and $(k(\tau_y g, \alpha))_{\alpha \in \Phi^+}$ belong to the same component. Since $\bar{\tau}_y g = \bar{g}$ we get that $(k(\bar{g}, \alpha))_{\alpha \in \Phi^+}$ and $(k(\tau_y g, \alpha))_{\alpha \in \Phi^+}$ are in the same component. But once again we know that $(k(\bar{g}, \alpha))_{\alpha \in \Phi^+}$ and $(k(g, \alpha))_{\alpha \in \Phi^+}$ are in the same component. It follows that $(k(g, \alpha))_{\alpha \in \Phi^+}$ and $(k(\tau_y g, \alpha))_{\alpha \in \Phi^+}$ are in the same component. Hence we have shown that $X_{W_a}[\lambda](\mathbb{Z})$ is stable under the action of $\mathbb{Z}\Phi$. It follows that $X_{W_a}[\lambda]$ is also stable under $\mathbb{Z}\Phi$. \square

Proposition 4.3. *Let $F : W_a \hookrightarrow \text{Isom}(\mathbb{R}^n)$ be the Φ^+ -representation of W_a . Then*

- (1) W_a acts naturally on the irreducible components of \widehat{X}_{W_a} via the action defined as $w \diamond X_{W_a}[\lambda] := F(w)(X_{W_a}[\lambda])$ for λ admitted. Furthermore if we assume that $w \in W_a$ decomposes as $w = \tau_x \bar{w}$, then $w \diamond X_{W_a}[\lambda] = \bar{w} \diamond X_{W_a}[\lambda]$. Finally this action is transitive.
- (2) The previous action induces an action on the admitted vectors by $w \diamond \lambda := \gamma$ such that $w \diamond X_{W_a}[\lambda] = X_{W_a}[\gamma]$. In other words we have $w \diamond X[\lambda] = X[w \diamond \lambda]$.

Proof. (1) Let $x \in W_a$ such that $(k(x, \alpha))_{\alpha \in \Phi^+} \in X_{W_a}[\lambda]$. For any $w \in W_a$ there exists an admitted vector γ such that $(k(wx, \alpha))_{\alpha \in \Phi^+} \in X_{W_a}[\gamma]$. However, since we know that $F(w)(x) = (k(wx, \alpha))_{\alpha \in \Phi^+}$, it follows that $F(w)$ sends at least one element of $X_{W_a}[\lambda]$ to $X_{W_a}[\gamma]$. Moreover $F(w)$ is an isometry and then $F(w)(X_{W_a}[\lambda])$ must be an affine space of the same dimension as $X_{W_a}[\lambda]$.

Let us show now that $(F(w)(X_{W_a}[\lambda]))(\mathbb{Z}) = F(w)(X_{W_a}[\lambda](\mathbb{Z}))$. First of all the inclusion $F(w)(X_{W_a}[\lambda](\mathbb{Z})) \subset (F(w)(X_{W_a}[\lambda]))(\mathbb{Z})$ is clear. With respect to the other inclusion, let us take $z \in (F(w)(X_{W_a}[\lambda]))(\mathbb{Z})$. Then we have $F(w)^{-1}(z) \in X_{W_a}[\lambda]$ but $F(w)^{-1} = F(w^{-1})$ and the coefficients of the matrix associated to $F(w^{-1})$ are integers together with the coordinates of the translation part. It follows that $F(w)^{-1}(z) \in X_{W_a}[\lambda](\mathbb{Z})$ and thus $z \in F(w)(X_{W_a}[\lambda](\mathbb{Z}))$. The equality follows.

Finally, $F(w)(X_{W_a}[\lambda])$ is an affine space of dimension $|\Delta|$ all of whose integer points are associated to elements of W_a . It follows that $F(w)(X_{W_a}[\lambda])$ must be an irreducible component of \widehat{X}_{W_a} . However $(k(wx, \alpha))_{\alpha \in \Phi^+} \in X_{W_a}[\gamma]$, then since the components of X_{W_a} have no intersection we must have $F(w)(X_{W_a}[\lambda]) = X_{W_a}[\gamma]$. Thus, the following map is well defined

$$\begin{aligned} W_a \times H^0(\widehat{X}_{W_a}) &\longrightarrow H^0(\widehat{X}_{W_a}) \\ (w, X_{W_a}[\lambda]) &\longmapsto w \diamond X_{W_a}[\lambda]. \end{aligned}$$

Since F is a morphism we have

$$e \diamond X_{W_a}[\lambda] = F(e)(X_{W_a}[\lambda]) = id(X_{W_a}[\lambda]) = X_{W_a}[\lambda],$$

and for all $w, w' \in W_a$ we also have

$$\begin{aligned} ww' \diamond X_{W_a}[\lambda] &= F(ww')(X_{W_a}[\lambda]) \\ &= F(w)F(w')(X_{W_a}[\lambda]) = F(w)(F(w')(X_{W_a}[\lambda])) \\ &= w \diamond (w' \diamond X_{W_a}[\lambda]). \end{aligned}$$

This concludes the first statement of 1). With respect to the second statement we just have to see that the action of $F(\tau_x)$ doesn't do anything on the components. By Lemma 4.2 we know that $X_{W_a}[\lambda]$ is stable under $\mathbb{Z}\Phi$. It follows that $F(\tau_x)(X_{W_a}[\lambda]) = X_{W_a}[\lambda]$.

To show the transitivity we just have to see that we can obtain any component of $H^0(\widehat{X}_{W_a})$ from the component $X_{W_a}[0]$. Let $X_{W_a}[\gamma]$ be such a component and $(k(g, \alpha))_{\alpha \in \Phi^+} \in X_{W_a}[\gamma]$. It follows that $(k(\bar{g}, \alpha))_{\alpha \in \Phi^+} \in X_{W_a}[\gamma]$. As $\bar{g} = \bar{g}e$ and since $(k(e, \alpha))_{\alpha \in \Phi^+} \in X_{W_a}[0]$ it follows that $F(\bar{g})(X_{W_a}[0]) = X_{W_a}[\gamma]$, that is $\bar{g} \diamond X_{W_a}[0] = X_{W_a}[\gamma]$.

(2) This is a straightforward consequence of the first point. \square

Proposition 4.4. *Let f_Φ be the index of connection of Φ and let λ be an admitted vector.*

- (1) *In W there are exactly f_Φ elements belonging to $X_{W_a}[\lambda]$.*
- (2) *$X_{W_a}[\lambda]$ has exactly f_Φ orbits under the action of $\mathbb{Z}\Phi$.*
- (3) *Each orbit of $X_{W_a}[\lambda]$ has only one element of W .*

Proof. (1) First of all we note that $W \cap \iota^{-1}(X_{W_a}[\lambda](\mathbb{Z})) \neq \emptyset$. Indeed, for any $w = \tau_x \bar{w} \in \iota^{-1}(X_{W_a}[\lambda](\mathbb{Z}))$ we know by Lemma 4.2 that $(k(\bar{w}, \alpha))_{\alpha \in \Phi^+}$ is also in $X_{W_a}[\lambda]$, that is $\bar{w} \in W \cap \iota^{-1}(X_{W_a}[\lambda](\mathbb{Z}))$. We need now to show that if we take another admitted vector γ then $|W \cap \iota^{-1}(X_{W_a}[\lambda](\mathbb{Z}))| = |W \cap \iota^{-1}(X_{W_a}[\gamma](\mathbb{Z}))|$. But this is true because any component can be sent to any other one via an element of W . In particular there exists $g \in W$ such that $g \diamond X_{W_a}[\lambda] = X_{W_a}[\gamma]$. Further, if $(k(w, \alpha))_{\alpha \in \Phi^+} \in X_{W_a}[\lambda]$ with $w \in W$, then $F(g)((k(w, \alpha))_{\alpha \in \Phi^+})$ is also associated to an element of W . The result follows since g is a bijection. By Theorem 4.3 we know that there are $n! \prod_{i=1}^n c_i$ irreducible components in \widehat{X}_{W_a} where $\tilde{\alpha} = \sum_{i=1}^n c_i \alpha_i$ is the longest root in Φ^+ . However it is well known that $|W| = n! (\prod_{i=1}^n c_i) f_\Phi$. Since all the irreducible components have same number of elements in W , we deduce that this number is the quotient of $|W|$ by the number of irreducible components, which is exactly f_Φ .

(2) This is a consequence of the point above. Let us take the f_Φ elements $\{w_1, \dots, w_{f_\Phi}\}$ of $W \cap \iota^{-1}(X_{W_a}[\lambda](\mathbb{Z}))$. Then the $\mathbb{Z}\Phi$ -orbits O_{w_i} and O_{w_j} are different for all $i, j \in \{1, \dots, f_\Phi\}$. Indeed, if there was an element belonging to both then it would follow that there exists $x \in \mathbb{Z}\Phi$ such that $w_i = \tau_x w_j$, but this is impossible because w_i and w_j are in the finite group W . We therefore have f_Φ distinct orbits. We claim that there is no other orbit. Indeed, let $x \in X_{W_a}[\lambda]$. Then there exists $w \in W_a$ such that $x = \iota(w)$. Moreover, one can write $w = \tau_y \bar{w}$ with $y \in \mathbb{Z}\Phi$ and $\bar{w} \in W$. This implies that $\iota(w)$ and $\iota(\bar{w})$ are in the same orbit O_x . However, since $\bar{w} \in W \cap \iota^{-1}(X_{W_a}[\lambda](\mathbb{Z}))$ it follows that $O_x = O_{w_i}$ for some $i = 1, \dots, f_\Phi$.

(3) Assume that we have an orbit in $X_{W_a}[\lambda]$ with at least two elements w_1, w_2 coming from W . Then there exists $x \in \mathbb{Z}\Phi$ such that $w_2 = \tau_x w_1$. Since w_1 and $w_2 \in W$ the previous equality cannot occur. It follows that each orbit has at most one element of W . Moreover, by the first point above we know that $X_{W_a}[\lambda]$ has f_Φ elements of W . Since there are f_Φ orbits in $X_{W_a}[\lambda]$ it follows that each orbit has a unique element of W . \square

Proposition 4.5. *Let W_a be an affine Weyl group such that $W_a \neq W(\tilde{A}_2)$. Let $s_i, s_j \in S$ such that $s_i \neq s_j$. Then $\widehat{X}_{W_a}[\lambda(s_i)] \neq \widehat{X}_{W_a}[\lambda(s_j)]$ (where the meaning of $X_{W_a}[\lambda(s_i)]$ is defined in Notation 4.2).*

Proof. For the $B_2 (= C_2)$ and G_2 cases one can directly see the result on Figures 7 and 8. We assume now that the rank of Φ is greater or equal than 3. Let us assume that s_i and s_j are in the same component $\widehat{X}_{W_a}[\lambda(s_i)]$. We claim that there exists a root $\theta \in \Phi^+ \setminus \Delta$ such that $P_\theta(s_i) \neq P_\theta(s_j)$. Assuming this claim, because of Theorem 4.1 and Corollary 3.1 we have $k(s_i, \theta) = P_\theta(s_i) + \lambda_\theta(s_i) = 0$ and $k(s_j, \theta) = P_\theta(s_j) + \lambda_\theta(s_j) = 0$, which implies that $\lambda_\theta(s_i) \neq \lambda_\theta(s_j)$. However, since s_i and s_j are in $\widehat{X}_{W_a}[\lambda(s_i)]$, we must have $\lambda(s_i) = \lambda(s_j)$ and then $\lambda_\alpha(s_i) = \lambda_\alpha(s_j)$ for all $\alpha \in \Phi^+ \setminus \Delta$. Therefore, s_i and s_j cannot lie in the same component.

Proof of the claim. Let Φ be a root system satisfying $\text{rank}(\Phi) \geq 3$. This is a general fact, which can be checked case by case, that in any such a crystallographic root system there always exists a root $\theta \in \Phi^+ \setminus \Delta$ such that either α_i is in the expression of θ (according to Δ) but α_j is not, or α_j is in the expression of θ (according to Δ) but α_i is not. We apply this statement on the dual root system of Φ . Let $\theta \in \Phi^+ \setminus \Delta$ such that α_i^\vee is in the expression of θ^\vee (according to Δ^\vee) but α_j^\vee is not, the other case being symmetric. Thus, because of Lemma 4.1 we have X_{α_i} in the expression of P_θ , whereas X_{α_j} doesn't appear in this expression. Again because of Lemma 4.1 we know that all the coefficients of the polynomial P_θ are positive. Therefore, since $k(s_i, \alpha) = 0$ for all $\alpha \in \Phi^+ \setminus \{\alpha_i\}$ and $k(s_i, \alpha_i) = -1$ we have $P_\theta(s_i) \leq -1$. Moreover, we also have $k(s_j, \alpha) = 0$ for all $\alpha \in \Phi^+ \setminus \{\alpha_j\}$ and $k(s_j, \alpha_j) = -1$, which implies that $P_\theta(s_j) = 0$. \square

Remark 4.2. The A_2 case fails because there are not enough roots. See Figure 6 where the generators $s_0 = (0, 0, 1)$, $s_1 = (-1, 0, 0)$, and $s_2 = (0, -1, 0)$ lie in the same component.

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