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Generalized backstepping and Gramian methods for spectral assignability in infinite dimension

Ludovick Gagnon* and Christophe Zhang†

February 27, 2026

Abstract

In this article, we explore connections between two stabilization methods: Gramian stabilization and backstepping (and more generally F -equivalence). These methods are related to the problem of spectral assignability (also known as the pole placement or pole shifting problem), which consists in finding a linear feedback such that the resulting closed-loop system has a prescribed spectrum. We provide a constructive solution to the spectral assignability problem for a broad class of infinite-dimensional controllable systems and target spectra, with unbounded but admissible control operators. We construct a feedback using a generalization of the backstepping method, and provide an extensive study of the resulting closed-loop operator. We prove that it has a Riesz basis of eigenvectors, which we explicitly describe, along with its biorthonormal basis. This allows us to introduce general Gramian operators, and give an alternative expression of the feedback using these operators.

Keywords: Rapid stabilization, control, backstepping method, Gramian stabilization, Gramian operators, spectral assignability

2020 MSC: 93C20, 93D15, 93B17.

Contents

1	Introduction	2
1.1	Main assumptions	3
1.2	Main results	5
1.2.1	Fredholm backstepping and stabilization Gramian	5
1.2.2	Spectral assignability and eigenfunctions	6
1.2.3	Comments on the main results	7
1.3	Structure of the paper	7
2	Overview of closely related stabilization methods	7
2.1	The backstepping method	7
2.2	The stabilization Gramian	9
2.3	Riccati equations	10
2.4	F -equivalence	11
2.5	Sylvester equations and generalized Gramians	12
2.6	Spectral assignability or pole shifting	13

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3	Functional setting	14
3.1	Nested spaces	14
3.2	Extensions of A	14
3.3	Admissible feedbacks and closed-loop operators	14
3.4	Notations	16
4	F-equivalence for a broad class of target systems	16
4.1	The eigenfunctions of the target operator	16
4.2	An existence result	17
4.3	Solving the F -equivalence problem	20
4.3.1	The transformation T	20
4.3.2	The target system	20
4.3.3	The equivalence equation and the uniqueness condition	21
5	Complete solution of the spectral assignability problem: eigenvectors and uniqueness properties	24
5.1	Properties of the $\{\chi_n\}$ and related families	24
5.2	Uniqueness properties	26
6	Generalized Gramian operators	28
6.1	A poleshifting Gramian operator	28
6.2	An integral operator formulation	30
7	Perspectives	32
A	Dealing with the case $\lambda_n = \tilde{\lambda}_n$	33
B	Definition and properties of T and K	34
B.1	Technical lemmas on the spectra	34
B.2	Definition of the F -equivalence transformation T	36
B.3	Properties of the feedback K	39
C	Riesz bases in Hilbert spaces	45

1 Introduction

This article studies the closed-loop control of autonomous control systems on an infinite dimensional Hilbert space H with scalar controls. For $T > 0$, given an open-loop system:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t \in (0, T) \tag{1}$$

with $x(t) \in H$, $u \in L^2((0, T); \mathbb{C})$, $A : D(A) \rightarrow H$, and $B : \mathbb{C} \rightarrow D(A^*)'$, our aim is to find a linear *feedback law* K , such that the resulting closed-loop system with control $u(t) := Kx(t)$

$$\dot{x}(t) = (A + BK)x(t), \quad t \in (0, T) \tag{2}$$

has prescribed properties. A typical goal is for the closed-loop system to be exponentially stable, *i.e.*, to find constants $C, \lambda > 0$ such that, for any initial condition $x(0) \in H$,

$$\|x(t)\|_H \leq Ce^{-\lambda t} \|x(0)\|_H, \quad t \in (0, T).$$

Another important and related problem is spectral assignability, or poleshifting. Given the control system (1) and prescribed eigenvalues, the goal is to find a feedback K such that the spectrum of the closed-loop operator $A + BK$ is equal to these eigenvalues. The spectral assignability is thus more general than the

exponential stabilization, the latter being achieved by prescribing the eigenvalues in the left-half plane of $-\lambda$.

A central focus of this article will be around two well-known stabilization methods to achieve the rapid stabilization of (2). The first is the **generalized backstepping method**, which consists in finding a feedback K such that the resulting closed-loop system can be invertibly mapped to a stable *target system*. This target system is chosen beforehand for the stability properties of its solutions. The main difficulty with the backstepping method is to prove by hand the existence of this invertible mapping. This method has undergone very active developments, with now necessary or sufficient conditions for the existence of Fredholm transformations [22, 24]. One of the main results of the present article (Theorem 1) is to prove that the stabilizing feedback K from the Fredholm backstepping method is the same than the one given by the **Gramian stabilization** [27, 43]. Our analysis carries further, with connections to optimal control and the Riccati equations through the stabilization Gramian. Indeed, in Theorem 2, the generalization of Theorem 1, we prove the F -equivalence between the control system (1) and another with prescribed eigenvalues. This provides a feedback and solves the corresponding **spectral assignability problem**. An in-depth study of this problem leads us to naturally define **generalized Gramian operators**. These operators are commonly used for model reduction, and sometimes stabilization. In our framework, they appear as crucial objects for spectral assignability problems, and yield another expression of this feedback, analogous to the well-known Gramian feedbacks used in stabilization.

Generalized backstepping and Gramian stabilization are both well-known and thoroughly explored stabilization methods. Our work uncovers profound connections between the two, through their relation to the spectral assignability problem. This raises perspectives on further generalizations of both methods. A quick review on the different stabilization problems or methods mentioned above is found in Section 2.

1.1 Main assumptions

Unless otherwise specified, we will make the following assumptions throughout this article, first on the operators A and B :

Assumption 1.1. *Let H be a complex Hilbert space. Denote by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ its scalar product and norm respectively.*

(H1) $A : D(A) \subset H \rightarrow H$ is an unbounded operator, generator of the C_0 -semigroup e^{tA} over the state space H , with dense domain $D(A)$ in H ;

(H2) $B : \mathbb{C} \rightarrow D(A^*)'$ is unbounded on H and admissible (see for instance [42]): there exists $C > 0$ such that for every $t > 0$,

$$\int_0^t |B^* e^{sA^*} u_0| ds \leq C \|u_0\|, \quad \forall u_0 \in D(A^*);$$

(H3) The operator A has a compact resolvent and is diagonalizable: there exists an orthonormal basis of H of eigenfunctions $\{\varphi_n\}_{n \in \mathbb{Z}}$ satisfying $A\varphi_n = \lambda_n \varphi_n, \forall n \in \mathbb{Z}$. We also assume that the eigenvalues $\{\lambda_n\}_{n \in \mathbb{Z}}$ are simple;

(H4) Superlinear growth of the eigenvalues : there exists $\alpha > 1$ and $0 < c < C$ such that,

$$c(|n| + 1)^{\alpha-1} \leq |\lambda_{n+1} - \lambda_n| \leq C(|n| + 1)^{\alpha-1}, \quad \forall n \in \mathbb{Z}, \quad (3)$$

and the gap estimate

$$c \max\{|k|^{\alpha-1}, |n|^{\alpha-1}\} |k - n| \leq |\lambda_k - \lambda_n|, \quad \forall k, n \in \mathbb{Z}; \quad (4)$$

(H5) There exist $c, C > 0$, independent of $n \in \mathbb{Z}$, such that

$$c \leq |B^* \varphi_n| \leq C. \quad (5)$$

We make an additional assumption on the target spectrum $\{\tilde{\lambda}_n\}$ for the spectral assignability problem:

Assumption 1.2. For $n \neq m \in \mathbb{Z}$, $\tilde{\lambda}_n \neq \tilde{\lambda}_m$. Moreover, there exists $\alpha > 1$ and $C > c > 0$ such that,

$$\begin{aligned} c(|n| + 1)^{\alpha-1} &\leq |\tilde{\lambda}_{n+1} - \lambda_n| \leq C(|n| + 1)^{\alpha-1}, \quad \forall n \in \mathbb{Z}, \\ c(|n| + 1)^{\alpha-1} &\leq |\tilde{\lambda}_{n+1} - \tilde{\lambda}_n| \leq C(|n| + 1)^{\alpha-1}, \quad \forall n \in \mathbb{Z}, \\ c &\leq |\tilde{\lambda}_n - \lambda_n| \leq C, \quad \forall n \in \mathbb{Z}. \end{aligned} \tag{6}$$

Moreover, we have the gap estimates

$$\begin{aligned} c \max\{|k|^{\alpha-1}, |n|^{\alpha-1}\} |k - n| &\leq |\tilde{\lambda}_k - \lambda_n|, \quad \forall k, n \in \mathbb{Z}, \\ c \max\{|k|^{\alpha-1}, |n|^{\alpha-1}\} |k - n| &\leq |\tilde{\lambda}_k - \tilde{\lambda}_n|, \quad \forall k, n \in \mathbb{Z}, \end{aligned} \tag{7}$$

Finally, we formulate additional assumptions on A and \tilde{A} , which will be required to give analytical expressions of the generalized Gramian operators. We first define the following spectral bounds:

$$\omega_+ := \sup_n \operatorname{Re}(\lambda_n), \quad \omega_- := \inf_n \operatorname{Re}(\lambda_n).$$

Recall that, since A generates a C_0 -semigroup by Assumption 1.1, $\omega_+ < \infty$. Symmetrically, in some instances, we will require $-A$ to generate C_0 -semigroup (see Proposition 6.3). This motivates our first additional assumption:

Assumption 1.3 (well-posedness of the backward semigroup).

$$\omega_- > -\infty. \tag{8}$$

In order to give analytical expressions of the generalized Gramian operators, we will require the following assumptions:

Assumption 1.4 (Complete shift to the left).

$$\sup_n \operatorname{Re}(\tilde{\lambda}_n) < \omega_-. \tag{9}$$

Assumption 1.5 (Complete shift to the right).

$$\inf_n \operatorname{Re}(\tilde{\lambda}_n) > \omega_+. \tag{10}$$

In order to properly state our main results, we use the notion of admissible feedback law [36]. We refer to Section 3.3 for properties of admissible feedback laws under Hypothesis 1.1 and 1.2.

Definition 1.1. Define the set of admissible feedback laws

$$\mathbb{K}_{\text{ad}} := \left\{ K : D(K) \subset H \rightarrow \mathbb{C} \mid \forall T > 0, \forall \phi \in D(A^*), Ke^{tA^*} \phi \in L^2((0, T); \mathbb{C}) \right\}.$$

Several remarks are in order.

Remark 1.1. • The first two assumptions of Assumption 1.1 are classical for the well-posedness of the control system 1. The Hille-Yosida theorem ensures that any densely defined operator A satisfying (H3) and (H4) generates a C_0 -semigroup. For the well-posedness of the closed-loop system (2), we use the notion of admissible feedbacks;

- Assumption (H5) is essentially the Hautus test for skew-adjoint operators [42], combined with the admissibility of B , to establish the controllability assumption of the control system. This assumption is classical in the Fredholm backstepping framework;

- Assumption (H4) excludes hyperbolic systems, as their treatment is very specific. In our proof, assumption (H4) is a cornerstone in establishing the existence and properties of the transformation T and the feedback K ;
- We underline that the assumption $\lambda_n \neq \tilde{\lambda}_n$ is not restrictive nor necessary. It is simply for convenience in this paper. Indeed, we prove in Lemma A.1 that the case $\lambda_n = \tilde{\lambda}_n$ is handled separately by setting $K\varphi_n = 0$ and $T\varphi_n = \varphi_n + C\varphi_n$ where $C : H \rightarrow H$ is a compact operator. The invertibility of T remains unchanged;
- In the literature around the Fredholm backstepping method, assumption (H3) may be relaxed to finite multiplicity for 1-dimensional PDEs (see for instance [22, 24]). We choose here to assume simple eigenvalues for 1-dimensional PDEs to simplify the presentation, but one can easily adapt the functional setting of [22, 24] to deal with finite and bounded multiplicity of the eigenvalues. This is discussed further in Section 7;
- Assumption 1.2 means that the spectra $\{\lambda_n\}_{n \in \mathbb{Z}}$ and $\{\tilde{\lambda}_n\}_{n \in \mathbb{Z}}$ have similar asymptotic behaviors. It includes the typical case of a uniform damping,

$$\tilde{\lambda}_n = \lambda_n - \lambda, \quad n \in \mathbb{Z},$$

as well as more involved target systems, such as the one used to stabilize the linearized water tank system [13].

1.2 Main results

To present our main results precisely, we introduce the space $H_s, s \geq 0$ associated to the operator A . Taking $\beta \in \rho(A)$, we define the following norms for $s \geq 0$:

$$\|u\|_s^2 = \sum_{n \in \mathbb{Z}} |\lambda_n - \beta|^{2s} |\langle u, \varphi_n \rangle|^2, \quad \forall u \in H,$$

Thus $H_s := \{u \in H \mid \|u\|_s < +\infty\} \subset H$.

Our main results fall into two category.

1.2.1 Fredholm backstepping and stabilization Gramian

We begin stating our first main result, which brings together two well-established stabilization methods. The most striking result of our article is to prove that the backstepping method and the Gramian stabilization method, yield the same feedback:

Theorem 1. *Let $A : D(A) \subset H \rightarrow H$ be a diagonalizable skew-adjoint operator and $B \in \mathcal{L}(\mathbb{C}; D(A^*)')$ be an admissible control operator. Under Assumptions 1.1 on the spectral properties of A and the controllability of (1), for $\lambda > 0$, there exist an invertible transformation $T \in \mathcal{L}(H; H)$ and an admissible feedback $K \in H'_{1/2}$ such that*

$$\begin{cases} T(A + BK) = (A - 2\lambda I)T, & T \text{ invertible} \\ TB = B. \end{cases} \quad (11)$$

Moreover, this feedback K , obtained by the backstepping method, can also be written

$$K = B^*(C_\lambda^\infty)^{-1} \text{ in } H'_{1/2}$$

where the stabilization Gramian $C_\lambda^\infty \in \mathcal{L}(H; H)$ satisfies

$$C_\lambda^\infty(A + \lambda I)^* + (A + \lambda I)C_\lambda^\infty = BB^*.$$

Finally, there exists an isomorphism $\tau \in \mathcal{L}(H; H)$ such that

$$\tau A = A\tau, \quad T = \tau(C_\lambda^\infty)^{-1}.$$

The previous theorem is the consequence of the broader result on the complete solution of the spectral assignability problem for $\{\lambda_n\}_{n \in \mathbb{Z}}$ and $\{\tilde{\lambda}_n\}_{n \in \mathbb{Z}}$ satisfying Assumption 1.1 and Assumption 1.2.

Theorem 2. *Let $A : D(A) \subset H \rightarrow H$ and $B \in \mathcal{L}(\mathbb{C}; D(A^*))'$ satisfy Assumptions 1.1. Let $\{\tilde{\lambda}_n\}$ satisfy Assumption 1.2. Then, there exists an admissible feedback $K \in H'_{1/2}$ such that $A+BK$ has the eigenvalues $\{\tilde{\lambda}_n\}$.*

Moreover, defining the operator $\tilde{A} : D(A) \rightarrow H$ by

$$\tilde{A}\varphi_n = \tilde{\lambda}_n\varphi_n, \quad \forall n \in \mathbb{Z},$$

(A, B) and (\tilde{A}, B) are F -equivalent, i.e., there exists an invertible transformation $T \in \mathcal{L}(H; H)$ such that

$$\begin{cases} T(A + BK) = \tilde{A}T, \\ TB = B. \end{cases} \quad (12)$$

Under the additional assumptions (8) and (9), (resp. (10)), the feedback K can also be written

$$K = B^*\Gamma_\infty^{-1} \text{ in } H'_{1/2}. \quad (13)$$

where the generalized Gramian $\Gamma_\infty \in \mathcal{L}(H; H)$ is defined by

$$\Gamma_\infty = \int_0^\infty e^{-tA}BB^*e^{t\tilde{A}} dt, \quad \text{on } H_{1/2} \quad \left(\text{resp. } - \int_0^\infty e^{tA}BB^*e^{-t\tilde{A}} dt, \quad \text{on } H_{1/2} \right).$$

Finally, there exists an isomorphism $\tau \in \mathcal{L}(H; H)$ such that

$$\tau A = A\tau, \quad T = \tau(\Gamma_\infty)^{-1}.$$

It should be noted that the space $H_{1/2}$ appears naturally in this present context for the admissibility of K . Notice that these results could have been written in the wider array of spaces H_s for $s \in (\frac{1}{2\alpha}, 1 - \frac{1}{2\alpha})$. We shall not go further in this direction, and refer the interested reader to [22, 24].

1.2.2 Spectral assignability and eigenfunctions

To prove these theorems, we revisit the classical results on spectral assignability, and more particularly the results of Rebarber [36]. We consider the family of eigenvectors introduced in [36], and propose a different approach, not relying on cardinal functions, to prove that it is a Riesz basis (see Appendix C for the definition of Riesz basis) of eigenfunctions of the operator $A + BK$. This theorem gathers the results of Corollary 5.1 and Proposition 5.3 in Section 5.1.

Theorem 3. *Under Assumption 1.1 and 1.2, the family*

$$\chi_n := (A - \tilde{\lambda}_n I)^{-1}B \in H,$$

is a Riesz basis of eigenfunctions of $A + BK$ with eigenvalues $\{\tilde{\lambda}_n\}_{n \in \mathbb{Z}}$

$$(A + BK)\chi_n = \tilde{\lambda}_n\chi_n, \quad \text{in } H,$$

where K is given by Theorem 2. Moreover, denoting $b_n = B^\varphi_n \in \mathbb{C}$, the bi-orthogonal family of $\{\chi_n\}_{n \in \mathbb{Z}}$ is given by*

$$\theta_n := -\overline{b_n \tilde{k}_n} ((A - \tilde{\lambda}_n I)^{-1})^* K^*, \quad \forall n \in \mathbb{Z},$$

where $\{\tilde{k}_n\}_{n \in \mathbb{Z}}$ is the unique sequence $\{\tilde{k}_n\} \in \ell^\infty(\mathbb{Z}; \mathbb{C})$ such that

$$\sum_n \frac{b_n \tilde{k}_n}{\lambda_p - \tilde{\lambda}_n} = 1. \quad (14)$$

An important point of this article is to prove, under Assumption 1.1 and 1.2, that the $\{\chi_n\}$ form a Riesz basis. Consequently, $\{\varphi_n\}$ and $\{\chi_n\}$ forms two fundamental basis, which are both explicit and intrinsic to the control system (A, B) and the target spectrum $\{\tilde{\lambda}_n\}$. These basis also allow different expressions of the feedback. Indeed, from the F -equivalence point of view, K is “naturally” defined on the $\{\varphi_n\}$, whereas from the generalized Gramian point of view, it is naturally defined on $\{\chi_n\}$. Our proof allows to define K on both and thus provide several expressions for the feedback law K .

1.2.3 Comments on the main results

These results are based on several novel ideas:

- We use infinite-dimensional extensions of Cauchy matrices to prove the existence of the feedback K , thus extending the range of the generalized backstepping method and solving a wide array of F -equivalence problems;
- the profound connection between Gramian stabilization (and its generalization) and F -equivalence comes from their mutual relationship to the spectral assignability problem, and the fact that they solve the same equivalence equation.

Our work therefore brings some important insights on the stabilization of controllable systems:

- we give two different expressions of the poleshifting feedback. The Gramian-type expression (13) is more compact but involves the inverse of the generalized Gramian operator; whereas the F -equivalence approach yields the feedback as the solution of an infinite dimensional linear system, which opens perspectives of numerical approximations.
- with this work, we address the common question: what type of target system is valid for the backstepping method? Assumption 1.2 gives insight on the target spectrum that one can expect from the backstepping method. Notice however that the weak setting of (11) and (12) does not transpose easily to target operators \tilde{A} with different eigenfunctions than ϕ_n , as they might not span $D(A)$, introducing many difficulties to properly define (11) and (12).

1.3 Structure of the paper

We begin this paper with an overview of different related stabilization methods in Section 2, allowing to give insights into different developments in the literature and giving perspective to our main results. The functional framework and relevant notations are presented in Section 3.

In Section 4 we solve the F -equivalence problem (12), in Section 5 we study the family $\{\chi_n\}$ comprehensively. Finally in Section 6 we then introduce the generalized Gramian operators and link them to the transformation and the feedback obtained in the previous section.

2 Overview of closely related stabilization methods

In this section, we present the different stabilization methods that will be discussed throughout this paper.

2.1 The backstepping method

The backstepping method¹ consists in proving the existence of a feedback law $w = Ku$ and an invertible operator T mapping the solution u of the equation to be stabilized,

$$\begin{cases} u_t(t) = Au(t) + Bw(t), \\ u(0) = u_0, \end{cases} \quad (15)$$

¹Here and below, we refer to the backstepping method as the infinite dimensional backstepping method, as introduced in [3, 7]. We refer for instance to [29] for the backstepping method for finite dimensional systems, allowing to construct feedback laws in a recursive way.

where $A : D(A) \subset H \rightarrow H$ is the infinitesimal generator of a semigroup of contractions e^{tA} and $B \in \mathcal{L}(\mathbb{C}; D(A^*))'$, onto the solution z of an exponentially stable equation (thanks to a well-chosen λ), called the target system,

$$\begin{cases} z_t(t) = (A - \lambda I)z(t), \\ z(0) = z_0, \end{cases} \quad (16)$$

Here, u_0 and z_0 belong to the Hilbert space H . The choice of $A - \lambda I$ as a target system introduces straightforward internal damping into the equation.

Denote the growth [42] of the semigroup e^{tA} ,

$$\omega_0 := \inf_{t \in (0, \infty)} \frac{\log(\|e^{tA}\|_{\mathcal{L}(H)})}{t},$$

then (16) is exponentially stable,

$$\|z(t)\| \leq C e^{(\omega_0 - \lambda)t} \|z_0\|,$$

for $\lambda > \omega_0$. If T is invertible from H to itself, then the exponential decay of u solution to (15) is obtained choosing $z_0 = Tu_0$

$$\|u(t)\| = \|T^{-1}z(t)\| \leq C \|T^{-1}\|_{\mathcal{L}(H)} e^{(\omega_0 - \lambda)t} \|Tu_0\| \leq C \|T\|_{\mathcal{L}(H)} \|T^{-1}\|_{\mathcal{L}(H)} e^{(\omega_0 - \lambda)t} \|u_0\|. \quad (17)$$

Properties on T and K , and in particular the invertibility of T , are studied by first taking, at least formally, the time derivative of $z(t) = Tu(t)$, which gives,

$$T(A + BK) = (A - \lambda I)T \quad (18)$$

This operator equality is the equivalent of the PDE on the kernel (more precisely, see Remark 2.1 below). The goal hence is to prove the existence isomorphism-operator pair (T, K) solving the previous equation.

Historically, the backstepping method for PDEs consists in writing T as a Volterra transformation of the second kind [2, 6]. Assuming that the equation to stabilize is defined on a one-dimensional spatial domain $\Omega = (0, 1)$

$$Tu(x) = u(x) - \int_0^x k(x, y)u(y)dy, \quad \forall x \in \Omega. \quad (19)$$

After integration by parts, equation (18) translates to a PDE on the kernel k . Properties of T are thus read off the properties of k . Moreover, the transformation T defined by (19) is particularly well-adapted for boundary controls at $x = 1$, and may also be used for internal controls [41, 46].

Fredholm alternative. More recently, the Fredholm-type transformation

$$z(t, \cdot) = Tu(t, \cdot) = (Id + \mathcal{T}_{comp})u(t, \cdot),$$

was introduced by Coron and Lü for the rapid stabilization of the Korteweg-de Vries equation [18] and Kuramoto-Sivashinsky equation [19] by means of the backstepping method. In this transformation, $T : H \rightarrow H$ is a Fredholm operator with an invertible and a compact part. Although much more technical, the Fredholm transformation provides a systematic approach to the backstepping method based on the spectral properties of the operator A and the controllability properties of B to prove the rapid stabilization for a large class of equations.

This extension of backstepping for Volterra transformation to more general invertible transformations has led to the backstepping stabilization of systems with an internal scalar control (see for instance [13, 17, 22, 24, 47]). In these works, the invertibility of the transformation T relies on the Fredholm alternative theory as well as controllability assumptions. For skew-adjoint operators A , the controllability assumption already points to a possible connection with the Gramian operators, which are invertible if and only if the system is controllable. Moreover, these successive works have paved the way to a more

comprehensive understanding of the backstepping method, in particular regarding the resulting closed-loop system and its spectral elements [13, 22, 24].

The Fredholm alternative is usually characterized by imposing to (18) the uniqueness condition $TB = B$, thereby resolving the degree of freedom in (18) and turning it into a system of two equations. It therefore reads,

$$TA + BK = (A - \lambda I)T, \quad \text{in } D(A^*)', \quad (20)$$

$$TB = B, \quad \text{in } D(A^*)'. \quad (21)$$

The uniqueness condition $TB = B$ was first introduced in [11] to prove the existence of (T, K) solving (20)-(21) in finite dimension, used implicitly in [14, 15] and finally stated explicitly in [12] to remove nonlocal terms involved in (18) for distributed controls.

We underline that equations (20)-(21) have to be considered in $D(A^*)'$, as B is an unbounded operator from \mathbb{C} to H . This introduces many technical difficulties, such as the solvability of the uniqueness condition $TB = B$. It is now standard to notice that this unboundedness of B is reflected to K and T , and that it is in fact necessary, as bounded B yields T to be a Hilbert-Schmidt operator from H to itself, resulting in a not-invertible transformation [22, 24].

Remark 2.1. *An interesting question is to find a $TB = B$ uniqueness-type condition characterizing the Volterra transformation T solving $T(A + BK) = (A - \lambda I)T$.*

2.2 The stabilization Gramian

We follow closely here the clear and concise presentation found in [43, 45].

Stabilization with the controllability Gramian. For finite or infinite dimensional systems, Gramian operators are fundamental objects in linear control theory. For finite dimensional systems under the form (1), with $A \in M_n(\mathbb{R})$, $B \in M_{n,m}(\mathbb{R})$, $n, m \in \mathbb{N}$, the *controllability Gramian* is given by

$$G_{A,B}^T = \int_0^T e^{(T-t)A} B B^* e^{(T-t)A^*} dt. \quad (22)$$

It appears notably in the expression of the minimal $L^2(0, T)$ -norm control [39, Chapter 4, Section 3.5]. In infinite dimension, the Gramian is defined by the following symmetric bilinear form on $D(A^*)$. Using the same notations as for finite dimensional systems,

$$G_{A,B}^T(f, g) = \int_0^T \left\langle B^* e^{(T-t)A^*} f, B^* e^{(T-t)A^*} g \right\rangle dt, \quad \forall f, g \in D(A^*). \quad (23)$$

In finite dimension, the first results of stabilization using the Gramian (22) to design a feedback law go back to [33, Theorem 3.1] and [26], where the feedback law is given by

$$K_G = -B^*(e^{-TA} G_{A,B}^T e^{-TA^*})^{-1}. \quad (24)$$

This feedback ensures asymptotic stabilization, but not exponential stabilization at a given rate.

Weighted stabilization Gramian. To enhance the stabilization provided by a Gramian feedback, a variant introducing a weight inside the integral:

$$C_\lambda^T := \int_0^T e^{-2\lambda t} e^{-tA} B B^* e^{-tA^*} dt,$$

and defining the feedback by

$$K_{\lambda,T} := -B^*(C_\lambda^T)^{-1}. \quad (25)$$

A strict Lyapunov function is then defined by

$$V(z) := \langle z, (C_\lambda^T)^{-1} z \rangle.$$

The decay of the Lyapunov function comes from the inequality (see [16, Section 13.1])

$$(C_\lambda^T)^{-1} A + A^* (C_\lambda^T)^{-1} - (C_\lambda^T)^{-1} B B^* (C_\lambda^T)^{-1} \leq -2\lambda (C_\lambda^T)^{-1},$$

so that in the end, $K_{\lambda,T}$ stabilizes the system exponentially with decay rate λ .

In infinite dimension, an analog feedback was introduced in [28], in the general case of A generating a C_0 -group, and B and admissible control operator such that (A, B) is controllable in H . Define

$$G_\lambda^T(f, g) = \int_0^{T+\frac{1}{2\lambda}} f_\lambda(t) \langle B^* e^{(T-t)A^*} f, B^* e^{(T-t)A^*} g \rangle dt, \quad \forall f, g \in D(A^*),$$

where

$$f_\lambda(t) = \begin{cases} e^{-2\lambda t} & \text{if } 0 \leq t \leq T, \\ 2\lambda e^{-2\lambda T} (T + \frac{1}{2\lambda} - t) & \text{if } T \leq t \leq T + \frac{1}{2\lambda}, \end{cases}$$

and define the following feedback:

$$K_\lambda := -B^* (G_\lambda^T)^{-1}.$$

One then has the following stabilization result (see [28]):

Theorem 2.1. *The operator $A + BK_\lambda$ is a restriction of the infinitesimal generator of a C_0 -semigroup $S(t)$, $t \geq 0$ on H , such that*

$$\|S(t)y\| \leq C e^{-\lambda t} \|y\|, \quad \forall y \in H.$$

Infinite-time stabilization Gramian. Instead of considering a weighted integral operator as above, one can consider an operator defined with an integral over $(0, \infty)$. This was introduced by Roger Bass, and extended in [43] to infinite dimensional systems with unbounded A and B , under the hypothesis that A is the infinitesimal generator of a group, and B is admissible and such that (A, B) is controllable in H . Define

$$C_\lambda^\infty(f, g) := \int_0^\infty e^{-2\lambda t} \langle B^* e^{-tA^*} f, B^* e^{-tA^*} g \rangle dt, \quad (26)$$

with $\lambda > \max \{\operatorname{Re}(\lambda) \mid \lambda \in \sigma(A) \cup \sigma(A^*)\}$. The corresponding linear operator $C_\lambda^\infty \in \mathcal{L}(D(A^*); D(A^*))$ is invertible, and can be extended to an isomorphism of H . One then defines in the same spirit the feedback law

$$K_\lambda^\infty := -B^* (C_\lambda^\infty)^{-1}. \quad (27)$$

2.3 Riccati equations

We briefly mention the link between the stabilization Gramian and the algebraic Riccati equation. The latter stems from optimal control problems and allows robust feedback to be obtained. There is extensive literature on this subject, and we refer the interested reader to [5] and the references therein, and also [30, 31]. A specific case of the algebraic Riccati equation corresponds to that satisfied by the stabilisation Gramian:

$$C_\lambda^\infty (A + \lambda I)^* + (A + \lambda I) C_\lambda^\infty = B B^*. \quad (28)$$

We recall the properties of this equation.

Theorem 2.2 ([25], Theorem 3). *Let H and U be Hilbert spaces. Suppose that $A : D(A) \subset H \rightarrow H$ is the infinitesimal generator of a C_0 semigroup $S(t)$ on H . Assume also $B^* : D(A) \rightarrow U$ is bounded. Then,*

for the existence of an operator $C \in \mathcal{L}(H; H)$ with $C = C^*$ and $C \geq 0$, satisfying the Lyapunov operator equation

$$\langle Au, Cv \rangle_H + \langle u, CAv \rangle_H = -\langle B^*u, B^*v \rangle_U, \quad \forall u, v \in D(A),$$

it is necessary and sufficient that a positive number $\gamma > 0$ exists such that

$$\int_0^\infty \|B^*S(t)u\|_U^2 dt \leq \gamma \|u\|_H^2, \quad \forall u \in D(A).$$

Notice that Theorem 2.2 does not ensure the invertibility of C from H to itself. The invertibility would follow from the observability inequality, if satisfied.

2.4 F -equivalence

These three previous stabilization methods enter a larger class of problems in control theory, known as the F -equivalence between control systems. As mentioned previously, sometimes the classical target system $A - \lambda I$ might not be the best suited (see for instance [13]). The F -equivalence allows for a wider choice of target systems.

Definition 2.1 (F -equivalence). *Let $(\tilde{A}, D(\tilde{A}))$ be an unbounded operator, and $\tilde{B} \in L(D(\tilde{A}^*), \mathbb{C})$ be an admissible control operator for \tilde{A} . The control systems (A, B) and (\tilde{A}, \tilde{B}) are F -equivalent if there exist an admissible feedback K and an invertible operator $T : D(\tilde{A}) \rightarrow D(A + BK)$ such that*

$$T(A + BK) = \tilde{A}T, \quad TB = \tilde{B}. \quad (29)$$

This notion was introduced by Pavol Brunovsky in [8] for finite-dimensional systems. We point out that the core of the F -equivalence problem is to find an invertible transformation T that maps the system $A + BK$ to the target system \tilde{A} . This is translated by the first equation of (29), while the second translates the equivalence between *control* systems. It is then interesting to consider only the first equation of (29), which we call the equivalence equation:

Definition 2.2 (Equivalence equation). *The operator-feedback pair (T, K) where $T \in L(H)$ and $K : D(K) \subset H \rightarrow \mathbb{C}$ is said to solve the equivalence equation between A and \tilde{A} if*

- (i) $T : H \rightarrow H$ is continuously invertible;
- (ii) K is an admissible feedback;
- (iii) $TD(A + BK) = D(\tilde{A})$ and:

$$T(A + BK)f = \tilde{A}Tf, \quad \forall f \in D(A + BK). \quad (30)$$

As mentioned in the introduction, the backstepping method and the Gramian stabilization method are two examples where an equivalence equation is solved, with $\tilde{A} = A - \lambda I$. It should be noted that, under the hypothesis of this paper, one may simplify the F -equivalence problem to the generalized Fredholm backstepping.

Lemma 2.1. *Assume (A, B) and (\tilde{A}, \tilde{B}) satisfy $(H_1) - (H_5)$. Assume there exists (T, K) such that*

$$T(A + BK) = \tilde{A}T, \quad TB = \tilde{B}. \quad (31)$$

Then there exists (\tilde{T}, K) such that

$$\tilde{T}(A + BK) = \tilde{A}\tilde{T}, \quad TB = \tilde{B}. \quad (32)$$

We prove Lemma 2.1

Proof. It suffices to notice that, under the hypothesis $0 < c < |\tilde{b}_n| < C$ where $\tilde{b}_n = \tilde{B}^* \varphi_n$, there exists an isomorphism $\tau = \tau(B, \tilde{B})$ such that $\tau B = \tilde{B}$. Then,

$$\tau T B = \tilde{B}.$$

Thus $\tilde{T} := \tau T$ satisfies $\tilde{T} B = \tilde{B}$. Moreover, as

$$\tau \varphi_n \mapsto \frac{\tilde{b}_n}{b_n} \varphi_n$$

is a diagonal and, by assumption, invertible operator,

$$(A - \lambda I) \tau^{-1} \tau T = \tau^{-1} (A - \lambda I) \tau T$$

and thus

$$\tau T (A + BK) = (A - \lambda I) \tau T$$

that is,

$$\tilde{T} (A + BK) = (A - \lambda I) \tilde{T}$$

□

2.5 Sylvester equations and generalized Gramians

The Riccati equation (28) and the F -equivalence problem (29) are both instances of a general class of operator equations called Sylvester equations (see [20] and the references therein for the infinite-dimensional case):

$$AX + XC = D.$$

The Riccati equation (28) arising in Gramian stabilization has a specific structure: $A = C^*$ and D is symmetric positive. This plays a crucial role in defining a Lyapunov function and proving stabilization. When these structural assumptions are relaxed, the solutions of these equations are typically non-symmetric Gramian operators.

For Sylvester equations with non-selfadjoint data,

$$A^* G + GA = -Q, \quad Q \neq Q^*,$$

bounded solutions are typically called **non-normal Lyapunov Gramians**. Such operators generally fail to be self-adjoint and do not define coercive quadratic Lyapunov functionals; instead, they encode stability and input–output properties through duality pairings [34].

Further relaxing the symmetries of the Riccati equation gives rise to **Petrov–Galerkin Gramians**. These are commonly used in projection-based approximations of infinite-dimensional control systems [1]. Let $V, W \subset X$ be non-orthogonal trial and test spaces, with associated oblique projections P_V and P_W . Projecting Lyapunov or Gramian equations onto $V \times W$ leads to reduced operators G_{VW} satisfying

$$P_W A^* P_W^* G_{VW} + G_{VW} P_V A P_V^* = -P_W Q P_V^*,$$

which are symmetric only with respect to the induced oblique duality between V and W , but not on the original state space. At the operator-theoretic level, these constructions are governed by infinite-dimensional Sylvester equations of the form

$$AG + GA^\sharp = -F,$$

whose well-posedness has been studied in abstract Banach and Hilbert space settings [20]. Such generalized Gramians lack the coercivity required for direct stabilization. In this article, we show that they appear naturally when considering the F -equivalence problem associated to a spectral assignability problem, and that they define the pole-shifting feedback through an analog of (24)–(25)–(27).

2.6 Spectral assignability or poleshifting

All the problems described above, and many stabilization problems, are related to the problem of assigning poles (or eigenvalues) to a given control system, also called **poleshifting**. This central question in control theory may be formulated as follows: *given a control system (A, B) , and desired eigenvalues $\{\tilde{\lambda}_n\}$, does there exist a linear feedback law K such that the closed-loop operator $A + BK$ has the eigenvalues $\{\tilde{\lambda}_n\}$? If so, is it possible to give an expression of K , and describe the eigenvectors?* The problem of poleshifting in general is broader than exponential stabilization. It is typically used in stabilization by assigning the eigenvalues $\tilde{\lambda}_n$ to the left half complex plane to ensure the rapid or exponential stability of the closed-loop operator $A + BK$.

Finite dimension. A very well-known result in **finite dimension**, using the control canonical form, gives a positive answer to the question of poleshifting (see [16] for details):

Theorem 4. *Let $n \geq 1$, $(A, B) \in \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times 1}$. If the control system (A, B) is controllable, then for any $\{\tilde{\lambda}_k\}_{1 \leq k \leq n} \in \mathbb{C}^n$, there exists a unique $K \in \mathbb{C}^{1 \times n}$ such that $A + BK$ has eigenvalues $\{\tilde{\lambda}_n\}$.*

The well-known Ackermann formula gives the expression of the feedback shifting the poles of A to the roots of a new polynomial $P_{\text{shifted}}(x)$, in which the Kalman matrix \mathfrak{K} of (A, B) is inverted:

$$K = (0 \quad \dots \quad 1) \mathfrak{K}^{-1} P_{\text{shifted}}(A). \quad (33)$$

Infinite dimensional spectral assignability Although some extensions of the canonical form to hyperbolic systems exist [10, 37], there is no general canonical form for infinite-dimensional control systems. However, the problem of spectral assignability has been extensively studied, even when A and B are not bounded [36, 40]. The problem is formulated as follows in the latter reference:

Definition 2.3. *Let $A : D(A) \subset H \rightarrow H$ be an unbounded, densely defined operator on H , and $B : \mathbb{C} \rightarrow D(A^*)'$ an admissible control operator. For a given sequence $\{\tilde{\lambda}_n\}_{n \in \mathbb{N}}$ of complex numbers, does there exist a linear feedback $K : D(K) \subset H \rightarrow \mathbb{C}$ such that the resulting closed-loop operator $A + BK$ has the eigenvalues $\{\tilde{\lambda}_n\}$?*

Interestingly, the Fredholm backstepping method, and the Gramian stabilization method, when they can be applied, yield feedbacks that assign the poles of the system to the eigenvalues $\{\lambda_n - \tilde{\lambda}_n\}$. In a broader variety of situations however, it seems less likely that the spectral assignability problem should have a solution. For example, it is well known (see [37, 40]) that for bounded control systems, poleshifting to $\{\tilde{\lambda}_n\}$ with a bounded feedback K can only be achieved if $(\lambda_n - \tilde{\lambda}_n)_n \in \ell^2$, which precludes exponential stabilization. Accordingly, with less regularity requirements on B and K , a wider range of poleshifting is possible, but one expects similar obstructions to arise.

In the unbounded case, the foundational paper [36] gives a solution to the spectral assignability problem, under assumptions on the spectrum of A and the target spectrum. In essence, if the spectrum of A and the target spectrum are the zero sets of two cardinal functions which share the same poles, and further satisfy additional asymptotic estimates, then the author provides a feedback, as well as a Riesz basis of eigenvectors and its biorthonormal family (along which the feedback is then expressed). All these objects are given somewhat constructively but involve the aforementioned cardinal functions. In contrast, while we assume similar estimates on the spectra involved, we do not resort to cardinal functions, and give several expressions for the feedback, which we believe to be more constructive than in previous works on the subject.

Lifting the boundedness assumption on B and K raises another issue. For any bounded B and K , $D(A + BK) = D(A)$ and so $A + BK$ always has dense domain. However, when B and K are no longer bounded, and the domain of the operator $A + BK$ has no relationship a priori to $D(A)$. Manipulating these unbounded operators and their dense domains requires special care, which we present in the next section.

3 Functional setting

We begin this section by stating this simple lemma, the proof of which can be found in Appendix B.1.

Lemma 3.1. *Under (H_4) of Assumption (1.1), we have*

$$c|k - n|^\alpha \leq |\lambda_k - \lambda_n|, \quad \forall k, n \in \mathbb{Z},$$

and,

$$\sum_{k \in \mathbb{Z}} \frac{1}{|\lambda_k|} < +\infty.$$

3.1 Nested spaces

We introduce a family of nested spaces particular case of our operator A . Taking $\beta \in \rho(A)$, we define the following norms for $s \geq 0$:

$$\|u\|_s^2 = \sum_{n \in \mathbb{Z}} |\lambda_n - \beta|^{2s} |\langle u, \varphi_n \rangle|^2, \quad \forall u \in H,$$

and the assorted spaces:

$$H_s = \{u \in H, \quad \|u\|_s < +\infty\} \subset H.$$

To accommodate the case $s < 0$, we define

$$H_\infty := \bigcap_{s \geq 0} H_s.$$

Since for all $n \in \mathbb{Z}$, $\varphi_n \in H_\infty$, then H_∞ is dense in H . We then define, for $s < 0$:

$$\|u\|_s^2 = \sum_{n \in \mathbb{Z}} |\lambda_n - \beta|^{2s} |\langle \varphi_n, u \rangle|^2, \quad \forall u \in H'_\infty,$$

and the spaces

$$H_s := \{u \in H'_\infty, \quad \|u\|_s < +\infty\} \supset H.$$

Since A has compact resolvent, for $s_1 < s_2$, the injection $H_{s_2} \hookrightarrow H_{s_1}$ is compact. Finally, note that, for $s \geq 0$, $H_{-s} = (H_s)'$. In particular, (5) implies that $B \in (H_{1/2})' = H_{-1/2}$.

3.2 Extensions of A

The operator A is unbounded, and it is clear by definition that $D(A) = H_1$, and more generally $AH_s = H_{s-1}$ for $s \geq 1$. On the other hand, for $s < 1$, we can define an extension A_s of the operator A on H_s , by writing

$$\langle A_s x, \varphi_n \rangle = \lambda_n \langle x, \varphi_n \rangle, \quad \forall x \in H_s.$$

Clearly, for $x \in H_s$, $A_s x \in H_{s-1}$, and for $x \in H_1$, $A_s x = Ax$. To alleviate notations, in the rest of this article, we denote by A both the unbounded operator A and its extensions A_s , $s < 1$.

3.3 Admissible feedbacks and closed-loop operators

We recall here the definition of admissible feedback laws to establish some important properties in this subsection.

Definition 3.1. *Define the set of admissible feedback laws*

$$\mathbb{K}_{\text{ad}} := \left\{ K : D(K) \subset H \rightarrow \mathbb{C} \mid \forall T > 0, \forall \phi \in D(A^*), K e^{tA^*} \phi \in L^2((0, T); \mathbb{C}) \right\}.$$

Then, for any $K \in \mathbb{K}_{\text{ad}}$, since $D(K) \subset H$, the closed-loop operator $A + BK$ has a meaning in H_{-1} , that is,

$$(A + BK)f = A_0f + (Kf)B \in H_{-1}, \quad \forall f \in D(K).$$

Notice that, above, $A_0 : D(A_0) = H \subset H_{-1} \rightarrow H_{-1}$ is used. The domain of the closed-loop operator is thus defined by,

$$D(A + BK) := \{f \in D(K) \mid A_0f + (Kf)B \in H\}.$$

In this paper, we work on a certain subset of admissible feedback laws, satisfying all the hypotheses of the following lemma.

Lemma 3.2. *Assume Assumption 1.1 and $\{k_n\}_{n \in \mathbb{Z}} \subset \ell^\infty(\mathbb{Z}; \mathbb{C})$. Then, for any $f \in H_{1/2}$, the feedback defined by*

$$Kf := \sum_{n \in \mathbb{Z}} k_n \langle f, \varphi_n \rangle$$

is well-defined in $\mathcal{L}(H_{1/2}; \mathbb{C})$. Moreover, if there exists $c > 0$ such that $c < |k_n|, \forall n \in \mathbb{Z}$, then for any $T > 0$, there exists $C > 0$ such that,

$$\int_0^T \left\| K e^{A^* t} u_0 \right\|_{\mathbb{C}} dt \leq C \|u_0\|_H, \quad \forall u_0 \in D(A^*).$$

Thus, such feedbacks are admissible in the sense of Definition 3.1.

Proof. By Cauchy-Schwarz inequality,

$$\begin{aligned} \left| \sum_{n \in \mathbb{Z}} k_n \langle f, \varphi_n \rangle \right|^2 &\leq \left(\sum_{n \in \mathbb{Z}} \frac{|k_n|}{|\lambda_n|} \right) \left(\sum_{n \in \mathbb{Z}} |\lambda_n| |\langle f, \varphi_n \rangle| \right) \\ &\leq \|k_n\|_{\ell^\infty(\mathbb{Z}; \mathbb{C})} \|f\|_{H_{1/2}} \left(\sum_{n \in \mathbb{Z}} \frac{1}{|\lambda_n|} \right) \\ &\leq C \|k_n\|_{\ell^\infty(\mathbb{Z}; \mathbb{C})} \|f\|_{H_{1/2}}. \end{aligned}$$

To prove the second assertion, introduce

$$\tau_{KB} : \varphi_n \mapsto \frac{b_n}{k_n} \varphi_n.$$

Since there exist $c, C > 0$ such that, $\forall n \in \mathbb{Z}$, $c < |k_n| < C$ and $c < |b_n| < C$, τ_{KB} is a continuous

isomorphism from H to itself. Since τ_{KB} commutes with A , for $u_0 \in D(A)$, we have,

$$\begin{aligned}
\int_0^T \left\| K e^{A^* t} u_0 \right\|_{\mathbb{C}} dt &= \int_0^T \left\| K \tau_{KB} \tau_{KB}^{-1} e^{A^* t} u_0 \right\|_{\mathbb{C}} dt \\
&= \int_0^T \left\| K \tau_{KB} e^{A^* t} \tau_{KB}^{-1} u_0 \right\|_{\mathbb{C}} dt \\
&= \int_0^T \left\| K \tau_{KB} \sum_{n \in \mathbb{Z}} \langle e^{A^* t} \tau_{KB}^{-1} u_0, \varphi_n \rangle \varphi_n \right\|_{\mathbb{C}} dt \\
&= \int_0^T \left\| K \sum_{n \in \mathbb{Z}} \frac{b_n \langle e^{A^* t} \tau_{KB}^{-1} u_0, \varphi_n \rangle}{k_n} \varphi_n \right\|_{\mathbb{C}} dt \\
&= \int_0^T \left\| \sum_{n \in \mathbb{Z}} b_n \langle e^{A^* t} \tau_{KB}^{-1} u_0, \varphi_n \rangle \right\|_{\mathbb{C}} dt \\
&= \int_0^T \left\| B^* e^{A^* t} \tau_{KB}^{-1} u_0 \right\|_{\mathbb{C}} dt \\
&\leq C \|\tau_{KB}^{-1} u_0\|_H \\
&\leq C \|u_0\|_H
\end{aligned}$$

from the admissibility of B and $\tau_{KB}^{-1} \in \mathcal{L}(H; H)$. □

3.4 Notations

Since we work with scalar controls and unbounded operators, we will consider control operators in the usual space $\mathcal{L}(\mathbb{C}, D(A^*)')$ (see for example [16, 42]). For a control operator $B : \mathbb{C} \rightarrow D(A^*)'$, there exists a unique vector $b \in D(A^*)'$ such that:

$$Bu = ub, \quad \forall u \in \mathbb{C}.$$

In the rest of this article, we will resort to the usual shorthand $b = B$. Depending on the context B will therefore refer either to the linear mapping from \mathbb{C} to $D(A^*)'$, or to the associated vector of $D(A^*)'$.

An important instance lies in the notation B^* , which clearly denotes the linear form:

$$B^* \varphi = \langle \varphi, B \rangle_{D(A^*), D(A^*)'}, \quad \forall \varphi \in D(A^*).$$

Conversely, the feedbacks are densely defined linear forms, and we will manipulate their adjoints, in the following way: let K be an admissible feedback. Then $K^* : \mathbb{C} \rightarrow D(K)'$ is represented by a unique vector $k \in D(K)'$ such that

$$K^* a = ak, \quad \forall a \in \mathbb{C}.$$

Similarly, we will use the notational shorthand $k = K^*$.

4 F -equivalence for a broad class of target systems

In this section, we solve the spectral assignability problem with target spectrum $\{\tilde{\lambda}_n\}$, by establishing F -equivalence between (A, B) and a well-chosen target system. We thus provide proofs for the first parts of Theorems 2 and 1, namely the existence of solutions to equations (11) and (12).

4.1 The eigenfunctions of the target operator

We begin by defining the family $\{\chi_n\}$ and discussing some of its basic properties. Given a sequence $\{\tilde{\lambda}_n\}$ in the resolvent set $\rho(A)$, we define the family of functions

$$\chi_n := (A - \tilde{\lambda}_n I)^{-1} B, \quad n \in \mathbb{Z}. \quad (34)$$

Given that $B \in D(A^*)'$, we immediately have $\chi_n \in H$. We even have

Lemma 4.1. *If $B \in H_{-1/2}$, then the family $\{\chi_n\}_{n \in \mathbb{Z}}$ belongs to $H_{1/2}$.*

Proof. It is a straightforward computation,

$$\begin{aligned} \|\chi_n\|_{1/2} &= \left\| (A - \tilde{\lambda}_n I)^{-1} B \right\|_{1/2} \\ &= \left\| A(A - \tilde{\lambda}_n I)^{-1} B \right\|_{-\frac{1}{2}} \\ &= \left\| (I - \tilde{\lambda}_n A^{-1})^{-1} B \right\|_{-\frac{1}{2}} \\ &\leq C_n \|B\|_{-\frac{1}{2}} \end{aligned}$$

□

This family of functions plays a central role in the properties of the control system (A, B) . The reason for this is that formally, given any linear form K , an eigenvector h of $A + BK$ with eigenvalue $\tilde{\lambda}_n$ should satisfy

$$Ah + (Kh)B = \tilde{\lambda}_n h \text{ i.e., } h = -(Kh)(A - \tilde{\lambda}_n I)^{-1}B,$$

and thus be colinear to χ_n . From this same computation, we also immediately see that formally, $K\chi_n = -1$. This is not sufficient to define K on a dense domain and ensure that $A + BK$ has a dense domain, unless the $\{\chi_n\}$ form a Riesz basis. This is the object of Corollary 5.1 in the next section. But many steps are required beforehand.

4.2 An existence result

We now prove a fundamental result which will allow us to properly define the aforementioned feedback K .

Proposition 4.1. *Under Assumptions 1.1 and 1.2, there exists a unique sequence $\{k_n\} \in \ell^\infty(\mathbb{Z}; \mathbb{C})$ such that*

$$\sum_{n \in \mathbb{Z}} \frac{b_n k_n}{\lambda_n - \tilde{\lambda}_p} = -1, \quad \forall p \in \mathbb{Z}, \quad (35)$$

where $b_n := \overline{B^* \varphi_n}$. Furthermore, there exist $c, C > 0$ such that

$$c \leq |k_n| \leq C, \quad \forall n \in \mathbb{Z}. \quad (36)$$

Thus, the feedback $K \in \mathcal{L}(H_{1/2}; \mathbb{C})$, defined by

$$Kf = K \sum_{n \in \mathbb{Z}} \langle f, \varphi_n \rangle \varphi_n = \sum_{n \in \mathbb{Z}} \langle f, \varphi_n \rangle K \varphi_n = \sum_{n \in \mathbb{Z}} f_n k_n, \quad \forall f \in H_{1/2} \quad (37)$$

where $K\varphi_n := k_n$, is admissible.

The proof of the proposition follows from the infinite Cauchy matrix, introduced in [21].

Proof. We recall that

$$\sum_{n \in \mathbb{Z}} \frac{b_n k_n}{\lambda_n - \tilde{\lambda}_p} = -1, \quad \forall p \in \mathbb{Z}, \quad (38)$$

may be expressed as

$$C \begin{pmatrix} \vdots \\ b_1 k_1 \\ \vdots \\ b_n k_n \\ \vdots \end{pmatrix} = - \begin{pmatrix} \vdots \\ 1 \\ \vdots \\ 1 \\ \vdots \end{pmatrix}$$

with

$$C_{pn} = \frac{1}{\lambda_n - \tilde{\lambda}_p}, \quad n, p \in \mathbb{Z}, \quad (39)$$

that is, (38) is equivalent to

$$\sum_{n \in \mathbb{Z}} C_{pn} b_n k_n = -1, \quad \forall p \in \mathbb{Z}.$$

Its inverse is formally given by

$$C_{pn}^{-1} = (\lambda_p - \tilde{\lambda}_n) \left(\prod_{\substack{k \in \mathbb{Z} \\ k \neq p}} \frac{\tilde{\lambda}_n - \lambda_k}{\lambda_p - \lambda_k} \right) \left(\prod_{\substack{m \in \mathbb{Z} \\ m \neq n}} \frac{\lambda_p - \tilde{\lambda}_m}{\tilde{\lambda}_n - \tilde{\lambda}_m} \right), \quad n, p \in \mathbb{Z},$$

the infinite products may be rearranged as

$$\begin{aligned} & \left(\prod_{k \neq p} \frac{\tilde{\lambda}_n - \lambda_k}{\lambda_p - \lambda_k} \right) \left(\prod_{m \neq n} \frac{\lambda_p - \tilde{\lambda}_m}{\tilde{\lambda}_n - \tilde{\lambda}_m} \right) = \left(\frac{\tilde{\lambda}_n - \lambda_n}{\lambda_p - \lambda_n} \right) \left(\frac{\lambda_p - \tilde{\lambda}_p}{\tilde{\lambda}_n - \tilde{\lambda}_p} \right) \left(\prod_{k \neq n, p} \left(\frac{\tilde{\lambda}_n - \lambda_k}{\lambda_p - \lambda_k} \right) \left(\frac{\lambda_p - \tilde{\lambda}_k}{\tilde{\lambda}_n - \tilde{\lambda}_k} \right) \right) \\ &= \left(\frac{\tilde{\lambda}_n - \lambda_n}{\lambda_p - \lambda_n} \right) \left(\frac{\lambda_p - \tilde{\lambda}_p}{\tilde{\lambda}_n - \tilde{\lambda}_p} \right) \left(\prod_{k \neq n, p} \left(\frac{\lambda_p - \tilde{\lambda}_k}{\lambda_p - \lambda_k} \right) \left(\frac{\tilde{\lambda}_n - \lambda_k}{\tilde{\lambda}_n - \tilde{\lambda}_k} \right) \right) \\ &= \left(\frac{\tilde{\lambda}_n - \lambda_n}{\lambda_p - \lambda_n} \right) \left(\frac{\lambda_p - \tilde{\lambda}_p}{\tilde{\lambda}_n - \tilde{\lambda}_p} \right) \left(\prod_{k \neq n, p} \left(1 + \frac{\lambda_k - \tilde{\lambda}_k}{\lambda_p - \lambda_k} \right) \left(1 + \frac{\tilde{\lambda}_k - \lambda_k}{\tilde{\lambda}_n - \tilde{\lambda}_k} \right) \right) \\ &= \left(\frac{\tilde{\lambda}_n - \lambda_n}{\lambda_p - \lambda_n} \right) \left(\frac{\lambda_p - \tilde{\lambda}_p}{\tilde{\lambda}_n - \tilde{\lambda}_p} \right) \left(\prod_{k \neq n, p} 1 + \frac{\lambda_k - \tilde{\lambda}_k}{\lambda_p - \lambda_k} \right) \left(\prod_{k \neq n, p} 1 + \frac{\tilde{\lambda}_k - \lambda_k}{\tilde{\lambda}_n - \tilde{\lambda}_k} \right) \\ &= \left(\frac{\tilde{\lambda}_n - \lambda_n}{\lambda_p - \lambda_n} \right) \left(\frac{\lambda_p - \tilde{\lambda}_p}{\tilde{\lambda}_n - \tilde{\lambda}_p} \right) \left(1 + \frac{\lambda_n - \tilde{\lambda}_n}{\lambda_p - \lambda_n} \right)^{-1} \left(\prod_{k \neq p} 1 + \frac{\lambda_k - \tilde{\lambda}_k}{\lambda_p - \lambda_k} \right) \left(1 + \frac{\tilde{\lambda}_p - \lambda_p}{\tilde{\lambda}_n - \tilde{\lambda}_p} \right)^{-1} \left(\prod_{k \neq n} 1 + \frac{\tilde{\lambda}_k - \lambda_k}{\tilde{\lambda}_n - \tilde{\lambda}_k} \right) \\ &= \left(\frac{\lambda_p - \tilde{\lambda}_p}{\tilde{\lambda}_n - \lambda_p} \right) \left(\frac{\tilde{\lambda}_n - \lambda_n}{\lambda_p - \tilde{\lambda}_n} \right) \left(\prod_{k \neq p} 1 + \frac{\lambda_k - \tilde{\lambda}_k}{\lambda_p - \lambda_k} \right) \left(\prod_{m \neq n} 1 + \frac{\tilde{\lambda}_m - \lambda_m}{\tilde{\lambda}_n - \tilde{\lambda}_m} \right) \end{aligned}$$

Notice that the infinite products in the computation above are all well defined since

$$\lim_{k \rightarrow \pm\infty} \left| \frac{\tilde{\lambda}_n - \lambda_k}{\lambda_p - \lambda_k} \right| = \lim_{m \rightarrow \pm\infty} \left| \frac{\lambda_p - \tilde{\lambda}_m}{\tilde{\lambda}_n - \tilde{\lambda}_m} \right| = 1,$$

and

$$\lim_{k \rightarrow \pm\infty} \left| 1 + \frac{\lambda_k - \tilde{\lambda}_k}{\lambda_p - \lambda_k} \right| = \lim_{m \rightarrow \pm\infty} \left| 1 + \frac{\tilde{\lambda}_m - \lambda_m}{\tilde{\lambda}_n - \tilde{\lambda}_m} \right| = 1,$$

as

$$\lim_{k \rightarrow \pm\infty} \left| \frac{\lambda_k - \tilde{\lambda}_k}{\lambda_p - \lambda_k} \right| = \lim_{m \rightarrow \pm\infty} \left| \frac{\tilde{\lambda}_m - \lambda_m}{\tilde{\lambda}_n - \tilde{\lambda}_m} \right| = 0.$$

Therefore,

$$C_{pn}^{-1} = \left(\frac{(\lambda_p - \tilde{\lambda}_p)(\tilde{\lambda}_n - \lambda_n)}{\tilde{\lambda}_n - \lambda_p} \right) \left(\prod_{\substack{k \in \mathbb{Z} \\ k \neq p}} \left(1 + \frac{\lambda_k - \tilde{\lambda}_k}{\lambda_p - \lambda_k} \right) \right) \left(\prod_{\substack{m \in \mathbb{Z} \\ m \neq n}} \left(1 + \frac{\tilde{\lambda}_m - \lambda_m}{\tilde{\lambda}_n - \tilde{\lambda}_m} \right) \right). \quad (40)$$

The following technical lemma allows us to define properly the infinite Cauchy matrix associated to the spectrum $\{\lambda_n\}_{n \in \mathbb{Z}}$ and $\{\tilde{\lambda}_n\}_{n \in \mathbb{Z}}$. The proof can be found in Appendix B.1.

Lemma 4.2. *For any given sequences $\{\lambda_n\}_{n \in \mathbb{Z}}$ and $\{\tilde{\lambda}_n\}_{n \in \mathbb{Z}}$ satisfying Assumption 1.1 and 1.2, there exists $C > 0$ such that, for any $n, p \in \mathbb{Z}$,*

$$(i) \left| \prod_{\substack{k \in \mathbb{Z} \\ k \neq p}} \left(1 + \frac{\lambda_k - \tilde{\lambda}_k}{\lambda_p - \lambda_k} \right) \right| \leq C, \quad \forall p \in \mathbb{Z};$$

$$(ii) \left| \prod_{\substack{m \in \mathbb{Z} \\ m \neq n}} \left(1 + \frac{\tilde{\lambda}_m - \lambda_m}{\tilde{\lambda}_n - \tilde{\lambda}_m} \right) \right| \leq C, \quad \forall n \in \mathbb{Z};$$

$$(iii) \sum_{n \in \mathbb{Z}} \left| \frac{1}{\tilde{\lambda}_n - \lambda_p} \right| < C, \quad \forall p \in \mathbb{Z};$$

$$(iv) \sum_{p \in \mathbb{Z}} \left| \frac{1}{\tilde{\lambda}_n - \lambda_p} \right| < C, \quad \forall n \in \mathbb{Z};$$

$$(v) \sum_{n \in \mathbb{Z}} \left| \frac{1}{\tilde{\lambda}_n - \lambda_p} \right|^2 < C, \quad \forall p \in \mathbb{Z};$$

$$(vi) \sum_{p \in \mathbb{Z}} \left| \frac{1}{\tilde{\lambda}_n - \lambda_p} \right|^2 < C, \quad \forall n \in \mathbb{Z}.$$

We define the auxiliary transformation $\check{T} : H \rightarrow H$,

$$\check{T}f = \check{T} \sum_{n \in \mathbb{Z}} \langle f, \varphi_n \rangle \varphi_n = \sum_{n \in \mathbb{Z}} \langle f, \varphi_n \rangle \check{T} \varphi_n = \sum_{p \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \frac{\langle f, \varphi_n \rangle}{\lambda_n - \tilde{\lambda}_p} \varphi_p = \sum_{p \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \langle f, \varphi_n \rangle C_{pn} \varphi_p \quad (41)$$

so that (38) is equivalent to,

$$\check{T} \tau_B K^* = \check{T} \sum_{n \in \mathbb{Z}} k_n b_n \varphi_n = \sum_{p \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \frac{k_n b_n}{\lambda_n - \tilde{\lambda}_p} \varphi_p = - \sum_{p \in \mathbb{Z}} \varphi_p, \quad \text{in } D(A^*)'$$

where $\tau_B : H \rightarrow H$ is the isomorphism, under Assumption 1.1, defined by

$$\tau_B : \varphi_n \mapsto b_n \varphi_n.$$

We therefore define $\check{T}^{-1} : H \rightarrow H$ by

$$\check{T}^{-1} f = \sum_{n \in \mathbb{Z}} \langle f, \varphi_n \rangle \check{T}^{-1} \varphi_n = \sum_{n \in \mathbb{Z}} \sum_{p \in \mathbb{Z}} \langle f, \varphi_n \rangle C_{pn}^{-1} \varphi_p \quad (42)$$

We verify that \check{T}^{-1} is indeed the inverse of \check{T} in H (see Appendix B.2 for the proof):

Proposition 4.2. *Assume Assumption 1.1 and 1.2. Let \check{T} and \check{T}^{-1} being defined by (41) and (42) respectively. These transformations are continuous from H to H . Moreover,*

$$\check{T} \circ \check{T}^{-1} = \check{T}^{-1} \circ \check{T} = Id_H$$

We are then able to prove (see Appendix B.3) that there exists $c, C > 0$ such that $c < |k_n| < C$, $\forall n \in \mathbb{Z}$. The feedback K defined by (37) is thus admissible by Lemma 3.2. \square

4.3 Solving the F -equivalence problem

We now relate Proposition 4.1 to the F -equivalence problem given in Definition 2.2.

4.3.1 The transformation T

We first define the isomorphism T : denote by $\check{T} : H \rightarrow H$ the isomorphism induced by the infinite Cauchy matrix $-C$ in the proof above:

$$\langle \check{T} \varphi_n, \varphi_p \rangle = \frac{1}{\tilde{\lambda}_p - \lambda_n} = -C_{pn}, \quad \forall n, p \in \mathbb{Z}.$$

Define the diagonal operators

$$\tau_K \varphi_n = k_n \varphi_n, \quad \tau_B \varphi_n = b_n \varphi_n, \quad \forall n \in \mathbb{Z}.$$

Clearly, both τ_K and τ_B are isomorphisms on H , given the estimates (36) and (5). We then define the following isomorphism:

$$T := \tau_B \check{T} \tau_K. \quad (43)$$

4.3.2 The target system

To that end, we now define the unbounded operator \tilde{A} by

$$\tilde{A} \varphi_n = \tilde{\lambda}_n \varphi_n, \quad \forall n \in \mathbb{Z}, \quad (44)$$

with the straightforward characterization of its domain

$$D(\tilde{A}) = \left\{ f \in H, \sum_{n \in \mathbb{Z}} |\tilde{\lambda}_n|^2 |\langle f, \varphi_n \rangle|^2 < \infty \right\}.$$

Lemma 4.3. *We have the asymptotic relationship*

$$\lambda_n \underset{n \rightarrow \infty}{\sim} \tilde{\lambda}_n,$$

and as a result,

$$D(\tilde{A}) = D(A).$$

Proof. The asymptotic relationship follows directly from the growth of $|\lambda_n|$ and the third estimate of (6). \square

In the backstepping framework, \tilde{A} is the operator of the so-called target system. It is straightforward that this operator generates a C_0 -semigroup:

Lemma 4.4. *The operator $(\tilde{A}, D(A))$ generates a C_0 -semigroup on H .*

Proof. Given the third estimate of (6) in Assumption 1.2, there exists $\tilde{\omega} \in \mathbb{R}$ such that for all $n \in \mathbb{Z}$, $\operatorname{Re}(\tilde{\lambda}_n) \leq \tilde{\omega}$. Therefore, $(\tilde{\omega}, \infty) \subset \rho(\tilde{A})$, and we have, for $\lambda > \tilde{\omega}$, and for any $f \in H$,

$$\begin{aligned} \|(\tilde{A} - \lambda I)^{-1} f\|^2 &= \sum_{n \in \mathbb{Z}} \frac{|\langle f, \varphi_n \rangle|^2}{|\tilde{\lambda}_n - \lambda|^2} \\ &= \sum_{n \in \mathbb{Z}} \frac{|\langle f, \varphi_n \rangle|^2}{|\operatorname{Im}(\tilde{\lambda}_n)|^2 + |\operatorname{Re}(\tilde{\lambda}_n) - \lambda|^2} \\ &\leq \frac{\|f\|^2}{|\lambda - \tilde{\omega}|^2}. \end{aligned}$$

Since \tilde{A} is clearly closed and densely defined, we conclude using a variant of the Hille-Yosida Theorem [35, page 12, Corollary 3.8]. \square

4.3.3 The equivalence equation and the uniqueness condition

We now turn to the proof of the F -equivalence between (A, B) and (\tilde{A}, B) . We begin by a spectral study of the operator $A + BK$:

Lemma 4.5. *Let K be defined by (37). Then,*

(i) *For $n \in \mathbb{Z}$,*

$$(A + BK)\chi_n = \tilde{\lambda}_n \chi_n, \quad \text{in } H_{1/2}.$$

(ii) *The operator $A + BK$ is densely defined and has the Riesz basis of eigenvectors $\{T^{-1}\varphi_n\}_{n \in \mathbb{Z}}$ with corresponding eigenvalues $\{\tilde{\lambda}_n\}_{n \in \mathbb{Z}}$.*

Proof. Since $\chi_n \in H_{1/2}$ by Lemma 4.1, we can compute,

$$\begin{aligned} K\chi_n &= K(A - \tilde{\lambda}_n I)^{-1} B \\ &= K \sum_{k \in \mathbb{Z}} \frac{b_k}{\lambda_k - \tilde{\lambda}_n} \varphi_k \\ &= \sum_{k \in \mathbb{Z}} \frac{k_k b_k}{\lambda_k - \tilde{\lambda}_n} \\ &= -1 \end{aligned} \tag{45}$$

by virtue of Proposition 4.1. Since $A\chi_n \in H_{-1/2}$, we can compute in $H_{-1/2}$,

$$\begin{aligned} (A + BK)\chi_n &= A\chi_n - B \\ &= A(A - \tilde{\lambda}_n I)^{-1} B - B \\ &= A(A - \tilde{\lambda}_n I)^{-1} B - \tilde{\lambda}_n (A - \tilde{\lambda}_n I)^{-1} B + \tilde{\lambda}_n (A - \tilde{\lambda}_n I)^{-1} B - B \\ &= (A - \tilde{\lambda}_n)(A - \tilde{\lambda}_n I)^{-1} B + \tilde{\lambda}_n (A - \tilde{\lambda}_n I)^{-1} B - B \\ &= B + \tilde{\lambda}_n (A - \tilde{\lambda}_n I)^{-1} B - B \\ &= \tilde{\lambda}_n (A - \tilde{\lambda}_n I)^{-1} B \\ &= \tilde{\lambda}_n \chi_n. \end{aligned}$$

Since $\chi_n \in H_{1/2}$, $\forall n \in \mathbb{Z}$, we have proved item (i). Now, to prove (ii), let us compute, for $n \in \mathbb{Z}$,

$$\begin{aligned}
T\chi_n &= T(A - \tilde{\lambda}_n I)^{-1}B \\
&= T \sum_{p \in \mathbb{Z}} \frac{b_p}{\lambda_p - \tilde{\lambda}_n} \varphi_p \\
&= \sum_{p \in \mathbb{Z}} \frac{b_p}{\lambda_p - \tilde{\lambda}_n} \sum_{\ell \in \mathbb{Z}} \frac{k_p b_\ell}{\tilde{\lambda}_\ell - \lambda_p} \varphi_\ell \\
&= \sum_{\ell \in \mathbb{Z}} b_\ell \left(\sum_{p \in \mathbb{Z}} \frac{k_p b_p}{(\lambda_p - \tilde{\lambda}_n)(\tilde{\lambda}_\ell - \lambda_p)} \right) \varphi_\ell
\end{aligned}$$

Now, if $\ell \neq p$,

$$\frac{1}{(\lambda_p - \tilde{\lambda}_n)(\tilde{\lambda}_\ell - \lambda_p)} = \frac{1}{\tilde{\lambda}_\ell - \tilde{\lambda}_n} \left(\frac{1}{\lambda_p - \tilde{\lambda}_n} + \frac{1}{\tilde{\lambda}_\ell - \lambda_p} \right)$$

so that

$$T\chi_n = -b_n \left(\sum_{p \in \mathbb{Z}} \frac{k_p b_p}{(\lambda_p - \tilde{\lambda}_n)^2} \right) \varphi_n + \sum_{\ell \neq n} \frac{b_\ell}{\tilde{\lambda}_\ell - \tilde{\lambda}_n} \left(\sum_{p \in \mathbb{Z}} \frac{b_p k_p}{\lambda_p - \tilde{\lambda}_n} - \sum_{p \in \mathbb{Z}} \frac{b_p k_p}{\lambda_p - \tilde{\lambda}_\ell} \right) \varphi_\ell.$$

Thanks to (35), we arrive at

$$T\chi_n = -b_n \left(\sum_{p \in \mathbb{Z}} \frac{k_p b_p}{(\lambda_p - \tilde{\lambda}_n)^2} \right) \varphi_n. \tag{46}$$

Clearly, $T^{-1}\varphi_n$ is an eigenvector of $A + BK$ with eigenvalue $\tilde{\lambda}_n$, since it is colinear to the eigenvector χ_n . Moreover, since T^{-1} is an isomorphism, and the family $\{\varphi_n\}_{n \in \mathbb{Z}}$ form a Riesz basis of H , the family $\{T^{-1}\varphi_n\}_{n \in \mathbb{Z}}$ form a Riesz basis of H as well. \square

Remark 4.1. *Given the explicit expression of the $\{\chi_n\}_{n \in \mathbb{Z}}$, one would be inclined to work with this family, rather than the more implicit $\{T^{-1}\varphi_n\}_{n \in \mathbb{Z}}$. However, at this stage, we cannot prove that the $\{\chi_n\}_{n \in \mathbb{Z}}$ form a Riesz basis. A way to prove this would be to estimate the factor $\left(\sum_{p \in \mathbb{Z}} \frac{k_p b_p}{(\lambda_p - \tilde{\lambda}_n)^2} \right)$ in (46), which seems difficult at this stage. We will see in Section 5.1 how one can obtain the Riesz basis property for the $\{\chi_n\}_{n \in \mathbb{Z}}$ in another way.*

Note that at this stage, the spectral assignability problem 2.3 is solved by the feedback K . Establishing F -equivalence will now help us gain more insight on the feedback K , and the properties of the closed-loop operator $A + BK$.

Proposition 4.3. *The operator $A + BK$ has dense domain $D(A + BK)$. Moreover, $TD(A + BK) = D(\tilde{A})$ and*

$$T(A + BK) = \tilde{A}T \text{ on } D(A + BK), \tag{47}$$

i.e., A and \tilde{A} are F -equivalent.

Proof. Given Lemma 4.5, we use the family $\{T^{-1}\varphi_n\}_{n \in \mathbb{Z}}$ to characterize $D(A + BK)$. First note that this Riesz basis has a bi-orthogonal basis, given $\{T^*\varphi_n\}_{n \in \mathbb{Z}}$, since $\{\varphi_n\}_{n \in \mathbb{Z}}$ is a biorthonormal basis of H . The domain $D(A + BK)$ can then be characterized as

$$D(A + BK) = \left\{ f \in H, \quad \|(A + BK)f\|^2 = \sum_{n \in \mathbb{Z}} |\tilde{\lambda}_n|^2 |\langle f, T^*\varphi_n \rangle|^2 < \infty \right\}. \tag{48}$$

Since this space contains the dense space $\text{Span}\{T^{-1}\varphi_n \mid n \in \mathbb{Z}\}$, it is dense in H . Moreover, the family $\{(1/\tilde{\lambda}_n)T^{-1}\varphi_n\}_{n \in \mathbb{Z}}$ (or simply $T^{-1}\varphi_n$ if $\tilde{\lambda}_n = 0$) is a Riesz basis of $D(A+BK)$. Its image $\{(1/\tilde{\lambda}_n)\varphi_n\}_{n \in \mathbb{Z}}$ is a Riesz basis of $D(\tilde{A})$. Hence,

$$TD(A+BK) = D(\tilde{A}).$$

Denote by $\|f\|_{A+BK} = \|(A+BK)f\|$, $f \in D(A+BK)$ the norm in (48), and let $f = \sum_{n \in \mathbb{Z}} (f_n/\tilde{\lambda}_n)T^{-1}\varphi_n \in D(A+BK)$. Then,

$$\begin{aligned} \|T(A+BK)f\|^2 &= \|T \sum_{n \in \mathbb{Z}} f_n T^{-1}\varphi_n\|^2 \\ &= \sum_{n \in \mathbb{Z}} |f_n|^2 \\ &= \|f\|_{A+BK}^2, \end{aligned}$$

and

$$\begin{aligned} \|\tilde{A}Tf\|^2 &= \left\| \tilde{A} \sum_n \frac{f_n}{\tilde{\lambda}_n} \varphi_n \right\|^2 \\ &= \|f_n\|_{A+BK}^2. \end{aligned}$$

Hence, both $T(A+BK) : D(A+BK) \rightarrow H$ and $\tilde{A}T : D(A+BK) \rightarrow H$ are continuous. We now write, using Lemma 4.5,

$$T(A+BK)(T^{-1}\varphi_n) = \tilde{\lambda}_n T(T^{-1}\varphi_n) = \tilde{\lambda}_n \varphi_n = \tilde{A}\varphi_n = \tilde{A}T(T^{-1}\varphi_n).$$

Hence, (47) holds on a Riesz basis of $D(A+BK)$, which proves the proposition by continuity. \square

Proposition 4.4. *The operator $A+BK$ generates a C^0 -semigroup on H , i.e., the closed-loop system*

$$\begin{cases} y_t = (A+BK)y, & t \in (0, T) \\ y(0) = y_0 \in H \end{cases}$$

is well-posed in H .

Proof. Define the following semigroup on H :

$$S(t)y := T^{-1}e^{t\tilde{A}}T, \quad \forall t \geq 0, \forall y \in H.$$

Then, for $y \in D(A+BK)$, $Ty \in D(\tilde{A})$ and we can write:

$$\begin{aligned} \frac{S(t)y - y}{t} &= \frac{T^{-1}e^{t\tilde{A}}Ty - y}{t} \\ &= T^{-1} \frac{e^{t\tilde{A}}(Ty) - Ty}{t} \\ &\xrightarrow{t \rightarrow 0^+} T^{-1}\tilde{A}Ty \\ &= (A+BK)y. \end{aligned}$$

\square

At this stage, we have established F -equivalence: we have an invertible transformation T defined by (43) which maps the closed-loop system generated by $A+BK$ (K being defined by Proposition 4.1) to the target system which is generated by \tilde{A} .

We stress that we have achieved this for a broader class of target systems than those previously considered in the literature. We point out that our result covers situations well outside the scope of rapid stabilization, as the target spectrum $\{\tilde{\lambda}_n\}$ can have a wide range of behaviors. In the specific scope of stabilization problems, with the notable exception of [13], the usual target system is given by $\tilde{A} = A - \lambda I$, i.e., $\tilde{\lambda}_n = \lambda_n - \lambda$.

5 Complete solution of the spectral assignability problem: eigenvectors and uniqueness properties

Even though we have constructively defined the feedback K and established equivalence between the closed-loop operator $A + BK$ and the target operator \tilde{A} , the fact that B and K are unbounded makes the closed-loop operator delicate to manipulate. In particular, the description of its domain $D(A + BK)$ given in (48) is implicit since it involves the operator T . In this section, we further our study of K and the closed-loop operator, to give a complete solution of the spectral assignability problem.

5.1 Properties of the $\{\chi_n\}$ and related families

To obtain further properties of the feedback K , we first study the family $\{\chi_n\}_{n \in \mathbb{Z}}$ and other related families of vectors. The goal of this subsection is to prove that the $\{\chi_n\}$ form a Riesz basis of H , and give an expression for its bi-orthogonal basis.

To that end, we now turn to another, closely related, F -equivalence problem. Notice that, exchanging the role of λ_p and $\tilde{\lambda}_n$ in Proposition 4.1 yields the existence of a unique sequence $\{\tilde{k}_n\} \in \ell^\infty(\mathbb{Z}; \mathbb{C})$ such that

$$\sum_{n \in \mathbb{Z}} \frac{b_n \tilde{k}_n}{\lambda_p - \tilde{\lambda}_n} = 1, \quad \forall p \in \mathbb{Z} \quad (49)$$

and there exist $c, C > 0$ such that

$$c \leq |\tilde{k}_n| \leq C. \quad (50)$$

This allows us to define the feedback $\tilde{K} \in (H_{1/2})'$ by

$$\tilde{K}f = \sum_{n \in \mathbb{Z}} \tilde{k}_n \langle f, \varphi_n \rangle. \quad (51)$$

Following the same path as in the above section, we can define the invertible operator $\tau_{\tilde{K}}$ by

$$\tau_{\tilde{K}}\varphi_n := \tilde{k}_n \varphi_n, \quad \forall n \in \mathbb{Z}$$

and the invertible operator \tilde{T}_1 induced by the Cauchy matrix associated to (49):

$$\tilde{T}_1\varphi_n := \sum_{p \in \mathbb{Z}} \frac{1}{\lambda_p - \tilde{\lambda}_n} \varphi_p,$$

and finally the invertible operator

$$\tilde{T} := \tau_B \tilde{T}_1 \tau_{\tilde{K}}.$$

Now, we introduce the family of functions

$$\xi_n := (\tilde{A} - \lambda_n I)^{-1} B, \quad n \in \mathbb{Z}. \quad (52)$$

As in Lemma 4.1, $\xi_n \in H_{1/2}$, $\forall n \in \mathbb{Z}$

Lemma 5.1. *Let \tilde{K} be defined by (51). Then,*

(i) *For $n \in \mathbb{Z}$,*

$$(\tilde{A} + B\tilde{K})\xi_n = \lambda_n \xi_n, \quad \text{in } H_{1/2}.$$

(ii) *The operator $\tilde{A} + B\tilde{K}$ is densely defined and has the Riesz basis of eigenvectors $\{\xi_n\}$ with corresponding eigenvalues $\{\lambda_n\}$ for all $n \in \mathbb{Z}$.*

Proof. Through a proof similar to that of Lemma 4.5, we obtain

$$\tilde{K}\xi_n = -1, \quad \forall n \in \mathbb{Z}.$$

Similar computations then lead to (i).

Notice then that, for $n \in \mathbb{Z}$, by definition of T and \tilde{T} ,

$$T\varphi_n = k_n\xi_n, \quad \tilde{T}\varphi_n = \tilde{k}_n\chi_n. \quad (53)$$

Thus, $\{k_n\xi_n\}_{n \in \mathbb{Z}}$ is a Riesz basis of H . It then follows from (36) that $\{\xi_n\}_{n \in \mathbb{Z}}$ is also a Riesz basis of H , and (ii) follows. \square

From the proof above we also extract the crucial corollary, which provides the first part of Theorem 3:

Corollary 5.1. *The family $\{\chi_n\}_{n \in \mathbb{Z}}$ defined by (34) (resp. $\{\xi_n\}_{n \in \mathbb{Z}}$ defined by (52)) forms a Riesz basis of H of eigenvectors of $A + BK$ (resp. $\tilde{A} + B\tilde{K}$).*

Proof. Combining (53) with (50), we obtain, with the same argument as above, that $\{\chi_n\} = \{(1/\tilde{k}_n)\tilde{T}\varphi_n\}$ is a Riesz basis of H . \square

Adapting Proposition 4.3, we then get

Proposition 5.1. *The operator $\tilde{A} + B\tilde{K}$ has dense domain $D(\tilde{A} + B\tilde{K})$. Moreover, $\tilde{T}D(\tilde{A} + B\tilde{K}) = D(A)$ and*

$$\tilde{T}(\tilde{A} + B\tilde{K}) = A\tilde{T}.$$

Now, in order to come full circle, we prove

Proposition 5.2. *There holds*

$$\tilde{K} = -KT^{-1} \quad (54)$$

and

$$\tilde{T} = T^{-1}. \quad (55)$$

Proof. Recall that $T^{-1}\varphi_n \in D(A + BK)$ is the eigenvector of $A + BK$ with eigenvalue $\tilde{\lambda}_n$, so that we can compute in H_{-1} :

$$AT^{-1}\varphi_n + (KT^{-1}\varphi_n)B = \tilde{\lambda}_n T^{-1}\varphi_n, \quad \text{i.e.,} \quad T^{-1}\varphi_n = -(KT^{-1}\varphi_n)\chi_n. \quad (56)$$

Since both $\{T^{-1}\varphi_n\}_{n \in \mathbb{Z}}$ and $\{\chi_n\}_{n \in \mathbb{Z}}$ form Riesz bases of H , we deduce that there exist $c, C > 0$ such that

$$c \leq |KT^{-1}\varphi_n| \leq C, \quad \forall n \in \mathbb{Z}. \quad (57)$$

The Riesz basis $\{T^{-1}\varphi_n\}_{n \in \mathbb{Z}}$ has a bi-orthogonal family given by $\{T^*\varphi_n\}_{n \in \mathbb{Z}}$, since $\{\varphi_n\}_{n \in \mathbb{Z}}$ is a Hilbertian basis. Then, for $n \in \mathbb{Z}$, φ_n decomposes as

$$\begin{aligned} \varphi_n &= \sum_{p \in \mathbb{Z}} \langle \varphi_n, T^*\varphi_p \rangle T^{-1}\varphi_p \\ &= \sum_{p \in \mathbb{Z}} \langle T\varphi_n, \varphi_p \rangle T^{-1}\varphi_p \\ &= k_n \sum_{p \in \mathbb{Z}} \frac{b_p}{\tilde{\lambda}_p - \lambda_n} T^{-1}\varphi_p. \end{aligned}$$

Now, by Assumption 1.2, (5), Lemma 4.2, and (57), for any $n \in \mathbb{Z}$, we have $\left\{ \frac{b_p(KT^{-1}\varphi_p)}{\tilde{\lambda}_p - \lambda_n} \right\}_{p \in \mathbb{Z}} \in \ell^1(\mathbb{Z}; \mathbb{C})$.

Hence, we may compute

$$k_n = K\varphi_n = k_n \sum_{p \in \mathbb{Z}} \frac{b_p(KT^{-1}\varphi_p)}{\tilde{\lambda}_p - \lambda_n}.$$

Hence, $-(KT^{-1}\varphi_n)$ solves (49), which has a unique solution given by (\tilde{k}_n) . This implies

$$\tilde{k}_n = -KT^{-1}\varphi_n, \quad \forall n \in \mathbb{Z},$$

which proves (54).

Now, for $n \in \mathbb{Z}$, combining (56), (54) and (53), we have

$$T^{-1}\varphi_n = -(KT^{-1}\varphi_n)\chi_n = \tilde{k}_n\chi_n = \tilde{T}\varphi_n,$$

which proves (55), since both operators are isomorphisms on H . \square

We now deduce the explicit expression of the bi-orthogonal family to the Riesz basis $\{\chi_n\}_{n \in \mathbb{Z}}$, which provides the second part of Theorem 3.

Proposition 5.3. *The $\{\chi_n\}_{n \in \mathbb{Z}}$ admit a bi-orthogonal Riesz basis $\{\theta_n\}_{n \in \mathbb{Z}}$ which is given by*

$$\theta_n := -\overline{b_n \tilde{k}_n} ((A - \tilde{\lambda}_n I)^{-1})^* K^*, \quad \forall n \in \mathbb{Z}.$$

Consequently, the feedback K may also be defined by

$$Kf = - \sum_{n \in \mathbb{Z}} \langle f, \theta_n \rangle, \quad \forall f \in H_{1/2}.$$

Proof. Since the $\{\chi_n\}_{n \in \mathbb{Z}}$ form a Riesz basis of H , there exists a bi-orthogonal family $\{\theta_n\}_{n \in \mathbb{Z}}$ which is also a Riesz basis of H . As mentioned above, the Riesz basis $\{T^{-1}\varphi_n\}_{n \in \mathbb{Z}}$ has the bi-orthogonal family $\{T^*\varphi_n\}$. Thus, the Riesz bases $\{(1/\tilde{k}_n)T^{-1}\varphi_n\}_{n \in \mathbb{Z}}$ and $\{\overline{\tilde{k}_n}T^*\varphi_n\}_{n \in \mathbb{Z}}$ are bi-orthogonal. On the one hand, (53) and (55) imply

$$\frac{1}{\tilde{k}_n} T^{-1}\varphi_n = \chi_n, \quad \forall n \in \mathbb{Z}, \quad (58)$$

and on the other hand,

$$\begin{aligned} \overline{\tilde{k}_n} T^*\varphi_n &= \tilde{k}_n \sum_{p \in \mathbb{Z}} \langle T^*\varphi_n, \varphi_p \rangle \varphi_p \\ &= \tilde{k}_n \sum_{p \in \mathbb{Z}} \langle \varphi_n, T\varphi_p \rangle \varphi_p \\ &= \tilde{k}_n \sum_{p \in \mathbb{Z}} \frac{\overline{b_n \tilde{k}_p}}{\overline{\tilde{\lambda}_n - \tilde{\lambda}_p}} \varphi_p \\ &= \overline{\tilde{k}_n \tilde{b}_n} \sum_{p \in \mathbb{Z}} \frac{\overline{\tilde{k}_p}}{\overline{\tilde{\lambda}_n - \tilde{\lambda}_p}} \varphi_p \\ &= -\overline{\tilde{k}_n \tilde{b}_n} (A^* - \overline{\tilde{\lambda}_n} I)^{-1} K^* \\ &= -\overline{\tilde{k}_n \tilde{b}_n} ((A - \tilde{\lambda}_n I)^{-1})^* K^*, \end{aligned}$$

where $K^* \in H_{-1}$ is defined by $\langle f, K^* \rangle = Kf$ for any $f \in H_1$. The definition of θ_n follows easily from the above computations.

It only remains to prove the equivalent formulation of K . But this comes from the orthogonality of the bi-orthogonal family, which concludes the proof. \square

5.2 Uniqueness properties

Proposition 5.4. *The feedback K defined in (37) is the unique admissible feedback in \mathbb{K}_{ad} such that $A + BK$ has eigenvalues $\{\tilde{\lambda}_n\}_{n \in \mathbb{Z}}$, and a Riesz basis of eigenvectors.*

Proof. Let K_1 be another such feedback in \mathbb{K}_{ad} . Then there exists a Riesz basis $\{h_n\}_{n \in \mathbb{Z}}$ of H such that, for any $n \in \mathbb{Z}$,

$$(A + BK_1)h_n = \tilde{\lambda}_n h_n,$$

in H_{-1} , and recalling that $\tilde{\lambda}_n$ is in the resolvent set of A , we have, still in H_{-1} ,

$$Ah_n - \tilde{\lambda}_n h_n = -(K_1 h_n)B \quad \text{i.e.,} \quad h_n = -(K_1 h_n)(A - \tilde{\lambda}_n I)^{-1}B = -(K_1 h_n)\chi_n.$$

In particular, $K_1 h_n \neq 0$, and $D(K_1) = D(K)$. Then, we have

$$K_1 h_n = -(K_1 h_n)(K_1 \chi_n)$$

which implies $K_1 \chi_n = -1 = K \chi_n, \forall n \in \mathbb{Z}$. We conclude the proof with Proposition 5.3. \square

This uniqueness property allows us to describe the set of solutions to the equivalence equation (30):

Proposition 5.5. *Let $S \in \mathcal{L}(H)$ be continuously invertible. Then, (S, K) (where K is defined by Proposition 4.1) solves the equivalence equation if and only if there exist a sequence $\{\mu_n\}_{n \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z}; \mathbb{C})$ and $c, C > 0$ such that*

$$c \leq |\mu_n| \leq C, \quad \begin{cases} T^{-1}\varphi_n = \mu_n S^{-1}\varphi_n \\ S\chi_n = \mu_n T\chi_n \end{cases}, \quad \forall n \in \mathbb{Z}, \quad (59)$$

where T is defined by (43).

Proof. Suppose that (S, K) solves the equivalence equation. We have, by continuous invertibility of S and T ,

$$\begin{cases} S(A + BK) = \tilde{A}S & \text{on } D(A + BK) \\ (A + BK)T^{-1} = T^{-1}\tilde{A} & \text{on } D(\tilde{A}) \end{cases}.$$

By Definition 2.2, for $n \in \mathbb{Z}$, $S\chi_n \in D(\tilde{A})$. Thus, for $n \in \mathbb{Z}$, $ST^{-1}\varphi_n \in D(\tilde{A})$ and $T^{-1}S\chi_n \in D(A + BK)$. From (4.5), we then obtain

$$\tilde{A}ST^{-1}\varphi_n = S(A + BK)T^{-1}\varphi_n = \tilde{\lambda}_n ST^{-1}\varphi_n = ST^{-1}\tilde{A}\varphi_n.$$

Similarly, we obtain

$$(A + BK)T^{-1}S\chi_n = T^{-1}\tilde{A}S\chi_n = \tilde{\lambda}_n T^{-1}S\chi_n = T^{-1}S(A + BK)\chi_n.$$

From the above we deduce that ST^{-1} is an isomorphism of H which commutes with \tilde{A} , and $T^{-1}S$ is an isomorphism of H which commutes with $A + BK$.

Since A and $A + BK$ have simple eigenvalues, this implies that ST^{-1} and $T^{-1}S$ are diagonal operators on $\{\varphi_n\}_{n \in \mathbb{Z}}$ and $\{\chi_n\}_{n \in \mathbb{Z}}$ respectively, and by their continuous invertibility, there exist $\{\mu_n\}_n, \{\nu_n\}_n \in \ell^\infty(\mathbb{Z}; \mathbb{C})$ and $c, C > 0$ such that

$$c \leq |\mu_n| + |\nu_n| \leq C, \quad \begin{cases} ST^{-1}\varphi_n = \mu_n \varphi_n, \\ T^{-1}S\chi_n = \nu_n \chi_n \end{cases} \quad (60)$$

Now, for $n \in \mathbb{Z}$, $\tilde{k}_n T\chi_n = \varphi_n$, and applying ST^{-1} to this equality we obtain

$$\tilde{k}_n S\chi_n = \mu_n \varphi_n,$$

i.e.,

$$S\chi_n = \mu_n T\chi_n.$$

On the other hand, applying T to the second equality of (60) yields

$$S\chi_n = \nu_n T\chi_n.$$

Hence, $\mu_n = \nu_n, \forall n \in \mathbb{Z}$ and the second equality of (59) follows. To obtain the first equality, we simply apply S^{-1} to the first equality of (60).

Conversely, (59) implies that ST^{-1} (resp. $T^{-1}S$) has the eigenvectors $\{\varphi_n\}_{n \in \mathbb{Z}}$ (resp. $\{\chi_n\}_{n \in \mathbb{Z}}$) and thus commutes with \tilde{A} (resp. $A + BK$). Moreover, the bounds on $\{|\mu_n|\}_{n \in \mathbb{Z}}$ implies that $ST^{-1}D(\tilde{A}) = D(\tilde{A})$. Then, for $f \in D(A + BK)$, $Tf \in D(\tilde{A})$ and so $Sf = ST^{-1}Tf \in D(\tilde{A})$. For $f \in D(A + BK)$, we can then write, using the commutation property of ST^{-1} ,

$$S(A + BK)f = ST^{-1}T(A + BK)f = (ST^{-1})\tilde{A}Tf = \tilde{A}(ST^{-1})Tf = \tilde{A}Sf,$$

hence, (S, K) solves the equivalence equation. \square

6 Generalized Gramian operators

In the previous section, a generalized form of the backstepping method has allowed us to construct a feedback, and prove several previously unknown properties concerning the families $\{\chi_n\}$ and $\{\xi_n\}$, as well as their bi-orthogonal families. The uniqueness of the feedback K has allowed us to describe all the solutions of the equivalence equation. We now exploit these new results to give another expression of the feedback K , through a generalization of the well-known Gramian stabilization method.

6.1 A poleshifting Gramian operator

We begin by considering a particular solution to the equivalence equation, which is easily given by Proposition 5.5.

Proposition 6.1. *[Generalized Gramian operator] Define the following linear mapping:*

$$\Gamma_\infty \varphi_n = \overline{b_n} \chi_n, \quad \forall n \in \mathbb{Z}. \quad (61)$$

Then, Γ_∞^{-1} satisfies the equivalence equation, with K defined by (37),

$$\Gamma_\infty^{-1}(A + BK) = \tilde{A}\Gamma_\infty^{-1} \text{ on } D(A + BK)$$

together with the uniqueness condition

$$-\Gamma_\infty^* K^* = B. \quad (62)$$

Moreover, for $s \in [0, 1 - \frac{1}{2\alpha})$, $\Gamma_\infty \in \mathcal{L}(H_s)$, and we have:

$$K = -B^* \Gamma_\infty^{-1} \in (H_s)'. \quad (63)$$

Proof. Combining (61) with (58) implies

$$\Gamma_\infty^{-1} \chi_n = \frac{1}{\overline{b_n}} \varphi_n = \frac{\tilde{k}_n}{\overline{b_n}} T \chi_n, \quad \forall n \in \mathbb{Z}.$$

Clearly, (5) and (50) imply that $|\tilde{k}_n/\overline{b_n}|$ is bounded away from 0. We can thus use the converse of Proposition 5.5, which proves that (Γ_∞^{-1}, K) solves the equivalence equation.

To prove (62), note that for $n \in \mathbb{Z}$ we have

$$\begin{aligned} \langle -\Gamma_\infty^* K^*, \varphi_n \rangle &= -\langle K^*, \Gamma_\infty \varphi_n \rangle \\ &= -\langle K^*, \overline{b_n} \chi_n \rangle \\ &= -b_n \langle K^*, \chi_n \rangle \\ &= -b_n \overline{\overline{K} \chi_n} \\ &= b_n. \end{aligned}$$

where we have used (45).

Finally, let $f = \sum_{n \in \mathbb{Z}} f_n \varphi_n \in H_s$. Then,

$$\begin{aligned} \Gamma_\infty f &= \sum_{n \in \mathbb{Z}} \overline{b_n} f_n \chi_n \\ &= \sum_{n \in \mathbb{Z}} \overline{b_n} f_n \sum_{p \in \mathbb{Z}} \frac{b_p}{\lambda_p - \tilde{\lambda}_n} \varphi_p \\ &= \sum_{p \in \mathbb{Z}} b_p \left(\sum_{n \in \mathbb{Z}} \frac{\overline{b_n} f_n}{\lambda_p - \tilde{\lambda}_n} \right) \varphi_p. \end{aligned}$$

Now, for $P_1 \leq P_2 \in \mathbb{Z}$, we have, recalling hypothesis (5) on B ,

$$\begin{aligned} \sum_{p=P_1}^{P_2} |\lambda_p|^{2s} |b_p|^2 \left| \sum_{n \in \mathbb{Z}} \frac{\overline{b_n} f_n}{\lambda_p - \tilde{\lambda}_n} \right|^2 &\leq \sum_{p=P_1}^{P_2} \sum_{n \in \mathbb{Z}} |\lambda_p|^{2s} |b_p|^2 \frac{|b_n|^2 |f_n|^2}{|\lambda_p - \tilde{\lambda}_n|^2} \\ &\leq \sum_{n \in \mathbb{Z}} |b_n|^2 |f_n|^2 \sum_{p=P_1}^{P_2} \frac{|\lambda_p|^{2s} |b_p|^2}{|\lambda_p - \tilde{\lambda}_n|^2} \\ &\leq C \sum_{n \in \mathbb{Z}} |f_n|^2 \sum_{p=P_1}^{P_2} \frac{|\lambda_p|^{2s}}{|\lambda_p - \tilde{\lambda}_n|^2} \end{aligned}$$

Interpolating between the gap estimates (7) and (71) (see Lemma B.1), we have, for any $\nu \in (0, 1)$,

$$|\lambda_p - \tilde{\lambda}_n| \geq C |p|^{\nu(\alpha-1)} |p-n|^\nu |p-n|^{\alpha(1-\nu)}, \quad \forall n, p \in \mathbb{Z}.$$

Taking $\nu = \frac{s\alpha}{\alpha-1}$, we then obtain the estimate

$$|\lambda_p - \tilde{\lambda}_n| \geq C |p|^{s\alpha} |p-n|^{(1-s)\alpha}.$$

Now, the spectral estimates (3)–(4) easily yield $|\lambda_p| \sim |p|^\alpha$. Hence, there exist constants $C, C' > 0$ such that

$$\frac{|\lambda_p|^{2s}}{|\lambda_p - \tilde{\lambda}_n|^2} \leq C \frac{|\lambda_p|^{2s}}{|p|^{2s\alpha}} \frac{1}{|p-n|^{2(1-s)\alpha}} \leq C' \frac{1}{|p-n|^{2(1-s)\alpha}}.$$

Then,

$$\begin{aligned} \sum_{p=P_1}^{P_2} |\lambda_p|^{2s} |b_p|^2 \left| \sum_{n \in \mathbb{Z}} \frac{\overline{b_n} f_n}{\lambda_p - \tilde{\lambda}_n} \right|^2 &\leq C \sum_{n \in \mathbb{Z}} |f_n|^2 \left(\frac{|\lambda_n|^{2s}}{|\lambda_n - \tilde{\lambda}_n|^2} + \sum_{\substack{p=P_1 \\ p \neq n}}^{P_2} \frac{C'}{|p-n|^{2(1-s)\alpha}} \right) \\ &\leq C \left(C'' \sum_{n \in \mathbb{Z}} |f_n|^2 |\lambda_n|^{2s} + \sum_{n \in \mathbb{Z}} |f_n|^2 \sum_{\substack{p=P_1 \\ p \neq n}}^{P_2} \frac{C'}{|p-n|^{2(1-s)\alpha}} \right) \\ &\leq C''' (\|f\|_s^2 + \|f\|), \end{aligned}$$

where the last two lines are obtained by using the third inequality of (6), the definition of the H_s norm $\|\cdot\|_s$, and the fact that, since $s < 1 - \frac{1}{2\alpha}$, then $2(1-s)\alpha > 1$,

$$\sum_{p \in \mathbb{Z}} \frac{1}{|p|^{2(1-s)\alpha}} < +\infty.$$

By continuity of the injection $H_s \rightarrow H$, the above proves that for $f \in H_s, \Gamma_\infty f \in H_s$ and there exists a constant $C > 0$ such that

$$\|\Gamma_\infty f\|_{H_s} \leq C \|f\|_{H_s},$$

which concludes the proof. \square

Considering only (61), its connection to the well-known stabilization Gramian (26) is not clear, and, without additional hypothesis, the analogy mainly rests on the relationship (62) between Γ_∞ and K .

We will now see that under further Assumptions on A and \tilde{A} , Γ_∞ can be written as an integral operator, thus recovering Petrov-Galerkin Gramians in a poleshifting context instead of the usual model reduction framework. This operator naturally reduces to the stabilization Gramian C_λ^∞ in the appropriate framework.

6.2 An integral operator formulation

Using a resolvent formula, we can give an integral expression of χ_n and Γ_∞ . Recall the following well-known result (see for example [32, Proposition 4.14]):

Lemma 6.1. *Let M be an unbounded operator which generates a C_0 -semigroup on $H_{-1/2}$. Suppose that M has a Riesz basis of eigenvectors with associated eigenvalues ρ_n , so that $\sup_n \operatorname{Re}(\rho_n) < \infty$. Then, for all $f \in H_{-1/2}$, for all $\rho \in \mathbb{C}$ such that $\operatorname{Re}(\rho) > \sup_n \operatorname{Re}(\rho_n)$,*

$$(M - \rho I)^{-1} f = - \int_0^\infty e^{-\rho t} e^{tM} f dt. \quad (64)$$

Poleshifting to the right. The above lemma leads to an integral expression of χ_n when the target spectrum $\{\tilde{\lambda}_n\}$ is “completely to the right” of $\{\lambda_n\}$.

To that end, recall that we defined

$$\omega_+ := \sup_n \operatorname{Re}(\lambda_n), \quad \omega_- := \inf_n \operatorname{Re}(\lambda_n).$$

Recall that, since A generates a C_0 -semigroup by Assumption 1.1, $\omega_+ < \infty$.

Proposition 6.2. *Let $n \in \mathbb{Z}$. Assume that*

$$\operatorname{Re}(\tilde{\lambda}_n) > \omega_+. \quad (65)$$

Then,

$$\chi_n = - \int_0^\infty e^{-\tilde{\lambda}_n t} e^{tA} B dt \quad (66)$$

Proof. We apply Lemma 6.1 with $M = A$ and $\rho = \tilde{\lambda}_n$, thanks to (65). \square

Poleshifting to the left. In practice, it is far more common to consider a target spectrum shifted “completely to the left” (which is typical of stabilization problems where one wants to damp the dominant eigenvalues), the generic example being $\tilde{\lambda}_n = \lambda_n - \lambda$ with $\lambda > 0$ large enough.

In that case, one can still apply Lemma 6.1 to obtain an integral expression for χ_n , but this requires the additional assumption (8) on A for the integral to be well-defined.

Proposition 6.3. *Let $n \in \mathbb{Z}$. Under the Assumption (8), the operator $-A$ generates a C_0 -semigroup on $H_{-1/2}$, which we denote by e^{-tA} . Moreover, if*

$$\operatorname{Re}(\tilde{\lambda}_n) < \omega_-, \quad (67)$$

then

$$\chi_n = \int_0^\infty e^{\tilde{\lambda}_n t} e^{-tA} B dt. \quad (68)$$

Proof. Under Assumption 1.1, $-A$ is densely defined and closed on $H_{-1/2}$. Moreover, the segment (ω_-, ∞) is in the resolvent set of $-A$. Then, for $f \in H_{-1/2}$ and for $\lambda > \omega_-$ we have

$$\|(-A - \lambda I)^{-1} f\|_{-\frac{1}{2}}^2 = \sum_n \frac{|\langle f, \varphi_n \rangle|^2}{|\lambda_n - \alpha| |\lambda_n - \lambda|^2} = \sum_n \frac{|\langle f, \varphi_n \rangle|^2}{|\lambda_n - \alpha| (\operatorname{Im}(\lambda_n)^2 + (\operatorname{Re}(\lambda_n) + \lambda)^2)}.$$

Now, $\lambda > \omega$ so that $\operatorname{Re}(\lambda_n) + \lambda \geq \inf_n \operatorname{Re}(\lambda_n) + \lambda = \lambda - \omega \geq 0$. We deduce

$$\|(-A - \lambda I)^{-1} f\|_{-\frac{1}{2}} \leq \frac{1}{\lambda - \omega} \|f\|_{-\frac{1}{2}}.$$

Applying a variant of the Hille-Yosida Theorem (see for example [35, page 12, Corollary 3.8]), we deduce that $-A$ generates a C_0 -semigroup on $H_{-1/2}$. We can then apply Lemma 6.1 with $M = -A$ and $\rho = -\tilde{\lambda}_n$ and deduce (68). \square

Generalized Gramian operators. We can now provide integral expressions for the invertible transformation Γ_∞ .

Proposition 6.4. *Under Assumption (9), there holds*

$$\Gamma_\infty f = \int_0^\infty e^{-tA} B B^* e^{t\tilde{A}} f dt, \quad \forall f \in H_{1/2}. \quad (69)$$

Under Assumption (10), there holds

$$\Gamma_\infty f = - \int_0^\infty e^{tA} B B^* e^{-t\tilde{A}} f dt, \quad \forall f \in H_{1/2}. \quad (70)$$

Proof. We prove only (69), the proof of (70) is essentially the same.

Let $f = \sum_n f_n \varphi_n \in H_{1/2}$. Note that for all $t \geq 0$,

$$e^{-tA} B B^* e^{t\tilde{A}} f = \left(\sum_n \bar{b}_n f_n e^{t\tilde{\lambda}_n} \right) e^{-tA} B = \sum_n \bar{b}_n f_n e^{t\tilde{\lambda}_n} e^{-tA} B.$$

Moreover,

$$\|e^{-tA} B\|_{-\frac{1}{2}} \leq e^{-t\omega_-} \|B\|_{-\frac{1}{2}}.$$

Let $n \in \mathbb{Z}$. By assumption (9), $\operatorname{Re}(\tilde{\lambda}_n) - \omega_- < 0$, thus

$$\int_0^\infty \|\bar{b}_n f_n e^{t\tilde{\lambda}_n} e^{-tA} B\|_{-\frac{1}{2}} dt \leq \frac{|b_n| |f_n|}{\omega_- - \operatorname{Re}(\tilde{\lambda}_n)} \|B\|_{-\frac{1}{2}} \leq \frac{|b_n| |f_n|}{\omega_- - \sup_p \operatorname{Re}(\tilde{\lambda}_p)} \|B\|_{-\frac{1}{2}}.$$

Since $f \in H_{1/2}$, and by Lemma 3.1, we have $(f_n) \in \ell^1(\mathbb{Z}; \mathbb{C})$. Thus the series $\sum_{n \in \mathbb{Z}} \int_0^\infty \|\bar{b}_n f_n e^{t\tilde{\lambda}_n} e^{-tA} B\|_{-\frac{1}{2}} dt$ converges and we can write

$$\begin{aligned} \int_0^\infty e^{-tA} B B^* e^{t\tilde{A}} f dt &= \int_0^\infty \sum_n \bar{b}_n f_n e^{t\tilde{\lambda}_n} e^{-tA} B dt \\ &= \sum_n \int_0^\infty \bar{b}_n f_n e^{t\tilde{\lambda}_n} e^{-tA} B dt \\ &= \sum_n \bar{b}_n f_n \int_0^\infty e^{t\tilde{\lambda}_n} e^{-tA} B dt. \end{aligned}$$

Thus, recalling (68), we obtain

$$\int_0^\infty e^{-tA} BB^* e^{t\tilde{A}} f dt = \sum_n \overline{b_n} f_n \chi_n = \sum_n f_n \Gamma_\infty \varphi_n = \Gamma_\infty f.$$

□

Combining Propositions 6.1 and 6.4 directly yields Theorem 2. We now focus on the special case where

- The operators $-A$ and $-A^*$ generate C_0 -semigroups, *i.e.*,

$$\omega_- > -\infty;$$

- The target operator is given by

$$\tilde{A} = -A^* - \lambda I, \quad \lambda > \omega_+ - \omega_-;$$

so that (9) is satisfied, and one can directly apply Proposition (6.4):

Corollary 6.1. *Under the assumptions listed above, we have*

$$\Gamma_\infty f = \int_0^\infty e^{-tA} BB^* e^{t(-A^* - \lambda I)} f dt = \int_0^\infty e^{-\lambda t} e^{-tA} BB^* e^{-tA^*} f dt = C_\lambda^\infty f, \quad \forall f \in H_{1/2}.$$

That is, the stabilization Gramian (26) is a particular case of the generalized Gramian introduced in Proposition 6.1. The latter Proposition, combined with Corollary 6.1 and Proposition 5.5, directly proves Theorem 1.

7 Perspectives

Multi-dimensional systems. This work sheds some light on the links between different stabilization methods. One of the main difficulties of the backstepping method for PDEs is the barrier of dimension due to the lack of spectral gap as the dimension of the physical domain grows. Indeed, for eigenvalues of finite multiplicities, the Fredholm backstepping method can be applied [22, 24]. This framework mainly covers PDEs in 1 space dimension. It has also been extended to some specific 2-dimensional geometries in [44]. More recently, [23] tackled the rapid stabilization of parabolic equations with the backstepping method in any dimension, but with some assumptions on the size of the control regions.

This leaves wide open the question of solving spectral assignability problems for PDEs in higher space dimension, or, spectrally speaking, for operators with a relaxed spectral gap condition. Of particular interest is the extension of the generalized Gramians introduced in Section 6. Indeed, the well-known stabilization Gramian (introduced in Section 2.2) can be defined for PDEs in any finite space dimension, and is invertible if and only if the system is controllable. It seems straightforward to extend the integral operator expressions of the generalized Gramians (69)–(70) to systems with a higher underlying space dimension (or a weaker spectral gap), but the study of their invertibility poses a challenging open problem.

Hyperbolic systems. Another well-known hurdle in extensions of the generalized backstepping method lies in the growth exponent for the spectrum of A . In (3), the bound $\alpha > 1$ is central to our computations, especially to compute the feedback K using generalized Cauchy matrices in Section 4. This critical exponent was believed to be 3/2 until it was established in [24] that it was at most 1.

This exponent 1 essentially corresponds to 1-D hyperbolic systems, for which backstepping results have been obtained only for particular cases [13, 47, 48]. The question of a generalized Fredholm backstepping method (as developed in [24], and in the present article) for hyperbolic systems is a longstanding open problem.

In this article, we introduced generalized Gramians after having established F -equivalence using generalized backstepping, and proved that both viewpoints are essentially equivalent. This begs the question of doing things in the reverse order: for 1-D hyperbolic systems, generalized Gramians can be readily defined. One could then consider using them to prove that F -equivalence can be established with a Fredholm backstepping type transformation satisfying

$$\begin{cases} T(A + BK) = \tilde{A}T \\ TB = B. \end{cases}$$

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A Dealing with the case $\lambda_n = \tilde{\lambda}_n$

We prove

Lemma A.1. *Under Assumption 1.1 and 1.2, and allowing the case $\lambda_n = \tilde{\lambda}_n$, then there exists (T, K) such that*

$$\begin{cases} TA + BK = \tilde{A}T \\ TB = B \end{cases}$$

Proof. The proof follows closely [23]. Denote $\mathcal{Z} := \{n \in \mathbb{Z} \mid \lambda_n = \tilde{\lambda}_n\}$ and

$$H = H_{\mathcal{Z}} \oplus H_{\mathbb{Z} \setminus \mathcal{Z}}$$

with $H_{\mathcal{Z}} := \overline{\{\varphi_n \mid n \in \mathcal{Z}\}}^{\|\cdot\|_H}$ and $H_{\mathbb{Z} \setminus \mathcal{Z}} := \overline{\{\varphi_n \mid n \in \mathbb{Z} \setminus \mathcal{Z}\}}^{\|\cdot\|_H}$. We write $T := T\Pi_{H_{\mathcal{Z}}} + T\Pi_{H_{\mathbb{Z} \setminus \mathcal{Z}}}$. For $n \in \mathbb{Z} \setminus \mathcal{Z}$, we use the classical definition,

$$\langle T\varphi_n, \varphi_k \rangle = -\frac{b_k k_n}{\lambda_n - \tilde{\lambda}_k}, \quad k \in \mathbb{Z}.$$

For $n \in \mathcal{Z}$, since

$$k_n b_k = (\lambda_n - \tilde{\lambda}_n) \langle T\varphi_n, \varphi_k \rangle = 0,$$

we impose $K\varphi_n = 0, \forall n \in \mathcal{Z}$. This yields,

$$\lambda_n T\varphi_n = \tilde{A}T\varphi_n$$

so $T\varphi_n = c_n \varphi_n, n \in \mathcal{Z}$ for $c_n \in \mathbb{C}^*$. We turn to the $TB = B$ condition :

$$\sum_{n \notin \mathcal{Z}} b_n T\varphi_n + \sum_{n \in \mathcal{Z}} b_n c_n \varphi_n = \sum_{n \in \mathbb{Z}} b_n \varphi_n.$$

For $k \in \mathbb{Z} \setminus \mathcal{Z}$

$$\sum_{n \notin \mathcal{Z}} b_n \langle T\varphi_n, \varphi_k \rangle = -b_k \sum_{n \notin \mathcal{Z}} \frac{b_n k_n}{\lambda_n - \tilde{\lambda}_k} = b_k, \quad \forall k \in \mathbb{Z} \setminus \mathcal{Z}.$$

Proposition 4.1 defines uniquely $\{k_n\}_{n \in \mathbb{Z} \setminus \mathcal{Z}}$. Consider now $k \in \mathcal{Z}$. The $TB = B$ condition writes,

$$c_k = 1 - \sum_{n \notin \mathcal{Z}} \frac{b_n}{b_k} \langle T\varphi_n, \varphi_k \rangle = 1 + \sum_{n \notin \mathcal{Z}} \frac{b_n k_n}{\tilde{\lambda}_n - \lambda_k}, \quad k \in \mathcal{Z}$$

We claim : for any $k \in \mathcal{Z}$,

$$\sum_{n \notin \mathcal{Z}} \frac{b_n k_n}{\tilde{\lambda}_n - \lambda_k} \neq -1.$$

Assume the contrary : there exists $k_0 \in \mathcal{Z}$ such that

$$\sum_{n \notin \mathcal{Z}} \frac{b_n k_n}{\tilde{\lambda}_n - \lambda_{k_0}} = -1$$

Then define C the Cauchy matrix

$$C_{nk} = \frac{1}{\tilde{\lambda}_n - \lambda_k}, \quad \text{for } n, k \in (\mathbb{Z} \setminus \mathcal{Z}) \cup \{k_0\}.$$

Thus, since $k_{k_0} = 0$ by definition,

$$C \begin{pmatrix} \vdots \\ k_{k_0-1} b_{k_0-1} \\ 0 \\ k_{k_0+1} b_{k_0+1} \\ \vdots \end{pmatrix} = - \begin{pmatrix} \vdots \\ 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix}$$

Then, since C satisfies the hypothesis in Proposition 4.1 applies. Therefore, there exists $c, C > 0$ such that

$$0 < c < |k_n| < C, \quad \forall n \in (\mathbb{Z} \setminus \mathcal{Z}) \cup \{k_0\}$$

But since $k_{k_0} = 0$, we find the contradiction, proving the claim. The invertibility of T the follows easily since

$$|c_n| \rightarrow 1, \quad \text{as } |n| \rightarrow +\infty$$

□

B Definition and properties of T and K

B.1 Technical lemmas on the spectra

First, we reformulate the lower uniform bounds on the spectral gap into a simpler form,

Lemma B.1. *Under Assumption 1.1 and 1.2, then there exists $c > 0$ such that,*

$$|\lambda_n - \lambda_k| \geq c|n - k|^\alpha, \quad \alpha > 1, \quad \forall k, n \in \mathbb{Z}, \quad (71)$$

$$|\lambda_n - \tilde{\lambda}_k| \geq c|n - k|^\alpha, \quad \alpha > 1, \quad \forall k, n \in \mathbb{Z}, \quad (72)$$

$$|\tilde{\lambda}_n - \tilde{\lambda}_k| \geq c|n - k|^\alpha, \quad \alpha > 1, \quad \forall k, n \in \mathbb{Z}, \quad (73)$$

Proof. Without loss of generality, assume $|k| > |n|$, the case $|n| = |k|$ being trivial. From Assumption 1.1 and 1.2 the left-hand side of each inequalities in (71) are lower bounded by

$$c|k|^{\alpha-1}|k - n|,$$

for some constant $c > 0$. We are able to conclude if there is a constant $c' > 0$ such that,

$$|k|^{\alpha-1} \geq c|k - n|^{\alpha-1},$$

but this comes from the monotonicity of the function $x \in \mathbb{R}^+ \mapsto x^{\alpha-1}$ with $\alpha > 1$ and $|k - n| \leq 2|k|$. □

We proceed with the proof of Lemma 3.1

Proof. The first part of Lemma 3.1 comes from the previous Lemma. The second part follows from the convergence of

$$\sum_{k \in \mathbb{Z}^*} \frac{1}{|k|^\alpha}$$

□

We can now prove Lemma 4.2

Proof. We prove assertions (i), (iii) and (v), as (ii), (iv) and (vi) follow similarly using the uniform upper bound on the gap $|\lambda_n - \tilde{\lambda}_n|$. We begin with (i). Using Lemma B.1, and the uniform lower bound on the spectral gaps,

$$\begin{aligned} \left| \prod_{\substack{k \in \mathbb{Z} \\ k \neq p}} \left(1 + \frac{\lambda_k - \tilde{\lambda}_k}{\lambda_p - \lambda_k} \right) \right| &\leq \prod_{\substack{k \in \mathbb{Z} \\ k \neq p}} \left(1 + \left| \frac{\lambda_k - \tilde{\lambda}_k}{\lambda_p - \lambda_k} \right| \right) \\ &\leq \prod_{\substack{k \in \mathbb{Z} \\ k \neq p}} \left(1 + \frac{C}{|p - k|^\alpha} \right) \\ &= \exp \left(\sum_{\substack{k \in \mathbb{Z} \\ k \neq p}} \ln \left(1 + \frac{C}{|p - k|^\alpha} \right) \right) \\ &\leq \exp \left(\ln \left(1 + \frac{C}{|2p|^\alpha} \right) + \sum_{\substack{k \in \mathbb{Z} \\ |k| < |p|}} \ln \left(1 + \frac{C}{\left| |p| - |k| \right|^\alpha} \right) + \sum_{\substack{k \in \mathbb{Z} \\ |k| > |p|}} \ln \left(1 + \frac{C}{\left| |p| - |k| \right|^\alpha} \right) \right) \\ &= \exp \left(\ln \left(1 + \frac{C}{|2p|^\alpha} \right) + 2 \sum_{j \in \mathbb{N}^*} \ln \left(1 + \frac{C}{|j|^\alpha} \right) \right) \\ &\leq \exp \left(2 \sum_{k \in \mathbb{N}^*} \ln \left(1 + \frac{C}{k^\alpha} \right) \right). \end{aligned}$$

Since $\ln(1 + 1/y)$ is decreasing over $y \in (1, \infty)$,

$$\begin{aligned} \sum_{k \in \mathbb{N}^*} \ln \left(1 + \frac{C}{k^\alpha} \right) &\leq \ln(1 + C) + \int_1^\infty \ln \left(1 + \frac{C}{x^\alpha} \right) dx \\ &= \ln(1 + C) + x \ln \left(1 + \frac{C}{x^\alpha} \right) \Big|_1^\infty + \alpha \int_1^\infty \frac{C}{x^\alpha + C} dx \\ &= \alpha(C)^{\frac{1}{\alpha}} \int_{C^{-\frac{1}{\alpha}}}^\infty \frac{1}{y^\alpha + 1} dy \\ &\leq \alpha(C)^{\frac{1}{\alpha}} \int_0^\infty \frac{1}{y^\alpha + 1} dy \\ &\leq C \end{aligned}$$

Hence, there exists $C > 0$ such that, for every $p \in \mathbb{Z}$,

$$\left| \prod_{\substack{k \in \mathbb{Z} \\ k \neq p}} \left(1 + \frac{\lambda_k - \tilde{\lambda}_k}{\lambda_p - \lambda_k} \right) \right| \leq C.$$

We prove (iii) and (v) using the same computations, for any $p \in \mathbb{Z}$,

$$\sum_{n \in \mathbb{Z}} \left| \frac{1}{\tilde{\lambda}_n - \lambda_p} \right| \leq C + \sum_{n \in \mathbb{Z} \setminus \{p\}} \frac{C}{\|n - p\|^\alpha} \leq C \quad (74)$$

and

$$\sum_{n \in \mathbb{Z}} \left| \frac{1}{\tilde{\lambda}_n - \lambda_p} \right|^2 \leq C + \sum_{n \in \mathbb{Z} \setminus \{p\}} \frac{C}{\|n - p\|^{2\alpha}} \leq C$$

□

B.2 Definition of the F -equivalence transformation T

We now prove Proposition 4.2.

Proof. For $f \in H$,

$$\|\tilde{T}f\|_H^2 = \sum_{p \in \mathbb{Z}} \left| \sum_{n \in \mathbb{Z}} \frac{f_n}{\lambda_n - \tilde{\lambda}_p} \right|^2 \quad (75)$$

$$\leq 2 \sum_{p \in \mathbb{Z}} \left| \frac{f_p}{\lambda_p - \tilde{\lambda}_p} \right|^2 + 2 \sum_{p \in \mathbb{Z}} \left| \sum_{n \neq p} \frac{f_n}{\lambda_n - \tilde{\lambda}_p} \right|^2 \quad (76)$$

$$\leq C \|f\|_H^2 + 2 \sum_{p \in \mathbb{Z}} \left(\sum_{n \neq p} \frac{|f_n|^2}{|\lambda_n - \tilde{\lambda}_p|} \right) \left(\sum_{n \neq p} \frac{1}{|\lambda_n - \tilde{\lambda}_p|} \right) \quad (77)$$

$$\leq C \|f\|_H^2 + C \sum_{p \in \mathbb{Z}} \sum_{n \neq p} \frac{|f_n|^2}{|\lambda_n - \tilde{\lambda}_p|} \quad (78)$$

$$\leq C \|f\|_H^2 + C \sum_{n \in \mathbb{Z}} \sum_{p \neq n} \frac{|f_n|^2}{|\lambda_n - \tilde{\lambda}_p|} \quad (79)$$

$$\leq C \|f\|_H^2 + C \sum_{n \in \mathbb{Z}} |f_n|^2 \quad (80)$$

$$\leq C \|f\|_H^2 \quad (81)$$

Moreover,

$$\|\tilde{T}^{-1}f\|_H^2 = \sum_{p \in \mathbb{Z}} \left| \sum_{n \in \mathbb{Z}} f_n \left(\frac{(\lambda_p - \tilde{\lambda}_p)(\tilde{\lambda}_n - \lambda_n)}{\tilde{\lambda}_n - \lambda_p} \right) \left(\prod_{k \neq p} \left(1 + \frac{\lambda_k - \tilde{\lambda}_k}{\lambda_p - \lambda_k} \right) \right) \left(\prod_{m \neq n} \left(1 + \frac{\tilde{\lambda}_m - \lambda_m}{\tilde{\lambda}_n - \tilde{\lambda}_m} \right) \right) \right|^2 \quad (82)$$

$$\leq C \sum_{p \in \mathbb{Z}} \left| \sum_{n \in \mathbb{Z}} \frac{f_n}{\tilde{\lambda}_n - \lambda_p} \right|^2 \quad (83)$$

$$\leq C \|f\|_H^2 \quad (84)$$

using similar reasoning than from the continuity of \tilde{T} .

Finally, we show that \tilde{T}^{-1} is the inverse of \tilde{T} , i.e.

$$\tilde{T} \circ \tilde{T}^{-1} = \tilde{T}^{-1} \circ \tilde{T} = Id : H \rightarrow H.$$

We focus on the case of $\check{T}^{-1} \circ \check{T}$, while the other case can be treated similarly. Thanks to the boundedness of \check{T} and \check{T}^{-1} , $\check{T}^{-1} \circ \check{T}$ is a bounded operator from H to itself. Since $\{\varphi_n\}_{n \in \mathbb{N}^*}$ is an orthogonal basis of H , it suffices to show that

$$(\check{T}^{-1} \circ \check{T})\varphi_n = \varphi_n, \quad \forall n \in \mathbb{N}^*.$$

Recalling the definition of C_{pn} and C_{pn}^{-1} from (39) and (40),

$$\begin{aligned} (\check{T}^{-1} \circ \check{T})\varphi_n &= \check{T}^{-1}(\check{T}\varphi_n) \\ &= \check{T}^{-1} \left(\sum_{p \in \mathbb{Z}} C_{pn} \varphi_p \right) \\ &= \sum_{p \in \mathbb{Z}} C_{pn} \check{T}^{-1} \varphi_p \\ &= \sum_{p \in \mathbb{Z}} C_{pn} \sum_{k \in \mathbb{Z}} C_{kp}^{-1} \varphi_k \\ &= \sum_{k \in \mathbb{Z}} \left(\sum_{p \in \mathbb{Z}} C_{kp}^{-1} C_{pn} \right) \varphi_k \end{aligned}$$

then $\check{T} \circ \check{T}^{-1} = Id_H$ if and only if

$$\sum_{p \in \mathbb{Z}} C_{kp}^{-1} C_{pn} = \delta_{kn}, \quad \forall k, n \in \mathbb{Z}, \quad (85)$$

where δ_{kn} denotes the Kronecker symbol. To prove (85), we prove, for fixed $k, n \in \mathbb{Z}$,

$$\sum_{p \in \mathbb{Z}} C_{kp}^{-1} C_{pn} = \lim_{N \rightarrow +\infty} \sum_{0 \leq |p| \leq N} (C_{kp}^{-1})^N C_{pn}^N = \delta_{kn}. \quad (86)$$

where, for any $N \in \mathbb{N}^*$, $(C_{kp}^{-1})^N$ and C_{pn}^N are the entries of the Cauchy matrices ([38])

$$C^N := (C_{pn}^N)_{0 \leq |n|, |p| \leq N}, \quad (C^N)^{-1} := ((C_{pn}^{-1})^N)_{0 \leq |n|, |p| \leq N} \in \mathbb{C}^{(2N+1) \times (2N+1)} \quad (87)$$

with,

$$\begin{aligned} C_{pn}^N &= \frac{1}{\lambda_n - \tilde{\lambda}_p}, \quad 0 \leq |n|, |p| \leq N, \\ (C_{pn}^{-1})^N &= \left(\frac{(\lambda_p - \tilde{\lambda}_p)(\tilde{\lambda}_n - \lambda_n)}{\tilde{\lambda}_n - \lambda_p} \right) \left(\prod_{\substack{k \neq p \\ |k| \leq N}} \left(1 + \frac{\lambda_k - \tilde{\lambda}_k}{\lambda_p - \lambda_k} \right) \right) \left(\prod_{\substack{m \neq n \\ |m| \leq N}} \left(1 + \frac{\tilde{\lambda}_m - \lambda_m}{\tilde{\lambda}_n - \tilde{\lambda}_m} \right) \right), \quad 0 \leq |n|, |p| \leq N. \end{aligned}$$

Since, by definition, C^N is a Cauchy matrix and $(C^N)^{-1}$ is its inverse, we have,

$$\sum_{0 \leq |p| \leq N} (C_{ip}^{-1})^N C_{pj}^N = \delta_{ij}, \quad \forall 0 \leq |i|, |j| \leq N \quad (88)$$

To prove (86), first fix $k, n \in \mathbb{Z}$. From now on, we consider $N \in \mathbb{N}^*$ such that $N \geq 2(|k| + |n|)$. For this N , (88) still holds and, from (39), we have $C_{pn} = C_{pn}^N$ for $|n|, |p| \leq N$, and thus,

$$\sum_{0 \leq |p| \leq N} (C_{kp}^{-1})^N C_{pn} = \sum_{0 \leq |p| \leq N} (C_{kp}^{-1})^N C_{pn}^N = \delta_{kn} \quad \forall N \geq 2(|k| + |n|). \quad (89)$$

Moreover, for any $N \geq 2(|k| + |n|)$ and any $p \in \mathbb{Z}$ such that $|p| \leq N$, we have

$$\begin{aligned}
|C_{pn}^{-1} - (C_{pn}^{-1})^N| &\leq |C_{pn}^{-1}| \left| 1 - \left(\prod_{|k|>N} \left| 1 + \frac{\lambda_k - \tilde{\lambda}_k}{\lambda_p - \lambda_k} \right|^{-1} \right) \left(\prod_{|m|>N} \left| 1 + \frac{\tilde{\lambda}_m - \lambda_m}{\tilde{\lambda}_n - \tilde{\lambda}_m} \right|^{-1} \right) \right| \\
&\leq |C_{pn}^{-1}| \left| 1 - \prod_{|k|>N} \left(1 + \left| \frac{\lambda_k - \tilde{\lambda}_k}{\lambda_p - \lambda_k} \right| \right)^{-1} \prod_{|m|>N} \left(1 + \left| \frac{\tilde{\lambda}_m - \lambda_m}{\tilde{\lambda}_n - \tilde{\lambda}_m} \right| \right)^{-1} \right| \\
&= |C_{pn}^{-1}| \left(1 - \exp \left(- \sum_{|k|>N} \ln \left(1 + \left| \frac{\lambda_k - \tilde{\lambda}_k}{\lambda_p - \lambda_k} \right| \right) - \sum_{|m|>N} \ln \left(1 + \left| \frac{\tilde{\lambda}_m - \lambda_m}{\tilde{\lambda}_n - \tilde{\lambda}_m} \right| \right) \right) \right).
\end{aligned}$$

From the uniform bounds of Assumption 1.1 and 1.2, we have,

$$\left| \frac{\lambda_k - \tilde{\lambda}_k}{\lambda_p - \lambda_k} \right| \leq \frac{C}{|k|^{\alpha-1}||k| - |p||} \leq \frac{C}{|k|^{\alpha-1}||k| - N|}, \quad \forall |k| > N \geq |p|, \quad (90)$$

for $C > 0$. With a similar estimate on,

$$\left| \frac{\tilde{\lambda}_m - \lambda_m}{\tilde{\lambda}_n - \tilde{\lambda}_m} \right|,$$

we deduce, that for fixed $k, n \in \mathbb{Z}$ and for any $N \geq 2(|n| + |k|)$ and any $|p| \leq N$,

$$\begin{aligned}
|C_{pn}^{-1} - (C_{pn}^{-1})^N| &\leq |C_{pn}^{-1}| \left(1 - \exp \left(- \sum_{|k|>N} \ln \left(1 + \frac{C}{|k|^{\alpha-1}||k| - |p||} \right) - \sum_{|m|>N} \ln \left(1 + \frac{C}{|m|^{\alpha-1}||m| - |n||} \right) \right) \right) \\
&\leq |C_{pn}^{-1}| \left(1 - \exp \left(-4 \sum_{k>N} \ln \left(1 + \frac{C}{|k|^{\alpha-1}|k - N|} \right) \right) \right) \\
&\leq 4|C_{pn}^{-1}| \sum_{k>N} \frac{C}{|k|^{\alpha-1}|k - N|} \\
&\leq \frac{C}{|\tilde{\lambda}_n - \lambda_p|} \sum_{k>N} \frac{1}{|k|^{\alpha-1}|k - N|},
\end{aligned}$$

where $C > 0$ is a constant, changing from line to line, and where we used $1 - e^{-x} \leq 2x$, $\forall x > 0$, which holds for N sufficiently large enough. Thus, using an series-integral comparison argument, it is easy to deduce,

$$\begin{aligned}
\sum_{k>N} \frac{1}{|k|^{\alpha-1}|k - N|} &\leq \frac{1}{N^{\alpha-1}} + \int_{N+1}^{\infty} \frac{1}{x^{\alpha-1}(x - N)} dx \\
&= \frac{1}{N^{\alpha-1}} + \int_1^{\infty} \frac{1}{x(x + N)^{\alpha-1}} dx \\
&= \frac{1}{N^{\alpha-1}} + \int_1^N \frac{1}{x(x + N)^{\alpha-1}} dx + \int_N^{\infty} \frac{1}{x(x + N)^{\alpha-1}} dx \\
&\leq \frac{1}{N^{\alpha-1}} + \left[\frac{\ln(x)}{(x + N)^{\alpha-1}} \right]_1^N + (\alpha - 1) \int_1^N \frac{\ln(x)}{(x + N)^{\alpha}} dx + \int_N^{\infty} \frac{1}{x^{\alpha}} dx \\
&\leq \frac{1}{N^{\alpha-1}} + \frac{\ln(N)}{(2N)^{\alpha-1}} + (\alpha - 1) \ln(N) \left[\frac{1}{(1 - \alpha)(x + N)^{\alpha-1}} \right]_1^N + \frac{1}{(\alpha - 1)N^{\alpha-1}} \\
&\leq C \frac{\ln(N)}{N^{\alpha-1}},
\end{aligned}$$

for some constant $C > 0$. Therefore,

$$|C_{pn}^{-1} - (C_{pn}^{-1})^N| \leq \frac{C}{|\tilde{\lambda}_n - \lambda_p|} \frac{\ln N}{N^{\alpha-1}}$$

We finally obtain that (86) holds using the previous estimate, as well as Assumption 1.1 and 1.2

$$\begin{aligned} & \left| \sum_{p \in \mathbb{Z}} C_{kp}^{-1} C_{pn} - \sum_{|p| \leq N} (C_{kp}^{-1})^N C_{pn} \right| \\ &= \left| \sum_{|p| > N} C_{kp}^{-1} C_{pn} + \sum_{|p| \leq N} (C_{kp}^{-1} - (C_{kp}^{-1})^N) C_{pn} \right| \\ &\leq \sum_{|p| > N} \frac{C}{|\tilde{\lambda}_p - \lambda_k| |\tilde{\lambda}_n - \lambda_p|} + \sum_{|p| \leq N} \left(\frac{C}{|\tilde{\lambda}_p - \lambda_k| |\lambda_n - \tilde{\lambda}_p|} \right) \frac{\ln N}{N^{\alpha-1}} \\ &\leq \sum_{|p| > N} \frac{C}{p^2 (|p| - |k|)^{\alpha-1} (|p| - |n|)^{\alpha-1}} + \sum_{\substack{|p| \leq N \\ p \neq n, k}} \left(\frac{C}{p^2 (|p| - |k|)^{\alpha-1} (|p| - |n|)^{\alpha-1}} \right) \frac{\ln N}{N^{\alpha-1}} + C \frac{\ln N}{N^{\alpha-1}} \\ &\leq \sum_{|p| > N} \frac{C}{p^2 (N/2)^{2(\alpha-1)}} + \left(\frac{\ln N}{N^{\alpha-1}} \right)^2 + C \frac{\ln N}{N^{\alpha-1}} \\ &\leq C \left(\frac{\ln N}{N^{3(\alpha-1)}} + \left(\frac{\ln N}{N^{\alpha-1}} \right)^2 + \frac{\log N}{N^{\alpha-1}} \right) \rightarrow 0, \end{aligned}$$

as $N \rightarrow +\infty$, where we used estimates of the form (90). Therefore, the equality (85) holds implying that $\check{T}^{-1} \circ \check{T} = Id_H$. Using the same method we can also deduce that

$$\sum_{p \in \mathbb{Z}} C_{kp} C_{pn}^{-1} = \delta_{kn}. \quad (91)$$

implying that $\check{T} \circ \check{T}^{-1} = Id_H$. This ends the proof of Proposition 4.2

□

B.3 Properties of the feedback K

We can now finish the proof of Proposition 4.1, by proving estimates on the (k_n) .

Proof. We have,

$$\begin{aligned} k_n b_n &= \sum_{p \in \mathbb{Z}} C_{np}^{-1} \\ &= \sum_{p \in \mathbb{Z}} \left(\frac{(\lambda_n - \tilde{\lambda}_n)(\tilde{\lambda}_p - \lambda_p)}{\tilde{\lambda}_p - \lambda_n} \right) \left(\prod_{k \neq n} \left(1 + \frac{\lambda_k - \tilde{\lambda}_k}{\lambda_n - \lambda_k} \right) \right) \left(\prod_{m \neq p} \left(1 + \frac{\tilde{\lambda}_m - \lambda_m}{\tilde{\lambda}_p - \tilde{\lambda}_m} \right) \right) \\ &= (\lambda_n - \tilde{\lambda}_n) \left(\prod_{k \neq n} \left(1 + \frac{\lambda_k - \tilde{\lambda}_k}{\lambda_n - \lambda_k} \right) \right) \sum_{p \in \mathbb{Z}} \left(\frac{\tilde{\lambda}_p - \lambda_p}{\tilde{\lambda}_p - \lambda_n} \right) \left(\prod_{m \neq p} \left(1 + \frac{\tilde{\lambda}_m - \lambda_m}{\tilde{\lambda}_p - \tilde{\lambda}_m} \right) \right) \\ &= (\lambda_n - \tilde{\lambda}_n) \left(\prod_{k \neq n} \left(1 + \frac{\lambda_k - \tilde{\lambda}_k}{\lambda_n - \lambda_k} \right) \right) \sum_{p \in \mathbb{Z}} \left(\frac{\tilde{\lambda}_p - \lambda_p}{\tilde{\lambda}_p - \lambda_n} \right) \left(\prod_{m \neq p} \left(\frac{\tilde{\lambda}_p - \lambda_m}{\tilde{\lambda}_p - \tilde{\lambda}_m} \right) \right) \end{aligned}$$

We therefore define

$$F_n(\{\lambda_\ell\}, \{\tilde{\lambda}_\ell\}) := \left(\prod_{k \neq n} \left(1 + \frac{\lambda_k - \tilde{\lambda}_k}{\lambda_n - \lambda_k} \right) \right), \quad J_n(\{\lambda_\ell\}, \{\tilde{\lambda}_\ell\}) := \sum_{p \in \mathbb{Z}} \left(\frac{\tilde{\lambda}_p - \lambda_p}{\tilde{\lambda}_p - \lambda_n} \right) \left(\prod_{m \neq p} \left(\frac{\tilde{\lambda}_p - \lambda_m}{\tilde{\lambda}_p - \tilde{\lambda}_m} \right) \right) \quad (92)$$

We first prove

Lemma B.2. *Assume Assumption 1.1 and 1.2. Then,*

$$c < |F_n(\{\lambda_\ell\}, \{\tilde{\lambda}_\ell\})| < C$$

uniformly with respect to $n \in \mathbb{Z}$.

Proof. The upper bound is given by Lemma 4.2 (i).

On the other hand,

$$\begin{aligned} |F_n(\{\lambda_\ell\}, \{\tilde{\lambda}_\ell\})| &= \prod_{k \neq n} \left| 1 + \frac{\lambda_k - \tilde{\lambda}_k}{\lambda_n - \lambda_k} \right| \\ &\geq \left| 1 + \frac{\lambda_{-n} - \tilde{\lambda}_{-n}}{\lambda_n - \lambda_{-n}} \right| \prod_{|k| \neq |n|} \left| 1 + \frac{\lambda_k - \tilde{\lambda}_k}{\lambda_n - \lambda_k} \right| \\ &\geq \left| 1 + \frac{\lambda_{-n} - \tilde{\lambda}_{-n}}{\lambda_n - \lambda_{-n}} \right| \exp \left(\sum_{|k| \neq |n|} \ln \left| 1 + \frac{\lambda_k - \tilde{\lambda}_k}{\lambda_n - \lambda_k} \right| \right) \end{aligned}$$

First, from Assumption 1.1 and 1.2, there exists $C > 0$, changing from line to line and independent of $n \in \mathbb{Z}$, such that,

$$\left| 1 + \frac{\lambda_{-n} - \tilde{\lambda}_{-n}}{\lambda_n - \lambda_{-n}} \right| = \left| \frac{\lambda_n - \tilde{\lambda}_{-n}}{\lambda_n - \lambda_{-n}} \right| \geq C \frac{|n|^\alpha}{|n|^\alpha} \geq C, \quad (93)$$

where we used (3) for $-n \leq j \leq n$:

$$|\lambda_n - \lambda_{-n}| \leq C \sum_{j=0}^{2n-1} (|j - n| + 1)^{\alpha-1} \leq C|n|^\alpha.$$

Moreover, for $|k| \neq |n|$, from Assumption 1.1 -1.2 and Lemma 3.1, there exists $C^* > 0$ such that,

$$\left| \frac{\lambda_k - \tilde{\lambda}_k}{\lambda_n - \lambda_k} \right| \leq \frac{C^*}{|k - n|^\alpha} \leq \frac{C^*}{||k| - |n||^\alpha}.$$

Then,

$$\frac{C^*}{||k| - |n||^\alpha} \leq \frac{1}{2} \iff \begin{cases} (2C^*)^{1/\alpha} + |n| \leq |k| \\ \text{or} \\ |k| \leq |n| - (2C^*)^{1/\alpha} \end{cases}$$

the second condition being void if $|n| - (2C^*)^{1/\alpha} < 0$. We begin with the first condition, using $\ln(1 - x) \geq$

$-2x, x \in [0, \frac{1}{2}]$:

$$\begin{aligned}
\exp \left(\sum_{\substack{|k| \geq |n| + (2C^*)^{1/\alpha} \\ |k| \neq |n|}} \ln \left| 1 + \frac{\lambda_k - \tilde{\lambda}_k}{\lambda_n - \lambda_k} \right| \right) &\geq \exp \left(\sum_{\substack{|k| \geq |n| + (2C^*)^{1/\alpha} \\ |k| \neq |n|}} \ln \left| 1 - \left| \frac{\lambda_k - \tilde{\lambda}_k}{\lambda_n - \lambda_k} \right| \right| \right) \\
&\geq \exp \left(-2 \sum_{\substack{|k| \geq |n| + (2C^*)^{1/\alpha} \\ |k| \neq |n|}} \left| \frac{\lambda_k - \tilde{\lambda}_k}{\lambda_n - \lambda_k} \right| \right) \\
&\geq \exp \left(-2 \sum_{\substack{|k| \geq |n| + (2C^*)^{1/\alpha} \\ |k| \neq |n|}} \left(\frac{C}{||k| - |n||^\alpha} \right) \right) \\
&\geq \exp \left(-2 \sum_{|m| > \lfloor (2C)^{1/\alpha} \rfloor} \left(\frac{C}{|m|^\alpha} \right) \right) \\
&\geq C
\end{aligned} \tag{94}$$

A similar computation also yields a uniform lower bound in the second condition $|n| - |k| \geq (2C)^{1/\alpha}$.

We finally consider the complementary case $|n| - (2C)^{1/\alpha} < |k| < |n| + (2C)^{1/\alpha}$ and $|k| \neq |n|$. Using again (3), this time for $\min(n, k) \leq j \leq \max(n, k)$, we have

$$\begin{aligned}
|\lambda_n - \lambda_k| &\leq C \sum_{j=0}^{k-n+1} (|n-1+j|+1)^{\alpha-1} \\
&\leq C(|n|+1)^\alpha - (|k|+1)^\alpha \\
&\leq C|\xi(n)|^{\alpha-1}|n-k| \\
&\leq C|n|^{\alpha-1}|n-k|
\end{aligned}$$

and where $\xi(n) \in (|n|, |n| + (2C^*)^{1/\alpha})$ and therefore $|\xi(n)| \leq C|n|$ for some constant $C > 0$. Using the above inequality together with (7), we get

$$\begin{aligned}
\exp \left(\sum_{\substack{|n| - (2C^*)^{1/\alpha} \leq |k| \leq |n| + (2C^*)^{1/\alpha} \\ |k| \neq |n|}} \ln \left| \frac{\lambda_n - \tilde{\lambda}_k}{\lambda_n - \lambda_k} \right| \right) &\geq \exp \left(\sum_{\substack{|n| - (2C^*)^{1/\alpha} \leq |k| \leq |n| + (2C^*)^{1/\alpha} \\ |k| \neq |n|}} \ln \left| \frac{C||n| - (2C^*)^{1/\alpha}|^{\alpha-1}|k-n|}{|n|^{\alpha-1}|k-n|} \right| \right) \\
&\geq \exp \left(\sum_{\substack{|n| - (2C^*)^{1/\alpha} \leq |k| \leq |n| + (2C^*)^{1/\alpha} \\ |k| \neq |n|}} \ln C \right) \\
&\geq \exp \left(2(2C^*)^{1/\alpha} \ln C \right) \\
&\geq C
\end{aligned} \tag{95}$$

with $C > 0$ independent of $n \in \mathbb{Z}$ and changing from line to line. Collecting (93)-(95), we deduce the uniform lower bound on $|F_n(\{\lambda_\ell\}, \{\tilde{\lambda}_\ell\})|$ with respect to $n \in \mathbb{Z}$, which concludes the proof. \square

We now turn to the behaviour of $J_n(\{\lambda_\ell\}, \{\tilde{\lambda}_\ell\})$.

Proof. The proof consists in two steps:

Step A: Show that a truncated version J_n^N of J_n is equal to 1 for any $\lambda \in \mathbb{C}$, any $N \in \mathbb{N}^*$ and any $1 \leq n \leq N$.

Step B: Pass to the limit $N \rightarrow \infty$ thanks to the distribution of $\{\lambda_n\}_{n \in \mathbb{N}^*}$, and leveraging that $\lambda_n \sim n^\alpha$ with $\alpha > 1$.

Step A: Note that we immediately derive from (92):

$$J_n(\{\lambda_\ell\}, \{\tilde{\lambda}_\ell\}) = \sum_{p \in \mathbb{Z}} \frac{\prod_{m \neq n} \tilde{\lambda}_p - \lambda_m}{\prod_{m \neq p} \tilde{\lambda}_p - \tilde{\lambda}_m}.$$

We therefore truncate J_n as follows, for any $N \in \mathbb{N}^*$ and any $|n| \leq N$:

$$J_n^N(\{\lambda_\ell\}, \{\tilde{\lambda}_\ell\}) := \sum_{|j| \leq N} \frac{\prod_{\substack{|m| \leq N \\ m \neq n}} (\tilde{\lambda}_j - \lambda_m)}{\prod_{\substack{|m| \leq N \\ m \neq j}} (\tilde{\lambda}_j - \tilde{\lambda}_m)}. \quad (96)$$

We now claim that, for any $N \in \mathbb{N}^*$, and for any complex sequences $\{a_n\}_{n \in \mathbb{Z}}$ and $\{\tilde{a}_n\}_{n \in \mathbb{Z}}$ such that for all $m, n \in \mathbb{Z}$, $a_n \neq \tilde{a}_n$, and $n \neq m \implies a_n \neq a_m$ and $\tilde{a}_n \neq \tilde{a}_m$,

$$J_n^N(\{a_\ell\}, \{\tilde{a}_\ell\}) = 1, \quad \forall |n| \leq N. \quad (97)$$

First, note the following symmetry in the definition of (96): for any $N \in \mathbb{N}^*$, and any permutation σ of $\llbracket -N, N \rrbracket$, extended by the identity mapping to the rest of \mathbb{Z} ,

$$\begin{aligned} J_{\sigma(n)}^N(\{a_{\sigma^{-1}(\ell)}\}, \{\tilde{a}_{\sigma^{-1}(\ell)}\}) &= \sum_{|j| \leq N} \frac{\prod_{\substack{|m| \leq N \\ m \neq \sigma(n)}} \tilde{a}_{\sigma^{-1}(j)} - a_{\sigma^{-1}(m)}}{\prod_{\substack{|m| \leq N \\ m \neq j}} \tilde{a}_{\sigma^{-1}(j)} - a_{\sigma^{-1}(m)}} \\ &= \sum_{|\sigma(j)| \leq N} \frac{\prod_{\substack{|\sigma(m)| \leq N \\ \sigma(m) \neq \sigma(n)}} \tilde{a}_j - a_m}{\prod_{\substack{|\sigma(m)| \leq N \\ \sigma(m) \neq \sigma(j)}} \tilde{a}_j - \tilde{a}_m} \\ &= \sum_{|j| \leq N} \frac{\prod_{\substack{|m| \leq N \\ m \neq n}} \tilde{a}_j - a_m}{\prod_{\substack{|m| \leq N \\ m \neq j}} \tilde{a}_j - \tilde{a}_m} \\ &= J_n^N(\{a_\ell\}, \{\tilde{a}_\ell\}). \end{aligned}$$

Hence, it is sufficient to prove that for all $N \in \mathbb{N}^*$, and all complex sequences $\{a_n\}, \{\tilde{a}_n\}$, satisfying the hypotheses listed above,

$$J_N^N(\{a_n\}, \{\tilde{a}_n\}) = \sum_{|j| \leq N} \frac{\prod_{m=-N}^{N-1} (\tilde{a}_j - a_m)}{\prod_{\substack{|m| \leq N \\ m \neq j}} (\tilde{a}_j - \tilde{a}_m)} = 1 \quad (98)$$

Remark further that (98) rewrites

$$\sum_{j=-N}^{N-1} \frac{\prod_{m=-N}^{N-1} (\tilde{a}_j - a_m)}{\prod_{\substack{|m| \leq N \\ m \neq j}} (\tilde{a}_j - \tilde{a}_m)} + \frac{\prod_{m=-N}^{N-1} (\tilde{a}_N - a_m)}{\prod_{m=-N}^{N-1} (\tilde{a}_N - \tilde{a}_m)} = 1$$

and, multiplying by $\prod_{m=-N}^{N-1} (\tilde{a}_N - \tilde{a}_m)$, we obtain,

$$\sum_{j=-N}^{N-1} \frac{\prod_{m=-N}^{N-1} (\tilde{a}_j - a_m)(\tilde{a}_N - \tilde{a}_m)}{\prod_{\substack{|m| \leq N \\ m \neq j}} (\tilde{a}_j - \tilde{a}_m)} + \prod_{m=-N}^{N-1} (\tilde{a}_N - a_m) - \prod_{m=-N}^{N-1} (\tilde{a}_N - \tilde{a}_m) = 0$$

which may finally be reformulated as,

$$\sum_{j=-N}^{N-1} (a_j - \tilde{a}_j) \prod_{\substack{|m| \leq N \\ m \neq j, N}} \frac{(\tilde{a}_j - a_m)(\tilde{a}_N - \tilde{a}_m)}{(\tilde{a}_j - \tilde{a}_m)} + \prod_{m=-N}^{N-1} (\tilde{a}_N - a_m) - \prod_{m=-N}^{N-1} (\tilde{a}_N - \tilde{a}_m) = 0$$

We are now in a position to define,

$$Q_N(\mu) := \sum_{j=-N}^{N-1} (a_j - \tilde{a}_j) \prod_{\substack{|m| \leq N \\ m \neq j, N}} \frac{(\tilde{a}_j - a_m)(\mu - \tilde{a}_m)}{(\tilde{a}_j - \tilde{a}_m)} + \prod_{m=-N}^{N-1} (\mu - a_m) - \prod_{m=-N}^{N-1} (\mu - \tilde{a}_m).$$

This polynomial is of degree at most $2N - 1$ in $\mu \in \mathbb{C}$. Indeed, in the last two products, the terms of order $2N$ cancel each other out. Moreover, for $-N \leq p \leq N - 1$, we have,

$$\begin{aligned} Q_N(\tilde{a}_p) &= \sum_{j=-N}^{N-1} (a_j - \tilde{a}_j) \prod_{\substack{|m| \leq N \\ m \neq j, N}} \frac{(\tilde{a}_j - a_m)(\tilde{a}_p - \tilde{a}_m)}{(\tilde{a}_j - \tilde{a}_m)} + \prod_{m=-N}^{N-1} (\tilde{a}_p - a_m) \\ &= (a_p - \tilde{a}_p) \prod_{\substack{|m| \leq N \\ m \neq p, N}} \frac{(\tilde{a}_p - a_m)(\tilde{a}_p - \tilde{a}_m)}{(\tilde{a}_p - \tilde{a}_m)} + \prod_{m=-N}^{N-1} (\tilde{a}_p - a_m) \\ &= - \prod_{\substack{|m| \leq N \\ m \neq N}} (\tilde{a}_p - a_m) + \prod_{m=-N}^{N-1} (\tilde{a}_p - a_m) \\ &= 0. \end{aligned}$$

Since Q_N is a polynomial of degree at most $2N - 1$ with $2N$ roots, this implies that $Q_N \equiv 0$. This proves (98) and therefore the claim (97).

Step B: This step is standard and consists in passing to the limit in (97). We first write J_n^N as,

$$J_n^N(\{\lambda_\ell\}, \{\tilde{\lambda}_\ell\}) := \sum_{|j| \leq N} \left(\frac{\tilde{\lambda}_j - \lambda_j}{\tilde{\lambda}_j - \lambda_n} \right) \prod_{\substack{|m| \leq N \\ m \neq j}} \left(\frac{\tilde{\lambda}_j - \lambda_m}{\tilde{\lambda}_j - \tilde{\lambda}_m} \right). \quad (99)$$

We apply the dominated convergence Theorem. Define

$$I_{j,n}^N := \left(\frac{\tilde{\lambda}_j - \lambda_j}{\tilde{\lambda}_j - \lambda_n} \right) \prod_{\substack{|m| \leq N \\ m \neq j}} \left(\frac{\tilde{\lambda}_j - \lambda_m}{\tilde{\lambda}_j - \tilde{\lambda}_m} \right), \quad I_{j,n} := \left(\frac{\tilde{\lambda}_j - \lambda_j}{\tilde{\lambda}_j - \lambda_n} \right) \prod_{\substack{m \in \mathbb{Z} \\ m \neq j}} \left(\frac{\tilde{\lambda}_j - \lambda_m}{\tilde{\lambda}_j - \tilde{\lambda}_m} \right). \quad (100)$$

Observe that,

$$\frac{I_{j,n}^N}{I_{j,n}} = \prod_{\substack{|m| > N \\ m \neq j}} \left(\frac{\tilde{\lambda}_j - \lambda_m}{\tilde{\lambda}_j - \tilde{\lambda}_m} \right)^{-1} = \prod_{\substack{|m| > N \\ m \neq j}} \left(\frac{\tilde{\lambda}_j - \tilde{\lambda}_m}{\tilde{\lambda}_j - \lambda_m} \right) = \prod_{\substack{|m| > N \\ m \neq j}} \left(1 + \frac{\lambda_m - \tilde{\lambda}_m}{\tilde{\lambda}_j - \lambda_m} \right). \quad (101)$$

Now, for any $j \in \mathbb{Z}$, when $m \rightarrow +\infty$ with $m \neq j$,

$$\log \left(1 + \frac{\lambda_m - \tilde{\lambda}_m}{\tilde{\lambda}_j - \lambda_m} \right) \sim \frac{\lambda}{C|m-j|^\alpha}$$

and therefore

$$\sum_{\substack{|m| > N \\ m \neq j}} \log \left(1 + \frac{\lambda_m - \tilde{\lambda}_m}{\tilde{\lambda}_j - \lambda_m} \right) \rightarrow 0, \quad N \rightarrow \infty.$$

This and (101) imply $I_{j,n}^N \rightarrow I_{j,n}$ when $N \rightarrow +\infty$ for any $j \in \mathbb{Z}$, independently of $n \in \mathbb{Z}$. Moreover, from (100), (101) and Assumption 1.2, we have,

$$\begin{aligned} |I_{j,n}^N \mathbf{1}_{|j| \leq N}| &= |I_{j,n} \mathbf{1}_{|j| \leq N}| \prod_{|m| > N} \left| 1 + \frac{\lambda_m - \tilde{\lambda}_m}{\tilde{\lambda}_j - \lambda_m} \right| \\ &\leq |I_{j,n} \mathbf{1}_{|j| \leq N}| \prod_{|m| > N} \left(1 + \frac{C}{|m|^{\alpha-1}|m-j|} \right) \\ &\leq |I_{j,n} \mathbf{1}_{|j| \leq N}| \prod_{|m| > N} \left(1 + \frac{C}{|m|^{\alpha-1}||m|-|j||} \right) \\ &\leq |I_{j,n} \mathbf{1}_{|j| \leq N}| \prod_{|m| > N} \left(1 + \frac{C}{|m|^{\alpha-1}||m|-N|} \right) \\ &\leq |I_{j,n} \mathbf{1}_{|j| \leq N}| \left(\prod_{k \in \mathbb{N}^*} 1 + \frac{C}{|k+N|^{\alpha-1}|k|} \right)^2 \\ &\leq |I_{j,n}| \left(\prod_{k \in \mathbb{N}^*} 1 + \frac{C}{|k|^\alpha} \right)^2, \end{aligned} \quad (102)$$

$$(103)$$

where we used the fact that if $j > N$ then $\mathbf{1}_{j \leq N}$ vanishes, yielding an easy estimate.

We can now show that the right-hand side of (102) is summable. From Lemma 4.2 (ii), there exists a constant $C > 0$ such that

$$\left| \prod_{\substack{m \in \mathbb{Z} \\ m \neq j}} \frac{\tilde{\lambda}_j - \lambda_m}{\tilde{\lambda}_j - \tilde{\lambda}_m} \right| = \left| \prod_{\substack{m \in \mathbb{Z} \\ m \neq j}} 1 + \frac{\tilde{\lambda}_m - \lambda_m}{\tilde{\lambda}_j - \tilde{\lambda}_m} \right| \leq C.$$

Then, using Assumption 1.2, we have, for $j \neq n$,

$$|I_{j,n}| \leq C \frac{C'}{|\tilde{\lambda}_j - \lambda_n|} \leq \frac{C''}{|j - n|^\alpha},$$

and there exists a constant C'''

$$|I_{n,n}| \leq C'''.$$

Moreover, there exists $\tilde{C} > 0$ such that,

$$\prod_{k \in \mathbb{N}^*} \left(1 + \frac{C}{|k|^\alpha} \right) \leq \tilde{C}.$$

Therefore, for $N \geq |n|$

$$\sum_{|j| \leq N} |I_{j,n}| \leq \tilde{C} \left(|I_{n,n}| + \sum_{\substack{|j| \leq N \\ j \neq n}} |I_{j,n}| \right) \leq \tilde{C} \left(C''' + \sum_{\substack{|j| \leq N \\ j \neq n}} \frac{C''}{|j - n|^\alpha} \right) \leq C_1 + 2C_2 \sum_{p \geq 1} \frac{1}{p^\alpha}.$$

Thus, by dominated convergence

$$J_n^N(\lambda_n, \tilde{\lambda}_n) = - \sum_{j \in \mathbb{N}^*} I_{j,n}^N \mathbf{1}_{j \leq N} \longrightarrow - \sum_{j \in \mathbb{N}^*} I_{j,n} = J_n(\lambda).$$

when $N \rightarrow +\infty$. □

□

C Riesz bases in Hilbert spaces

We recall here some definitions about vector families in Hilbert spaces (see for instance [4, 9, 12, 22]).

Definition C.1 (Vector family). *Let X be a Hilbert space. A family of vectors $\{\xi_n\}_{n \in \mathcal{I}}$, where $\mathcal{I} = \mathbb{Z}$, \mathbb{N} , or \mathbb{N}^* is said to be*

(1) **Dense** in X , if $\overline{\text{span}\{\xi_i; i \in \mathcal{I}\}} = X$.

(2) **ω -independent** in X , if

$$\sum_{k \in \mathcal{I}} c_k \xi_k = 0 \text{ in } X \text{ with } \{c_n\}_{n \in \mathcal{I}} \in \ell^2(\mathcal{I}) \implies c_n = 0, \forall n \in \mathcal{I}.$$

(3) **Quadratically close** to a family of vector $\{e_n\}_{n \in \mathcal{I}}$, if

$$\sum_{k \in \mathcal{I}} \|\xi_k - e_k\|_X^2 < +\infty.$$

(4) **Riesz basis** of X , if it is the image of an isomorphism (on X) of some orthonormal basis.

An equivalent definition of Riesz basis can also be stated as follows

Definition C.2 (Riesz basis). *A family of vectors $\{\xi_n\}_{n \in \mathcal{I}}$, where $\mathcal{I} = \mathbb{Z}, \mathbb{N}$, or \mathbb{N}^* of X is called a Riesz basis of X , if it is dense in X and if there exist $C_1, C_2 > 0$ such that for any $\{a_n\}_{n \in \mathcal{I}} \in \ell^2(\mathcal{I})$ we have*

$$C_1 \sum_{k \in \mathcal{I}} |a_k|^2 \leq \left\| \sum_{k \in \mathcal{I}} a_k \xi_k \right\|_X^2 \leq C_2 \sum_{k \in \mathcal{I}} |a_k|^2. \quad (104)$$

The following lemma has been heavily used in the literature as a criterion for Riesz basis.

Lemma C.1. *Let $\{\xi_n\}_{n \in \mathcal{I}}$ be quadratically close to an orthonormal basis $\{e_n\}_{n \in \mathcal{I}}$. Suppose that $\{\xi_n\}_{n \in \mathcal{I}}$ is either dense in X or ω -independent in X , then $\{\xi_n\}_{n \in \mathcal{I}}$ is a Riesz basis of X .*

Finally in this paper we also make use of the following Lemma

Lemma C.2. *Let X, Y be Hilbert spaces. Let $T : X \rightarrow Y$ be an isomorphism. Suppose that $\{\xi_n\}_{n \in \mathcal{I}}$ is a Riesz basis of X , then with $\zeta_n := T\xi_n$, the family $\{\zeta_n\}_{n \in \mathcal{I}}$ is a Riesz basis of Y .*

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