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Koopman-Based Sliding Mode Control for Data-Driven Stabilization of Nonlinear Systems

Moussa Labbadi¹ and Zhengang Zhong²

Abstract— This paper presents a data-driven method for the design of stabilizing controllers for nonlinear systems with exponential stability guarantees. The approach leverages Koopman operator theory to obtain a bilinear representation of the system dynamics directly from measurement data, while explicitly accounting for approximation errors arising from the use of finitely many data samples. On top of this lifted representation, a sliding mode control (SMC) strategy is developed to robustly stabilize the system and compensate for residual modeling error. The proposed framework focuses on bilinear single-input systems, for which rigorous stability guarantees are provided. Furthermore, the methodology naturally extends to multi-input systems, opening the way for broader applications in data-driven robust control.

Keywords— Koopman operator theory, sliding mode control, data-driven stabilization.

I. INTRODUCTION

Data-driven approaches have recently gained considerable attention in control theory as measurement data have become increasingly accessible. This trend is particularly notable for nonlinear systems, where classical techniques such as system identification or first-principles modeling are often time-consuming, require expert knowledge, and may lead to non-convex optimization problems [1], [2], [3], [4]. By leveraging available data, these methods can construct representations of complex systems that enable convex controller design [5], [6], [7]. While data-driven feedback design with closed-loop guarantees has been extensively studied for linear systems [8], [5], [6], [7], providing rigorous guarantees for general nonlinear systems remains challenging [1], [2], [3], [4]. Stability guarantees have been obtained for specific classes, including Hammerstein and Wiener systems [9], bilinear systems [10], polynomial systems [11], [12], and rational dynamics [13], typically via semidefinite programming (SDP). For more general nonlinear systems, SDP-based designs can be implemented using polynomial approximations [14], [15], kernel regression [16], [17], or linear parameter-varying embeddings [18], [19]. A comprehensive overview of these methods is provided in [1].

The Koopman operator [20] provides a linear, infinite-dimensional representation of nonlinear systems, enabling linear analysis and control via observables that evolve linearly. Finite-dimensional approximations, obtained through EDMD [21], [22], [23], [24] or machine learning [25], [26],

allow practical control design, with error bounds analyzed in [27], [28]. While many methods assume exact lifted dynamics, Koopman-based control has been successfully applied across diverse applications [29], [30]. For further details on Koopman-based control, including error bounds and closed-loop guarantees, we refer the reader to the recent survey [31].

Inspired by [31], [28], this paper leverages a finite-dimensional Koopman approximation to formulate a bilinear system for sliding mode control, proposing a controller that guarantees fixed-time convergence of the sliding variables and exponential stability of the closed-loop system for single-input cases. Guidelines for multi-input extensions are also provided.

Notation: \mathbb{R} : the set of real numbers. For $n \in \mathbb{N}$, \mathbb{R}^n denotes the n -dimensional real vector space. $\text{sgn}(x)$: the sign function, defined componentwise for $x \in \mathbb{R}$ as $\text{sgn}(x) = 1$ if $x > 0$, 0 if $x = 0$, -1 if $x < 0$. $\|\cdot\|$, $\|\cdot\|_F$: Euclidean and Frobenius norms, respectively. \otimes : Kronecker product. \mathbb{I}_m : $m \times m$ identity matrix.

II. PROBLEM SETTING

Consider an unknown nonlinear control-affine system:

$$\dot{x}(t) = f(x(t)) + \sum_{i=1}^m g_i(x(t))u_i(t), \quad (1)$$

where $x(t) \in \mathbb{R}^n$ denotes the state, and the input $u \in \mathbb{R}^m$ represents the control. The mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is referred to as the drift, whereas $g_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $i \in [m]$, are the input vector fields. For an initial condition $x(0) = \hat{x} \in \mathbb{R}^n$ and control input u , we denote the corresponding solution of (1) at time t by $x(t; \hat{x}, u)$, assuming existence and uniqueness with $f(0) = 0$. The dynamics (1) are unknown; only data samples of the form are available $f(x_j) + \sum_{i=1}^m g_i(x_j)u_i$, collected at state points $x_j \in X \subset \mathbb{R}^n$, $j \in [d]$, for a finite number $d \in \mathbb{N}$ of suitably chosen inputs u_i . The set X is assumed to be compact and convex.

Our objective is to construct a sliding mode control law $\mu : \mathbb{R}^n \rightarrow \mathbb{R}^m$ using the surrogate Koopman-based model, in such a way that the closed-loop dynamics of (1) achieve exponential stability of the origin within a specified Lyapunov sublevel set, even in the presence of modeling error.

III. KOOPMAN-BASED DATA-DRIVEN REPRESENTATION

In this section, we introduce the Koopman operator framework and its data-driven approximation, followed by the

¹M. Labbadi is with IRDL, UMR CNRS 6027, Bretagne INP, ENSTA, Institut Polytechnique de Paris, Univ. Brest, Univ. Bretagne Sud, Brest, France moussa.labbadi@enib.fr

²Z. Zhong is with the Department of Statistics, University of Warwick, Coventry, United Kingdom. zhengang.zhong@warwick.ac.uk

derivation of suitable error bounds characterizing the resulting approximation error.

A. Koopman operator

Let $\mathcal{V} := L^2(\mathcal{X}, \mu)$ denote the space of square-integrable real-valued functions on \mathcal{X} , where μ is the probability measure induced by the data. Given the autonomous system $\dot{x}(t) = f(x(t))$, $x(t_0) = x_0$, we define the Koopman semigroup of operators $\mathcal{K}^{\Delta t} : \mathcal{V} \rightarrow \mathcal{V}$ as

$$(\mathcal{K}^{\Delta t} \psi)(x) = \psi(\Phi^{\Delta t}(x)),$$

where $\Phi^{\Delta t} : \mathcal{X} \rightarrow \mathcal{X}$ is the flow map generated by the dynamics f , satisfying $x(t + \Delta t) = \Phi^{\Delta t}(x(t))$ over the time horizon Δt and $\psi \in \mathcal{V}$ is a real-valued observable of the system. It is known that the Koopman operator $\mathcal{K}^{\Delta t}$ is linear but usually acts on an infinite-dimensional space of observables [32]. This property indicates that the evolution of observables is invariant in the space \mathcal{V} despite the inherent nonlinearity of dynamical systems. Also, as the Koopman operator is a linear operator [33], it can hence be applied as a linear predictor for the observables of nonlinear dynamics [34]. The idea behind the Koopman operator is applying the eigenfunctions and eigenvalues of $\mathcal{K}^{\Delta t}$ to study the flow map $\Phi^{\Delta t}$ corresponding to the dynamics f . Let $\varphi_i : \mathcal{X} \rightarrow \mathbb{R}$ and $\lambda_i \in \mathbb{C}$ denote the i -th eigenfunction and eigenvalue of the Koopman operator, respectively, i.e. $\mathcal{K}^{\Delta t} \varphi_i(x) = \lambda_j \varphi_i(x)$. If the function ψ lives in the span of the eigenfunctions φ_i , i.e. $\psi = \sum_{i=1}^{\infty} a_i \varphi_i$, then the observable evaluated along the flow map is $\psi \circ \Phi^{\Delta t} = \mathcal{K}^{\Delta t} \psi = \sum_{i=1}^{\infty} a_i \lambda_i \varphi_i$ [35], which enables spectral identification of the dynamics.

B. Koopman operator for controlled systems

For the control-affine systems (1), we characterize the dynamics of the observables via Koopman generator following the work [36]. For the fixed-input control system with \bar{u} , the family of Koopman operators $\mathcal{K}_{\bar{u}}^{\Delta t}$ is generated by the operator $\mathcal{K}_{\bar{u}} : \mathcal{V} \rightarrow \mathcal{V}$, which is defined by $\mathcal{K}_{\bar{u}} \psi := \lim_{\Delta t \rightarrow 0^+} \frac{\psi \circ \Phi_{\bar{u}}^{\Delta t} - \psi}{\Delta t}$. For differentiable observables, the Koopman generator can be characterized via $\mathcal{K}_{\bar{u}} \psi = f \cdot \nabla_x \psi + (g \cdot \bar{u}) \cdot \nabla_x \psi$ with $g = [g_1, \dots, g_m]^T$ and $\bar{u} = [\bar{u}_1, \dots, \bar{u}_m]^T$. With $\mathcal{B}_{\bar{u}} := \mathcal{K}_{\bar{u}} - \mathcal{K}_0$, we acquire $\mathcal{B}_{\bar{u}} \psi = (g \cdot \bar{u}) \cdot \nabla_x \psi$, which allows the dynamics to be formulated in terms of the control affine operator $\mathcal{K}_{u(t)} = \mathcal{K}_0 + \sum_{i=1}^m u_i(t) \mathcal{B}_{e_i}$:

$$\frac{d}{dt} \psi(x(t)) = f(x(t), u(t)) \cdot \nabla_x \psi(x(t)) = (\mathcal{K}_{u(t)} \psi)(x(t)).$$

Although the Koopman operator is defined on the infinite-dimensional function space \mathcal{V} , in practice we do not have access to the full space. Instead, we identify a finite-dimensional subspace $\mathcal{V}_n \subset \mathcal{V}$ from state-input data, which serves as a surrogate space for capturing the behavior of the nonlinear dynamics. Given a subspace $\mathcal{V}_n := \text{span}\{\psi_j, j = 1, \dots, N\}$ of \mathcal{V} , the Koopman generators projected to the finite-dimensional space \mathcal{V}_n are denoted as $\mathcal{K}_{u(t)}^{\mathcal{V}_n}$.

With a bit abuse of notation, we denote $\psi := [\psi_1, \dots, \psi_N]$ and $\hat{z}(x) = \psi(x(t))$ for the remainder of paper. Given a training data set of state-input-derivative tuples $\{x^j, u^j, \dot{x}^j\}_{j=1}^n$, we construct the Extended Dynamic Mode Decomposition (EDMD)-based approximation \mathbf{K}_u through

$$\hat{z}(x) \approx \mathbf{K}_{u(t)} \hat{z}(x) = \left(\mathbf{K}_0 + \sum_{i=1}^m u_i(t) \mathbf{B}_{e_i} \right) \hat{z}(x),$$

where $\begin{bmatrix} \mathbf{K}_0 & \mathbf{B}_{e_1} & \dots & \mathbf{B}_{e_m} \end{bmatrix} = \dot{\psi}_{x,u} (\psi_{x,u})^+$, $\psi_{x,u} = \begin{bmatrix} \psi(x^1) & \dots & \psi(x^n) \\ u^1 \otimes \psi(x^1) & \dots & u^n \otimes \psi(x^n) \end{bmatrix}$ and $\dot{\psi}_{x,u} = [\dot{\psi}^1 \ \dot{\psi}^2 \ \dots \ \dot{\psi}^n]$ with the j -th element defined via $\dot{\psi}_j := \dot{\psi}_{u_j}(x_j)$.

C. EDMD error bounds

This subsection considers the modeling error between the lifted dynamics of the true system and those obtained from the data-driven approximation. The error arises from the approximation of the Koopman generator. To address this, we leverage the following lemma, adapted from [37, Theorem 3], which establishes a probabilistic error bound for the lifted system constructed using the approximated Koopman generator.

A₁: Assume that the data, for each autonomous system with constant control $u \equiv e_i, i \in \{0, \dots, m\}$, is sampled i.i.d. on X w.r.t. the Lebesgue measure.

Lemma 1 ([37, Theorem 3]): Suppose that **A₁** holds and $\mathbb{U} \subset \mathbb{R}^{n_u}$ is bounded. Then, for error bound $\varepsilon > 0$ and probabilistic tolerance $\delta \in (0, 1)$, the probabilistic error bound $\mathbb{P}\left(\left\| \mathcal{K}_{u(t)}^{\mathcal{V}_n} - \mathbf{K}_{u(t)} \right\|_F \leq \varepsilon\right) \geq 1 - \delta \quad \forall t \geq 0$ holds for all measurable control functions $u : [0, \infty) \rightarrow \mathbb{U}$ if $n \geq \bar{n} = \mathcal{O}(N^2/\varepsilon^2\delta)$ holds for the number of data points.

Similar to [38], we further make the following standard assumption to ensure that no projection error occurs.

A₂: For any $\psi \in \mathcal{V}$, the relation $\psi(x(t; \hat{x}, u)) \in \mathcal{V}$ holds for all $u(t) \equiv \hat{u} \in U$, $\hat{x} \in X$, and $t \geq 0$.

By Lemma 1 and **A₂** above, we readily obtain $\hat{\gamma}(x, t) = \left(\mathcal{K}_{u(t)}^{\mathcal{V}_n} - \mathbf{K}_{u(t)} \right) \psi(x) \leq c_r \|\psi(x)\|$, which in turn justifies our assumption **A₂** below.

IV. DATA-DRIVEN CONTROL OF NONLINEAR SYSTEMS

In this section, we design a data-driven sliding mode controller for (1) within the Koopman framework, focusing on the single-input case ($m = 1$), where the lifted dynamics yield a robust bilinear control problem; extensions to multi-input systems and broader parametrizations follow naturally.

A. Robust bilinear reformulation

Based on the section III, the lifted dynamics for the scalar-input case ($m = 1$):

$$\dot{\hat{z}}(x) = A \hat{z}(x) + (\mathbf{K}_0 + \mathbf{B}_{e_1} \hat{z}(x)) u(t) + \hat{\gamma}(t, x), \quad (2)$$

where $\hat{\gamma}(t, x)$ denotes the nonlinear remainder.

A representative solution for (2) is proposed in [28], where the bilinear term is treated as an uncertain bilinear system

and handled via robust convex optimization. While this approach ensures stability, it suffers from several intrinsic drawbacks: i) Conservatism: modeling the bilinear term purely as uncertainty limits achievable performance; ii) Scalability: convex formulations scale poorly with the lifted state dimension or number of data samples; iii) Robustness: structural properties of the nonlinear remainder $\hat{\gamma}(x, t)$ are not exploited, leading to overly restrictive stability conditions. To overcome these limitations, we advocate a sliding mode control design in Koopman-lifted coordinates, leveraging invariance and robustness to guarantee exponential stability while reducing conservatism and enhancing robustness. We adopt the following standing assumption.

A₃: The nonlinear remainder $\hat{\gamma}(t, x)$ satisfies a proportional bound of the form

$$\|\hat{\gamma}(t, x)\| \leq c_r \|\hat{z}(x)\|, \quad \forall (t, x) \in [0, \infty) \times X,$$

for some constant $c_r > 0$ and compact set $X \subset \mathbb{R}^n$.

B. Proposed approach

The forthcoming analysis relies on the following definitions and notations.

Definition 1: The singularity hyperplane is defined as

$$\mathcal{S} := \left\{ \hat{z}(x) \in \mathbb{R}^n \mid \Lambda(\mathbf{K}_0 + \mathbf{B}_{\mathbf{e}_1} \hat{z}(x)) = 0 \right\}, \quad (3)$$

where $\mathbf{K}_0 + \mathbf{B}_{\mathbf{e}_1} \hat{z}(x)$ denotes the bilinear input operator of (2) and $\Lambda \in \mathbb{R}^{1 \times n}$ is chosen such that $\Lambda B \neq 0$.

Definition 2: For $\sigma > 0$, the boundary layer associated with (3) is

$$\mathcal{L}_\sigma := \left\{ \hat{z}(x) \in \mathbb{R}^n \mid |\Lambda(\mathbf{K}_0 + \mathbf{B}_{\mathbf{e}_1} \hat{z}(x))| < \frac{\sigma}{2} \right\}. \quad (4)$$

Definition 3: The augmented sliding mode control input is

$$\mu(t) = u_s(t) + u_{\text{eq}}(t) + u_a(t), \quad (5)$$

where $u_s(t)$, $u_{\text{eq}}(t)$, and $u_a(t)$ denote the discontinuous sliding, equivalent, and auxiliary corrective components, respectively.

Under (5), system (2) admits Filippov solutions [39] for any locally bounded measurable feedback.

The control law is structured as

$$\Delta(\hat{z}) = \begin{cases} \delta_0, & \hat{z}(x) \in \mathcal{L}_\sigma, \\ \Lambda(\mathbf{K}_0 + \mathbf{B}_{\mathbf{e}_1} \hat{z}(x)), & \text{otherwise,} \end{cases} \quad (6a)$$

$$\Gamma(\hat{z}) = \Delta(\hat{z}) - \Lambda + \mathbf{B}_{\mathbf{e}_1} \hat{z}(x), \quad (6b)$$

$$\varrho(t) = c_r \|\Lambda\| \|\hat{z}(x)\| + \xi \mathbf{e}^{s^2}, \quad (6c)$$

with

$$\lambda(t) = (\|\Lambda A\| + c_r \|\Lambda\|) \|\hat{z}(x)\| + \xi \mathbf{e}^{s^2}, \quad (7)$$

and design constants $\eta, \rho, \xi, \delta_0 > 0$. The parameters Λ, δ_0 , and σ satisfy

$$|\Lambda \mathbf{K}_0| > \frac{\sigma}{2}, \quad |\Lambda \mathbf{K}_0 - \delta_0| > \frac{\sigma}{2}, \quad (8)$$

ensuring nonsingular control behavior.

The control operates in two regimes: i) Outside the boundary layer: $\hat{z}(x) \notin \mathcal{L}_\sigma$. The singularity plane does not

influence the system. Active branches of u_a , u_s , and u_{eq} drive $\hat{z}(x)$ toward $s(t) = 0$ and maintain it there. ii) Inside the boundary layer: $\hat{z}(x) \in \mathcal{L}_\sigma$. Two cases arise: i) If $s(t) \neq 0$, u_a ensures reaching $s(t) = 0$, while u_s and u_{eq} are zero. ii) If $s(t) = 0$, u_{eq} is zero, and u_a and u_s maintain the manifold and ensure asymptotic stability. Singularity is avoided as $\Delta(\hat{z})$ remains finite.

Inequalities (8) guarantee the plane $\Lambda \mathbf{B}_{\mathbf{e}_1} \hat{z}(x) = 0$ does not intersect \mathcal{L}_σ and $\Gamma(\hat{z})$ remains invertible.

V. SLIDING MODE CONTROL

Classical SMC [40] offers three main advantages: reduction of dynamics to a lower-order manifold, finite-time convergence to the sliding surface, and robustness to matched uncertainties. However, a well-known drawback is the possible singularity when the input gain vanishes [41]. Define the sliding variable:

$$s(t) = \Lambda \hat{z}(x) - \int_0^t \Lambda \mathbf{K}_0 u_a(\tau) d\tau, \quad (9)$$

which corresponds to the integral sliding mode concept [42]. On the sliding manifold $s(t) = 0$, $\dot{s}(t) = 0$ implies

$$\Lambda \dot{\hat{z}}(x) - \Lambda \mathbf{K}_0 u_a(t) = 0.$$

Substituting the system dynamics (2) and the composite control $\mu(t) = u_a(t) + u_s(t) + u_{\text{eq}}(t)$ gives

$$\Lambda A \hat{z}(x) + \Lambda \mathbf{B}_{\mathbf{e}_1} \hat{z}(x) u_a(t) + \Lambda(\mathbf{K}_0 + \mathbf{B}_{\mathbf{e}_1} \hat{z}(x)) u_s(t) = 0.$$

Solving for the equivalent control produces the singular term

$$u_{\text{eq}}(t) = -\frac{\Lambda A \hat{z}(x) + \Lambda \mathbf{B}_{\mathbf{e}_1} \hat{z}(x) u_a(t)}{\Lambda(\mathbf{K}_0 + \mathbf{B}_{\mathbf{e}_1} \hat{z}(x))}, \quad (10)$$

while the standard discontinuous sliding law is

$$u_s(t) = -\frac{\varrho(t)}{\Lambda(\mathbf{K}_0 + \mathbf{B}_{\mathbf{e}_1} \hat{z}(x))} \text{sgn}(s(t)), \quad (11)$$

with $\varrho(t)$ given in (6c). Both expressions become unbounded on the singularity hyperplane \mathcal{S} .

To regularize the control law, the boundary layer \mathcal{L}_σ is introduced. The regularized control components are:

$$u_{\text{eq}}(t) = \begin{cases} 0, & \text{if } \hat{z}(x) \in \mathcal{L}_\sigma \text{ and } s(t) \neq 0, \\ -\Gamma(\hat{z})^{-1} \left(\Lambda A \hat{z}(x) + \Lambda \mathbf{B}_{\mathbf{e}_1} \hat{z}(x) u_a(t) \right), & \text{otherwise,} \end{cases}$$

$$u_s(t) = \begin{cases} 0, & \text{if } \hat{z}(x) \in \mathcal{L}_\sigma, \\ -\Gamma(\hat{z})^{-1} \lambda(t) \text{sgn}(s(t)), & \text{otherwise,} \end{cases}$$

$$u_a(t) = \begin{cases} -\frac{\lambda(t)}{\Lambda \mathbf{B}_{\mathbf{e}_1} \hat{z}(x)} \text{sgn}(s(t)), & \text{if } \hat{z}(x) \in \mathcal{L}_\sigma \text{ and } s(t) \neq 0, \\ \Gamma(\hat{z})^{-1} \left[\Lambda A \hat{z}(x) - \eta \Delta(\hat{z})(\mathbf{K}_0 + \mathbf{B}_{\mathbf{e}_1} \hat{z}(x))^T \hat{Q} \hat{z}(x) \right]. & \end{cases} \quad (12)$$

This design guarantees bounded control near \mathcal{S} , ensures finite-time convergence of $s(t)$ to the sliding manifold, and preserves asymptotic stability of the closed-loop system. A formal stability analysis is presented in the next section.

VI. STABILITY ANALYSIS

This section establishes the fundamental properties of the proposed control scheme: fixed-time reachability of the sliding manifold and exponential stability of the closed-loop system.

A. Fixed-time stability of the sliding manifold

Theorem 1: Under \mathcal{A}_1 , \mathcal{A}_2 and \mathcal{A}_3 , consider the bilinear system (2) subject to the matched disturbance bound $\|\hat{\gamma}(\hat{z}, t)\| \leq k\|\hat{z}(x)\|$. Under the control law defined by Definitions 2–3 and equations (6a)–(7), the sliding manifold $s(t) = 0$, defined in (12), is reached in fixed time, $T(s(0))$, bounded by

$$T(s(0)) \leq T_{\max} = \frac{2}{\xi}, \quad (13)$$

where $\xi > 0$ is the design parameter from (7).

Proof: The proof proceeds by analyzing the Lyapunov function candidate $V(t) = \frac{1}{2}s(t)^2$ and considering the two operational regimes of the controller.

Taking the time derivative of $V(t)$ and substituting the dynamics (2) and the switching function derivative yields:

$$\begin{aligned} \dot{V}(t) = s(t)\dot{s}(t) = s(t) \left\{ \Lambda A \hat{z}(x) + \Lambda \mathbf{B}_{e_1} \hat{z}(x) u_a(t) \right. \\ \left. + \Lambda (\mathbf{K}_0 + \mathbf{B}_{e_1} \hat{z}(x)) \mu(t) + \Lambda \hat{\gamma}(\hat{z}, t) \right\}. \end{aligned}$$

Case 1: $\hat{z}(x) \notin \mathcal{L}_\sigma$ (Nominal SMC operation).

Substituting the control components $u_s(t)$, $u_{eq}(t)$, and the definition of $\Gamma(\hat{z})$ leads to:

$$\dot{V}(t) \leq -\varrho(t)|s(t)| + c_r \|\Lambda\| \|\hat{z}(x)\| |s(t)|. \quad (14)$$

Invoking the definition of $\varrho(t)$ from (7), $\varrho(t) = c_r \|\Lambda\| \|\hat{z}(x)\| + \xi e^{s^2}$, we obtain:

$$\dot{V}(t) \leq -\xi e^{s^2} |s(t)| < 0 \quad \text{for all } s(t) \neq 0. \quad (15)$$

Case 2: $\hat{z}(x) \in \mathcal{L}_\sigma$ (Boundary layer operation).

In this regime, $u_s(t) = u_{eq}(t) = 0$. Substituting the auxiliary control $u_a(t)$ yields:

$$\dot{V}(t) = s(t) \{ \Lambda A \hat{z}(x) + \Lambda \mathbf{B}_{e_1} \hat{z}(x) u_a(t) + \Lambda \hat{\gamma}(\hat{z}, t) \}.$$

Substituting $u_a(t)$ inside the boundary layer and using the bound on $\varrho(t)$ yields

$$\dot{V}(t) \leq -\xi e^{s^2} |s(t)| < 0, \quad s(t) \neq 0.$$

This guarantees that the sliding manifold $s(t) = 0$ is reached in fixed time.

The global evolution of the Lyapunov function is governed by a differential inequality whose structure depends on the value of $V(t)$:

$$\dot{V}(t) \leq \begin{cases} -2^{3/2} \xi V(t)^{3/2}, & \text{if } V(t) \geq 1/2, \\ -2^{1/2} \xi V(t)^{1/2}, & \text{if } V(t) < 1/2. \end{cases} \quad (16)$$

This is derived from the following analysis:

- **Case 1:** ($V(t) \geq 1/2 \implies |s(t)| \geq 1$). In this case, $e^{s^2} |s(t)| \geq |s(t)|^3$. Therefore,

$$\begin{aligned} \dot{V}(t) &\leq -\xi e^{s^2} |s(t)| \leq -\xi |s(t)|^3 \\ &= -\xi (2V(t))^{3/2} = -2^{3/2} \xi V(t)^{3/2}. \end{aligned}$$

- **Case 2:** ($V(t) < 1/2 \implies |s(t)| < 1$). In this case, $e^{s^2} |s(t)| \geq |s(t)|$. Therefore,

$$\begin{aligned} \dot{V}(t) &\leq -\xi e^{s^2} |s(t)| \leq -\xi |s(t)| \\ &= -\xi (2V(t))^{1/2} = -2^{1/2} \xi V(t)^{1/2}. \end{aligned}$$

To compute the settling time T , we integrate the differential inequality (16) over two phases. Let T_1 be the time to go from $V(0)$ to $V = 1/2$, and T_2 the time to go from $V = 1/2$ to $V = 0$. The total settling time is bounded by $T \leq T_1 + T_2$.

For the first phase ($V \geq 1/2$), we have $\dot{V} \leq -2^{3/2} \xi V^{3/2}$. Separating variables and integrating:

$$\begin{aligned} \int_{V(0)}^{1/2} \frac{dV}{V^{3/2}} &\leq -2^{3/2} \xi \int_0^{T_1} dt, \\ -2 \left[(1/2)^{-1/2} - V(0)^{-1/2} \right] &\leq -2^{3/2} \xi T_1. \end{aligned}$$

Since $V(0)^{-1/2} \geq 0$, this yields $T_1 \leq \frac{2}{2^{3/2} \xi} (1/2)^{-1/2} = \frac{1}{2^{1/2} \xi}$.

For the second phase ($V < 1/2$), we have $\dot{V} \leq -2^{1/2} \xi V^{1/2}$. Integrating:

$$\begin{aligned} \int_{1/2}^0 \frac{dV}{V^{1/2}} &\leq -2^{1/2} \xi \int_0^{T_2} dt, \\ -2 \left[0 - (1/2)^{1/2} \right] &\leq -2^{1/2} \xi T_2. \end{aligned}$$

Thus, $T_2 \leq \frac{2(1/2)^{1/2}}{2^{1/2} \xi} = \frac{1}{2^{1/2} \xi}$.

The total settling time is bounded by $T \leq T_1 + T_2 \leq \frac{1}{2^{1/2} \xi} + \frac{1}{2^{1/2} \xi} = \frac{2}{\xi}$, which is independent of the initial condition $s(0)$. ■

Remark 1: The fixed-time stability property ensures that the sliding manifold is reached not only in finite time but within a bounded duration $T_{\max} = 2/\xi$, regardless of the initial state $s(0)$. This is a stronger robustness property compared to conventional finite-time stability, as it provides a guaranteed upper bound on the convergence time [43]. ▽

Remark 2: In this paper, we omit the formal definitions of finite-time and fixed-time stability, as our focus is on presenting a data-driven control design based on the Koopman operator. For detailed discussions of these stability properties, the reader is referred to [43] and the references therein. ▽

B. Exponential stability of the closed-loop system

Theorem 2 (Exponential Stability): Under \mathcal{A}_1 , \mathcal{A}_2 and \mathcal{A}_3 and the conditions of Theorem 1, the closed-loop system is exponentially stable. There exist constants $\kappa > 0$ and $\beta > 0$ such that

$$\|\hat{z}(x)\| \leq \kappa \|\hat{z}(0)\| e^{-\beta t}, \quad t \geq 0, \quad (17)$$

with decay rate

$$\beta \geq \frac{\lambda_{\min}(\hat{R}) - 2c_r \|\hat{Q}\|}{2\lambda_{\max}(\hat{Q})}. \quad (18)$$

Proof: Under sliding mode conditions ($s(t) \equiv 0$), the control components simplify to $u_s(t) = 0$ and $u_{eq}(t) = 0$. The dynamics are governed solely by the auxiliary control $u_a(t)$. The closed-loop dynamics are given by:

$$\begin{aligned} \dot{\hat{z}}(x) &= A \hat{z}(x) - \eta (\mathbf{K}_0 + \mathbf{B}_{e_1} \hat{z}(x)) (\mathbf{K}_0 \\ &\quad + \mathbf{B}_{e_1} \hat{z}(x))^\top \hat{Q} \hat{z}(x) + \hat{\gamma}(\hat{z}, t). \end{aligned}$$

Consider the quadratic Lyapunov function candidate:

$$W(t) = \hat{z}(x)^\top \hat{Q} \hat{z}(x). \quad (19)$$

Taking its time derivative along the trajectories of the closed-loop system yields:

$$\begin{aligned} \dot{W}(t) &= 2\hat{z}(x)^\top \hat{Q} \dot{\hat{z}}(x) = 2\hat{z}(x)^\top \hat{Q} \\ &\quad \left[A \hat{z}(x) - \eta (\mathbf{K}_0 + \mathbf{B}_{e_1} \hat{z}(x)) (\mathbf{K}_0 + \mathbf{B}_{e_1} \hat{z}(x))^\top \hat{Q} \hat{z}(x) \right. \\ &\quad \left. + \hat{\gamma}(\hat{z}, t) \right] = \hat{z}(x)^\top (\hat{Q} A + A^\top \hat{Q}) \hat{z}(x) + 2\hat{z}(x)^\top \hat{Q} \hat{\gamma}(\hat{z}, t) \\ &\quad - 2\eta \hat{z}(x)^\top \hat{Q} (\mathbf{K}_0 + \mathbf{B}_{e_1} \hat{z}(x)) (\mathbf{K}_0 + \mathbf{B}_{e_1} \hat{z}(x))^\top \hat{Q} \hat{z}(x). \end{aligned}$$

Substituting the Lyapunov equation $\hat{Q}A + A^\top \hat{Q} = -\hat{R}$ and noting that the second term is a negative semi-definite quadratic form, we obtain:

$$\begin{aligned} \dot{W}(t) &= -\hat{z}(x)^\top \hat{R} \hat{z}(x) \\ &\quad - 2\eta \|(\mathbf{K}_0 + \mathbf{B}_{e_1} \hat{z}(x))^\top \hat{Q} \hat{z}(x)\|^2 + 2\hat{z}(x)^\top \hat{Q} \hat{\gamma}(\hat{z}, t). \end{aligned}$$

We now bound the indefinite disturbance term. Applying the Cauchy-Schwarz inequality and the given disturbance bound $\|\hat{\gamma}(\hat{z}, t)\| \leq c_r \|\hat{z}(x)\|$ yields:

$$\begin{aligned} 2\hat{z}(x)^\top \hat{Q} \hat{\gamma}(\hat{z}, t) &\leq 2\|\hat{Q}^{1/2} \hat{z}(x)\| \cdot \|\hat{Q}^{1/2} \hat{\gamma}(\hat{z}, t)\| \\ &\leq 2\|\hat{Q}^{1/2}\| \cdot \|\hat{z}(x)\| \cdot \|\hat{Q}^{1/2}\| \cdot \|\hat{\gamma}(\hat{z}, t)\| \\ &\leq 2\|\hat{Q}\| \cdot \|\hat{z}(x)\| \cdot c_r \|\hat{z}(x)\| = 2c_r \|\hat{Q}\| \|\hat{z}(x)\|^2. \end{aligned}$$

Furthermore, the first term is bounded by its quadratic form:

$$-\hat{z}(x)^\top \hat{R} \hat{z}(x) \leq -\lambda_{\min}(\hat{R}) \|\hat{z}(x)\|^2. \quad (20)$$

Since the second term, $-2\eta \|(\mathbf{K}_0 + \mathbf{B}_{e_1} \hat{z}(x))^\top \hat{Q} \hat{z}(x)\|^2$, is always non-positive, it can be dropped from the inequality. Combining these results, we establish the following upper bound for $\dot{W}(t)$:

$$\dot{W}(t) \leq -\left(\lambda_{\min}(\hat{R}) - 2c_r \|\hat{Q}\|\right) \|\hat{z}(x)\|^2. \quad (21)$$

Follows directly from the Lyapunov inequality $\dot{W}(t) \leq -\alpha \|\hat{z}(x)\|^2$ where $\alpha = \lambda_{\min}(\hat{R}) - 2c_r \|\hat{Q}\| > 0$, and the fact that $\lambda_{\min}(\hat{Q}) \|\hat{z}\|^2 \leq W(t) \leq \lambda_{\max}(\hat{Q}) \|\hat{z}\|^2$. Therefore, if the condition $c_r < \lambda_{\min}(\hat{R}) / (2\|\hat{Q}\|)$ is satisfied, then $\dot{W}(t) < 0$ for all $\hat{z}(x) \neq 0$, proving exponential stability. ■

Remark 3: When the lifting map is chosen such that the original state is included as part of the lifted coordinates, i.e., $\hat{z} = \psi(x) = \begin{bmatrix} x \\ \phi(x) \end{bmatrix}$, we have $\hat{z}_k \rightarrow 0$ implies $x_k \rightarrow 0$. Additionally, the exponential stability guarantee provides a stronger characterization of system behavior than asymptotic stability alone, ensuring a minimum rate of convergence $\beta > 0$ that is independent of initial conditions. This result follows directly from the quadratic form of the Lyapunov function and the linear dissipation inequality. ▽

VII. GENERAL SMC LAWS REDUCING CONSERVATISM

We extend the previous scalar-input controller to general input dimensions $m \geq 1$, introducing a multi-input sliding mode control law that reduces conservatism. Consider the lifted dynamics

$$\dot{\hat{z}}(x) = A\hat{z}(x) + \mathbf{K}_0 u + \mathbf{B}_{e_1}(u \otimes \hat{z}(x)) + \hat{\gamma}(t, x), \quad (22)$$

where $\hat{z} \in \mathbb{R}^n$ is the lifted state, $u \in \mathbb{R}^m$ is the control input, $A \in \mathbb{R}^{n \times n}$, $\mathbf{K}_0 \in \mathbb{R}^{n \times m}$, $\mathbf{B}_{e_1} \in \mathbb{R}^{n \times nm}$ denotes the bilinear operator in Kronecker form, and $\hat{\gamma}(t, x)$ is the approximation remainder satisfying a finite-gain bound. The lifted state is constrained to $\hat{z} \in \mathbf{D}_z \subset \mathbb{R}^n$, and $\hat{\gamma}(t, x)$ satisfies $\|\hat{\gamma}(t, x)\| \leq \bar{\gamma}$ for all $x \in \mathbb{X}$ and $u \in \mathbb{U}$.

A. Multi-input SMC: robust bilinear formulation

For $m > 1$, the singularity hyperplane is

$$\mathcal{S} := \left\{ \hat{z} \in \mathbb{R}^n \mid \det(\Lambda(\mathbf{K}_0 + \mathbf{B}_{e_1}(\mathbb{I}_m \otimes \hat{z}))) = 0 \right\}, \quad (23)$$

with $\Lambda \in \mathbb{R}^{m \times n}$ a design matrix ensuring nonsingularity. Define the adaptive boundary layer

$$\mathcal{L}_\sigma := \left\{ \hat{z} \in \mathbb{R}^n \mid \|\Lambda(\mathbf{K}_0 + \mathbf{B}_{e_1}(\mathbb{I}_m \otimes \hat{z}))\|_F < \frac{\sigma}{2} \right\}. \quad (24)$$

The multi-input augmented sliding mode control law is

$$\mu(t) = u_s(t) + u_{eq}(t) + u_a(t), \quad (25)$$

with $u_s(t), u_{eq}(t), u_a(t) \in \mathbb{R}^m$, and the vector sliding variable $s \in \mathbb{R}^m$.

$$\Delta(\hat{z}) = \begin{cases} \delta_0 \mathbb{I}_m, & \hat{z} \in \mathcal{L}_\sigma, \\ \Lambda(\mathbf{K}_0 + \mathbf{B}_{e_1}(\mathbb{I}_m \otimes \hat{z})), & \text{otherwise,} \end{cases} \quad (26)$$

$$\Gamma(\hat{z}) = \Delta(\hat{z}) - \Lambda \mathbf{B}_{e_1}(\mathbb{I}_m \otimes \hat{z}), \quad (27)$$

$$\varrho(t) = c_r \|\Lambda\|_F \|\hat{z}\| + \xi e^{s^\top s}, \quad (28)$$

with design constants $\eta, \rho, \xi, \delta_0 > 0$ chosen to satisfy

$$\|\Lambda \mathbf{K}_0\|_F > \frac{\sigma}{2}, \quad \|\Lambda \mathbf{K}_0 - \delta_0 \mathbb{I}_m\|_F > \frac{\sigma}{2}.$$

Remark 4: i) The multi-input control law generalizes the scalar-input design while reducing conservatism and improving scalability; ii) The law guarantees boundedness of u_s, u_{eq} , and u_a near the singularity hyperplane \mathcal{S} ; iii) Guidelines for the general solution are presented; the proof of closed-loop exponential stability is omitted due to page limitations. The proof of sliding mode convergence is identical to the scalar-input case. ▽

VIII. CONCLUSIONS

In this paper, we presented a robust SMC for bilinear systems using finite-dimensional Koopman approximations. The proposed design ensures fixed-time convergence of the sliding variables and exponential stability of the closed-loop system, while mitigating singularities inherent to classical SMC. The framework extends naturally to multi-input systems and remains compatible with data-driven Koopman models, providing a scalable and practical control solution.

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