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STRONG RATE OF CONVERGENCE IN THE SMOLUCHOWSKI–KRAMERS APPROXIMATION FOR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

CHARLES-EDOUARD BRÉHIER AND ZIYI LEI

ABSTRACT. We consider a class of stochastic damped semilinear wave equations, in the small-mass limit. It has previously been established that the solution converges to the solution of a stochastic semilinear heat equation. In this work we exhibit a rate of convergence in this Smoluchowski–Kramers approximation result, which is shown to depend on the regularity of the noise. For instance, for trace-class noise the strong rate of convergence is 1, and for space-time white noise (in dimension 1) the strong rate of convergence is 1/2.

1. INTRODUCTION

In this work, we are interested in establishing rates of convergence in the Smoluchowski–Kramers approximation for the following class of stochastic partial differential equations:

$$(1) \quad \epsilon^2 \partial_{tt} u^\epsilon(t, x) = \Delta_x u^\epsilon(t, x) - \partial_t u^\epsilon(t, x) + f(u^\epsilon(t, x)) + \dot{W}^Q(t, x),$$

for $t \geq 0$ and $x \in \mathcal{D} \subset \mathbb{R}^d$ ($d \in \mathbb{N}$), with homogeneous Dirichlet boundary conditions, and initial values $u^\epsilon(0, x)$ and $v^\epsilon(0, x) = \partial_t u^\epsilon(0, x)$. The nonlinearity f is assumed to be a globally Lipschitz continuous mapping. Finally, the evolution is driven by Gaussian noise which is white in time and possibly colored in space, with a covariance operator denoted by Q .

As explained below, we consider (1) and other stochastic partial differential equations, as stochastic evolution equations with solutions taking values in Hilbert spaces. We refer for instance to the more rigorous formulation (21) of (1) in Section 2.2. In that setting, the evolution is driven by a Q -Wiener process. It is assumed that the covariance operator Q commutes with the linear second-order operator driving the evolution (Δ_x with homogeneous Dirichlet boundary conditions for (1) and (2)).

The Smoluchowski–Kramers approximation result provides convergence of u^ϵ to the solution u^0 to the semilinear heat equation

$$(2) \quad \partial_t u^0(t, x) = \Delta_x u^0(t, x) + f(u^0(t, x)) + \dot{W}^Q(t, x),$$

for $t \geq 0$ and $x \in \mathcal{D}$, with homogeneous Dirichlet boundary conditions and initial value $u^0(0, x) = \lim_{\epsilon \rightarrow 0} u^\epsilon(0, x)$.

The literature contains many results on such convergence results for various stochastic partial differential equations. We refer to the seminal articles [5, 4] by Cerrai and Freidlin, and to [25, 11, 7, 6, 10, 9, 8, 23, 32] for more articles on the Smoluchowski–Kramers approximation for stochastic partial differential equations and infinite dimensional stochastic systems, and to the recent preprint [17]. Many recent works have been devoted to analyze systems with constraints [2, 15], and systems with non-constant friction [3, 14, 13, 12], following the works [18, 19, 16] for stochastic differential equations. We also refer to the recent articles [1, 27, 29, 22, 28] and preprints [26, 24, 21, 31, 20, 30] for an overview of recent activity on proving Smoluchowski–Kramers approximation results for stochastic differential equations of different types.

To the best of our knowledge, rates of convergence in the Smoluchowski–Kramers approximation for stochastic partial differential equations have not been exhibited so far. The objective of this work is to provide strong rates of convergence.

For the system (1), the main result of this work can be described as follows. Let $\beta = 1$ if the equation is driven by trace-class noise, and $\beta = 1/2$ if the equation is driven by space-time white noise in dimension $d = 1$ (see Assumption 2.3 for a more general version). Under appropriate conditions on the initial values

and on the nonlinearity, one has the following convergence result: for all $T \in (0, \infty)$ and $\alpha \in (0, \beta)$, there exists $C_\alpha(T) \in (0, \infty)$ such that one has

$$(3) \quad \sup_{t \in [0, T]} (\mathbb{E}[\|u^\epsilon(t, \cdot) - u^0(t, \cdot)\|_{L^2(\mathcal{D})}^2])^{\frac{1}{2}} \leq C_\alpha(T) \epsilon^\alpha.$$

We refer to Theorem 1 in Section 3 for a more rigorous and general statement of our main result, using the notation and assumptions introduced in Section 2.

Therefore, according to the strong error estimates (3), the strong rate of convergence in the Smoluchowski–Kramers approximation is the parameter β which describes the spatial regularity of the noise.

In the near future, this manuscript will be completed with weak error analysis. Under appropriate conditions, we will prove that the weak order is equal to 1 for the system (3) driven by either trace-class noise ($\beta = 1$) or by space-time white noise in dimension $d = 1$ ($\beta = 1/2$). More generally, the strong and weak orders of convergence are equal to β and to $\min(2\beta, 1)$ respectively. The weak error analysis exploits many of the auxiliary properties obtained below (in Section 5), and additional auxiliary results, especially on Kolmogorov equations. Note that the situation may differ from the analysis of the Smoluchowski–Kramers approximation for finite-dimensional stochastic differential equations, where both strong and weak rates of convergence are equal to 1.

This manuscript is organized as follows. Section 2 provides the necessary notation and assumptions, and contains a description of the considered stochastic evolution equations. The main result, Theorem 1, is stated in Section 3. Section 4 presents technical properties of the semigroup associated with the heat and the damped wave equations, where the dependence with respect to the parameter ϵ is analyzed carefully. Section 5 is devoted to present moment bounds and regularity properties of solutions to the stochastic partial differential equations (1) (uniformly with respect to ϵ) and (2). The proof of Theorem 1 is provided in Section 6.

2. PRELIMINARIES

2.1. Setting.

2.1.1. *Notation.* Let $d \in \mathbb{N}$ be a positive integer and let $\mathcal{D} \subset \mathbb{R}^d$ be an open bounded domain with polygonal boundary $\partial\mathcal{D}$, for instance $\mathcal{D} = (0, 1)^d$. Define the separable Hilbert space $H = L^2(\mathcal{D})$. The inner product and the norm in H are denoted by $\langle \cdot, \cdot \rangle_H$ and $\|\cdot\|_H$ respectively. Owing to the Riesz theorem, the dual space of H is identified with H .

Let us denote by $-\Lambda$ the Laplace operator endowed with homogeneous Dirichlet boundary conditions. Then Λ is a self-adjoint unbounded linear operator from H to H , with domain $D(\Lambda) = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$, and there exists a complete orthonormal system $(e_n)_{n \in \mathbb{N}}$ of H and a nondecreasing sequence $(\lambda_n)_{n \in \mathbb{N}}$ of positive real numbers such that one has

$$(4) \quad \Lambda e_n = \lambda_n e_n.$$

Note that the eigenvalue λ_1 is positive. In addition, there exists $c_{\Lambda, d} \in (0, \infty)$ such that $\lambda_n \sim c_{\Lambda, d} n^{2/d}$ when $n \rightarrow \infty$.

For any nonnegative real number $\alpha \in [0, \infty)$, define

$$H^\alpha = \{u \in H; \sum_{n=1}^{\infty} \lambda_n^\alpha \langle u, e_n \rangle_H^2 < \infty\}.$$

Moreover, for all $u_1, u_2, u \in H^\alpha$, set

$$\langle u_1, u_2 \rangle_{H^\alpha} = \sum_{n=1}^{\infty} \lambda_n^\alpha \langle u_1, e_n \rangle_H \langle u_2, e_n \rangle_H, \quad \|u\|_{H^\alpha}^2 = \langle u, u \rangle_{H^\alpha} = \sum_{n=1}^{\infty} \lambda_n^\alpha \langle u, e_n \rangle_H^2.$$

For any $\alpha \in [0, \infty)$, the space H^α , equipped with the inner product $\langle \cdot, \cdot \rangle_{H^\alpha}$ and the norm $\|\cdot\|_{H^\alpha}$, is a separable Hilbert space.

For any nonnegative real number $\alpha \in [0, \infty)$, the space $H^{-\alpha}$ is defined as the closure of the space

$$\{u \in H; \sum_{n=1}^{\infty} \lambda_n^{-\alpha} \langle u, e_n \rangle_H^2 < \infty\},$$

with inner product $\langle \cdot, \cdot \rangle_{H^{-\alpha}}$ and norm $\|\cdot\|_{H^{-\alpha}}$ defined as follows: for all $u_1, u_2, u \in H^{-\alpha}$, set

$$\langle u_1, u_2 \rangle_{H^{-\alpha}} = \sum_{n=1}^{\infty} \lambda_n^{-\alpha} \langle u_1, e_n \rangle_H \langle u_2, e_n \rangle_H, \quad \|u\|_{H^{-\alpha}}^2 = \langle u, u \rangle_{H^{-\alpha}} = \sum_{n=1}^{\infty} \lambda_n^{-\alpha} \langle u, e_n \rangle_H^2.$$

For any $\alpha \in [0, \infty)$, the space $H^{-\alpha}$ equipped with the inner product $\langle \cdot, \cdot \rangle_{H^{-\alpha}}$ and the norm $\|\cdot\|_{H^{-\alpha}}$ is a separable Hilbert space. It can be identified with the dual space of H^α . Note also that $H^0 = H$.

For any real number $\alpha \in \mathbb{R}$, introduce the product space

$$\mathcal{H}^\alpha = H^\alpha \times H^{\alpha-1}.$$

Moreover, for all $x_1 = (u_1, v_1), x_2 = (u_2, v_2), x = (u, v) \in \mathcal{H}^\alpha$, define

$$\begin{aligned} \langle x_1, x_2 \rangle_{\mathcal{H}^\alpha} &= \langle (u_1, v_1), (u_2, v_2) \rangle_{\mathcal{H}^\alpha} = \langle u_1, u_2 \rangle_{H^\alpha} + \langle v_1, v_2 \rangle_{H^{\alpha-1}}, \\ \|x\|_{\mathcal{H}^\alpha}^2 &= \langle x, x \rangle_{\mathcal{H}^\alpha} = \langle (u, v), (u, v) \rangle_{\mathcal{H}^\alpha} = \|u\|_{H^\alpha}^2 + \|v\|_{H^{\alpha-1}}^2. \end{aligned}$$

For any $\alpha \in \mathbb{R}$, the space \mathcal{H}^α , equipped with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}^\alpha}$ and the norm $\|\cdot\|_{\mathcal{H}^\alpha}$, is a separable Hilbert space. When $\alpha = 0$, the notation $\mathcal{H} = \mathcal{H}^0$, $\langle \cdot, \cdot \rangle_{\mathcal{H}} = \langle \cdot, \cdot \rangle_{\mathcal{H}^0}$ and $\|\cdot\|_{\mathcal{H}} = \|\cdot\|_{\mathcal{H}^0}$ is used in the sequel.

Observe that if $\alpha_1 < \alpha_2$, one has $H^{\alpha_2} \subset H^{\alpha_1}$ and $\mathcal{H}^{\alpha_2} \subset \mathcal{H}^{\alpha_1}$, with continuous embeddings: for all $u \in H^{\alpha_2}$ and all $x \in \mathcal{H}^{\alpha_2}$, one has

$$(5) \quad \|u\|_{H^{\alpha_1}} \leq \lambda_1^{-\frac{\alpha_2 - \alpha_1}{2}} \|u\|_{H^{\alpha_2}}, \quad \|x\|_{\mathcal{H}^{\alpha_1}} \leq \lambda_1^{-\frac{\alpha_2 - \alpha_1}{2}} \|x\|_{\mathcal{H}^{\alpha_2}}.$$

Let $\alpha \in \mathbb{R}$ be an arbitrary real number. Define the projection operators $\Pi_u: \mathcal{H}^\alpha \rightarrow H^\alpha$ and $\Pi_v: \mathcal{H}^\alpha \rightarrow H^{\alpha-1}$ such that for all $(u, v) \in \mathcal{H}^\alpha$ one has

$$(6) \quad \Pi_u(u, v) = u \quad \text{and} \quad \Pi_v(u, v) = v.$$

The operators Π_u and Π_v depend on α , however the dependence is omitted to simplify notation. Note that for all $\alpha \in \mathbb{R}$ and all $x = (u, v) \in \mathcal{H}^\alpha$, one has

$$(7) \quad \max(\|\Pi_u x\|_{H^\alpha}, \|\Pi_v x\|_{H^{\alpha-1}}) \leq \|x\|_{\mathcal{H}^\alpha}.$$

For all $\alpha \in \mathbb{R}$, let the linear operator Λ^α be defined as follows: for all $u \in H^{2\alpha}$, set

$$\Lambda^\alpha u = \sum_{n \in \mathbb{N}} \lambda_n^\alpha \langle u, e_n \rangle_H e_n.$$

For any positive real number $\alpha \in (0, \infty)$, the linear operator Λ^α is unbounded from H to H , and has domain $D(\Lambda^\alpha) = H^{2\alpha}$. Moreover, for all $u \in H^{2\alpha}$, one has $\|\Lambda^\alpha u\|_H = \|u\|_{H^{2\alpha}}$.

For any $\epsilon \in (0, \infty)$, the linear operator A_ϵ defined as follows: for all $\alpha \in \mathbb{R}$ and for all $x = (u, v) \in \mathcal{H}^\alpha$, set

$$(8) \quad A_\epsilon x = \left(\frac{v}{\epsilon}, -\frac{\Lambda u}{\epsilon} - \frac{v}{\epsilon^2} \right) \in \mathcal{H}^{\alpha-1}.$$

Note that the linear operator A_ϵ can be considered as an unbounded linear operator on \mathcal{H} with domain \mathcal{H}^1 .

2.1.2. Semigroups. The linear operator Λ generates a semigroup $(e^{-t\Lambda})_{t \geq 0}$ of bounded linear operators, defined as follows:

$$(9) \quad e^{-t\Lambda} u = \sum_{n \in \mathbb{N}} e^{-t\lambda_n} \langle u, e_n \rangle_H e_n, \quad \forall t \geq 0, \quad \forall u \in H,$$

where for all $n \in \mathbb{N}$ the mapping $t \mapsto u_n(t) := e^{-t\lambda_n} \langle u, e_n \rangle_H$ is the solution of the linear ordinary differential equation

$$u_n'(t) = -\lambda_n u_n(t), \quad t \geq 0; \quad u_n(0) = \langle u, e_n \rangle.$$

In addition, for any $\epsilon \in (0, \infty)$, the linear operator A_ϵ generates a semigroup $(e^{tA_\epsilon})_{t \geq 0}$ of bounded linear operators, defined as follows:

$$(10) \quad e^{tA_\epsilon} x = \sum_{n \in \mathbb{N}} (f_n^\epsilon(t) e_n, g_n^\epsilon(t) e_n), \quad \forall t \geq 0, \quad \forall x = (u, v) \in \mathcal{H},$$

where for all $n \in \mathbb{N}$ the mapping $t \mapsto (f_n^\epsilon(t), g_n^\epsilon(t))$ is the solution of the linear two-dimensional system of ordinary differential equations

$$(11) \quad \frac{d}{dt} \begin{pmatrix} f_n^\epsilon(t) \\ g_n^\epsilon(t) \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\epsilon} \\ -\frac{\lambda_n}{\epsilon} & -\frac{1}{\epsilon^2} \end{pmatrix} \begin{pmatrix} f_n^\epsilon(t) \\ g_n^\epsilon(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{\epsilon} g_n^\epsilon(t) \\ -\frac{\lambda_n}{\epsilon} f_n^\epsilon(t) - \frac{1}{\epsilon^2} g_n^\epsilon(t) \end{pmatrix}, \quad t \geq 0; \quad \begin{pmatrix} f_n^\epsilon(0) \\ g_n^\epsilon(0) \end{pmatrix} = \begin{pmatrix} \langle u, e_n \rangle_H \\ \langle v, e_n \rangle_H \end{pmatrix}.$$

Properties of semigroups $(e^{-t\Lambda})_{t \geq 0}$ and $(e^{tA_\epsilon})_{t \geq 0}$ are studied in Section 4.1 and Section 4.2 respectively.

2.1.3. Nonlinearity. The nonlinearity is given by a mapping $f : u \in H \mapsto f(u) \in H$. Given such a function f , the mapping $F : \mathcal{H} \rightarrow \mathcal{H}^1$ is then defined as follows: for all $(u, v) \in \mathcal{H}$,

$$(12) \quad F(u, v) = (0, f(u)).$$

First, the mapping f is assumed to satisfy the following Lipschitz continuity condition.

Assumption 2.1. *The mapping f is globally Lipschitz continuous, with Lipschitz constant L_f : one has*

$$L_f = \sup_{u_1, u_2 \in H, u_1 \neq u_2} \frac{\|f(u_2) - f(u_1)\|_H}{\|u_2 - u_1\|_H} \in [0, \infty).$$

In particular, observe that $f : H \rightarrow H$ has at most linear growth. Moreover, note that under Assumption 2.1, the mapping $F : \mathcal{H} \rightarrow \mathcal{H}^1$ is also globally Lipschitz continuous. Indeed, using the definitions of the norms $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}^1}$ and the inequality (5), for all $x_1 = (u_1, v_1), x_2 = (u_2, v_2) \in \mathcal{H}$, one has

$$\|F(x_2) - F(x_1)\|_{\mathcal{H}} \leq \lambda_1^{-\frac{1}{2}} \|F(x_2) - F(x_1)\|_{\mathcal{H}^1} \leq \lambda_1^{-\frac{1}{2}} \|f(u_2) - f(u_1)\|_H \leq \frac{L_f}{\sqrt{\lambda_1}} \|u_2 - u_1\|_H \leq \frac{L_f}{\sqrt{\lambda_1}} \|x_2 - x_1\|_{\mathcal{H}}.$$

Supplementary regularity conditions are imposed on the nonlinearity f .

Assumption 2.2. *The mapping $f : H \rightarrow H$ is 4 times Fréchet differentiable, with bounded derivatives of order 1 to 4. For all $k \in \{1, 2, 3, 4\}$, set*

$$\|f\|_k = \sum_{\ell=1}^k \sup_{u \in H} \sup_{h_1, \dots, h_\ell \in H \setminus \{0\}} \frac{\|D^\ell f(u).(h_1, \dots, h_\ell)\|_H}{\|h_1\|_H \cdots \|h_\ell\|_H} \in [0, \infty),$$

where $D^\ell f$ denotes the Fréchet derivative of f of order ℓ .

In addition, it is assumed that $f(H^1) \subset H^1$ and that $f : H^1 \rightarrow H^1$ has at most linear growth, i.e., there exists $C_1 \in (0, \infty)$ such that for all $u \in H^1$ one has

$$(13) \quad \|f(u)\|_{H^1} \leq C_\alpha (1 + \|u\|_{H^1}),$$

and that for all $\alpha \in (0, 1)$, all $\varepsilon \in (0, 1 - \alpha]$, there exists $C_{\alpha, \varepsilon} \in (0, \infty)$ such that for all $u \in H^{\alpha + \varepsilon}$ one has

$$(14) \quad \|f(u)\|_{H^\alpha} \leq C_{\alpha, \varepsilon} (1 + \|u\|_{H^{\alpha + \varepsilon}}).$$

Finally, for all $\alpha \in [0, 1/2)$, there exists $C_\alpha \in (0, \infty)$ such that for all $u_1, u_2 \in H^\alpha$ one has

$$(15) \quad \|f(u_2) - f(u_1)\|_{H^{-1}} \leq C_\alpha (1 + \|u_1\|_{H^\alpha} + \|u_2\|_{H^\alpha}) \|u_2 - u_1\|_{H^{-\alpha}}.$$

Note that the condition (15) from Assumption 2.2 implies the following additional regularity result: if $\beta \in [\frac{1}{2}, 1]$, for all $u_1, u_2 \in H^\beta$, one has

$$(16) \quad \|f(u_2) - f(u_1)\|_{H^{-1}} \leq C_\beta (1 + \|u_1\|_{H^\beta} + \|u_2\|_{H^\beta}) \|u_2 - u_1\|_{H^{\beta-1}}.$$

That result follows from applying (15) with $\alpha = 1 - \beta \in [0, \frac{1}{2}]$, noting that $\beta \geq 1 - \beta$ and using the inequality (5).

2.1.4. *The Wiener process.* Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where the expectation operator is denoted by $\mathbb{E}[\cdot]$. A filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions is considered. Let $(\beta_n)_{n \in \mathbb{N}}$ be a family of independent standard real-valued Wiener processes, adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$.

Given a sequence $(q_n)_{n \in \mathbb{N}}$ of nonnegative real numbers, let us define for all $t \geq 0$

$$(17) \quad W^Q(t) = \sum_{n \in \mathbb{N}} \sqrt{q_n} \beta_n(t) e_n.$$

The stochastic process $(W^Q(t))_{t \geq 0}$ is then a Q -Wiener process, with the covariance operator Q defined by

$$Qh = \sum_{n \in \mathbb{N}} q_n \langle h, e_n \rangle_H e_n.$$

More precisely, the following conditions are imposed.

Assumption 2.3. *The sequence $(q_n)_{n \in \mathbb{N}}$ is bounded. Moreover, there exists $\beta \in (0, 1]$ such that, one has*

$$(18) \quad \sum_{n \in \mathbb{N}} q_n \lambda_n^{\alpha-1} < \infty, \quad \forall \alpha \in [0, \beta).$$

Under Assumption 2.3, the covariance operator Q is a bounded linear self-adjoint operator on H . In addition, its square root $Q^{\frac{1}{2}}$ is defined as the bounded linear operator on H given by

$$Q^{\frac{1}{2}}h = \sum_{n \in \mathbb{N}} \sqrt{q_n} \langle h, e_n \rangle_H e_n, \quad \forall h \in H.$$

The condition (18) can then be rewritten as

$$\|\Lambda^{\frac{\alpha-1}{2}} Q^{\frac{1}{2}}\|_{\mathcal{L}_2(H)}^2 = \sum_{n \in \mathbb{N}} \|\Lambda^{\frac{\alpha-1}{2}} Q^{\frac{1}{2}} e_n\|_H^2 = \sum_{n \in \mathbb{N}} q_n \lambda_n^{\alpha-1} < \infty$$

where $\mathcal{L}_2(H)$ denotes the space of Hilbert–Schmidt operators from H to H , and $\|\cdot\|_{\mathcal{L}_2(H)}$ denotes the associated norm. The condition (18) ensures that the Wiener process $(W^Q(t))_{t \geq 0}$ takes values in $H^{\alpha-1}$ for all $\alpha \in [0, \beta)$.

In addition, note that the covariance operator Q commutes with the linear operator Λ .

The stochastic evolution equation considered in this paper is driven by the \mathcal{H} -valued Wiener process $(\mathcal{W}^Q(t))_{t \geq 0}$ given by

$$(19) \quad \mathcal{W}^Q(t) = (0, W^Q(t)) = \sum_{n \in \mathbb{N}} \sqrt{q_n} \beta_n(t) (0, e_n), \quad \forall t \geq 0.$$

Under the condition (18), the Wiener process $(\mathcal{W}^Q(t))_{t \geq 0}$ takes values in \mathcal{H}^α for all $\alpha \in [0, \beta)$. Moreover, for all $\alpha \in [0, \beta)$ and $p \in [1, \infty)$, there exists $C_{\alpha,p} \in (0, \infty)$ such that for all $t \geq s \geq 0$ one has

$$(20) \quad \mathbb{E}[\|\mathcal{W}^Q(t) - \mathcal{W}^Q(s)\|_{\mathcal{H}^\alpha}^p] = C_{\alpha,p} (t-s)^{\frac{p}{2}}.$$

For instance, one may consider the identity operator $Q = I$ in dimension $d = 1$, in which case the condition (18) is satisfied with $\beta = 1/2$. If Q is a trace-class operator, i.e. if $\sum_{n=1}^{\infty} q_n < \infty$, then the condition (18) is satisfied with $\beta = 1$.

2.2. Stochastic partial differential equations. The aim of this article is to study the behavior when $\epsilon \rightarrow 0$ of the solution of the stochastic semilinear damped wave equation driven by additive noise

$$(21) \quad \begin{cases} du^\epsilon(t) = \frac{v^\epsilon(t)}{\epsilon} dt, & t \geq 0, \\ dv^\epsilon(t) = \left(-\frac{\Lambda u^\epsilon(t)}{\epsilon} - \frac{v^\epsilon(t)}{\epsilon^2} \right) dt + \frac{f(u^\epsilon(t))}{\epsilon} dt + \frac{1}{\epsilon} dW^Q(t), & t \geq 0, \\ u^\epsilon(0) = u_0^\epsilon, & v^\epsilon(0) = v_0^\epsilon. \end{cases}$$

The linear operator Λ , the mapping f and the Wiener process $(W^Q(t))_{t \geq 0}$ are described in Section 2.1. For all $\epsilon \in (0, 1)$, the unknowns $(u^\epsilon(t))_{t \geq 0}$ and $(v^\epsilon(t))_{t \geq 0}$ are H and H^{-1} -valued continuous stochastic processes, adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$. The initial values $u_0^\epsilon \in H$ and $v_0^\epsilon \in H^{-1}$ are \mathcal{F}_0 -measurable random

variables and are thus independent of the Wiener process $(W^Q(t))_{t \geq 0}$. They are allowed to depend on ϵ , more precise conditions are given below.

Employing the notation introduced in Section 2.1, the stochastic evolution system (21) can be equivalently rewritten as the stochastic evolution equation for the unknown $X^\epsilon(t) = (u^\epsilon(t), v^\epsilon(t)) \in \mathcal{H}$

$$(22) \quad \begin{cases} dX^\epsilon(t) = A_\epsilon X^\epsilon(t)dt + \frac{1}{\epsilon} F(X^\epsilon(t))dt + \frac{1}{\epsilon} dW^Q(t), & t \geq 0, \\ X^\epsilon(0) = x_0^\epsilon, \end{cases}$$

with A_ϵ given by (8), F given by (12) and the Wiener process $(W^Q(t))_{t \geq 0}$ given by (19). For all $\epsilon \in (0, 1)$, the initial value is given by $x_0^\epsilon = (u_0^\epsilon, v_0^\epsilon) \in \mathcal{H}$. For all $\epsilon \in (0, 1)$, the unknown $(X^\epsilon(t))_{t \geq 0}$ is a \mathcal{H} -valued continuous stochastic process, adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$.

Recall that a \mathcal{H} -valued stochastic process $(X^\epsilon(t))_{t \geq 0}$ (adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$ and with continuous trajectories) is a mild solution of the stochastic evolution equation (22) with initial value $X^\epsilon(0) = x_0^\epsilon$, if for all $t \geq 0$ almost surely one has

$$(23) \quad X^\epsilon(t) = e^{tA_\epsilon} x_0^\epsilon + \frac{1}{\epsilon} \int_0^t e^{(t-s)A_\epsilon} F(X^\epsilon(s)) ds + \frac{1}{\epsilon} \int_0^t e^{(t-s)A_\epsilon} dW^Q(s),$$

where $(e^{tA_\epsilon})_{t \geq 0}$ is the semigroup generated by the linear operator A_ϵ , given by (10), see Section 2.1.2.

For all $\epsilon \in (0, 1)$ and all $t \geq 0$, set

$$(24) \quad \mathcal{Z}^\epsilon(t) = \frac{1}{\epsilon} \int_0^t e^{(t-s)A_\epsilon} dW^Q(s).$$

The process $(\mathcal{Z}^\epsilon(t))_{t \geq 0}$ is called the stochastic convolution. Under Assumption 2.3, it will be seen below that this process is a Gaussian continuous \mathcal{H} -valued stochastic process. In addition, the mapping $F : \mathcal{H} \rightarrow \mathcal{H}$ is globally Lipschitz continuous, therefore by a standard fixed point procedure, it is straightforward to prove that for all $\epsilon \in (0, 1)$, there exists a unique global mild solution (23) of the stochastic evolution equation (22). More details, in particular regularity properties and moment bounds, are provided in Section 5 below. It is worth mentioning that properties which are uniform with respect to $\epsilon \in (0, 1)$ are required, and obtaining them requires to employ the results presented in Section 4.

When $\epsilon \rightarrow 0$, assuming that $u_0^\epsilon \rightarrow u_0^0$ in H , then for all $t \geq 0$, $u^\epsilon(t) = \Pi_u X^\epsilon(t)$ converges to the solution $u^0(t)$ at time t of the stochastic semilinear heat equation driven by additive noise

$$(25) \quad \begin{cases} du^0(t) = -\Lambda u^0(t)dt + f(u^0(t))dt + dW^Q(t), \\ u^0(0) = u_0^0. \end{cases}$$

The objective of this article is to provide strong error estimates, see Section 3 for a precise statement.

Recall that an H -valued stochastic process $(u^0(t))_{t \geq 0}$ (adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$ and with continuous trajectories) is a mild solution of the stochastic evolution equation (25) with initial value $u^0(0) = u_0^0$, if for all $t \geq 0$ almost surely one has

$$(26) \quad u^0(t) = e^{-t\Lambda} u_0^0 + \int_0^t e^{-(t-s)\Lambda} f(u^0(s)) ds + \int_0^t e^{-(t-s)\Lambda} dW^Q(s),$$

where $(e^{-t\Lambda})_{t \geq 0}$ is the semigroup generated by the linear operator $-\Lambda$, given by (9), see Section 2.1.2.

For all $t \geq 0$, set

$$(27) \quad Z(t) = \int_0^t e^{-(t-s)\Lambda} dW^Q(s).$$

The process $(Z(t))_{t \geq 0}$ is called the stochastic convolution. Under Assumption 2.3, it will be seen below that this process is a Gaussian continuous H -valued stochastic process. In addition, the mapping $f : H \rightarrow H$ is globally Lipschitz continuous (see Assumption 2.1), therefore by a standard fixed point procedure, it is straightforward to prove that there exists a unique global mild solution (26) of the stochastic evolution equation (25). We refer to Section 5 for additional details, in particular on regularity properties and moment bounds.

3. MAIN RESULTS

Before stating the main result, let us provide assumptions in the initial values $x_0^\epsilon = (u_0^\epsilon, v_0^\epsilon)$.

Assumption 3.1. *For all $\alpha \in [0, \beta)$, there exists $M_\alpha \in (0, \infty)$ such that*

$$(28) \quad \sup_{\epsilon \in (0,1)} \mathbb{E}[\|x_0^\epsilon\|_{\mathcal{H}^\alpha}^2] \leq M_\alpha,$$

$$(29) \quad \mathbb{E}[\|u_0^\epsilon - u_0^0\|_H^2] \leq M_\alpha \epsilon^{2\alpha}, \quad \forall \epsilon \in (0, 1),.$$

Note that under Assumption 3.1, owing to the Fatou lemma, for all $\alpha \in [0, \beta)$ one obtains

$$(30) \quad \mathbb{E}[\|u_0^0\|_{H^\alpha}^2] \leq M_\alpha.$$

Theorem 1. *Let Assumptions 2.1, 2.3 and 3.1 be satisfied.*

For all $T \in (0, \infty)$ and $\alpha \in [0, \beta)$ (where $\beta \in (0, 1]$ is given by the condition (18) from Assumption 2.3), there exists $C_\alpha(T) \in (0, \infty)$ such that for all $\epsilon \in (0, 1)$ one has

$$(31) \quad \sup_{0 \leq t \leq T} (\mathbb{E}[\|u^\epsilon(t) - u^0(t)\|_H^2])^{\frac{1}{2}} \leq C_\alpha(T) \epsilon^\alpha.$$

4. AUXILIARY RESULTS

This section presents several auxiliary results which play important roles in the error analysis.

4.1. Properties of the heat semigroup. Let us first recall standard properties concerning the heat semigroup $(e^{-t\Lambda})_{t \geq 0}$ defined by (9) in Section 2.1.2.

Lemma 4.1. *For all $\alpha \in \mathbb{R}$, all $u \in H^\alpha$ and all $t \geq 0$, one has*

$$(32) \quad \|e^{-t\Lambda}u\|_{H^\alpha} \leq e^{-t\lambda_1} \|u\|_{H^\alpha}.$$

In addition, the following smoothing property is satisfied: for all $\alpha \geq 0$, there exists $C_\alpha \in (0, \infty)$ such that for all $t > 0$ and all $u \in H$ one has

$$(33) \quad \|e^{-t\Lambda}u\|_{H^\alpha} \leq C_\alpha t^{-\frac{\alpha}{2}} \|u\|_H.$$

The results stated in Lemma 4.1 are standard and the elements of proof below are given for completeness.

Proof of Lemma 4.1. First, owing to the definitions (9) of $e^{-t\Lambda}u$ and of the norm $\|\cdot\|_{H^\alpha}$ on H^α (see Section 2.1.1), one obtains

$$\|e^{-t\Lambda}u\|_{H^\alpha}^2 = \sum_{n=1}^{\infty} \lambda_n^\alpha e^{-2t\lambda_n} \langle u, e_n \rangle_H^2 \leq e^{-2t\lambda_1} \sum_{n=1}^{\infty} \lambda_n^\alpha \langle u, e_n \rangle_H^2 = e^{-2t\lambda_1} \|u\|_{H^\alpha}^2.$$

This concludes the proof of (32). In order to prove (33), note that for all $\alpha \geq 0$ the mapping $x \geq 0 \mapsto x^\alpha e^{-x}$ is bounded. As a result, there exists $C_\alpha \in (0, \infty)$ such that for all $u \in H$ and all $t > 0$ one has

$$\|e^{-t\Lambda}u\|_{H^\alpha}^2 = t^{-\alpha} \sum_{n=1}^{\infty} t^\alpha \lambda_n^\alpha e^{-2t\lambda_n} \langle u, e_n \rangle_H^2 \leq C_\alpha^2 t^{-\alpha} \sum_{n=1}^{\infty} \langle u, e_n \rangle_H^2 = C_\alpha^2 t^{-\alpha} \|u\|_H^2.$$

This concludes the proof of (33). The proof of Lemma 4.1 is thus completed. \square

4.2. Properties of the damped wave semigroup. The objective of this section is to study properties of the damped wave semigroup $(e^{tA_\epsilon})_{t \geq 0}$ given by (10) in Section 2.1.2. The main difficulty is to obtain upper bounds which are either uniform with respect to the parameter $\epsilon \in (0, 1)$, or which contain terms which vanish when $\epsilon \rightarrow 0$. We obtain variants of the smoothing property (33).

First, the semigroup satisfies the following bounds which are uniform with respect to $\epsilon \in (0, 1)$ and $t \in [0, \infty)$.

Lemma 4.2. *For all $\alpha \in \mathbb{R}$ and all $x \in \mathcal{H}^\alpha$, one has*

$$(34) \quad \sup_{\epsilon \in (0,1)} \sup_{t \geq 0} \|e^{tA_\epsilon}x\|_{\mathcal{H}^\alpha} \leq \|x\|_{\mathcal{H}^\alpha}.$$

Proof of Lemma 4.2. Recall that for all $x = (u, v) \in \mathcal{H}^\alpha$ and all $t \geq 0$ one has

$$e^{tA_\epsilon} x = \sum_{n \in \mathbb{N}} (f_n^\epsilon(t) e_n, g_n^\epsilon(t) e_n)$$

where for all $n \in \mathbb{N}$ the mapping $t \geq 0 \mapsto (f_n^\epsilon(t), g_n^\epsilon(t))$ solves (11).

Let $n \in \mathbb{N}$. Observe that $f \mapsto f_n^\epsilon(t)$ solves the second-order linear differential equation

$$(35) \quad \epsilon^2 (f_n^\epsilon)''(t) + (f_n^\epsilon)'(t) + \lambda_n f_n^\epsilon(t) = 0.$$

It then follows that one has

$$\epsilon^2 \frac{d}{dt} |(f_n^\epsilon(t))'|^2 + 2 |(f_n^\epsilon(t))'|^2 + \lambda_n \frac{d}{dt} |f_n^\epsilon(t)|^2 = 2 \langle \epsilon^2 (f_n^\epsilon(t))'' + (f_n^\epsilon(t))' + \lambda_n f_n^\epsilon(t), (f_n^\epsilon(t))' \rangle = 0.$$

By integration one obtains for all $t \geq 0$

$$\epsilon^2 |(f_n^\epsilon(t))'|^2 + 2 \int_0^t |(f_n^\epsilon(s))'|^2 ds + \lambda_n |f_n^\epsilon(t)|^2 = \epsilon^2 |(f_n^\epsilon(0))'|^2 + \lambda_n |f_n^\epsilon(0)|^2.$$

For all $t \geq 0$ one has $g_n^\epsilon(t) = \epsilon (f_n^\epsilon(t))'$ owing to (11). Therefore multiplying both sides by $\lambda_n^{\alpha-1}$ and omitting the integral term in the left-hand side, the equality above yields the following inequality: for all $n \in \mathbb{N}$ and all $t \geq 0$, one has

$$\lambda_n^{\alpha-1} |g_n^\epsilon(t)|^2 + \lambda_n^\alpha |f_n^\epsilon(t)|^2 \leq \lambda_n^{\alpha-1} |g_n^\epsilon(0)|^2 + \lambda_n^\alpha |f_n^\epsilon(0)|^2.$$

Finally, using the identity

$$(36) \quad \|e^{tA_\epsilon} x\|_{\mathcal{H}^\alpha}^2 = \sum_{n \in \mathbb{N}} \lambda_n^\alpha |f_n^\epsilon(t)|^2 + \sum_{n \in \mathbb{N}} \lambda_n^{\alpha-1} |g_n^\epsilon(t)|^2$$

and the inequality above gives

$$\|e^{tA_\epsilon} x\|_{\mathcal{H}^\alpha}^2 \leq \|x\|_{\mathcal{H}^\alpha}^2.$$

This concludes the proof of Lemma 4.2. \square

Let us now state two further auxiliary results, which are variants of the smoothing property (33) for the heat semigroup, but require more attention. Those results play an important role in the analysis below.

Lemma 4.3. *For all $\delta \in [0, \frac{1}{2}]$ and $\rho \in [2\delta, 1]$, there exists $C_{\delta, \rho} \in (0, \infty)$ such that, for all $\alpha \in \mathbb{R}$, all $\epsilon \in (0, 1)$, all $t > 0$ and all $u \in H^{\alpha + \rho - 2\delta - 1}$,*

$$(37) \quad \|e^{tA_\epsilon}(0, u)\|_{\mathcal{H}^\alpha} \leq C_{\delta, \rho} t^{-\delta} e^\rho \left(1 + \epsilon^{2\delta - \rho} e^{-\frac{t}{2\epsilon^2}}\right) \|u\|_{H^{\alpha + \rho - 2\delta - 1}}.$$

As stated in Proposition 5.2 below, the result (37) will be used to analyze the moment bounds of stochastic convolution \mathcal{Z}^ϵ in the moment estimates of X^ϵ , with $\delta = \frac{1 - \delta_0}{2}$, $\rho = 1$, where $\delta_0 \in (0, \beta - \alpha]$ is arbitrarily small. In addition to this, for each application of Lemma 4.3, the appropriate values of the parameters δ, ρ will be specified.

Lemma 4.4. *For all $\alpha, \alpha_1, \alpha_2 \in [0, 1]$ and all $\delta \in [0, \frac{\alpha}{2}]$, there exist $C_{\alpha, \delta}, C_{\alpha_1}, C_{\alpha_2} \in (0, \infty)$ such that, for all $\epsilon \in (0, 1)$ and all $t > 0$, one has*

$$(38) \quad \|\Pi_u e^{tA_\epsilon}(0, v)\|_H \leq C_{\alpha, \delta} \epsilon^{\alpha t - \delta} \|v\|_{H^{\alpha - 1 - 2\delta}}, \quad \forall v \in H^{\alpha - 1 - 2\delta},$$

$$(39) \quad \|\Pi_u e^{tA_\epsilon}(u, 0)\|_H \leq C_{\alpha_1} t^{-\frac{\alpha_1}{2}} \epsilon^{\alpha_1} \|u\|_H + C_{\alpha_2} t^{-\frac{\alpha_2}{2}} \|u\|_{H^{-\alpha_2}}, \quad \forall u \in H.$$

The proofs of Lemmas 4.3 and 4.4 are based on a refined analysis of properties of the solutions of the systems (11), depending on the values of $\epsilon \in (0, 1)$ and $n \in \mathbb{N}$. For arbitrary $n \in \mathbb{N}$ and $\epsilon \in (0, 1)$, let $t \geq 0 \mapsto (f_n^{\epsilon, 0, 1}(t), g_n^{\epsilon, 0, 1}(t))$ and $t \geq 0 \mapsto (f_n^{\epsilon, 1, 0}(t), g_n^{\epsilon, 1, 0}(t))$ denote the solutions of the linear two-dimensional system of ordinary differential equations (11) with initial values $(f_n^{\epsilon, 0, 1}(0), g_n^{\epsilon, 0, 1}(0)) = (0, 1)$ and $(f_n^{\epsilon, 1, 0}(0), g_n^{\epsilon, 1, 0}(0)) = (1, 0)$ respectively.

Lemma 4.5. *There exists $C \in (0, \infty)$ such that the following holds.*

- If $1 - 4\lambda_n \epsilon^2 < 0$, then for all $t \geq 0$,

$$(40) \quad |f_n^{\epsilon, 0, 1}(t)| \leq C \lambda_n^{-1/2} e^{-\frac{t}{4\epsilon^2}}, \quad |g_n^{\epsilon, 0, 1}(t)| \leq C e^{-\frac{t}{4\epsilon^2}},$$

$$(41) \quad |f_n^{\epsilon, 1, 0}(t)| \leq C e^{-\frac{t}{4\epsilon^2}}.$$

- If $1 - 4\lambda_n\epsilon^2 \geq 0$, then for all $t \geq 0$ one has

$$(42) \quad |f_n^{\epsilon,0,1}(t)| \leq C\epsilon e^{-\lambda_n t}, \quad |g_n^{\epsilon,0,1}(t)| \leq C e^{-\frac{t}{\epsilon^2}} + C\lambda_n\epsilon^2 e^{-\lambda_n t},$$

$$(43) \quad |f_n^{\epsilon,1,0}(t)| \leq C e^{-t\lambda_n}.$$

4.2.1. *Proof of Lemma 4.5.* A few supplementary ingredients are required for the proof of Lemma 4.5.

First, one can provide the expressions of $f_n^\epsilon(t)$, for all $n \in \mathbb{N}$, all $\epsilon \in (0, 1)$, and for all $t \geq 0$, for any initial value $(f_n^\epsilon(0), g_n^\epsilon(0))$. The expressions depend on the sign of $1 - 4\lambda_n\epsilon^2$. In fact, using (35) it is straightforward to check that the mapping $t \geq 0 \mapsto \tilde{f}_n^\epsilon(t) = e^{\frac{t}{2\epsilon^2}} f_n^\epsilon(t)$ is solution of the second order linear differential equation

$$(\tilde{f}_n^\epsilon)''(t) + \frac{4\epsilon^2\lambda_n - 1}{4\epsilon^4} \tilde{f}_n^\epsilon(t) = 0,$$

with initial values $\tilde{f}_n^\epsilon(0) = f_n^\epsilon(0)$ and $(\tilde{f}_n^\epsilon)'(0) = (f_n^\epsilon)'(0) + \frac{1}{2\epsilon^2} f_n^\epsilon(0) = \frac{2\epsilon g_n^\epsilon(0) + f_n^\epsilon(0)}{2\epsilon^2}$. Solving the second order linear differential equation above yields the following expressions for $f_n^\epsilon(t) = e^{-\frac{t}{2\epsilon^2}} \tilde{f}_n^\epsilon(t)$ and for $g_n^\epsilon(t) = \epsilon (f_n^\epsilon)'(t)$, see (11).

- If $1 - 4\lambda_n\epsilon^2 < 0$, then for all $t \geq 0$ one has

$$(44) \quad \begin{cases} f_n^\epsilon(t) = e^{-\frac{t}{2\epsilon^2}} \left[f_n^\epsilon(0) \cos\left(\frac{\sqrt{4\lambda_n\epsilon^2 - 1}}{2\epsilon^2} t\right) + \frac{2\epsilon g_n^\epsilon(0) + f_n^\epsilon(0)}{\sqrt{4\lambda_n\epsilon^2 - 1}} \sin\left(\frac{\sqrt{4\lambda_n\epsilon^2 - 1}}{2\epsilon^2} t\right) \right], \\ g_n^\epsilon(t) = e^{-\frac{t}{2\epsilon^2}} \left[g_n^\epsilon(0) \cos\left(\frac{\sqrt{4\lambda_n\epsilon^2 - 1}}{2\epsilon^2} t\right) - \frac{2\lambda_n\epsilon f_n^\epsilon(0) + g_n^\epsilon(0)}{\sqrt{4\lambda_n\epsilon^2 - 1}} \sin\left(\frac{\sqrt{4\lambda_n\epsilon^2 - 1}}{2\epsilon^2} t\right) \right]. \end{cases}$$

- If $1 - 4\lambda_n\epsilon^2 = 0$, then for all $t \geq 0$ one has

$$(45) \quad \begin{cases} f_n^\epsilon(t) = e^{-\frac{t}{2\epsilon^2}} \left[f_n^\epsilon(0) \left(1 + \frac{t}{2\epsilon^2}\right) + \frac{t}{\epsilon} g_n^\epsilon(0) \right], \\ g_n^\epsilon(t) = e^{-\frac{t}{2\epsilon^2}} \left[g_n^\epsilon(0) \left(1 - \frac{t}{2\epsilon^2}\right) - \frac{t}{4\epsilon^3} f_n^\epsilon(0) \right]. \end{cases}$$

- If $1 - 4\lambda_n\epsilon^2 > 0$, then for all $t \geq 0$ one has

$$(46) \quad \begin{cases} f_n^\epsilon(t) = e^{-\frac{t}{2\epsilon^2}} \left[f_n^\epsilon(0) \cosh\left(\frac{\sqrt{1 - 4\lambda_n\epsilon^2}}{2\epsilon^2} t\right) + \frac{2\epsilon g_n^\epsilon(0) + f_n^\epsilon(0)}{\sqrt{1 - 4\lambda_n\epsilon^2}} \sinh\left(\frac{\sqrt{1 - 4\lambda_n\epsilon^2}}{2\epsilon^2} t\right) \right], \\ g_n^\epsilon(t) = e^{-\frac{t}{2\epsilon^2}} \left[g_n^\epsilon(0) \cosh\left(\frac{\sqrt{1 - 4\lambda_n\epsilon^2}}{2\epsilon^2} t\right) - \frac{2\lambda_n\epsilon f_n^\epsilon(0) + g_n^\epsilon(0)}{\sqrt{1 - 4\lambda_n\epsilon^2}} \sinh\left(\frac{\sqrt{1 - 4\lambda_n\epsilon^2}}{2\epsilon^2} t\right) \right]. \end{cases}$$

To prove some inequalities from Lemma 4.5, the following expression of $g_n^\epsilon(t)$ is employed: for all $n \in \mathbb{N}$, $\epsilon \in (0, 1)$ and $t \geq 0$, one has

$$(47) \quad g_n^\epsilon(t) = e^{-\frac{t}{\epsilon^2}} g_n^\epsilon(0) - \int_0^t \frac{\lambda_n}{\epsilon} e^{-\frac{t-s}{\epsilon^2}} f_n^\epsilon(s) ds.$$

The proof of the identity (47) is straightforward: from (11) one obtains

$$\frac{d}{dt} \left(\epsilon e^{\frac{t}{\epsilon^2}} g_n^\epsilon(t) \right) = \frac{d}{dt} \left(\epsilon^2 e^{\frac{t}{\epsilon^2}} (f_n^\epsilon)'(t) \right) = -\lambda_n \epsilon e^{\frac{t}{\epsilon^2}} f_n^\epsilon(t),$$

which yields by integration for all $t \geq 0$

$$\epsilon e^{\frac{t}{\epsilon^2}} g_n^\epsilon(t) = \epsilon g_n^\epsilon(0) - \int_0^t \lambda_n \epsilon e^{\frac{s}{\epsilon^2}} f_n^\epsilon(s) ds,$$

and the identity (47) then follows.

Finally, for all $n \in \mathbb{N}$ and $\epsilon \in (0, 1)$, let us define the auxiliary parameters

$$\theta_{n,\epsilon} = \min\left(\frac{1}{4\epsilon^2}, \lambda_n\right) = \begin{cases} \frac{1}{4\epsilon^2}, & \text{if } 1 - 4\lambda_n\epsilon^2 \leq 0 \\ \lambda_n, & \text{if } 1 - 4\lambda_n\epsilon^2 \geq 0 \end{cases},$$

$$\Theta_{n,\epsilon} = \lambda_n - \theta_{n,\epsilon} + \epsilon^2 \theta_{n,\epsilon}^2.$$

In addition, for all $t \geq 0$ set

$$\tilde{w}_{n,\epsilon}(t) = e^{\theta_{n,\epsilon} t} f_n^\epsilon(t).$$

By elementary computations, using (11) it is straightforward to check that the auxiliary mapping $\tilde{w}_{n,\epsilon}$ is solution of the linear second-order ordinary differential equation

$$(48) \quad \tilde{w}_{n,\epsilon}''(t) + \left(\frac{1}{\epsilon^2} - 2\theta_{n,\epsilon}\right)\tilde{w}_{n,\epsilon}'(t) + \frac{1}{\epsilon^2}\Theta_{n,\epsilon}\tilde{w}_{n,\epsilon}(t) = 0.$$

with initial values

$$\tilde{w}_{n,\epsilon}(0) = f_n^\epsilon(0), \quad \tilde{w}_{n,\epsilon}'(0) = \theta_{n,\epsilon}f_n^\epsilon(0) + (f_n^\epsilon)'(0) = \theta_{n,\epsilon}f_n^\epsilon(0) + \frac{1}{\epsilon}g_n^\epsilon(0).$$

Two useful identities are obtained using (48). First, one has

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} |\tilde{w}_{n,\epsilon}'(t)|^2 + \left(\frac{1}{\epsilon^2} - 2\theta_{n,\epsilon}\right) |\tilde{w}_{n,\epsilon}'(t)|^2 + \frac{\Theta_{n,\epsilon}}{2\epsilon^2} \frac{d}{dt} |\tilde{w}_{n,\epsilon}(t)|^2 \\ &= \left\langle \tilde{w}_{n,\epsilon}''(t) + \left(\frac{1}{\epsilon^2} - 2\theta_{n,\epsilon}\right)\tilde{w}_{n,\epsilon}'(t) + \frac{\Theta_{n,\epsilon}}{\epsilon^2}\tilde{w}_{n,\epsilon}(t), \tilde{w}_{n,\epsilon}'(t) \right\rangle \\ &= 0. \end{aligned}$$

One then obtains the following result by integration and using the initial values for $\tilde{w}_{n,\epsilon}(0)$ and $\tilde{w}_{n,\epsilon}'(0)$: for all $t \geq 0$ one has

$$(49) \quad \begin{aligned} \epsilon^2 |\tilde{w}_{n,\epsilon}'(t)|^2 + 2(1 - 2\epsilon^2\theta_{n,\epsilon}) \int_0^t |\tilde{w}_{n,\epsilon}'(s)|^2 ds + \Theta_{n,\epsilon} |\tilde{w}_{n,\epsilon}(t)|^2 &= \epsilon^2 |\tilde{w}_{n,\epsilon}'(0)|^2 + \Theta_{n,\epsilon} |\tilde{w}_{n,\epsilon}(0)|^2 \\ &= |\epsilon\theta_{n,\epsilon}f_n^\epsilon(0) + g_n^\epsilon(0)|^2 + \Theta_{n,\epsilon} |f_n^\epsilon(0)|^2. \end{aligned}$$

Second, one has

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |\epsilon^2 \tilde{w}_{n,\epsilon}'(t) + (1 - 2\epsilon^2\theta_{n,\epsilon})\tilde{w}_{n,\epsilon}(t)|^2 &= \left\langle \epsilon^2 \tilde{w}_{n,\epsilon}''(t) + (1 - 2\epsilon^2\theta_{n,\epsilon})\tilde{w}_{n,\epsilon}'(t), \epsilon^2 \tilde{w}_{n,\epsilon}'(t) + (1 - 2\epsilon^2\theta_{n,\epsilon})\tilde{w}_{n,\epsilon}(t) \right\rangle \\ &= - \left\langle \Theta_{n,\epsilon}\tilde{w}_{n,\epsilon}(t), \epsilon^2 \tilde{w}_{n,\epsilon}'(t) + (1 - 2\epsilon^2\theta_{n,\epsilon})\tilde{w}_{n,\epsilon}(t) \right\rangle \\ &= -\frac{\epsilon^2}{2} \Theta_{n,\epsilon} \frac{d}{dt} |\tilde{w}_{n,\epsilon}(t)|^2 - (1 - 2\epsilon^2\theta_{n,\epsilon})\Theta_{n,\epsilon} |\tilde{w}_{n,\epsilon}(t)|^2. \end{aligned}$$

One then obtains the following result by integration and using the initial values for $\tilde{w}_{n,\epsilon}(0)$ and $\tilde{w}_{n,\epsilon}'(0)$: for all $t \geq 0$ one has

$$(50) \quad \begin{aligned} & |\epsilon^2 \tilde{w}_{n,\epsilon}'(t) + (1 - 2\epsilon^2\theta_{n,\epsilon})\tilde{w}_{n,\epsilon}(t)|^2 + \epsilon^2 \Theta_{n,\epsilon} |\tilde{w}_{n,\epsilon}(t)|^2 + 2(1 - 2\epsilon^2\theta_{n,\epsilon})\Theta_{n,\epsilon} \int_0^t |\tilde{w}_{n,\epsilon}(s)|^2 ds \\ &= |\epsilon^2 \tilde{w}_{n,\epsilon}'(0) + (1 - 2\epsilon^2\theta_{n,\epsilon})\tilde{w}_{n,\epsilon}(0)|^2 + \epsilon^2 \Theta_{n,\epsilon} |\tilde{w}_{n,\epsilon}(0)|^2 \\ &= |\epsilon g_n^\epsilon(0) + (1 - \epsilon^2\theta_{n,\epsilon})f_n^\epsilon(0)|^2 + \epsilon^2 (\lambda_n - \theta_{n,\epsilon} + \epsilon^2\theta_{n,\epsilon}^2) |f_n^\epsilon(0)|^2. \end{aligned}$$

Set $\tilde{w}_{n,\epsilon}^{0,1}(t) = e^{\theta_{n,\epsilon}t} f_n^{\epsilon,0,1}(t)$ and $\tilde{w}_{n,\epsilon}^{1,0}(t) = e^{\theta_{n,\epsilon}t} f_n^{\epsilon,1,0}(t)$ for all $t \geq 0$.

The proof of Lemma 4.5 can now be performed using the auxiliary tools introduced above.

Proof of Lemma 4.5. Let $n \in \mathbb{N}$ and $\epsilon \in (0, 1)$. First, let us assume that $1 - 4\lambda_n\epsilon^2 < 0$.

- Proof of the inequality (40). It is straightforward to check that for all $z \in \mathbb{R}$ one has

$$|\sin(z)| \leq \frac{|z|}{\max(1, |z|)},$$

by combining the upper bounds $|\sin(z)| \leq 1$ and $|\sin(z)| \leq |z|$. Using the expression (44) of $f_n^{\epsilon,0,1}(t)$ and the inequality above with $z = \frac{\sqrt{4\lambda_n\epsilon^2 - 1}}{2\epsilon^2}t$, one obtains

$$\begin{aligned} |f_n^{\epsilon,0,1}(t)|^2 &= \left| e^{-\frac{t}{2\epsilon^2}} \frac{2\epsilon}{\sqrt{4\lambda_n\epsilon^2 - 1}} \sin\left(\frac{\sqrt{4\lambda_n\epsilon^2 - 1}}{2\epsilon^2}t\right) \right|^2 \\ &\leq e^{-\frac{t}{\epsilon^2}} \frac{t^2}{\epsilon^2} \frac{1}{\max(1, \frac{(4\lambda_n\epsilon^2 - 1)t^2}{4\epsilon^4})}. \end{aligned}$$

Then writing

$$\frac{t^2}{\epsilon^2} = \frac{1}{4\lambda_n} \frac{4\lambda_n\epsilon^2 t^2}{\epsilon^4} = \frac{1}{4\lambda_n} \frac{(4\lambda_n\epsilon^2 - 1)t^2}{\epsilon^4} + \frac{1}{4\lambda_n} \frac{t^2}{\epsilon^4},$$

and the upper bound $\sup_{x \geq 0} (1+x^2)e^{-\frac{x}{2}} < \infty$, one obtains the inequality

$$|f_n^{\epsilon,0,1}(t)|^2 \leq \frac{1}{4\lambda_n} e^{-\frac{t}{2}} \left(1 + \frac{t^2}{\epsilon^4}\right) \leq \frac{C}{\lambda_n} e^{-\frac{t}{2\epsilon^2}}.$$

This concludes the proof of the first inequality in (40). To prove the second inequality, note that due to the assumption $1 - 4\lambda_n\epsilon^2 < 0$ one has

$$\theta_{n,\epsilon} = \frac{1}{4\epsilon^2} > 0, \quad \Theta_{n,\epsilon} = \lambda_n - \theta_{n,\epsilon} + \epsilon^2\theta_{n,\epsilon}^2 = \lambda_n - \frac{3}{16\epsilon^2} > \frac{\lambda_n}{4}, \quad 1 - 2\epsilon^2\theta_{n,\epsilon} = \frac{1}{2}.$$

As a result, using the inequalities (49) and (50), with the initial values $f_n^{\epsilon,0,1}(0) = 0$ and $g_n^{\epsilon,0,1}(0) = 1$ one obtains for all $t \geq 0$

$$\begin{aligned} \epsilon |(\tilde{w}_{n,\epsilon}^{0,1})'(t)| &\leq 1 \\ |\epsilon^2(\tilde{w}_{n,\epsilon}^{0,1})'(t) + \frac{1}{2}\tilde{w}_{n,\epsilon}^{0,1}(t)| &\leq \epsilon, \end{aligned}$$

and using the triangle inequality then gives

$$|\tilde{w}_{n,\epsilon}^{0,1}(t)| \leq 2 \left(|\epsilon^2(\tilde{w}_{n,\epsilon}^{0,1})'(t) + \frac{1}{2}\tilde{w}_{n,\epsilon}^{0,1}(t)| + \epsilon^2 |(\tilde{w}_{n,\epsilon}^{0,1})'(t)| \right) \leq 4\epsilon.$$

Finally, using the identity

$$g_n^{\epsilon,0,1}(t) = \epsilon(f_n^{\epsilon,0,1})'(t) = \epsilon e^{-\theta_{n,\epsilon}t} \left((\tilde{w}_{n,\epsilon}^{0,1})'(t) - \theta_{n,\epsilon}\tilde{w}_{n,\epsilon}^{0,1}(t) \right),$$

and the value $\theta_{n,\epsilon} = \frac{1}{4\epsilon^2}$, one obtains for all $t \geq 0$

$$|g_n^{\epsilon,0,1}(t)| \leq \epsilon e^{-\frac{t}{4\epsilon^2}} |(\tilde{w}_{n,\epsilon}^{0,1})'(t)| + \frac{1}{4\epsilon} e^{-\frac{t}{4\epsilon^2}} |\tilde{w}_{n,\epsilon}^{0,1}(t)| \leq 2e^{-\frac{t}{4\epsilon^2}}.$$

This concludes the proof of the second inequality in (40). The proof of (40) is thus completed.

- Proof of the inequality (41). Using the expression (44) of $f_n^{\epsilon,1,0}(t)$ and the inequalities $|\cos(z)| \leq 1$ and $|\sin(z)| \leq |z|$ for all $z \in \mathbb{R}$, one has

$$\begin{aligned} |f_n^{\epsilon,1,0}(t)| &= e^{-\frac{t}{2\epsilon^2}} \left| \cos\left(\frac{\sqrt{4\lambda_n\epsilon^2 - 1}}{2\epsilon^2}t\right) + \frac{1}{\sqrt{4\lambda_n\epsilon^2 - 1}} \sin\left(\frac{\sqrt{4\lambda_n\epsilon^2 - 1}}{2\epsilon^2}t\right) \right| \\ &\leq e^{-\frac{t}{2\epsilon^2}} \left(1 + \frac{t}{2\epsilon^2}\right). \end{aligned}$$

Using then the upper bound $\sup_{x \geq 0} (1+x)e^{-\frac{x}{2}} < \infty$ then gives

$$|f_n^{\epsilon,1,0}(t)| \leq C e^{-\frac{t}{4\epsilon^2}}.$$

The proof of (41) is thus completed.

Let us now assume that $1 - 4\lambda_n\epsilon^2 \geq 0$.

- Proof of the inequality (42). Note that due to the assumption $1 - 4\lambda_n\epsilon^2 \geq 0$ one has

$$\theta_{n,\epsilon} = \lambda_n > 0, \quad \Theta_{n,\epsilon} = \lambda_n - \theta_{n,\epsilon} + \epsilon^2\theta_{n,\epsilon}^2 = \lambda_n^2\epsilon^2 > 0, \quad 1 - 2\epsilon^2\theta_{n,\epsilon} \geq 2\lambda_n\epsilon^2.$$

As a result, using the inequalities (49) and (50) with the initial values $f_n^{\epsilon,0,1}(0) = 0$ and $g_n^{\epsilon,0,1}(0) = 1$, one obtains for all $t \geq 0$

$$\begin{aligned} \epsilon |(\tilde{w}_{n,\epsilon}^{0,1})'(t)| &\leq 1 \\ |\epsilon^2(\tilde{w}_{n,\epsilon}^{0,1})'(t) + (1 - 2\epsilon^2\theta_{n,\epsilon})\tilde{w}_{n,\epsilon}^{0,1}(t)| &\leq \epsilon. \end{aligned}$$

Moreover, using the assumption $1 - 4\lambda_n\epsilon^2 \geq 0$, one has the lower bound

$$1 - 2\lambda_n\epsilon^2 \geq \frac{1}{2}.$$

Therefore, using the triangle inequality and the two inequalities above, for all $t \geq 0$ one obtains

$$\begin{aligned} |\tilde{w}_{n,\epsilon}^{0,1}(t)| &\leq 2(1 - 2\lambda_n\epsilon^2) |\tilde{w}_{n,\epsilon}^{0,1}(t)| \\ &\leq 2 \left(|\epsilon^2(\tilde{w}_{n,\epsilon}^{0,1})'(t) + (1 - 2\lambda_n\epsilon^2)\tilde{w}_{n,\epsilon}^{0,1}(t)| + \epsilon^2 |(\tilde{w}_{n,\epsilon}^{0,1})'(t)| \right) \end{aligned}$$

$$\leq 4\epsilon.$$

Thus for all $t \geq 0$ one has

$$|f_n^{\epsilon,0,1}(t)| = e^{-\theta_{n,\epsilon}t} |\tilde{w}_{n,\epsilon}^{0,1}(t)| \leq 4\epsilon e^{-\lambda_n t}.$$

This provides the first inequality from (42). To prove the second inequality, owing to the equality (47), recalling that $g_n^{\epsilon,0,1}(0) = 1$ and using the inequality proved above for $|f_n^{\epsilon,0,1}(t)|$, one has for all $t \geq 0$

$$\begin{aligned} |g_n^{\epsilon,0,1}(t)| &\leq e^{-\frac{t}{\epsilon^2}} + \int_0^t \frac{\lambda_n}{\epsilon} e^{-\frac{t-s}{\epsilon^2}} |f_n^{\epsilon,0,1}(s)| ds \\ &\leq e^{-\frac{t}{\epsilon^2}} + 4\lambda_n \int_0^t e^{-\frac{t-s}{\epsilon^2}} e^{-\lambda_n s} ds \\ &\leq e^{-\frac{t}{\epsilon^2}} + \frac{4\lambda_n \epsilon^2}{1 - \lambda_n \epsilon^2} (e^{-\lambda_n t} - e^{-\frac{t}{\epsilon^2}}) \\ &\leq e^{-\frac{t}{\epsilon^2}} + C\lambda_n \epsilon^2 e^{-\lambda_n t}, \end{aligned}$$

using the fact that $1 - \lambda_n \epsilon^2 \geq \frac{3}{4}$ owing to the assumption $1 - 4\lambda_n \epsilon^2 \geq 0$. This concludes the proof of the second inequality in (42). The proof of (42) is thus completed.

- Proof of the inequality (43). Using the inequality (49), with the initial values $f_n^{\epsilon,1,0}(0) = 1$ and $g_n^{\epsilon,1,0}(0) = 0$, one obtains for all $t \geq 0$

$$\Theta_{n,\epsilon} |\tilde{w}_{n,\epsilon}^{1,0}(t)|^2 \leq (\epsilon \theta_{n,\epsilon})^2 + \Theta_{n,\epsilon}.$$

Recalling that $\theta_{n,\epsilon} = \lambda_n$ and $\Theta_{n,\epsilon} = \lambda^2 \epsilon^2$ owing to the assumption $1 - 4\lambda_n \epsilon^2 \geq 0$, one has for all $t \geq 0$

$$|\tilde{w}_{n,\epsilon}^{1,0}(t)|^2 \leq 2.$$

Therefore for all $t \geq 0$ one has

$$|f_n^{\epsilon,1,0}(t)| = e^{-\theta_{n,\epsilon}t} |\tilde{w}_{n,\epsilon}^{1,0}(t)| \leq \sqrt{2} e^{-t\lambda_n}.$$

The proof of (43) is thus completed.

The proof of Lemma 4.5 is now completed. \square

4.2.2. Proofs of Lemma 4.3 and Lemma 4.4.

Proof of Lemma 4.3. Let $u \in H^{\alpha+\rho-2\delta-1}$, then one has the following expression for $e^{tA_\epsilon}(0, u)$: for all $t \geq 0$,

$$e^{tA_\epsilon}(0, u) = \sum_{n \in \mathbb{N}} \langle u, e_n \rangle f_n^{\epsilon,0,1}(t) (e_n, 0) + \sum_{n \in \mathbb{N}} \langle u, e_n \rangle g_n^{\epsilon,0,1}(t) (0, e_n),$$

using (10) and the notation introduced above the statement of Lemma 4.5. Alternatively, this may be written as

$$\Pi_u \left(e^{tA_\epsilon}(0, u) \right) = \sum_{n \in \mathbb{N}} \langle u, e_n \rangle f_n^{\epsilon,0,1}(t) e_n, \quad \Pi_v \left(e^{tA_\epsilon}(0, u) \right) = \sum_{n \in \mathbb{N}} \langle u, e_n \rangle g_n^{\epsilon,0,1}(t) e_n,$$

and as a result one has

$$\begin{aligned} \|e^{tA_\epsilon}(0, u)\|_{\mathcal{H}^\alpha}^2 &= \|\Pi_u e^{tA_\epsilon}(0, u)\|_{H^\alpha}^2 + \|\Pi_v e^{tA_\epsilon}(0, u)\|_{H^{\alpha-1}}^2 \\ &= \sum_{n \in \mathbb{N}} \lambda_n^\alpha \langle u, e_n \rangle^2 |f_n^{\epsilon,0,1}(t)|^2 + \sum_{n \in \mathbb{N}} \lambda_n^{\alpha-1} \langle u, e_n \rangle^2 |g_n^{\epsilon,0,1}(t)|^2 \\ &= \sum_{n \in \mathbb{N}} \lambda_n^{\alpha+\rho-2\delta-1} \langle u, e_n \rangle^2 \lambda_n^{1+2\delta-\rho} |f_n^{\epsilon,0,1}(t)|^2 + \sum_{n \in \mathbb{N}} \lambda_n^{\alpha+\rho-2\delta-1} \langle u, e_n \rangle^2 \lambda_n^{2\delta-\rho} |g_n^{\epsilon,0,1}(t)|^2 \\ &\leq \|u\|_{H^{\alpha-2\delta+\rho-1}}^2 \sup_{n \in \mathbb{N}} \left(\lambda_n^{1+2\delta-\rho} |f_n^{\epsilon,0,1}(t)|^2 + \lambda_n^{2\delta-\rho} |g_n^{\epsilon,0,1}(t)|^2 \right). \end{aligned}$$

To proceed, one needs to treat separately the cases $1 - 4\lambda_n \epsilon^2 < 0$ and $1 - 4\lambda_n \epsilon^2 \geq 0$: for all $n \in \mathbb{N}$, $\epsilon \in (0, 1)$ and $t \in (0, \infty)$, using the inequalities (40) and (42) from Lemma 4.5, one obtains

$$\begin{aligned} \lambda_n^{1+2\delta-\rho} |f_n^{\epsilon,0,1}(t)|^2 + \lambda_n^{2\delta-\rho} |g_n^{\epsilon,0,1}(t)|^2 &= (\lambda_n^{1+2\delta-\rho} |f_n^{\epsilon,0,1}(t)|^2 + \lambda_n^{2\delta-\rho} |g_n^{\epsilon,0,1}(t)|^2) \mathbb{1}_{1-4\lambda_n \epsilon^2 < 0} \\ &\quad + (\lambda_n^{1+2\delta-\rho} |f_n^{\epsilon,0,1}(t)|^2 + \lambda_n^{2\delta-\rho} |g_n^{\epsilon,0,1}(t)|^2) \mathbb{1}_{1-4\lambda_n \epsilon^2 \geq 0} \end{aligned}$$

$$\begin{aligned} &\leq C\lambda_n^{2\delta-\rho}e^{-\frac{t}{2\epsilon^2}}\mathbb{1}_{1-4\lambda_n\epsilon^2<0} \\ &+ C\left(\epsilon^2\lambda_n^{1+2\delta-\rho}e^{-2\lambda_n t} + \lambda_n^{2\delta-\rho}e^{-\frac{2t}{\epsilon^2}} + \lambda_n^{2\delta-\rho+2}\epsilon^4e^{-2\lambda_n t}\right)\mathbb{1}_{1-4\lambda_n\epsilon^2\geq 0}. \end{aligned}$$

On the one hand, using the inequality $\sup_{x\geq 0} x^{2\delta}e^{-x} < \infty$, one obtains

$$\begin{aligned} C\lambda_n^{2\delta-\rho}e^{-\frac{t}{2\epsilon^2}}\mathbb{1}_{1-4\lambda_n\epsilon^2<0} &\leq C\lambda_n^{2\delta-\rho}\frac{2\epsilon^{4\delta}}{t^{2\delta}}\mathbb{1}_{1-4\lambda_n\epsilon^2<0} \\ &\leq Ct^{-2\delta}\epsilon^{2\rho}(\lambda_n\epsilon^2)^{2\delta-\rho}\mathbb{1}_{1-4\lambda_n\epsilon^2<0} \\ &\leq Ct^{-2\delta}\epsilon^{2\rho}, \end{aligned}$$

using the conditions $2\delta - \rho \leq 0$ and $4\lambda_n\epsilon^2 > 1$ in the last step.

On the other hand, using the inequality $\sup_{x\geq 0} x^{2\delta}e^{-x} < \infty$, one obtains

$$\begin{aligned} \epsilon^2\lambda_n^{1+2\delta-\rho}e^{-2\lambda_n t}\mathbb{1}_{1-4\lambda_n\epsilon^2\geq 0} &\leq Ct^{-2\delta}\epsilon^2\lambda_n^{1-\rho}\mathbb{1}_{1-4\lambda_n\epsilon^2\geq 0} \\ &\leq Ct^{-2\delta}\epsilon^{2\rho}(\lambda_n\epsilon^2)^{1-\rho}\mathbb{1}_{1-4\lambda_n\epsilon^2\geq 0} \\ &\leq Ct^{-2\delta}\epsilon^{2\rho}. \end{aligned}$$

Finally, observe that the assumption $1 - 4\lambda_n\epsilon^2 \geq 0$ ensures that

$$\lambda_n^{2\delta-\rho+2}\epsilon^4e^{-2\lambda_n t}\mathbb{1}_{1-4\lambda_n\epsilon^2\geq 0} \leq \epsilon^2\lambda_n^{1+2\delta-\rho}e^{-2\lambda_n t}\mathbb{1}_{1-4\lambda_n\epsilon^2\geq 0} \leq Ct^{-2\delta}\epsilon^{2\rho},$$

and using again the inequality $\sup_{x\geq 0} x^{2\delta}e^{-x} < \infty$, one obtains

$$\begin{aligned} \lambda_n^{2\delta-\rho}e^{-\frac{2t}{\epsilon^2}}\mathbb{1}_{1-4\lambda_n\epsilon^2\geq 0} &\leq C\lambda_n^{2\delta-\rho}\frac{\epsilon^{4\delta}}{t^{2\delta}}e^{-\frac{t}{\epsilon^2}}\mathbb{1}_{1-4\lambda_n\epsilon^2\geq 0} \\ &\leq Ct^{-2\delta}\epsilon^{2\rho}\lambda_n^{2\delta-\rho}\epsilon^{2(2\delta-\rho)}e^{-\frac{t}{\epsilon^2}}\mathbb{1}_{1-4\lambda_n\epsilon^2\geq 0} \\ &\leq Ct^{-2\delta}\epsilon^{2\rho}\epsilon^{2(2\delta-\rho)}e^{-\frac{t}{\epsilon^2}}, \end{aligned}$$

using the condition $2\delta - \rho \leq 0$.

Gathering the estimates yields for all $t \in (0, \infty)$ the upper bound

$$\sup_{n \in \mathbb{N}} \left(\lambda_n^{1+2\delta-\rho} |f_n^{\epsilon,0,1}(t)|^2 + \lambda_n^{2\delta-\rho} |g_n^{\epsilon,0,1}(t)|^2 \right) \leq Ct^{-2\delta}\epsilon^{2\rho} \left(1 + \epsilon^{2(2\delta-\rho)}e^{-\frac{t}{\epsilon^2}} \right).$$

Due to the inequality above, one obtains (37) and the proof of Lemma 4.3 is thus completed. \square

Proof of Lemma 4.4. Like in the proof of Lemma 4.3 above, for all $v \in H^{\alpha-1-2\delta}$ and for all $t \geq 0$ one has

$$\Pi_u \left(e^{tA_\epsilon}(0, v) \right) = \sum_{n \in \mathbb{N}} \langle v, e_n \rangle f_n^{\epsilon,0,1}(t) e_n$$

and as a result one has

$$\begin{aligned} \|\Pi_v \left(e^{tA_\epsilon}(0, v) \right)\|_H^2 &= \sum_{n \in \mathbb{N}} \langle v, e_n \rangle^2 |f_n^{\epsilon,0,1}(t)|^2 \\ &= \sum_{n \in \mathbb{N}} \lambda_n^{\alpha-1-2\delta} \langle v, e_n \rangle^2 \lambda_n^{1+2\delta-\alpha} |f_n^{\epsilon,0,1}(t)|^2 \\ &\leq \|v\|_{H^{\alpha-1-2\delta}}^2 \sup_{n \in \mathbb{N}} \left(\lambda_n^{1+2\delta-\alpha} |f_n^{\epsilon,0,1}(t)|^2 \right). \end{aligned}$$

To proceed, one needs to treat separately the cases $1 - 4\lambda_n\epsilon^2 < 0$ and $1 - 4\lambda_n\epsilon^2 \geq 0$: for all $n \in \mathbb{N}$, $\epsilon \in (0, 1)$ and $t \in (0, \infty)$, using the inequalities (40) and (42) from Lemma 4.5, one obtains

$$\begin{aligned} \lambda_n^{1+2\delta-\alpha} |f_n^{\epsilon,0,1}(t)|^2 &= \lambda_n^{1+2\delta-\alpha} |f_n^{\epsilon,0,1}(t)|^2 \mathbb{1}_{1-4\lambda_n\epsilon^2<0} + \lambda_n^{1+2\delta-\alpha} |f_n^{\epsilon,0,1}(t)|^2 \mathbb{1}_{1-4\lambda_n\epsilon^2\geq 0} \\ &\leq C\lambda_n^{2\delta-\alpha} e^{-\frac{t}{2\epsilon^2}} \mathbb{1}_{1-4\lambda_n\epsilon^2<0} + C\lambda_n^{1+2\delta-\alpha} \epsilon^2 e^{-2\lambda_n t} \mathbb{1}_{1-4\lambda_n\epsilon^2\geq 0}. \end{aligned}$$

On the one hand, using the inequality $\sup_{x \geq 0} x^{2\delta} e^{-x} < \infty$, one obtains

$$\begin{aligned} \lambda_n^{2\delta-\alpha} e^{-\frac{t}{2\epsilon^2}} \mathbb{1}_{1-4\lambda_n\epsilon^2 < 0} &\leq C \lambda_n^{2\delta-\alpha} \frac{\epsilon^{4\delta}}{t^{2\delta}} \mathbb{1}_{1-4\lambda_n\epsilon^2 < 0} \\ &\leq C t^{-2\delta} \epsilon^{2\alpha} (\lambda_n \epsilon^2)^{2\delta-\alpha} \mathbb{1}_{1-4\lambda_n\epsilon^2 < 0} \\ &\leq C t^{-2\delta} \epsilon^{2\alpha}, \end{aligned}$$

using the conditions $2\delta - \alpha \leq 0$ and $4\lambda_n\epsilon^2 > 1$ in the last step.

On the other hand, using the inequality $\sup_{x \geq 0} x^{2\delta} e^{-x} < \infty$, one obtains

$$\begin{aligned} \lambda_n^{1+2\delta-\alpha} \epsilon^2 e^{-2\lambda_n t} \mathbb{1}_{1-4\lambda_n\epsilon^2 \geq 0} &\leq C t^{-2\delta} \lambda_n^{1-\alpha} \epsilon^2 \mathbb{1}_{1-4\lambda_n\epsilon^2 \geq 0} \\ &\leq C t^{-2\delta} \epsilon^{2\alpha} (\lambda_n \epsilon^2)^{1-\alpha} \mathbb{1}_{1-4\lambda_n\epsilon^2 \geq 0} \\ &\leq C t^{-2\delta} \epsilon^{2\alpha}. \end{aligned}$$

Gathering the estimates yields for all $t \in (0, \infty)$ the upper bound

$$\sup_{n \in \mathbb{N}} \left(\lambda_n^{1+2\delta-\alpha} |f_n^{\epsilon,0,1}(t)|^2 \right) \leq C t^{-2\delta} \epsilon^{2\alpha}.$$

Due to the inequality above, one obtains (38).

It remains to prove the inequality (39). Observe that for all $u \in H$ and all $t \geq 0$ one has

$$\Pi_u \left(e^{tA_\epsilon}(u, 0) \right) = \sum_{n \in \mathbb{N}} \langle u, e_n \rangle f_n^{\epsilon,1,0}(t) e_n$$

and as a result one has

$$\begin{aligned} \|\Pi_u \left(e^{tA_\epsilon}(u, 0) \right)\|_H^2 &= \sum_{n \in \mathbb{N}} \langle u, e_n \rangle^2 |f_n^{\epsilon,1,0}(t)|^2 \\ &= \sum_{n \in \mathbb{N}} \langle u, e_n \rangle^2 |f_n^{\epsilon,1,0}(t)|^2 \mathbb{1}_{1-4\lambda_n\epsilon^2 < 0} + \sum_{n \in \mathbb{N}} \langle u, e_n \rangle^2 |f_n^{\epsilon,1,0}(t)|^2 \mathbb{1}_{1-4\lambda_n\epsilon^2 \geq 0}. \end{aligned}$$

On the one hand, using the inequality (41) from Lemma 4.5 and the inequality $\sup_{x \geq 0} x^{\alpha_1} e^{-x} < \infty$, for all $n \in \mathbb{N}$, $\epsilon \in (0, 1)$ and $t \in (0, \infty)$ one has

$$|f_n^{\epsilon,1,0}(t)|^2 \mathbb{1}_{1-4\lambda_n\epsilon^2 < 0} \leq C e^{-\frac{t}{2\epsilon^2}} \leq C \epsilon^{2\alpha_1} t^{-\alpha_1}.$$

On the other hand, using the inequality (43) from Lemma 4.5 and the inequality $\sup_{x \geq 0} x^{\alpha_2} e^{-x} < \infty$, for all $n \in \mathbb{N}$, $\epsilon \in (0, 1)$ and $t \in (0, \infty)$ one has

$$|f_n^{\epsilon,1,0}(t)|^2 \mathbb{1}_{1-4\lambda_n\epsilon^2 \geq 0} \leq C e^{-2\lambda_n t} \mathbb{1}_{1-4\lambda_n\epsilon^2 \geq 0} \leq C t^{-\alpha_2} \lambda_n^{-\alpha_2}.$$

Therefore one obtains for all $t \in (0, \infty)$ and all $u \in H$

$$\begin{aligned} \|\Pi_u \left(e^{tA_\epsilon}(u, 0) \right)\|_H^2 &\leq C \epsilon^{2\alpha_1} t^{-\alpha_1} \sum_{n \in \mathbb{N}} \langle u, e_n \rangle^2 + C t^{-\alpha_2} \sum_{n \in \mathbb{N}} \lambda_n^{-\alpha_2} \langle u, e_n \rangle^2 \\ &\leq C \epsilon^{\alpha_1} t^{-\alpha_1} \|u\|_H^2 + C t^{-\alpha_2} \|u\|_{H^{-\alpha_2}}^2. \end{aligned}$$

This concludes the proof of (39). The proof of Lemma 4.4 is thus completed. \square

4.3. Convergence and error bounds on the semigroups. The objective of this section is to provide precise results to justify the convergence properties

$$\Pi_u e^{tA_\epsilon}(u, 0) \xrightarrow{\epsilon \rightarrow 0} e^{-t\Lambda} u$$

and

$$\Pi_u e^{tA_\epsilon} \left(0, \frac{v}{\epsilon} \right) \xrightarrow{\epsilon \rightarrow 0} e^{-t\Lambda} v,$$

and to obtain error bounds, depending on the regularity properties of u and v .

Lemma 4.6. For all $\alpha \in [0, 1]$ and all $\delta \in [0, \frac{\alpha}{2}]$, there exists $C_{\alpha, \delta} \in (0, \infty)$ such that, for all $\epsilon \in (0, 1)$, all $t > 0$ and all $u \in H^{\alpha-2\delta}$ one has

$$(51) \quad \|\Pi_u e^{tA_\epsilon}(u, 0) - e^{-t\Lambda}u\|_H \leq C_{\alpha, \delta} \epsilon^\alpha t^{-\delta} \|u\|_{H^{\alpha-2\delta}}.$$

Moreover, for all $\alpha \in [0, 1]$ and all $\delta \in [\frac{\alpha}{2}, \frac{1}{2}]$, there exists $C_{\alpha, \delta} \in (0, \infty)$ such that, for all $\epsilon \in (0, 1)$, all $t > 0$ and all $v \in H^{\alpha-2\delta}$ one has

$$(52) \quad \|\Pi_u e^{tA_\epsilon}(0, \frac{v}{\epsilon}) - e^{-t\Lambda}v\|_H \leq C_{\alpha, \delta} \epsilon^\alpha t^{-\delta} \|v\|_{H^{\alpha-2\delta}}.$$

Notice that one has $\alpha - 2\delta \geq 0$ for (51) but $\alpha - 2\delta \leq 0$ for (52).

The proof of Lemma 4.6 is a straightforward consequence of the following auxiliary result. The statement and the proof of Lemma 4.7 exploit the notation introduced in Section 4.2 for the proof of Lemma 4.5.

Lemma 4.7. For all $\alpha \in [0, 1]$ and all $\delta \in [0, \frac{\alpha}{2}]$, there exists $C_{\alpha, \delta} \in (0, \infty)$ such that, for all $\epsilon \in (0, 1)$ and all $t > 0$ one has

$$(53) \quad \sup_{n \in \mathbb{N}} \left(\lambda_n^{2\delta-\alpha} |f_n^{\epsilon, 1, 0}(t) - e^{-t\lambda_n}|^2 \right) \leq Ct^{-2\delta} \epsilon^{2\alpha}.$$

Moreover, for all $\alpha \in [0, 1]$ and all $\delta \in [\frac{\alpha}{2}, \frac{1}{2}]$, there exists $C_{\alpha, \delta} \in (0, \infty)$ such that, for all $\epsilon \in (0, 1)$ and all $t > 0$ one has

$$(54) \quad \sup \left(\lambda_n^{2\delta-\alpha} \left| \frac{f_n^{\epsilon, 0, 1}(t)}{\epsilon} - e^{-t\lambda_n} \right|^2 \right) \leq Ct^{-2\delta} \epsilon^{2\alpha}.$$

Proof of Lemma 4.7. Let us first prove the inequality (53), and assume that $\delta \in [0, \frac{\alpha}{2}]$. Let $n \in \mathbb{N}$ and $\epsilon \in (0, 1)$.

On the one hand, if $1 - 4\lambda_n \epsilon^2 < 0$, owing to the inequality (41) and to the lower bound $\lambda_n \geq \epsilon^2/4$, one has for all $t \geq 0$

$$\begin{aligned} |f_n^{\epsilon, 1, 0}(t) - e^{-t\lambda_n}|^2 &\leq 2|f_n^{\epsilon, 1, 0}(t)|^2 + 2e^{-2t\lambda_n} \\ &\leq Ce^{-\frac{t}{2\epsilon^2}}. \end{aligned}$$

As a result, using the inequality $\sup_{x \in \mathbb{R}} x^{2\delta} e^{-x} < \infty$, one obtains for all $t > 0$

$$\begin{aligned} \lambda_n^{2\delta-\alpha} |f_n^{\epsilon, 1, 0}(t) - e^{-t\lambda_n}|^2 \mathbb{1}_{1-4\lambda_n \epsilon^2 < 0} &\leq C \lambda_n^{2\delta-\alpha} \frac{\epsilon^{4\delta}}{t^{2\delta}} \mathbb{1}_{1-4\lambda_n \epsilon^2 < 0} \\ &\leq Ct^{-2\delta} \epsilon^{2\alpha} (\lambda_n \epsilon^2)^{2\delta-\alpha} \mathbb{1}_{1-4\lambda_n \epsilon^2 < 0} \\ &\leq Ct^{-2\delta} \epsilon^{2\alpha}, \end{aligned}$$

using the conditions $2\delta - \alpha \leq 0$ and $4\lambda_n \epsilon^2 > 1$ in the last step.

On the other hand, if $1 - 4\lambda_n \epsilon^2 \geq 0$, recalling that $\tilde{w}_{n, \epsilon}^{1, 0}(t) = e^{\theta_{n, \epsilon} t} f_n^{\epsilon, 1, 0}(t)$ with $\theta_{n, \epsilon} = \lambda_n$, one has for all $t \geq 0$

$$f_n^{\epsilon, 1, 0}(t) - e^{-t\lambda_n} = e^{-t\lambda_n} (\tilde{w}_{n, \epsilon}^{1, 0}(t) - 1) = e^{-t\lambda_n} (\tilde{w}_{n, \epsilon}^{1, 0}(t) - \tilde{w}_{n, \epsilon}^{1, 0}(0)).$$

Using the inequality (49) and the Cauchy–Schwarz inequality, one has

$$\begin{aligned} |\tilde{w}_{n, \epsilon}^{1, 0}(t) - \tilde{w}_{n, \epsilon}^{1, 0}(0)|^2 &= \left| \int_0^t (\tilde{w}_{n, \epsilon}^{1, 0})'(s) ds \right|^2 \\ &\leq t \int_0^t |(\tilde{w}_{n, \epsilon}^{1, 0})'(s)|^2 ds \\ &\leq t \frac{\lambda_n^2 \epsilon^2}{1 - 2\lambda_n \epsilon^2} \\ &\leq 2t\lambda_n^2 \epsilon^2, \end{aligned}$$

using the lower bound $1 - 2\lambda_n \epsilon^2 \geq \frac{1}{2}$ in the last step.

As a result, using the inequality $\sup_{x \in \mathbb{R}} x^{1+2\delta} e^{-x} < \infty$, one obtains for all $t > 0$

$$\lambda_n^{2\delta-\alpha} |f_n^{\epsilon, 1, 0}(t) - e^{-t\lambda_n}|^2 \mathbb{1}_{1-4\lambda_n \epsilon^2 \geq 0} \leq Ct \lambda_n^{2+2\delta-\alpha} \epsilon^2 e^{-2t\lambda_n} \mathbb{1}_{1-4\lambda_n \epsilon^2 \geq 0}$$

$$\begin{aligned}
&\leq Ct^{-2\delta}\lambda_n^{1-\alpha}\epsilon^2\mathbb{1}_{1-4\lambda_n\epsilon^2\geq 0} \\
&\leq Ct^{-2\delta}\epsilon^{2\alpha}(\lambda_n\epsilon^2)^{1-\alpha}\mathbb{1}_{1-4\lambda_n\epsilon^2\geq 0} \\
&\leq Ct^{-2\delta}\epsilon^{2\alpha},
\end{aligned}$$

using the condition $4\lambda_n\epsilon^2 \leq 1$ in the last step.

Gathering the estimates, one thus obtains for all $t > 0$ and all $\epsilon \in (0, 1)$

$$\sup_{n \in \mathbb{N}} \left(\lambda_n^{2\delta-\alpha} |f_n^{\epsilon,1,0}(t) - e^{-t\lambda_n}|^2 \right) \leq Ct^{-2\delta}\epsilon^{2\alpha}.$$

This concludes the proof of the inequality (53).

It remains to prove the inequality (54). Assume that $\delta \in [\frac{\alpha}{2}, \frac{1}{2}]$ and let $n \in \mathbb{N}$ and $\epsilon \in (0, 1)$.

On the one hand, if $1 - 4\lambda_n\epsilon^2 < 0$, owing to the inequality (40) from Lemma 4.5, one has for all $t \geq 0$

$$\left| \frac{f_n^{\epsilon,0,1}(t)}{\epsilon} - e^{-t\lambda_n} \right|^2 \leq \frac{2}{\epsilon^2} |f_n^{\epsilon,0,1}(t)|^2 + 2e^{-2t\lambda_n} \leq C \frac{1}{\lambda_n\epsilon^2} e^{-\frac{t}{2\epsilon^2}} + 2e^{-2t\lambda_n}.$$

As a result, using the inequality $\sup_{x \in \mathbb{R}} x^{2\delta} e^{-x} < \infty$, one obtains for all $t > 0$

$$\begin{aligned}
\lambda_n^{2\delta-\alpha} |f_n^{\epsilon,0,1}(t) - e^{-t\lambda_n}|^2 \mathbb{1}_{1-4\lambda_n\epsilon^2 < 0} &\leq C \left(\lambda_n^{2\delta-\alpha-1} t^{-2\delta} \epsilon^{4\delta-2} + \lambda_n^{-\alpha} t^{-2\delta} \right) \mathbb{1}_{1-4\lambda_n\epsilon^2 < 0} \\
&\leq Ct^{-2\delta}\epsilon^{2\alpha} \left((\lambda_n\epsilon^2)^{2\delta-\alpha-1} + (\lambda_n\epsilon^2)^{-\alpha} \right) \mathbb{1}_{1-4\lambda_n\epsilon^2 < 0} \\
&\leq Ct^{-2\delta}\epsilon^{2\alpha},
\end{aligned}$$

using the conditions $1 + \alpha - 2\delta \geq 0$, $\alpha \geq 0$ and $4\lambda_n\epsilon^2 > 1$ in the last step.

On the other hand, if $1 - 4\lambda_n\epsilon^2 \geq 0$, set

$$F_n^{\epsilon,0,1}(t) = f_n^{\epsilon,0,1}(t) + \epsilon g_n^{\epsilon,0,1}(t).$$

Note that owing to (11) for all $t \geq 0$ one has

$$\begin{aligned}
(F_n^{\epsilon,0,1})'(t) &= (f_n^{\epsilon,0,1})'(t) + \epsilon (g_n^{\epsilon,0,1})'(t) \\
&= -\lambda_n f_n^{\epsilon,0,1}(t) \\
&= -\lambda_n F_n^{\epsilon,0,1}(t) + \epsilon \lambda_n g_n^{\epsilon,0,1}(t).
\end{aligned}$$

Since $F_n^{\epsilon,0,1}(0) = f_n^{\epsilon,0,1}(0) + \epsilon g_n^{\epsilon,0,1}(0) = \epsilon$, one has for all $t \geq 0$

$$F_n^{\epsilon,0,1}(t) = \epsilon e^{-\lambda_n t} + \epsilon \lambda_n \int_0^t e^{-\lambda_n(t-s)} g_n^{\epsilon,0,1}(s) ds.$$

Moreover, one has $g_n^{\epsilon,0,1}(s) = \epsilon (f_n^{\epsilon,0,1})'(s)$ for all $s \geq 0$, therefore integrating by parts one obtains

$$\begin{aligned}
F_n^{\epsilon,0,1}(t) - \epsilon e^{-\lambda_n t} &= \epsilon \lambda_n \int_0^t e^{-\lambda_n(t-s)} g_n^{\epsilon,0,1}(s) ds \\
&= \epsilon^2 \lambda_n \int_0^t e^{-\lambda_n(t-s)} (f_n^{\epsilon,0,1})'(s) ds \\
&= \epsilon^2 \lambda_n \left(f_n^{\epsilon,0,1}(t) - \lambda_n \int_0^t e^{-\lambda_n(t-s)} f_n^{\epsilon,0,1}(s) ds \right).
\end{aligned}$$

As a result, using the inequality (42) from Lemma 4.5, for all $t \geq 0$ one has

$$\begin{aligned}
\left| \frac{1}{\epsilon} F_n^{\epsilon,0,1}(t) - e^{-\lambda_n t} \right| &= \frac{1}{\epsilon} \left| F_n^{\epsilon,0,1}(t) - \epsilon e^{-\lambda_n t} \right| \leq \epsilon \lambda_n \left(|f_n^{\epsilon,0,1}(t)| + \lambda_n \int_0^t e^{-\lambda_n(t-s)} |f_n^{\epsilon,0,1}(s)| ds \right) \\
&\leq C \epsilon^2 \lambda_n e^{-\lambda_n t} + C \epsilon^2 \lambda_n^2 \int_0^t e^{-\lambda_n(t-s)} e^{-\lambda_n s} ds \\
&\leq C \epsilon^2 \lambda_n e^{-\lambda_n t} + C \epsilon^2 \lambda_n^2 t e^{-\lambda_n t}.
\end{aligned}$$

Finally, using the identity $f_n^{\epsilon,0,1}(t) = F_n^{\epsilon,0,1}(t) - \epsilon g_n^{\epsilon,0,1}(t)$, the inequality above and the inequality (42) from Lemma 4.5, one obtains for all $t \geq 0$

$$\begin{aligned} \left| \frac{1}{\epsilon} f_n^{\epsilon,0,1}(t) - e^{-t\lambda_n} \right|^2 &\leq 2 \left| \frac{1}{\epsilon} F_n^{\epsilon,0,1}(t) - e^{-t\lambda_n} \right|^2 + 2 |g_n^{\epsilon,0,1}(t)|^2 \\ &\leq C \left(\epsilon^4 \lambda_n^2 e^{-2\lambda_n t} + \epsilon^4 \lambda_n^4 t^2 e^{-2\lambda_n t} + e^{-\frac{2t}{\epsilon^2}} \right). \end{aligned}$$

As a result, using the inequalities $\sup_{x \in \mathbb{R}} x^{2\delta} e^{-x} < \infty$ and $\sup_{x \in \mathbb{R}} x^{2+2\delta} e^{-x} < \infty$, one obtains for all $t > 0$

$$\begin{aligned} \lambda_n^{2\delta-\alpha} \left| \frac{f_n^{\epsilon,0,1}(t)}{\epsilon} - e^{-t\lambda_n} \right|^2 \mathbb{1}_{1-4\lambda_n \epsilon^2 \geq 0} &\leq C \lambda_n^{2\delta-\alpha} \left(\epsilon^4 \lambda_n^2 e^{-2\lambda_n t} + \epsilon^4 \lambda_n^4 t^2 e^{-2\lambda_n t} + e^{-\frac{2t}{\epsilon^2}} \right) \mathbb{1}_{1-4\lambda_n \epsilon^2 \geq 0} \\ &\leq C t^{-2\delta} \left(\epsilon^4 \lambda_n^{2-\alpha} + \lambda_n^{2\delta-\alpha} \epsilon^{4\delta} \right) \mathbb{1}_{1-4\lambda_n \epsilon^2 \geq 0} \\ &\leq C t^{-2\delta} \epsilon^{2\alpha} \left((\lambda_n \epsilon^2)^{2-\alpha} + (\lambda_n \epsilon^2)^{2\delta-\alpha} \right) \mathbb{1}_{1-4\lambda_n \epsilon^2 \geq 0} \\ &\leq C t^{-2\delta} \epsilon^{2\alpha}, \end{aligned}$$

using the conditions $2 - \alpha \geq 0$, $2\delta - \alpha \geq 0$ and $4\lambda_n \epsilon^2 \leq 1$ in the last step.

Gathering the estimates, one thus obtains for all $t > 0$ and all $\epsilon \in (0, 1)$

$$\sup_{n \in \mathbb{N}} \left(\lambda_n^{2\delta-\alpha} \left| \frac{f_n^{\epsilon,0,1}(t)}{\epsilon} - e^{-t\lambda_n} \right|^2 \right) \leq C t^{-2\delta} \epsilon^{2\alpha}.$$

This concludes the proof of the inequality (54).

The proof of Lemma 4.7 is thus completed. \square

Proof of Lemma 4.6. Let us first prove the inequality (51). Assume that $\delta \in [0, \frac{\alpha}{2}]$. Let $u \in H^{\alpha-2\delta}$, then for all $t \geq 0$ one has

$$\Pi_u \left(e^{tA_\epsilon}(u, 0) \right) = \sum_{n \in \mathbb{N}} \langle u, e_n \rangle f_n^{\epsilon,1,0}(t) e_n$$

where $t \geq 0 \mapsto (f_n^{\epsilon,1,0}(t), g_n^{\epsilon,1,0}(t))$ is the solution of (11) with initial values $f_n^{\epsilon,1,0}(0) = 1$ and $g_n^{\epsilon,1,0}(0) = 0$. As a result, one has

$$\begin{aligned} \|\Pi_u \left(e^{tA_\epsilon}(u, 0) \right) - e^{-t\Lambda} u\|_H^2 &= \sum_{n \in \mathbb{N}} |f_n^{\epsilon,1,0}(t) - e^{-t\lambda_n}|^2 \langle u, e_n \rangle^2 \\ &= \sum_{n \in \mathbb{N}} \lambda_n^{2\delta-\alpha} |f_n^{\epsilon,1,0}(t) - e^{-t\lambda_n}|^2 \lambda_n^{\alpha-2\delta} \langle u, e_n \rangle^2 \\ &\leq \sup_{n \in \mathbb{N}} \left(\lambda_n^{2\delta-\alpha} |f_n^{\epsilon,1,0}(t) - e^{-t\lambda_n}|^2 \right) \|u\|_{H^{\alpha-2\delta}}^2. \end{aligned}$$

Employing the inequality (53) from Lemma 4.7 then yields the inequality (51).

It remains to prove the inequality (52). Assume that $\delta \in [\frac{\alpha}{2}, \frac{1}{2}]$. Let $v \in H^{\alpha-2\delta}$, then for all $t \geq 0$ one has

$$\Pi_u \left(e^{tA_\epsilon} \left(0, \frac{v}{\epsilon} \right) \right) = \sum_{n \in \mathbb{N}} \langle v, e_n \rangle \frac{f_n^{\epsilon,0,1}(t)}{\epsilon} e_n$$

where $t \geq 0 \mapsto (f_n^{\epsilon,0,1}(t), g_n^{\epsilon,0,1}(t))$ is the solution of (11) with initial values $f_n^{\epsilon,0,1}(0) = 0$ and $g_n^{\epsilon,0,1}(0) = 1$. As a result, one has

$$\begin{aligned} \|\Pi_u \left(e^{tA_\epsilon} \left(0, \frac{v}{\epsilon} \right) \right) - e^{-t\Lambda} v\|_H^2 &= \sum_{n \in \mathbb{N}} \left| \frac{f_n^{\epsilon,0,1}(t)}{\epsilon} - e^{-t\lambda_n} \right|^2 \langle v, e_n \rangle^2 \\ &= \sum_{n \in \mathbb{N}} \lambda_n^{2\delta-\alpha} \left| \frac{f_n^{\epsilon,0,1}(t)}{\epsilon} - e^{-t\lambda_n} \right|^2 \lambda_n^{\alpha-2\delta} \langle v, e_n \rangle^2 \\ &\leq \sup_{n \in \mathbb{N}} \left(\lambda_n^{2\delta-\alpha} \left| \frac{f_n^{\epsilon,0,1}(t)}{\epsilon} - e^{-t\lambda_n} \right|^2 \right) \|v\|_{H^{\alpha-2\delta}}^2. \end{aligned}$$

Employing the inequality (54) from Lemma 4.7 then yields the inequality (52).

The proof of Lemma 4.6 is thus completed. \square

5. PROPERTIES OF THE SOLUTIONS OF THE STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

In Section 2.2, we have introduced stochastic evolution equations and stated well-posedness results for mild solutions. Using the properties on the semigroups presented in Sections 4.1 and 4.2, it is possible to prove rigorously these results by standard fixed point approaches, the details are omitted. In this section, the aim is to provide some moment bounds and regularity properties, for which the properties given by Lemma 4.1 for $(e^{-t\Lambda})_{t \geq 0}$ and Lemmas 4.2 and 4.3 for $(e^{tA_\varepsilon})_{t \geq 0}$ play an important role.

5.1. Properties of the solutions of the stochastic heat equation. Let u_0^0 be \mathcal{H} -valued \mathcal{F}_0 -measurable random variable. Recall that $(u^0(t))_{t \geq 0}$ denotes the unique mild solution of (25) – the mild formulation is given by (26) – with initial value $u^0(0) = u_0^0$.

Proposition 5.1. *Let $\beta \in (0, 1]$ such that the condition (18) from Assumption 2.3 holds. For all $T \in (0, \infty)$ and all $\alpha \in [0, \beta)$, there exists $C_\alpha(T) \in (0, \infty)$ such that, if the initial value u_0^0 is a \mathcal{F}_0 -measurable random variable, which satisfies $\mathbb{E}[\|u_0^0\|_{H^\alpha}^2] < \infty$, then one has*

$$(55) \quad \sup_{0 \leq t \leq T} \mathbb{E}[\|u^0(t)\|_{H^\alpha}^2] \leq C_\alpha(T)(1 + \mathbb{E}[\|u_0^0\|_{H^\alpha}^2]).$$

The result stated in Proposition 5.1 is standard and may be found in many articles and books. For completeness and for pedagogical reasons, a short proof is given below.

Proof. We recall that the stochastic convolution $Z(t)$ is given by (27).

Using the Itô isometry property, one has, for all $t \geq 0$ and all $\alpha \in [0, \beta)$,

$$\begin{aligned} \mathbb{E}[\|Z(t)\|_{H^\alpha}^2] &= \sum_{n=1}^{\infty} q_n \int_0^t \|e^{-(t-s)\Lambda} e_n\|_{H^\alpha}^2 ds \\ &= \sum_{n=1}^{\infty} q_n \lambda_n^\alpha \int_0^t e^{-2(t-s)\lambda_n} ds \\ &= \frac{1}{2} \sum_{n \in \mathbb{N}} q_n \lambda_n^{\alpha-1} (1 - e^{-2t\lambda_n}). \end{aligned}$$

Therefore one has for all $\alpha \in [0, \beta)$

$$(56) \quad \sup_{t \geq 0} \mathbb{E}[\|Z(t)\|_{H^\alpha}^2] < \infty.$$

Let us first prove the inequality (55) for $u^0(t)$, when $\alpha = 0$.

Using the mild formulation (26), the inequality (32) from Lemma 4.1 and the Lipschitz continuity of f (Assumption 2.1, one obtains for all $t \geq 0$

$$\begin{aligned} \|u^0(t)\|_H &\leq \|e^{-t\Lambda} u_0^0\|_H + \int_0^t \|e^{-(t-s)\Lambda} f(u^0(s))\|_H ds + \|Z(t)\|_H \\ &\leq \|u_0^0\|_H + \int_0^t \|f(u^0(s))\|_H ds + \|Z(t)\|_H \\ &\leq \|u_0^0\|_H + C \int_0^t (1 + \|u^0(s)\|_H) ds + \|Z(t)\|_H. \end{aligned}$$

Applying the Minkowski inequality and using the moment bounds (56) obtained above for the stochastic convolution $Z(t)$, one obtains for all $t \geq 0$

$$\begin{aligned} (\mathbb{E}[\|u^0(t)\|_H^2])^{\frac{1}{2}} &\leq (\mathbb{E}[\|u_0^0\|_H^2])^{\frac{1}{2}} + C \int_0^t (1 + (\mathbb{E}[\|u^0(s)\|_H^2])^{\frac{1}{2}}) ds + (\mathbb{E}[\|Z(t)\|_H^2])^{\frac{1}{2}} \\ &\leq C(1 + (\mathbb{E}[\|u_0^0\|_H^2])^{\frac{1}{2}}) + C \int_0^t (\mathbb{E}[\|u^0(s)\|_H^2])^{\frac{1}{2}} ds. \end{aligned}$$

Finally, applying the Grönwall inequality, there exists $C_0(T) \in (0, \infty)$ such that for all $t \in [0, T]$ one has

$$(\mathbb{E}[\|u^0(t)\|_H^2])^{\frac{1}{2}} \leq C_0(T)(1 + (\mathbb{E}[\|u_0^0\|_H^2])^{\frac{1}{2}}).$$

This concludes the proof of the inequality (55) when $\alpha = 0$. Let us then deal with the case $\alpha \in (0, \beta)$. Using the mild formulation (26), the inequality (32) and the smoothing inequality (33) from Lemma 4.1 and the Lipschitz continuity property of f (Assumption 2.1), one has, for all $t \geq 0$

$$\begin{aligned} \|u^0(t)\|_{H^\alpha} &\leq \|e^{-t\Lambda}u_0^0\|_{H^\alpha} + \int_0^t \|e^{-(t-s)\Lambda}f(u^0(s))\|_{H^\alpha} ds + \|Z(t)\|_{H^\alpha} \\ &\leq \|u_0\|_{H^\alpha} + C \int_0^t (t-s)^{-\frac{\alpha}{2}} \|f(u^0(s))\|_H ds + \|Z(t)\|_{H^\alpha} \\ &\leq \|u_0\|_{H^\alpha} + C \int_0^t (t-s)^{-\frac{\alpha}{2}} (1 + \|u^0(s)\|_H) ds + \|Z(t)\|_{H^\alpha}. \end{aligned}$$

Applying the Minkowski inequality and using the moment bounds (56) obtained above for the stochastic convolution $Z(t)$ and the moment bounds for $\|u^0(t)\|_H$ proved above in the case $\alpha = 0$, one obtains for all $t \in [0, T]$

$$\begin{aligned} (\mathbb{E}[\|u^0(t)\|_{H^\alpha}^2])^{\frac{1}{2}} &\leq (\mathbb{E}[\|u_0\|_{H^\alpha}^2])^{\frac{1}{2}} + C \int_0^t (t-s)^{-\frac{\alpha}{2}} (1 + (\mathbb{E}[\|u^0(s)\|_H^2])^{\frac{1}{2}}) ds + C_\alpha \\ &\leq (\mathbb{E}[\|u_0\|_{H^\alpha}^2])^{\frac{1}{2}} + C_0(T)(1 + (\mathbb{E}[\|u_0\|_H^2])^{\frac{1}{2}}) \int_0^t (t-s)^{-\frac{\alpha}{2}} ds + C_\alpha \\ &\leq C_\alpha(T)(1 + (\mathbb{E}[\|u_0\|_{H^\alpha}^2])^{\frac{1}{2}}), \end{aligned}$$

for some $C_\alpha(T) \in (0, \infty)$. This concludes the proof of the inequality (55) when $\alpha \in (0, \beta)$.

The proof of Proposition 5.1 is thus completed. \square

5.2. Properties of the solutions of the stochastic damped wave equation. For all $\epsilon \in (0, 1)$ let $x_0^\epsilon = (u_0^\epsilon, v_0^\epsilon)$ be an \mathcal{H} -valued \mathcal{F}_0 -measurable random variable. Recall that $(X^\epsilon(t))_{t \geq 0}$ denotes the unique mild solution of (22), with the mild formulation given by (23), with initial value $X^\epsilon(0) = x_0^\epsilon$.

Proposition 5.2. *Let $\beta \in (0, 1]$ such that the condition (18) from Assumption 2.3 holds. For all $T \in (0, \infty)$ and all $\alpha \in [0, \beta)$, there exists $C_\alpha(T) \in (0, \infty)$ such that, if for all $\epsilon \in (0, 1)$ the initial value x_0^ϵ is a \mathcal{F}_0 -measurable random variable, which satisfies $\sup_{\epsilon \in (0, 1)} \mathbb{E}[\|x_0^\epsilon\|_{\mathcal{H}^\alpha}^2] < \infty$, then one has*

$$(57) \quad \sup_{\epsilon \in (0, 1)} \sup_{0 \leq t \leq T} \mathbb{E}[\|X^\epsilon(t)\|_{\mathcal{H}^\alpha}^2] \leq C_\alpha(T) \left(1 + \sup_{\epsilon \in (0, 1)} \mathbb{E}[\|x_0^\epsilon\|_{\mathcal{H}^\alpha}^2]\right).$$

Proof. We recall that the stochastic convolution $\mathcal{Z}^\epsilon(t)$ is given by (24).

Using the Itô isometry property, one has

$$\mathbb{E}[\|\mathcal{Z}^\epsilon(t)\|_{\mathcal{H}^\alpha}^2] = \frac{1}{\epsilon^2} \sum_{n=1}^{\infty} q_n \int_0^t \|e^{(t-s)A_\epsilon}(0, e_n)\|_{\mathcal{H}^\alpha}^2 ds.$$

Let $\alpha \in [0, \beta)$, and let $\delta_0 \in (0, \beta - \alpha)$. Applying the inequality (37) from Lemma 4.3 with $\delta = \frac{1-\delta_0}{2}$ and $\rho = 1$, one obtains

$$\begin{aligned} \|e^{(t-s)A_\epsilon}(0, e_n)\|_{\mathcal{H}^\alpha}^2 &\leq C_{\delta_0} (t-s)^{\delta_0-1} \epsilon^2 (\epsilon^{-2\delta_0} e^{-\frac{t-s}{\epsilon^2}} + 1) \|e_n\|_{H^{\alpha+\delta_0-1}}^2 \\ &\leq C_{\delta_0} (t-s)^{\delta_0-1} \epsilon^2 (\epsilon^{-2\delta_0} e^{-\frac{t-s}{\epsilon^2}} + 1) \lambda_n^{\alpha+\delta_0-1}. \end{aligned}$$

Owing to the condition $\delta_0 \in (0, \beta - \alpha)$ and to Assumption 2.3, one has

$$\sum_{n \in \mathbb{N}} q_n \lambda_n^{\alpha+\delta_0-1} < \infty.$$

As a result, for all $t \geq 0$ one has

$$\begin{aligned} \mathbb{E}[\|\mathcal{Z}^\epsilon(t)\|_{\mathcal{H}^\alpha}^2] &\leq C_{\alpha, \delta_0} \left(\int_0^t (t-s)^{\delta_0-1} \epsilon^{-2\delta_0} e^{-\frac{t-s}{\epsilon^2}} ds + \int_0^t (t-s)^{\delta_0-1} ds \right) \\ &\leq C_{\alpha, \delta_0} \left(\int_0^\infty r^{\delta_0-1} e^{-r} dr + \int_0^t r^{\delta_0-1} dr \right). \end{aligned}$$

Therefore one has for all $T \in (0, \infty)$, $\alpha \in [0, \beta)$

$$(58) \quad \sup_{\epsilon \in (0,1)} \sup_{t \in [0,T]} \mathbb{E}[\|\mathcal{Z}^\epsilon(t)\|_{H^\alpha}^2] < \infty.$$

Let us first prove the inequality (57) for $X^{\epsilon,N}(t)$, when $\alpha = 0$.

Using the mild formulation (23), the inequality (34) from Lemma 4.2 and the inequality (37) from Lemma 4.3 with $\delta = 1/2$ and $\rho = 1$, one obtains for all $t \geq 0$

$$\begin{aligned} \|X^\epsilon(t)\|_{\mathcal{H}} &\leq \|e^{tA_\epsilon} P_N x_0^\epsilon\|_{\mathcal{H}} + \frac{1}{\epsilon} \int_0^t \|e^{(t-s)A_\epsilon} (0, f(u^\epsilon(s)))\|_{\mathcal{H}} ds + \|\mathcal{Z}^\epsilon(t)\|_{\mathcal{H}} \\ &\leq \|x_0^\epsilon\|_{\mathcal{H}} + C \int_0^t (t-s)^{-\frac{1}{2}} \|f(u^\epsilon(s))\|_{H^{-1}} ds + \|\mathcal{Z}^\epsilon(t)\|_{\mathcal{H}} \\ &\leq \|x_0^\epsilon\|_{\mathcal{H}} + C \int_0^t (t-s)^{-\frac{1}{2}} (1 + \|X^\epsilon(s)\|_{\mathcal{H}}) ds + \|\mathcal{Z}^\epsilon(t)\|_{\mathcal{H}}, \end{aligned}$$

using the Lipschitz continuity of f in the last step, more precisely using the inequalities

$$\|f(u)\|_{H^{-1}} \leq C\|f(u)\|_H \leq C(1 + \|u\|_H) \leq C(1 + \|x\|_{\mathcal{H}}), \quad \forall x = (u, v) \in \mathcal{H}.$$

Applying the Minkowski inequality and using the moment bounds (58) obtained above for the stochastic convolution $\mathcal{Z}^{\epsilon,N}(t)$, one obtains for all $t \in [0, T]$

$$\begin{aligned} (\mathbb{E}[\|X^\epsilon(t)\|_{\mathcal{H}}^2])^{\frac{1}{2}} &\leq (\mathbb{E}[\|x_0^\epsilon\|_{\mathcal{H}}^2])^{\frac{1}{2}} + C \int_0^t (t-s)^{-\frac{1}{2}} (1 + (\mathbb{E}[\|X^\epsilon(s)\|_{\mathcal{H}}^2])^{\frac{1}{2}}) ds + (\mathbb{E}[\|\mathcal{Z}^\epsilon(t)\|_{\mathcal{H}}^2])^{\frac{1}{2}} \\ &\leq C(T) (1 + (\mathbb{E}[\|x_0^\epsilon\|_{\mathcal{H}}^2])^{\frac{1}{2}}) + C \int_0^t (t-s)^{-\frac{1}{2}} (\mathbb{E}[\|X^\epsilon(s)\|_{\mathcal{H}}^2])^{\frac{1}{2}} ds. \end{aligned}$$

Using that inequality, the Fubini theorem, and a scaling property of the beta function, one has for all $t \geq 0$

$$\begin{aligned} (\mathbb{E}[\|X^\epsilon(t)\|_{\mathcal{H}}^2])^{\frac{1}{2}} &\leq C(T) (1 + (\mathbb{E}[\|x_0^\epsilon\|_{\mathcal{H}}^2])^{\frac{1}{2}}) \left(1 + \int_0^t (t-s)^{-\frac{1}{2}} ds\right) \\ &\quad + C \int_0^t \int_0^s (t-s)^{-\frac{1}{2}} (s-r)^{-\frac{1}{2}} (\mathbb{E}[\|X^\epsilon(r)\|_{\mathcal{H}}^2])^{\frac{1}{2}} dr ds \\ &\leq C(T) (1 + (\mathbb{E}[\|x_0^\epsilon\|_{\mathcal{H}}^2])^{\frac{1}{2}}) + C \int_0^t \int_r^t (t-s)^{-\frac{1}{2}} (s-r)^{-\frac{1}{2}} ds (\mathbb{E}[\|X^\epsilon(r)\|_{\mathcal{H}}^2])^{\frac{1}{2}} dr \\ &\leq C(T) (1 + (\mathbb{E}[\|x_0^\epsilon\|_{\mathcal{H}}^2])^{\frac{1}{2}}) + C \int_0^t (\mathbb{E}[\|X^\epsilon(r)\|_{\mathcal{H}}^2])^{\frac{1}{2}} dr. \end{aligned}$$

Finally, applying the Grönwall inequality, there exists $C_0(T) \in (0, \infty)$ such that for all $\epsilon \in (0, 1)$ and for all $t \in [0, T]$ one has

$$(\mathbb{E}[\|X^\epsilon(t)\|_{\mathcal{H}}^2])^{\frac{1}{2}} \leq C_0(T) (1 + (\mathbb{E}[\|X_0^\epsilon\|_H^2])^{\frac{1}{2}}).$$

This concludes the proof of the inequality (57) when $\alpha = 0$, and let us then deal with the case $\alpha \in (0, \beta)$. Using the mild formulation (23), the inequality (34) from Lemma 4.2 and the inequality (37) from Lemma 4.3 with $\delta = 1/2$ and $\rho = 1$, one obtains for all $t \geq 0$

$$\begin{aligned} \|X^\epsilon(t)\|_{\mathcal{H}^\alpha} &\leq \|e^{tA_\epsilon} x_0^\epsilon\|_{\mathcal{H}^\alpha} + \frac{1}{\epsilon} \int_0^t \|e^{(t-s)A_\epsilon} (0, f(u^\epsilon(s)))\|_{\mathcal{H}^\alpha} ds + \|\mathcal{Z}^\epsilon(t)\|_{\mathcal{H}^\alpha} \\ &\leq \|x_0^\epsilon\|_{\mathcal{H}^\alpha} + C \int_0^t (t-s)^{-\frac{1}{2}} \|f(u^\epsilon(s))\|_{H^{\alpha-1}} ds + \|\mathcal{Z}^\epsilon(t)\|_{\mathcal{H}^\alpha} \\ &\leq \|x_0^\epsilon\|_{\mathcal{H}^\alpha} + C \int_0^t (t-s)^{-\frac{1}{2}} (1 + \|X^\epsilon(s)\|_{\mathcal{H}}) ds + \|\mathcal{Z}^{\epsilon,N}(t)\|_{\mathcal{H}^\alpha}, \end{aligned}$$

using the condition $\alpha \leq 1$ to have the inequality

$$\|f(u)\|_{H^{\alpha-1}} \leq C\|f(u)\|_H \leq C(1 + \|u\|_H) \leq C(1 + \|x\|_{\mathcal{H}}), \quad \forall x = (u, v) \in \mathcal{H}$$

in the last step. Applying the Minkowski inequality and using the moment bounds (58) obtained above for the stochastic convolution $\mathcal{Z}^\epsilon(t)$ and the moment bounds for $\|X^\epsilon(t)\|_{\mathcal{H}}$ proved above in the case $\alpha = 0$, one obtains for all $t \in [0, T]$

$$\begin{aligned} (\mathbb{E}[\|X^\epsilon(t)\|_{\mathcal{H}^\alpha}^2])^{\frac{1}{2}} &\leq C_\alpha(T)(1 + (\mathbb{E}[\|x_0^\epsilon\|_{\mathcal{H}^\alpha}^2])^{\frac{1}{2}}) + C \int_0^t (t-s)^{-\frac{1}{2}} (\mathbb{E}[\|X^\epsilon(s)\|_{\mathcal{H}}^2])^{\frac{1}{2}} ds \\ &\leq C_\alpha(T)(1 + (\mathbb{E}[\|x_0^\epsilon\|_{\mathcal{H}^\alpha}^2])^{\frac{1}{2}}), \end{aligned}$$

for some $C_\alpha(T) \in (0, \infty)$. This concludes the proof of the inequality (57) when $\alpha \in (0, \beta)$.

The proof of Proposition 5.2 is thus completed. \square

6. PROOF OF THEOREM 1

Proof of Theorem 1. Let $T \in (0, \infty)$. For all $\epsilon \in (0, 1)$, owing to the mild formulation (23) for the solution $(X^\epsilon(t))_{t \geq 0}$ of (22), one has for all $t \in [0, T]$

$$\begin{aligned} u^\epsilon(t) &= \Pi_u X^\epsilon(t) \\ &= \Pi_u e^{tA_\epsilon} x_0^\epsilon + \frac{1}{\epsilon} \int_0^t \Pi_u e^{(t-s)A_\epsilon} F(X^\epsilon(s)) ds + \frac{1}{\epsilon} \int_0^t \Pi_u e^{(t-s)A_\epsilon} d\mathcal{W}^Q(s). \end{aligned}$$

Recall that $F(X^\epsilon(s)) = (0, f(u^\epsilon(s)))$. As a result, owing to the mild formulation (26) for the solution $(u^0(t))_{t \geq 0}$ of (25), one obtains the following decomposition of the error $r^\epsilon(t) = u^\epsilon(t) - u^0(t)$: for all $t \geq 0$ one has

$$(59) \quad r^\epsilon(t) = u^\epsilon(t) - u^0(t) = r_1^\epsilon(t) + r_2^\epsilon(t) + r_3^\epsilon(t),$$

with the error terms defined by

$$(60) \quad r_1^\epsilon(t) = \Pi_u e^{tA_\epsilon} x_0^\epsilon - e^{-t\Lambda} u_0,$$

$$(61) \quad r_2^\epsilon(t) = \frac{1}{\epsilon} \int_0^t \Pi_u e^{(t-s)A_\epsilon} (0, f(u^\epsilon(s))) ds - \int_0^t e^{-(t-s)\Lambda} f(u^0(s)) ds,$$

$$(62) \quad r_3^\epsilon(t) = \frac{1}{\epsilon} \int_0^t \Pi_u e^{(t-s)A_\epsilon} d\mathcal{W}^Q(s) - \int_0^t e^{-(t-s)\Lambda} dW^Q(s).$$

We claim that the following error bounds hold: for all $\alpha \in [0, \beta)$, there exists $C, C_\alpha(T), C_{\alpha, \beta}(T) \in (0, \infty)$ such that for all $\epsilon \in (0, 1)$ one has

$$(63) \quad \sup_{0 \leq t \leq T} (\mathbb{E}[\|r_1^\epsilon(t)\|_H^2])^{\frac{1}{2}} \leq C_\alpha(T) \epsilon^\alpha,$$

$$(64) \quad \sup_{0 \leq t \leq T} (\mathbb{E}[\|r_2^\epsilon(t)\|_H^2])^{\frac{1}{2}} \leq C_\alpha(T) \epsilon^\alpha + C \int_0^t (\mathbb{E}[\|r^\epsilon(s)\|_H^2])^{\frac{1}{2}} ds,$$

$$(65) \quad \sup_{0 \leq t \leq T} (\mathbb{E}[\|r_3^\epsilon(t)\|_H^2])^{\frac{1}{2}} \leq C_{\alpha, \beta}(T) \epsilon^\alpha.$$

- Proof of the inequality (63).

The error term $r_1^\epsilon(t)$ defined by (60) is decomposed as

$$r_1^\epsilon(t) = r_{1,1}^\epsilon(t) + r_{1,2}^\epsilon(t) + r_{1,3}^\epsilon(t)$$

where for all $\epsilon \in (0, 1)$ and all $t \geq 0$ one has

$$\begin{aligned} r_{1,1}^\epsilon(t) &= \Pi_u e^{tA_\epsilon} (u_0^\epsilon, 0) - \Pi_u e^{tA_\epsilon} (u_0^0, 0), \\ r_{1,2}^\epsilon(t) &= \Pi_u e^{tA_\epsilon} (u_0^0, 0) - e^{-t\Lambda} u_0^0, \\ r_{1,3}^\epsilon(t) &= \Pi_u e^{tA_\epsilon} (0, v_0^\epsilon). \end{aligned}$$

First, owing to the inequality (34) from Lemma 4.2, one obtains for all $t \geq 0$

$$\|r_{1,1}^\epsilon(t)\|_H = \|\Pi_u e^{tA_\epsilon} (u_0^\epsilon - u_0^0, 0)\|_H \leq \|e^{tA_\epsilon} (u_0^\epsilon - u_0^0, 0)\|_{\mathcal{H}} \leq \|(u_0^\epsilon - u_0^0, 0)\|_{\mathcal{H}} = \|u_0^\epsilon - u_0\|_H.$$

Therefore using the condition (29) from Assumption 3.1, for all $\epsilon \in (0, 1)$ one obtains

$$\sup_{t \geq 0} (\mathbb{E}[\|r_{1,1}^\epsilon(t)\|_H^2])^{\frac{1}{2}} \leq C_\alpha \epsilon^\alpha.$$

Second, using the inequality (51) from Lemma 4.6 with $\delta = 0$, one obtains for all $t \geq 0$

$$\|r_{1,2}^\epsilon(t)\|_H = \|\Pi_u e^{tA_\epsilon}(u_0^0, 0) - e^{-t\Lambda}u_0^0\|_H \leq C_\alpha \epsilon^\alpha \|u_0^0\|_{H^\alpha}.$$

Therefore using the inequality (30) which follows from Assumption 3.1, for all $\epsilon \in (0, 1)$ one obtains

$$\sup_{t \geq 0} (\mathbb{E}[\|r_{1,2}^\epsilon(t)\|_H^2])^{\frac{1}{2}} \leq C_\alpha \epsilon^\alpha.$$

Finally, using the inequality (38) from Lemma 4.4 with $\delta = 0$, one obtains for all $t \geq 0$

$$\|r_{1,3}^\epsilon(t)\|_H = \|\Pi_u e^{tA_\epsilon}(0, v_0^\epsilon)\|_H \leq C_\alpha \epsilon^\alpha \|v_0^\epsilon\|_{H^{\alpha-1}}.$$

Therefore using the inequality (28) from Assumption 3.1, for all $\epsilon \in (0, 1)$ one obtains

$$\sup_{t \geq 0} (\mathbb{E}[\|r_{1,3}^\epsilon(t)\|_H^2])^{\frac{1}{2}} \leq C_\alpha \epsilon^\alpha.$$

Gathering the three estimates obtained above, for all $\epsilon \in (0, 1)$ one has

$$\sup_{t \geq 0} (\mathbb{E}[\|r_1^\epsilon(t)\|_H^2])^{\frac{1}{2}} \leq \sup_{t \geq 0} (\mathbb{E}[\|r_{1,1}^\epsilon(t)\|_H^2])^{\frac{1}{2}} + \sup_{t \geq 0} (\mathbb{E}[\|r_{1,2}^\epsilon(t)\|_H^2])^{\frac{1}{2}} + \sup_{t \geq 0} (\mathbb{E}[\|r_{1,3}^\epsilon(t)\|_H^2])^{\frac{1}{2}} \leq C_\alpha \epsilon^\alpha.$$

This concludes the proof of the inequality (63).

- Proof of the inequality (64).

The error term $r_2^\epsilon(t)$ defined by (61) is decomposed as

$$r_2^\epsilon(t) = r_{2,1}^\epsilon(t) + r_{2,2}^\epsilon(t),$$

where for all $\epsilon \in (0, 1)$ and all $t \geq 0$ one has

$$\begin{aligned} r_{2,1}^\epsilon(t) &= \int_0^t \left(\frac{1}{\epsilon} \Pi_u e^{(t-s)A_\epsilon}(0, f(u^\epsilon(s))) - e^{-(t-s)\Lambda} f(u^\epsilon(s)) \right) ds, \\ r_{2,2}^\epsilon(t) &= \int_0^t e^{-(t-s)\Lambda} (f(u^\epsilon(s)) - f(u^0(s))) ds. \end{aligned}$$

On the one hand, using the inequality (52) from Lemma 4.6 with $\delta = \frac{\alpha}{2}$, the Lipschitz continuity of f from Assumption 2.1 and the inequality (7), one obtains

$$\begin{aligned} \|r_{2,1}^\epsilon(t)\|_H &\leq \int_0^t \left\| \frac{1}{\epsilon} \Pi_u e^{(t-s)A_\epsilon}(0, f(u^\epsilon(s))) - e^{-(t-s)\Lambda} f(u^\epsilon(s)) \right\|_H ds \\ &\leq C_\alpha \epsilon^\alpha \int_0^t (t-s)^{-\frac{\alpha}{2}} \|f(u^\epsilon(s))\|_H ds \\ &\leq C_\alpha \epsilon^\alpha \int_0^t (t-s)^{-\frac{\alpha}{2}} (1 + \|u^\epsilon(s)\|_H) ds \\ &\leq C_\alpha \epsilon^\alpha \int_0^t (t-s)^{-\frac{\alpha}{2}} (1 + \|X^\epsilon(s)\|_{\mathcal{H}}) ds. \end{aligned}$$

Using the Minkowski inequality, the moment bounds (57) from Proposition 5.2 and the condition (28) from Assumption 3.1, there exists $C_\alpha(T) \in (0, \infty)$ such that for all $t \in [0, T]$ and all $\epsilon \in (0, 1)$ one has

$$\begin{aligned} (\mathbb{E}[\|r_{2,1}^\epsilon(t)\|_H^2])^{\frac{1}{2}} &\leq C_\alpha(T) \epsilon^\alpha \left(1 + (\mathbb{E}[\|x_0^\epsilon\|_{\mathcal{H}}^2])^{\frac{1}{2}} \right) \int_0^t (t-s)^{-\frac{\alpha}{2}} ds \\ &\leq C_\alpha(T) \epsilon^\alpha. \end{aligned}$$

On the other hand, using the inequality (32) from Lemma 4.1 and the Lipschitz continuity of f from Assumption 2.1, one obtains for all $t \geq 0$ and all $\epsilon \in (0, 1)$

$$\|r_{2,2}^\epsilon(t)\|_H \leq \int_0^t \|e^{-(t-s)\Lambda} (f(u^\epsilon(s)) - f(u^0(s)))\|_H ds$$

$$\begin{aligned}
&\leq C \int_0^t \|u^\epsilon(s) - u^0(s)\|_H ds \\
&\leq C \int_0^t \|r^\epsilon(s)\|_H ds.
\end{aligned}$$

Using the Minkowski inequality, one then has for all $t \geq 0$ and all $\epsilon \in (0, 1)$

$$(\mathbb{E}[\|r_{2,2}^\epsilon(t)\|_H^2])^{\frac{1}{2}} \leq C \int_0^t (\mathbb{E}[\|r^\epsilon(s)\|_H^2])^{\frac{1}{2}} ds.$$

Gathering the two estimates obtained above, for all $\epsilon \in (0, 1)$, one has

$$\sup_{t \in [0, T]} (\mathbb{E}[\|r_2^\epsilon(t)\|_H^2])^{\frac{1}{2}} \leq \sup_{t \in [0, T]} (\mathbb{E}[\|r_{2,1}^\epsilon(t)\|_H^2])^{\frac{1}{2}} + \sup_{t \in [0, T]} (\mathbb{E}[\|r_{2,2}^\epsilon(t)\|_H^2])^{\frac{1}{2}} \leq C_\alpha(T)\epsilon^\alpha + C \int_0^t (\mathbb{E}[\|r^\epsilon(s)\|_H^2])^{\frac{1}{2}} ds.$$

This concludes the proof of the inequality (64).

- Proof of the inequality (65).

Using the expressions (17) and (19) for $W^Q(s)$ and $\mathcal{W}^Q(s)$ respectively, the error term $r_3^\epsilon(t)$ defined by (62) is written as

$$\begin{aligned}
r_3^\epsilon(t) &= \frac{1}{\epsilon} \int_0^t \Pi_u e^{(t-s)A_\epsilon} d\mathcal{W}^Q(s) - \int_0^t e^{-(t-s)\Lambda} dW^Q(s) \\
&= \sum_{n \in \mathbb{N}} \sqrt{q_n} \int_0^t \left(\Pi_u e^{(t-s)A_\epsilon} \left(0, \frac{e_n}{\epsilon}\right) - e^{-(t-s)\Lambda} e_n \right) d\beta_n(s).
\end{aligned}$$

Therefore using the Itô isometry property, for all $t \geq 0$ and all $\epsilon \in (0, 1)$ one has

$$\mathbb{E}[\|r_3^\epsilon(t)\|_H^2] = \sum_{n \in \mathbb{N}} q_n \int_0^t \|\Pi_u e^{(t-s)A_\epsilon} \left(0, \frac{e_n}{\epsilon}\right) - e^{-(t-s)\Lambda} e_n\|_H^2 ds,$$

and using then the inequality (52) from Lemma 4.6 with $\delta = \frac{1}{2} - \frac{\beta - \alpha}{4}$, one obtains for all $t \in [0, T]$

$$\begin{aligned}
\mathbb{E}[\|r_3^\epsilon(t)\|_H^2] &\leq C_{\alpha, \delta} \epsilon^{2\alpha} \sum_{n \in \mathbb{N}} q_n \int_0^t (t-s)^{-2\delta} \|e_n\|_{H^{\alpha-2\delta}}^2 ds \\
&\leq C_{\alpha, \beta} \epsilon^{2\alpha} \sum_{n \in \mathbb{N}} q_n \lambda_n^{\frac{\alpha+\beta}{2}-1} \int_0^t (t-s)^{-1+\frac{\beta-\alpha}{2}} ds \\
&\leq C_{\alpha, \beta}(T) \epsilon^{2\alpha},
\end{aligned}$$

using the condition (18) from Assumption 2.3 in the last step.

Therefore for all $\epsilon \in (0, 1)$ one has

$$\sup_{0 \leq t \leq T} (\mathbb{E}[\|r_3^\epsilon(t)\|_H^2])^{\frac{1}{2}} \leq C_{\alpha, \beta}(T) \epsilon^\alpha.$$

This concludes the proof of the inequality (65).

- Conclusion.

Recalling the decomposition (59) of the error, applying the Minkowski inequality and the auxiliary error estimates (63), (64) and (65), one obtains for all $\epsilon \in (0, 1)$ and all $t \in [0, T]$

$$\begin{aligned}
(\mathbb{E}[\|u^\epsilon(t) - u^0(t)\|_H^2])^{\frac{1}{2}} &= (\mathbb{E}[\|r^\epsilon(t)\|_H^2])^{\frac{1}{2}} \leq (\mathbb{E}[\|r_1^\epsilon(t)\|_H^2])^{\frac{1}{2}} + (\mathbb{E}[\|r_2^\epsilon(t)\|_H^2])^{\frac{1}{2}} + (\mathbb{E}[\|r_3^\epsilon(t)\|_H^2])^{\frac{1}{2}} \\
&\leq C_{\alpha, \beta}(T) \epsilon^\alpha + C \int_0^t (\mathbb{E}[\|u^\epsilon(s) - u^0(s)\|_H^2])^{\frac{1}{2}} ds.
\end{aligned}$$

Finally, applying the Grönwall inequality, there exists $C_{\alpha, \beta}(T) \in (0, \infty)$ such that

$$\sup_{0 \leq t \leq T} (\mathbb{E}[\|u^\epsilon(t) - u^0(t)\|_H^2])^{\frac{1}{2}} \leq C_{\alpha, \beta}(T) \epsilon^\alpha.$$

The proof of Theorem 1 is thus completed. \square

REFERENCES

- [1] C.-E. Bréhier. Uniform strong and weak error estimates for numerical schemes applied to multiscale SDEs in a Smoluchowski-Kramers diffusion approximation regime. *J. Comput. Dyn.*, 10(3):387–424, 2023.
- [2] Z. a. Brzeźniak and S. Cerrai. Stochastic wave equations with constraints: well-posedness and Smoluchowski-Kramers diffusion approximation. *Comm. Math. Phys.*, 406(9):Paper No. 223, 59, 2025.
- [3] S. Cerrai and A. Debussche. Smoluchowski-Kramers diffusion approximation for systems of stochastic damped wave equations with nonconstant friction. *Ann. Appl. Probab.*, 35(6):4106–4171, 2025.
- [4] S. Cerrai and M. Freidlin. On the Smoluchowski-Kramers approximation for a system with an infinite number of degrees of freedom. *Probab. Theory Related Fields*, 135(3):363–394, 2006.
- [5] S. Cerrai and M. Freidlin. Smoluchowski-Kramers approximation for a general class of SPDEs. *J. Evol. Equ.*, 6(4):657–689, 2006.
- [6] S. Cerrai, M. Freidlin, and M. Salins. On the Smoluchowski-Kramers approximation for SPDEs and its interplay with large deviations and long time behavior. *Discrete Contin. Dyn. Syst.*, 37(1):33–76, 2017.
- [7] S. Cerrai and N. Glatt-Holtz. On the convergence of stationary solutions in the Smoluchowski-Kramers approximation of infinite dimensional systems. *J. Funct. Anal.*, 278(8):108421, 38, 2020.
- [8] S. Cerrai and M. Salins. Smoluchowski-Kramers approximation and large deviations for infinite dimensional gradient systems. *Asymptot. Anal.*, 88(4):201–215, 2014.
- [9] S. Cerrai and M. Salins. Smoluchowski-Kramers approximation and large deviations for infinite-dimensional nongradient systems with applications to the exit problem. *Ann. Probab.*, 44(4):2591–2642, 2016.
- [10] S. Cerrai and M. Salins. On the Smoluchowski-Kramers approximation for a system with infinite degrees of freedom exposed to a magnetic field. *Stochastic Process. Appl.*, 127(1):273–303, 2017.
- [11] S. Cerrai, J. Wehr, and Y. Zhu. An averaging approach to the Smoluchowski-Kramers approximation in the presence of a varying magnetic field. *J. Stat. Phys.*, 181(1):132–148, 2020.
- [12] S. Cerrai and G. Xi. A Smoluchowski-Kramers approximation for an infinite dimensional system with state-dependent damping. *Ann. Probab.*, 50(3):874–904, 2022.
- [13] S. Cerrai and M. Xie. On the small noise limit in the Smoluchowski-Kramers approximation of nonlinear wave equations with variable friction. *Trans. Amer. Math. Soc.*, 376(11):7651–7689, 2023.
- [14] S. Cerrai and M. Xie. On the small-mass limit for stationary solutions of stochastic wave equations with state dependent friction. *Appl. Math. Optim.*, 90(1):Paper No. 7, 48, 2024.
- [15] S. Cerrai and M. Xie. The small-mass limit for some constrained wave equations with nonlinear conservative noise. *Electron. J. Probab.*, 30:Paper No. 25, 27, 2025.
- [16] M. Freidlin and W. Hu. Smoluchowski-Kramers approximation in the case of variable friction. volume 179, pages 184–207. 2011. *Problems in mathematical analysis*. No. 61.
- [17] B. Guelmame and J. Vovelle. A smoluchowski-kramers approximation for the stochastic variational wave equation, 2025.
- [18] S. Hottovy, A. McDaniel, G. Volpe, and J. Wehr. The Smoluchowski-Kramers limit of stochastic differential equations with arbitrary state-dependent friction. *Comm. Math. Phys.*, 336(3):1259–1283, 2015.
- [19] S. Hottovy, G. Volpe, and J. Wehr. Noise-induced drift in stochastic differential equations with arbitrary friction and diffusion in the Smoluchowski-Kramers limit. *J. Stat. Phys.*, 146(4):762–773, 2012.
- [20] S. Liu, W. Liu, and L. Xu. Long term convergence rate of smoluchowski-kramers approximation by stein’s method, 2026.
- [21] X. Liu, Q. Jiang, and W. Wang. The smoluchowski-kramers approximation for a system with arbitrary friction depending on both state and distribution, 2024.
- [22] M. Rousset, Y. Xu, and P.-A. Zitt. A weak overdamped limit theorem for Langevin processes. *ALEA Lat. Am. J. Probab. Math. Stat.*, 17(1):1–21, 2020.
- [23] M. Salins. Smoluchowski-Kramers approximation for the damped stochastic wave equation with multiplicative noise in any spatial dimension. *Stoch. Partial Differ. Equ. Anal. Comput.*, 7(1):86–122, 2019.
- [24] C. Shi, Y. Lv, and W. Wang. The smoluchowski-kramers approximation for a mckean-vlasov equation subject to environmental noise with state-dependent friction, 2024.
- [25] C. Shi and W. Wang. Small mass limit and diffusion approximation for a generalized Langevin equation with infinite number degrees of freedom. *J. Differential Equations*, 286:645–675, 2021.
- [26] C. Shi and W. Wang. The smoluchowski-kramers approximation with distribution-dependent potential and highly oscillating force, 2024.
- [27] T. C. Son, D. Q. Le, and M. H. Duong. Rate of convergence in the Smoluchowski-Kramers approximation for mean-field stochastic differential equations. *Potential Anal.*, 60(3):1031–1065, 2024.
- [28] N. V. Tan and N. T. Dung. A Berry-Esseen bound in the Smoluchowski-Kramers approximation. *J. Stat. Phys.*, 179(4):871–884, 2020.
- [29] L. Xie and L. Yang. The Smoluchowski-Kramers limits of stochastic differential equations with irregular coefficients. *Stochastic Process. Appl.*, 150:91–115, 2022.
- [30] Q. Zhao, X. Liu, and W. Wang. Smoluchowski-kramers approximation with state-dependent friction in rough path topology, 2025.
- [31] Q. Zhao and W. Wang. Convergence rate of smoluchowski-kramers approximation with stable levy noise, 2024.
- [32] Y. Zine. Smoluchowski-Kramers approximation for singular stochastic wave equations in two dimensions. *Electron. J. Probab.*, 30:Paper No. 88, 49, 2025.

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