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Rigidity and Structural Asymmetry of Bounded Solutions

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Abstract

In this manuscript, we introduce a family of parametrized non-homogeneous linear complex differential equations on $[1, \infty)$, depending on a complex parameter. We identify a *Rotation number hypothesis* on the non-homogeneous term, which establishes a structural asymmetry: if two solutions with the same initial condition equal to 1, corresponding respectively to the parameters s and $1 - \bar{s}$ lying in the critical strip, are both bounded on $[1, +\infty)$, then $\Re(s) = \frac{1}{2}$.

Keywords: Non-homogeneous complex linear differential equation, bounded solutions, Euler differential equation, Volterra integral equation.

MSC subject classifications: 34A30, 34E05

1 Introduction

In this manuscript, we study the following non-homogeneous linear complex differential equations:

$$\dot{\theta} = w t^{-1} \theta + t^{-1} \eta(t), \quad \theta(1) = \frac{1}{1-w}, \quad \theta : [1, +\infty) \rightarrow \mathbb{C}, \quad (1)$$

where $w \in \mathbb{C}$ such that $\Im(w) \neq 0$ is the parameter and the non-homogeneous term $\eta \in \mathcal{B}_{\text{loc}}([1, \infty), \mathbb{C})$ where $\mathcal{B}_{\text{loc}}([1, \infty), \mathbb{C})$ is the set of bounded and

locally integrable functions from $[1, +\infty)$ to \mathbb{C} . We are interested in the initial conditions of the bounded solutions of the previous differential equation. To this end, we analyze the behavior of the transformed solution $t \mapsto (1-w)t^{-i\Im(w)}\theta(t)$, which leads us to consider the following differential equation:

$$\begin{aligned} \dot{x} &= \Re(w)t^{-1}x + (1-w)t^{-1-i\Im(w)}\eta(t), \\ t &\in [1, +\infty), \quad x(1) = 1, \quad x : [1, +\infty) \rightarrow \mathbb{C}. \end{aligned} \quad (2)$$

Since $\eta \in \mathcal{B}_{\text{loc}}([1, \infty), \mathbb{C})$, the function

$$t \mapsto t^{\Re(w)} \int_1^t u^{-1-w} \eta(u) du, \quad t \geq 1.$$

is absolutely continuous on $[1, +\infty)$. The differential equation (2) is a non-homogeneous linear differential equation. Then, there exists a unique continuous solution $\psi_{\eta,w}(\cdot)$ of (2) such that $\psi_w(1) = 1$, which is given by

$$\psi_{\eta,w}(t) = t^{\Re(w)} \left[1 + (1-w) \int_1^t u^{-1-w} \eta(u) du \right], \quad \forall t \geq 1. \quad (3)$$

Let us introduce some notations.

Notation 1. For every $\eta \in \mathcal{B}_{\text{loc}}([1, \infty), \mathbb{C})$ and $w \in \mathbb{C}$, we denote by $\psi_{\eta,w}(\cdot)$ the unique continuous solution of the differential equation (2) given by equation (3).

Notation 2. Denote by \mathbb{C}_+ the right half-plane defined as

$$\mathbb{C}_+ := \left\{ w \in \mathbb{C} : \Re(w) > 0 \right\}.$$

For every $f \in \mathcal{B}_{\text{loc}}([1, \infty), \mathbb{C})$, let μ_f denote the function $\mu_f : \mathbb{C}_+ \rightarrow \mathbb{C}$, defined as

$$\mu_f(w) = -1 - (1-w) \int_1^{+\infty} u^{-1-w} f(u) du, \quad \forall w \in \mathbb{C}_+.$$

The function μ_f is defined for all $s \in \mathbb{C}_+$. Indeed, the integral is absolutely convergent, since $f \in \mathcal{B}_{\text{loc}}([1, \infty), \mathbb{C})$ and since for all $w \in \mathbb{C}_+$ we have $\Re(w) > 0$. In the following lemma, we are interested in bounded solutions on $[1, +\infty)$ and their initial condition.

Lemma 3. *Let $\eta \in \mathcal{B}_{\text{loc}}([1, \infty), \mathbb{C})$ and $w \in \mathbb{C}_+$. Consider the continuous solution $\psi_{\eta,w}(\cdot)$ of the differential equation (2), as introduced in Notation 1. Then,*

$$\sup_{t \geq 1} |\psi_{\eta,w}(t)| < +\infty \quad \text{if and only if} \quad \mu_\eta(w) = 0,$$

where the function μ_η is defined in Notation 2.

Proof. Let $w \in \mathbb{C}_+$ be fixed. Suppose that $\sup_{t \geq 1} |\psi_{\eta,w}(t)| < +\infty$. By equation (3), this implies

$$\sup_{t \geq 1} \left[t^{\Re(w)} \left| 1 + (1-w) \int_1^t u^{-1-w} \eta(u) du \right| \right] < +\infty.$$

Since $\Re(w) > 0$, we must have

$$\left| 1 + (1-w) \int_1^{+\infty} u^{-1-w} \eta(u) du \right| = 0,$$

which gives $\mu_\eta(w) = 0$. Now, prove that $\sup_{t \geq 1} |\psi_{\eta,w}(t)| < +\infty$. By definition of $\mu_\eta(w)$, we have

$$\mu_\eta(w) + 1 + (1-w) \int_1^t u^{-1-w} \eta(u) du = -(1-w) \int_t^{+\infty} u^{-1-w} \eta(u) du, \quad \forall t \geq 1.$$

Suppose that $\mu_\eta(w) = 0$, using equation (3), we obtain

$$\psi_{\eta,w}(t) = -(1-w)t^{\Re(w)} \int_t^{+\infty} u^{-1-w} \eta(u) du, \quad \forall t \geq 1. \quad (4)$$

By Hypothesis $\eta \in \mathcal{B}_{\text{loc}}([1, \infty), \mathbb{C})$ then there exists a constant $c > 0$ such that $|\eta(t)| < c$ for all $t \geq 1$, we obtain

$$\sup_{t \geq 1} |\psi_{\eta,w}(t)| \leq c \frac{|1-w|}{\Re(w)} < +\infty.$$

□

2 Main result

As formulated in the Dynamical Conjecture in [1], we assume that the function η satisfies the following *Rotation number hypothesis*:

$$\exists \rho_\eta \in \mathbb{C}^* : \sup_{t \geq 1} \left| \int_1^t (\eta(u) - \rho_\eta) du \right| < +\infty, \quad (\mathbf{H})$$

The complex number ρ_η is called the *rotation number of η* . Notice that if ρ_η exists, then it is unique; this can be proved by contradiction. Denote by $\mathcal{L}[\eta]$ the following transformation

$$\mathcal{L}[\eta](t) := \int_1^t (\eta(u) - \rho_\eta) du, \quad \forall t \geq 1.$$

The Rotation number hypothesis implies that $\mathcal{L}[\eta] \in \mathcal{B}_{\text{loc}}([1, \infty), \mathbb{C})$. Hence, the function $\mu_{\mathcal{L}[\eta]}$ is defined on \mathbb{C}_+ as given in the Notation 2.

We denote by $B \subset \mathbb{C}$ the critical strip, excluding the critical line and the real axis, defined as

$$B := \left\{ w \in \mathbb{C} : \Re(w) \in (0, 1), \Re(w) \neq \frac{1}{2}, \Im(w) \neq 0 \right\}.$$

The following main result provides an answer to the Dynamical Conjecture formulated in [1]. We shows that for a function η satisfying the Rotation number hypothesis **(H)**, for every $s \in B$, the continuous solutions $\psi_{\eta,s}(\cdot)$ and $\psi_{\eta,1-\bar{s}}(\cdot)$ of the differential equation (2) cannot both be bounded on $[1, +\infty)$.

Theorem 4. *Let $\eta \in \mathcal{B}_{\text{loc}}([1, \infty), \mathbb{C})$ satisfy the Rotation Number Hypothesis **(H)**. Suppose that the functions μ_η and $\mathcal{L}[\eta]$ satisfies:*

$$\forall x \in \mathbb{R}^* : \mu_\eta(1 + ix) \neq 0 \quad \text{and} \quad \mu_{\mathcal{L}[\eta]}(1 + ix) \neq \frac{\rho_\eta - ix}{1 + ix} - \frac{1}{|1 + ix|^2}.$$

Then, for every $s \in B$, we have

$$(\mu_\eta(s), \mu_\eta(1 - \bar{s})) \neq (0, 0).$$

Remark 5. In the proof of the Lemma 9, we note that

$$\frac{|1 + ix|^2}{ix} \left[\frac{\rho_\eta - ix}{1 + ix} - \mu_{\mathcal{L}[\eta]}(1 + ix) - \frac{1}{|1 + ix|^2} \right] = \lim_{r \rightarrow > 0} \mu_\eta(r + ix),$$

which is the analytic extension of the function μ_η to the line $\Re(z) = 0$. Consequently, the condition of the previous theorem is equivalent to

$$\forall x \in \mathbb{R}^* : \mu_\eta(1 + ix) \neq 0 \quad \text{and} \quad \mu_\eta(ix) \neq 0.$$

3 Main proposition

Let us introduce the following notation.

Notation 6. We denote by $I \subset \mathbb{C}$ the following set

$$I := \left\{ w \in \mathbb{C} : \Re(w) \in [0, 1], \Re(w) \neq \frac{1}{2}, \Im(w) \neq 0 \right\}.$$

Let $\eta \in \mathcal{B}_{\text{loc}}([1, \infty), \mathbb{C})$. Consider the continuous solutions of the differential equation (2) as given in Notation 1. For every $w \in I$, we denote by δ_w the function

$$\delta_{\eta,w}(t) := \frac{1}{2\Re(w) - 1} \left(\psi_{\eta,w}(t) - \psi_{\eta,1-\bar{w}}(t) \right), \quad \forall t \geq 1.$$

The differential equation (2) can be viewed as a differential equation with the parameter $\Re(w)$. Intuitively, this introduces an order structure within the set of solutions with a same initial condition and parameterized by $\Re(w)$. We start to exhibit this order between solutions through the following proposition.

Proposition 7. *Let $\eta \in \mathcal{B}_{\text{loc}}([1, \infty), \mathbb{C})$. For every $s := \sigma + i\tau \in I$, the functions $\delta_{\eta,s}$ and $\delta_{\eta,1+i\tau}$, defined in Notation (6), satisfies the following Volterra integral equation:*

$$\begin{aligned} \delta_{\eta,1+i\tau}(t) &= \sigma(1-\sigma)t^{1-\sigma} \int_1^t u^{-2(1-\sigma)} \int_1^u v^{-1-\sigma} \delta_{\eta,1+i\tau}(v) dv du \\ &+ \delta_{\eta,s}(t), \quad \forall t \geq 1, \end{aligned}$$

Lemma 8. *Let $\eta \in \mathcal{B}_{\text{loc}}([1, \infty), \mathbb{C})$. For every $s := \sigma + i\tau \in I$, the functions $\delta_{\eta,s}$ and $\delta_{\eta,1+i\tau}$, defined in Notation (6), satisfies the following Euler differential equation:*

$$\frac{d^2}{dt^2} \left(\delta_{\eta,s}(t) - \delta_{\eta,1+i\tau}(t) \right) = -\sigma(1-\sigma)t^{-2} \delta_{\eta,s}(t), \quad \forall t \geq 1,$$

with initial conditions $\delta_{\eta,s}(1) = \delta_{\eta,1+i\tau}(1) = 0$ and $\dot{\delta}_{\eta,s}(1) - \dot{\delta}_{\eta,1+i\tau}(1) = 0$.

Proof. Let $w \in \mathbb{C}_+$ and denote $w := \alpha + i\beta$. For every $t \geq 1$ denote

$$x_w(t) := t^{-\frac{1}{2}} \psi_{\eta,w}(t). \tag{5}$$

According to Notation 1, we obtain the following differential equation

$$\dot{x}_w(t) = \left(\alpha - \frac{1}{2}\right) t^{-1} x_w(t) + (1-w) t^{-\frac{3}{2}-i\beta} \eta(t), \quad x_w(1) = 1. \quad (6)$$

Let $s := \sigma + i\tau \in I$. From the previous equation, we have

$$\dot{x}_s(t) + \dot{x}_{1-\bar{s}}(t) = \left(\sigma - \frac{1}{2}\right) t^{-1} (x_s(t) - x_{1-\bar{s}}(t)) + (1-2i\tau) t^{-\frac{3}{2}-i\tau} \eta(t).$$

Use the fact that $x_s(1) = x_{1-\bar{s}}(1) = 1$ and integrate, we obtain

$$\begin{aligned} x_s(t) + x_{1-\bar{s}}(t) &= \left(\sigma - \frac{1}{2}\right) \int_1^t u^{-1} (x_s(u) - x_{1-\bar{s}}(u)) du \\ &\quad + 2 + (1-2i\tau) \int_1^t u^{-\frac{3}{2}-i\tau} \eta(u) du. \end{aligned}$$

From the differential equation (6), we have

$$\begin{aligned} \dot{x}_s(t) - \dot{x}_{1-\bar{s}}(t) &= \left(\sigma - \frac{1}{2}\right) t^{-1} (x_s(t) + x_{1-\bar{s}}(t)) \\ &\quad - (2\sigma - 1) t^{-\frac{3}{2}-i\tau} \eta(t). \end{aligned}$$

The two previous differential equations, gives

$$\begin{aligned} \dot{x}_s(t) - \dot{x}_{1-\bar{s}}(t) &= \left(\sigma - \frac{1}{2}\right)^2 t^{-1} \int_1^t u^{-1} (x_s(u) - x_{1-\bar{s}}(u)) du \\ &\quad + (2\sigma - 1) t^{-1} \left[1 + \left(\frac{1}{2} - i\tau\right) \int_1^t u^{-\frac{3}{2}-i\tau} \eta(u) du \right] \\ &\quad + (2\sigma - 1) t^{-\frac{3}{2}-i\tau} \eta(t). \end{aligned}$$

Use the fact that $x_s(1) - x_{1-\bar{s}}(1) = 0$ and integrate, we obtain

$$\begin{aligned} x_s(t) - x_{1-\bar{s}}(t) &= \left(\sigma - \frac{1}{2}\right)^2 \int_1^t u^{-1} \int_1^u v^{-1} (x_s(v) - x_{1-\bar{s}}(v)) dv du \\ &\quad + (2\sigma - 1) \int_1^t u^{-1} \left[1 + \left(\frac{1}{2} - i\tau\right) \int_1^u v^{-\frac{3}{2}-i\tau} \eta(v) dv \right] du \\ &\quad + (2\sigma - 1) \int_1^t u^{-\frac{3}{2}-i\tau} \eta(u), \quad du. \end{aligned} \quad (7)$$

By the notation (5), we have

$$x_s(t) - x_{1-\bar{s}}(t) = t^{-\frac{1}{2}}(\psi_{\eta,s}(t) - \psi_{\eta,1-\bar{s}}(t)).$$

By the Notation (6), we have

$$\delta_{\eta,s}(t) = \frac{1}{2\sigma - 1}(\psi_{\eta,s}(t) - \psi_{\eta,1-\bar{s}}(t)).$$

Then

$$x_s(t) - x_{1-\bar{s}}(t) = (2\sigma - 1)t^{-\frac{1}{2}}\delta_{\eta,s}(t),$$

In term of δ_s the integral equation (7) becomes

$$\begin{aligned} t^{-\frac{1}{2}}\delta_{\eta,s}(t) &= (\sigma - \frac{1}{2})^2 \int_1^t u^{-1} \int_1^u v^{-\frac{3}{2}} \delta_{\eta,s}(v) dv du \\ &\quad + \int_1^t u^{-1} \left[1 + \left(\frac{1}{2} - i\tau \right) \int_1^u v^{-\frac{3}{2}-i\tau} \eta(v) dv \right] du \\ &\quad + \int_1^t u^{-\frac{3}{2}-i\tau} \eta(u) du. \end{aligned}$$

By definition of the set I in Notation 6, we have in particular,

$$\begin{aligned} t^{-\frac{1}{2}}\delta_{\eta,1+i\tau}(t) &= \frac{1}{4} \int_1^t u^{-1} \int_1^u v^{-\frac{3}{2}} \delta_{\eta,1+i\tau}(v) dv du \\ &\quad + \int_1^t u^{-1} \left[1 + \left(\frac{1}{2} - i\tau \right) \int_1^u v^{-\frac{3}{2}-i\tau} \eta(v) dv \right] du \\ &\quad + \int_1^t u^{-\frac{3}{2}-i\tau} \eta(u) du. \end{aligned}$$

Subtracting the two previous equations gives:

$$\begin{aligned} \delta_{\eta,s}(t) - \delta_{\eta,1+i\tau}(t) &= (\sigma - \frac{1}{2})^2 t^{\frac{1}{2}} \int_1^t u^{-1} \int_1^u v^{-\frac{3}{2}} \delta_{\eta,s}(v) dv du \\ &\quad - \frac{1}{4} t^{\frac{1}{2}} \int_1^t u^{-1} \int_1^u v^{-\frac{3}{2}} \delta_{\eta,1+i\tau}(v) dv du. \end{aligned}$$

After two differentiations, to the second-order linear differential equation

$$\frac{d^2}{dt^2}(\delta_{\eta,s}(t) - \delta_{\eta,1+i\tau}(t)) = -\sigma(1 - \sigma)t^{-2}\delta_{\eta,s}(t),$$

with initial conditions $\delta_{\eta,s}(1) = \delta_{\eta,1+i\tau}(1) = 0$ and $\dot{\delta}_{\eta,s}(1) - \dot{\delta}_{\eta,1+i\tau}(1) = 0$. \square

Proof of the Proposition 7. By the Lemma 8 every $s := \sigma + i\tau \in I$, we have

$$\frac{d^2}{dt^2} (\delta_{\eta,s}(t) - \delta_{\eta,1+i\tau}(t)) = -\sigma(1-\sigma)t^{-2}\delta_{\eta,s}(t),$$

which can be written as

$$t^{1-\sigma} \frac{d^2}{dt^2} (\delta_{\eta,s}(t) - \delta_{\eta,1+i\tau}(t)) = -\sigma(1-\sigma)t^{-1-\sigma}\delta_{\eta,s}(t),$$

Use the fact that $\delta_{\eta,s}(1) - \delta_{\eta,1+i\tau}(1) = 0$ and $\dot{\delta}_{\eta,s}(1) - \dot{\delta}_{\eta,1+i\tau}(1) = 0$ and use the integration by part formula,

$$\begin{aligned} t^{1-\sigma} \frac{d}{dt} (\delta_{\eta,s}(t) - \delta_{\eta,1+i\tau}(t)) - (1-\sigma) \int_1^t u^{-\sigma} \frac{d}{du} (\delta_{\eta,s}(u) - \delta_{\eta,1+i\tau}(u)) du \\ = -\sigma(1-\sigma) \int_1^t u^{-1-\sigma} \delta_{\eta,s}(u) du, \end{aligned}$$

$$\begin{aligned} t^{1-\sigma} \frac{d}{dt} (\delta_{\eta,s}(t) - \delta_{\eta,1+i\tau}(t)) - (1-\sigma)t^{-\sigma} (\delta_{\eta,s}(t) - \delta_{\eta,1+i\tau}(t)) \\ - \sigma(1-\sigma) \int_1^t u^{-1-\sigma} (\delta_{\eta,s}(u) - \delta_{\eta,1+i\tau}(u)) du \\ = -\sigma(1-\sigma) \int_1^t u^{-1-\sigma} \delta_{\eta,s}(u) du, \end{aligned}$$

We obtained

$$\begin{aligned} t^{1-\sigma} \frac{d}{dt} (\delta_{\eta,s}(t) - \delta_{\eta,1+i\tau}(t)) - (1-\sigma)t^{-\sigma} (\delta_{\eta,s}(t) - \delta_{\eta,1+i\tau}(t)) \\ + \sigma(1-\sigma) \int_1^t u^{-1-\sigma} \delta_{\eta,1+i\tau}(u) du \\ = 0, \quad \forall t \geq 1. \end{aligned}$$

That can be written as

$$\begin{aligned} \frac{d}{dt} \left(t^{-1+\sigma} (\delta_{\eta,1+i\tau}(t) - \delta_{\eta,s}(t)) \right) = \sigma(1-\sigma)t^{-2(1-\sigma)} \int_1^t u^{-1-\sigma} \delta_{\eta,1+i\tau}(u) du \\ + \frac{d}{dt} (t^{-1+\sigma} \delta_{\eta,s}(t)), \quad \forall t \geq 1, \end{aligned}$$

Integrate

$$\begin{aligned} \delta_{\eta,1+i\tau}(t) &= \sigma(1-\sigma)t^{1-\sigma} \int_1^t u^{-2(1-\sigma)} \int_1^u v^{-1-\sigma} \delta_{\eta,1+i\tau}(v) dv du \\ &\quad + \delta_{\eta,s}(t), \quad \forall t \geq 1, \end{aligned}$$

□

4 Proof of the Theorem 4

Lemma 9. *Let $\eta \in \mathcal{B}_{\text{loc}}([1, \infty), \mathbb{C})$ satisfy the Rotation Number Hypothesis (H). For every $x \in \mathbb{R}^*$, the function $\delta_{\eta,1+ix}$, defined in Notation (6), satisfies the following equation*

$$\sup_{t \geq 1} \left| \int_1^t u^{-1} \delta_{\eta,1+ix}(u) du + \alpha_x t - \beta_x \ln(t) \right| < +\infty,$$

where

$$\alpha_x := \mu_\eta(1+ix) \quad \text{and} \quad \beta_x := \frac{|1+ix|^2}{ix} \left[\frac{\rho_\eta - ix}{1+ix} - \mu_{\mathcal{L}[\eta]}(1+ix) - \frac{1}{|1+ix|^2} \right].$$

Proof. Let $x \in \mathbb{R}^*$. By the Notation 6 and equation (3), we have

$$\begin{aligned} \delta_{\eta,1+ix}(t) &= t \left[1 - ix \int_1^t u^{-2-ix} \eta(u) du \right] \\ &\quad - 1 - (1-ix) \int_1^t u^{-1-ix} \eta(u) du, \quad \forall t \geq 1, \end{aligned}$$

which can be written as

$$\begin{aligned} \delta_{\eta,1+ix}(t) &= -t\mu_\eta(1+ix) + ixt \int_t^{+\infty} u^{-2-ix} \eta(u) du \\ &\quad - 1 - \rho \frac{1-ix}{ix} (1-t^{-ix}) \\ &\quad - (1-ix) \int_1^t u^{-1-ix} (\eta(u) - \rho) du, \quad \forall t \geq 1. \end{aligned}$$

Use the Rotation number hypothesis (**H**) and use the integration by formula

$$\begin{aligned}
\delta_{\eta,1+ix}(t) &= -t\mu_\eta(1+ix) - \rho \frac{1-ix}{ix} (1-t^{-ix}) + \rho \frac{ix}{1+ix} t^{-ix} \\
&\quad - 1 - |1+ix|^2 \int_1^{+\infty} u^{-2-ix} \int_1^u (\eta(v) - \rho) dv du \\
&\quad + ixt \int_t^{+\infty} u^{-2-ix} (\eta(u) - \rho) du \\
&\quad + (1-ix) \int_t^{+\infty} u^{-1-ix} (\eta(u) - \rho) du, \quad \forall t \geq 1.
\end{aligned}$$

Equivalent to,

$$\begin{aligned}
\delta_{\eta,1+ix}(t) &= -t\mu_\eta(1+ix) - \rho \frac{1-ix}{ix} (1-t^{-ix}) + \rho \frac{ix}{1+ix} t^{-ix} \\
&\quad - 1 - |1+ix|^2 \frac{1}{ix} (1 + \mu_{\mathcal{L}[\eta]}(1+ix)) \\
&\quad + ixt \int_t^{+\infty} u^{-2-ix} (\eta(u) - \rho) du \\
&\quad + (1-ix) \int_t^{+\infty} u^{-1-ix} (\eta(u) - \rho) du, \quad \forall t \geq 1.
\end{aligned}$$

or,

$$\begin{aligned}
\delta_{\eta,1+ix}(t) &= -t\alpha_x + \beta_x \\
&\quad - \rho \frac{1}{ix(1+ix)} t^{-ix} \\
&\quad + ix(2+ix)t \int_t^{+\infty} u^{-3-ix} \int_t^u (\eta(v) - \rho) dv du \\
&\quad + |1+ix|^2 \int_t^{+\infty} u^{-2-ix} \int_t^u (\eta(v) - \rho) dv du, \quad \forall t \geq 1.
\end{aligned}$$

where

$$\alpha_x := \mu_\eta(1+ix) \text{ and } \beta_x := \frac{|1+ix|^2}{ix} \left[\frac{\rho_\eta - ix}{1+ix} - \mu_{\mathcal{L}[\eta]}(1+ix) - \frac{1}{|1+ix|^2} \right].$$

Using the hypothesis (**H**), we have

$$\sup_{u \geq t \geq 1} \left| \int_t^u (\eta(v) - \rho) dv \right| < +\infty,$$

Then

$$\sup_{t \geq 1} \left| \int_1^t u^{-1} \delta_{\eta, 1+ix}(u) du + \alpha_x t - \beta_x \ln(t) \right| < +\infty,$$

□

Proof of the Theorem 4. Let $s := \sigma + i\tau \in B$. By the Lemma 9, there exists $M_\tau > 0$ such that

$$\left| \int_1^t u^{-1} \delta_{\eta, 1+i\tau}(u) du + \alpha_\tau t - \beta_\tau \ln(t) \right| < M_\tau, \quad \forall t \geq 1, \quad (8)$$

where by the hypothesis of the present Theorem we have $\alpha_\tau \neq 0$ and $\beta_\tau \neq 0$. Then there exists $\omega_\tau \in \mathbb{C}^*$ such that

$$\Re(\omega_\tau \alpha_\tau) \Re(\omega_\tau \beta_\tau) > 0 \quad \text{or} \quad \Im(\omega_\tau \alpha_\tau) \Im(\omega_\tau \beta_\tau) > 0.$$

Suppose that

$$\Re(\omega_\tau \alpha_\tau) \Re(\omega_\tau \beta_\tau) > 0. \quad (9)$$

The proof is analogous for the imaginary part. Fix ω_τ satisfying the previous equation and define

$$\tilde{\delta}_{1+i\tau}(t) := \Re(\omega_\tau \phi_{\eta, \tau}(t)) \quad \text{and} \quad \tilde{\delta}(t) := \Re(\omega_\tau \delta_{\eta, s}(t)). \quad (10)$$

Equation (8) implies

$$\left| \int_1^t u^{-1} \tilde{\delta}_{1+i\tau}(u) du + \Re(\omega_\tau \alpha_\tau) t - \Re(\omega_\tau \beta_\tau) \ln(t) \right| < M_\tau |\omega_\tau|, \quad \forall t \geq 1, \quad (11)$$

For every $c > 1$, denote

$$\mathcal{A}_c(t) := \frac{1}{2} + \frac{1}{\pi} \arctan(t - c); \quad \forall t \geq 1.$$

For every $c > 1$ and $g \in \mathcal{B}_{\text{loc}}([1, \infty), \mathbb{C})$, consider the following transformation

$$\Phi_c[g](t) := c t^{-1} \int_1^t \mathcal{A}_c(u) u^{-1} g(u) du, \quad \forall t \geq 1.$$

Since $\lim_{t \rightarrow +\infty} \mathcal{A}_c(t) = 1$ for every $c > 1$, by equation (11), we have

$$\forall c > 1: \quad \lim_{t \rightarrow +\infty} \left| t^{-1} \int_1^t \mathcal{A}_c(u) u^{-1} \tilde{\delta}_{1+i\tau}(u) du \right| = |\Re(\omega_\tau \alpha_\tau)| > 0. \quad (12)$$

Using equation (9), there exist a sufficiently large $c_s > 1$ and $t_{c_s} > 1$ such that for all $c > c_s$ and $t > t_{c_s}$ we have

$$\left| t^{-1} \int_1^t \mathcal{A}_c(u) u^{-1} \tilde{\delta}_{1+i\tau}(u) du \right| = \max_{z \in [1, t]} \left| z^{-1} \int_1^z \mathcal{A}_c(u) u^{-1} \tilde{\delta}_{1+i\tau}(u) du \right|, \quad (13)$$

Implies

$$\left| \Phi_c[\tilde{\delta}_{1+i\tau}](t) \right| = \max_{z \in [1, t]} \left| \Phi_c[\tilde{\delta}_{1+i\tau}](z) \right|, \quad \forall c > c_s, \quad \forall t > t_{c_s}. \quad (14)$$

The purpose of introducing \mathcal{A}_c is to obtain the previous equality. It ensures that the peaks of the function $t \mapsto \left| \Phi_c[\tilde{\delta}_{1+i\tau}](t) \right|$ are bounded above by the limiting value which is equal to $c|\Re(\omega_\tau \alpha_\tau)|$.

By the Proposition 7, for all $t \geq 1$ we have

$$\begin{aligned} \frac{t^{-1}}{\sigma(1-\sigma)} \left(\tilde{\delta}_{1+i\tau}(t) - \tilde{\delta}_s(t) \right) \\ = t^{-\sigma} \int_1^t u^{-2(1-\sigma)} \int_1^u v^{-1-\sigma} \tilde{\delta}_{1+i\tau}(v) dv du, \end{aligned}$$

Applying the transformation Φ_c ,

$$\begin{aligned} \frac{t}{\sigma(1-\sigma)} \left(\Phi_c[\tilde{\delta}_{1+i\tau}](t) - \Phi_c[\tilde{\delta}_s](t) \right) \\ = c \int_1^t \mathcal{A}_c(u) u^{-\sigma} \int_1^u v^{-2(1-\sigma)} \int_1^v \frac{z^{-\sigma}}{\mathcal{A}_c(z)} \mathcal{A}_c(z) z^{-1} \tilde{\delta}_{1+i\tau}(z) dz dv du, \end{aligned}$$

Use the integration by part formula,

$$\begin{aligned} \frac{t}{\sigma(1-\sigma)} \left(\Phi_c[\tilde{\delta}_{1+i\tau}](t) - \Phi_c[\tilde{\delta}_s](t) \right) \\ = \int_1^t \mathcal{A}_c(u) u^{-\sigma} \int_1^u \frac{v^{-1+\sigma}}{\mathcal{A}_c(v)} \Phi_c[\tilde{\delta}_{1+i\tau}](v) dv du \\ - \int_1^t \mathcal{A}_c(u) u^{-\sigma} \int_1^u v^{-2(1-\sigma)} \int_1^v z \left(\frac{d}{dz} \frac{z^{-\sigma}}{\mathcal{A}_c(z)} \right) \Phi_c[\tilde{\delta}_{1+i\tau}](z) dz dv du, \end{aligned}$$

We have

$$\frac{4}{\pi} \frac{d}{dz} \frac{z^{-\sigma}}{\mathcal{A}_c(z)} = - \frac{\sigma z^{-\sigma-1} \left(\frac{\pi}{2} + \arctan(z-c) \right) + \frac{z^{-\sigma}}{1+(z-c)^2}}{\left(\frac{\pi}{2} + \arctan(z-c) \right)^2} < 0, \quad \forall z \geq 1.$$

By equation (14), for all $c > c_s$ and for all $t \geq t_{c_s}$, we obtain

$$t \left| \Phi_c[\tilde{\delta}_{1+i\tau}](t) \right| < t \left| \Phi_c[\tilde{\delta}_s](t) \right| + \sigma(1-\sigma) \left| \Phi_c[\tilde{\delta}_{1+i\tau}](t) \right| K(\sigma, c, t), \quad (15)$$

where

$$\begin{aligned} K(\sigma, c, t) := & \int_1^t \mathcal{A}_c(u) u^{-\sigma} \int_1^u \frac{v^{-1+\sigma}}{\mathcal{A}_c(v)} dv du \\ & - \int_1^t \mathcal{A}_c(u) u^{-\sigma} \int_1^u v^{-2(1-\sigma)} \int_1^v z \left(\frac{d}{dz} \frac{z^{-\sigma}}{\mathcal{A}_c(z)} \right) dz dv du, \quad \forall t \geq 1. \end{aligned}$$

We prove the theorem by contradiction. Suppose that $\mu_\eta(s) = \mu_\eta(1 - \bar{s}) = 0$. Using the notation of $\delta_{\eta,s}$ given in Notation 6, it follows from the second item of the Lemma 3 that there exists a constant $C_s > 0$ such that

$$|\delta_{\eta,s}(t)| < C_s, \quad \forall t \geq 1.$$

Using the following compact convergence

$$\lim_{c \rightarrow +\infty} c \mathcal{A}_c(u) = \frac{1}{\pi},$$

We obtain

$$\lim_{c \rightarrow +\infty} \left| t \Phi_c[\delta_{\eta,s}](t) \right| = \lim_{c \rightarrow +\infty} \left| c \int_1^t \mathcal{A}_c(u) u^{-1} \delta_{\eta,s}(u) du \right| \leq \frac{1}{\pi} C_s \ln(t), \quad \forall t \geq 1.$$

By the notation of $\tilde{\delta}_s$ in equation (10), we get

$$\lim_{c \rightarrow +\infty} \left| t \Phi_c[\tilde{\delta}_s](t) \right| < \frac{1}{\pi} |\omega_\tau| C_s \ln(t), \quad \forall t \geq 1.$$

Equation (15) gives

$$\lim_{c \rightarrow +\infty} \frac{\left| \Phi_c[\tilde{\delta}_{1+i\tau}](t) \right|}{\ln(t)} \left(t - \sigma(1-\sigma) K(\sigma, c, t) \right) < \frac{1}{\pi} |\omega_\tau| C_s, \quad \forall t \geq t_{c_s}. \quad (16)$$

Now, let us examine the asymptotic behavior of $K(\sigma, c, t)$. We recall the definition

$$\mathcal{A}_c(t) = \frac{1}{2} + \frac{1}{\pi} \arctan(t - c), \quad t \geq 1,$$

and

$$K(\sigma, c, t) = \int_1^t \mathcal{A}_c(u) u^{-\sigma} \int_1^u \frac{v^{-1+\sigma}}{\mathcal{A}_c(v)} dv du \\ - \int_1^t \mathcal{A}_c(u) u^{-\sigma} \int_1^u v^{-2(1-\sigma)} \int_1^v z \left(\frac{d}{dz} \frac{z^{-\sigma}}{\mathcal{A}_c(z)} \right) dz dv du, \quad \forall t \geq 1.$$

Use the integration by parts formula to obtain

$$K(\sigma, c, t) = \frac{1}{2\sigma - 1} \int_1^t \mathcal{A}_c(u) u^{-\sigma} W(u) du, \quad (17)$$

with

$$W(u) := \frac{1}{\mathcal{A}_c(1)} (u^{-1+2\sigma} - 1) + u^{-1+2\sigma} \int_1^u \frac{v^{-\sigma}}{\mathcal{A}_c(v)} dv - \int_1^u \frac{v^{-1+\sigma}}{\mathcal{A}_c(v)} dv.$$

Use the integration by part formula, to get

$$\mathcal{A}_c(u) u^{-\sigma} W(u) = \frac{1}{1-\sigma} - \frac{1}{\sigma} \\ - \frac{\mathcal{A}_c(u)}{1-\sigma} u^{-1+\sigma} \left[\frac{\sigma}{\mathcal{A}_c(1)} + \int_1^u v^{1-\sigma} \frac{d}{dv} \frac{1}{\mathcal{A}_c(v)} dv \right] \\ + \frac{\mathcal{A}_c(u)}{\sigma} u^{-\sigma} \left[\frac{1-\sigma}{\mathcal{A}_c(1)} + \int_1^u v^{\sigma} \frac{d}{dv} \frac{1}{\mathcal{A}_c(v)} dv \right]$$

Since

$$\frac{d}{du} \mathcal{A}_c(u) = \frac{1}{\pi} \frac{1}{1 + (u - c)^2}, \quad \forall u \geq 1,$$

It follows that

$$\mathcal{A}_c(u) u^{-\sigma} W(u) = \frac{1}{1-\sigma} - \frac{1}{\sigma} \\ - \frac{\mathcal{A}_c(u)}{1-\sigma} u^{-1+\sigma} \left[\frac{\sigma}{\mathcal{A}_c(1)} - \frac{1}{\pi} \int_1^u \frac{1}{(\mathcal{A}_c(v))^2} \frac{v^{1-\sigma}}{1 + (v - c)^2} dv \right] \\ + \frac{\mathcal{A}_c(u)}{\sigma} u^{-\sigma} \left[\frac{1-\sigma}{\mathcal{A}_c(1)} - \frac{1}{\pi} \int_1^u \frac{1}{(\mathcal{A}_c(v))^2} \frac{v^{\sigma}}{1 + (v - c)^2} dv \right]$$

Then

$$\lim_{c \rightarrow +\infty} \mathcal{A}_c(u) u^{-\sigma} W(u) = \frac{2\sigma - 1}{\sigma(1 - \sigma)} - \frac{\sigma}{1 - \sigma} u^{-1+\sigma} + \frac{1 - \sigma}{\sigma} u^{-\sigma}$$

Inserting these into the integral for $K(\sigma, c, t)$ given by equation (17), we obtain the following equation

$$\lim_{c \rightarrow +\infty} \left(t - \sigma(1 - \sigma) K(\sigma, c, t) \right) = \frac{1}{2\sigma - 1} \left(\sigma t^\sigma - (1 - \sigma) t^{1-\sigma} \right).$$

Using the following compact convergence

$$\lim_{c \rightarrow +\infty} c \mathcal{A}_c(u) = \frac{1}{\pi},$$

Thanks to equation (11), we get

$$\lim_{t \rightarrow +\infty} \lim_{c \rightarrow +\infty} \left| \Phi_c[\tilde{\delta}_{1+i\tau}](t) \right| = \frac{1}{\pi} |\Re(\omega_\tau \alpha_\tau)| > 0.$$

Since $\sigma \in (0, 1) \setminus \frac{1}{2}$, we obtain

$$\lim_{t \rightarrow +\infty} \lim_{c \rightarrow +\infty} \frac{\left| \Phi_c[\tilde{\delta}_{1+i\tau}](t) \right|}{\ln(t)} \left(t - \sigma(1 - \sigma) K(\sigma, c, t) \right) = +\infty.$$

We obtain a contradiction with equation (16). \square

References

- [1] W. Oukil, Bounded Solutions of a Complex Differential Equation for the Riemann Hypothesis. Version 34, HAL Id : hal-03849860. (2025).