



**HAL**  
open science

# Finite convergence of the Moment-SOS hierarchy under hidden convexity

Srećko Đurašinović, Jean B Lasserre

► **To cite this version:**

Srećko Đurašinović, Jean B Lasserre. Finite convergence of the Moment-SOS hierarchy under hidden convexity. 2026. <hal-05530832>

**HAL Id: hal-05530832**

**<https://hal.science/hal-05530832v1>**

Preprint submitted on 27 Feb 2026

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons CC BY 4.0 - Attribution - International License

# Finite convergence of the Moment-SOS hierarchy under hidden convexity

Srećko Durašinović

CNRS@CREATE Singapore and College of Computing and Data Science, Nanyang Technological University, Singapore

Create Tower #08-01

1 Create Way

Singapore 138602

[srecko001@e.ntu.edu.sg](mailto:srecko001@e.ntu.edu.sg)

Jean B. Lasserre

LAAS-CNRS and Toulouse School of Economics (TSE)

University of Toulouse

LAAS, 7 avenue du Colonel Roche

31077 Toulouse Cédex 4, France

[lasserre@laas.fr](mailto:lasserre@laas.fr)

---

## Abstract

One considers polynomial optimization problems with compact feasible set  $\Omega$  defined by SOS-concave polynomials  $g_j$ , and with a globally non-convex polynomial objective  $f$ . We show that if  $f$  is strongly convex on  $\Omega$ , or SOS-convex on  $\Omega$  when the constraints  $g_j$  are at most quadratic, then the Moment-SOS hierarchy converges in finitely many steps, without a priori knowledge of this hidden (local) convexity. In addition, in the latter case, the exact order for which the relaxation is exact is provided by the degree of a Putinar-like certificate of convexity. This demonstrates that a general-purpose hierarchy can adapt to favorable hidden properties of a specific instance without being informed of them, yielding certified global minimizers.

**Keywords** Polynomial optimization; Moment-SOS hierarchy; Semidefinite relaxations; Hidden convexity.

**Acknowledgments** This research is supported by the AI Interdisciplinary Institute ANITI funding through the ANITI AI Cluster program under the Grant agreement ANR-23-IACL-0002, and also by the Marie-Sklodowska-Curie European doctoral network TENORS, grant 101120296. This research is also part of the programme DesCartes and is supported by the National Research Foundation, Prime Minister's Office, Singapore under its Campus for Research Excellence and Technological Enterprise (CREATE) programme.

## 1 Introduction

Given multivariate polynomials  $f, g_j \in \mathbb{R}[\mathbf{x}]$ ,  $j = 1, \dots, m$ , consider the polynomial optimization problem (POP)

$$f^* = \min_{\mathbf{x} \in \Omega} f(\mathbf{x}), \quad (1.1)$$

where the feasible set  $\Omega = \{\mathbf{x} \in \mathbb{R}^d : g_j(\mathbf{x}) \geq 0, j = 1, \dots, m\}$  is compact and convex, but  $f$  is *not* convex. If  $f$  is convex on  $\Omega$ , Problem (1.1) is convex but not a “convex program” as the latter definition usually assumes that  $f$  and the  $-g_j$ 's are convex *on the whole space*  $\mathbb{R}^d$ . This distinction is not a subtlety. Indeed, suppose that the  $g_j$ 's are concave polynomials and that the Slater's condition holds. Then the KKT-optimality conditions

$$\nabla f(\mathbf{x}^*) - \sum_{j=1}^m \lambda_j^* \nabla g_j(\mathbf{x}^*) = 0, \quad \lambda_j^* g_j(\mathbf{x}^*) = 0, \quad j = 1, \dots, m, \quad \lambda^* \geq 0, \quad (1.2)$$

at local minimizer  $\mathbf{x}^* \in \Omega$  (for some  $\lambda^* \geq 0$ ) have a quite different interpretation depending on whether  $f$  is convex on  $\mathbb{R}^d$  or on  $\Omega$  only. In the former case,

$$\mathcal{L}(\mathbf{x}, \lambda^*) - f^* := f(\mathbf{x}) - f^* - \sum_{j=1}^m \lambda_j^* g_j(\mathbf{x}) \geq 0, \quad \forall \mathbf{x} \in \mathbb{R}^d, \quad (1.3)$$

that is,  $\mathbf{x}^*$  is also a global minimizer of the Lagrangian  $\mathcal{L}(\cdot, \lambda^*)$  on the whole  $\mathbb{R}^d$ , because  $\mathcal{L}(\cdot, \lambda^*)$  is convex and  $\nabla \mathcal{L}(\mathbf{x}^*, \lambda^*) = 0$ . In the latter case, while convex on  $\Omega$ ,  $\mathcal{L}(\cdot, \lambda^*)$  is not necessarily convex on  $\mathbb{R}^d$  (typically when  $\deg(f) > \max_j \{\deg(g_j)\}$ ). Consider, for instance,  $f^* = \min\{x^3 : x(1-x) \geq 0\}$  with  $(x^*, \lambda^*) = (0, 0)$ .

For algorithms which converge to a local minimizer  $\mathbf{x}^* \in \Omega$  that satisfies (1.2), global optimality of  $\mathbf{x}^*$  is guaranteed if  $f$  is convex on  $\Omega$ . But if this (local) convexity on  $\Omega$  is hidden, one is left with an à priori local minimizer only. Recall that checking whether a given polynomial is convex is NP-hard [1].

## The Moment-SOS hierarchy

Let  $g_0 : \mathbf{x} \mapsto 1$  be a constant polynomial. Recall that the degree- $n$  semidefinite relaxation of moment-SOS hierarchy to solve the POP (1.1), reads:

$$\rho_n = \min_{\phi} \{ \phi(f) : \phi(1) = 1, \mathbf{M}_{n-d_j}(g_j \phi) \succeq 0, j = 0, \dots, m \}, \quad (1.4)$$

where  $\phi = (\phi_{\alpha})_{\alpha \in \mathbb{N}_{2n}^d}$ ,  $\mathbf{M}_{n-d_j}(g_j \phi)$  is the degree- $(n-d_j)$  localizing matrix associated with  $\phi$  and the polynomial  $g_j$ , and  $d_j = \lceil \deg(g_j)/2 \rceil$ ,  $j = 0, \dots, m$ ,  $d_f = \lceil \deg(f)/2 \rceil$ . Then,  $\rho_n \leq f^*$  for all  $n \geq \max[d_f, \max_j d_j]$ , and if the quadratic module defined by the  $g_j$ 's is Archimedean (see discussion around (2.8)),  $\rho_n \uparrow f^*$  as  $n \rightarrow +\infty$ . For more details on the Moment-SOS hierarchy and its applications, the interested reader is referred to, for example, [9].

▷ Remark 1.1. As proved in [10], finite convergence is a *generic* property of the Moment-SOS hierarchy in the general case, with no convexity assumption, neither on  $f$  nor on the  $g_j$ 's. However, when considering a specific instance  $(f, g_1, \dots, g_m)$  of POP, one does not know whether that instance is regular, even if theoretically, given the class  $(m, s)$  of POPs (1.1) with  $m$  inequality constraints and with  $\sup[\deg(f), \deg(g_j)] < s$ , genericity could be tested (but at a prohibitive computational cost).

## Convex and SOS-convex POPs

Convex or SOS-convex problems form a typical class for which the Moment-SOS hierarchy has *finite* convergence. For instance, if  $f$  and  $-g_j$  are SOS-convex (see (1.5)), finite convergence takes place at the first semidefinite relaxation of the hierarchy [7]. If  $f$  is only convex, finite convergence is guaranteed, but without knowing at which step of the hierarchy convergence takes place. We remind that, while convexity is hard to detect [1], SOS-convexity can be, in principle, checked efficiently. Of course, when the information that  $f$  and  $-g_j$  are convex is known à priori, some algorithms can be more efficient than the Moment-SOS hierarchy, but the interesting observation is when the convexity of  $f$  is *hidden*.

Indeed, the Moment-SOS hierarchy blindly *recognizes* SOS-convex POPs, *i.e.*, *without à priori knowledge of this SOS-convexity*, as it converges at the first step of the hierarchy. It is a nice feature of a general purpose method to be able to adapt its behavior by benefiting from strong properties of some specific instances, without being “informed” of such properties.

In [5], the authors have extended some results of [7]. They prove finite convergence of the Moment-SOS hierarchy for convex programs when (i) Slater’s condition holds, (ii) the quadratic module associated with the  $g_j$ 's is Archimedean, and (iii)  $\nabla^2 f(\mathbf{x}^*) \succ 0$  at every minimizer  $\mathbf{x}^*$ . Also, if  $\Omega \subset \mathbf{B}(0, R)$ , the same result holds under conditions (ii) and (iii) (but now satisfied by the Lagrangian  $\mathcal{L}$  rather than by  $f$ ). Two key ingredients in [5] are a result of Scheiderer [11] and the (convex) Farkas Lemma [3, Prop. 5.3.1]. Finally, it is shown in [6] that if both  $f$  and  $\Omega$  are convex (without the  $g_j$ 's are not necessarily concave), all points that satisfy the KKT-optimality conditions are global minimizers.

## Contribution

Let us start by introducing the notion of *SOS-convexity on  $\Omega$* . Namely, a polynomial  $f \in \mathbb{R}[\mathbf{x}]$  is said to be SOS-convex on  $\Omega$  if

$$\nabla^2 f(\mathbf{x}) = \sum_{j=0}^m \mathbf{L}_j(\mathbf{x}) \mathbf{L}_j(\mathbf{x})^\top g_j(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^d, \quad (1.5)$$

for some integers  $r_j \in \mathbb{N}$  and matrix polynomials  $\mathbf{L}_j \in \mathbb{R}[\mathbf{x}]^{d \times r_j}$ ,  $j = 0, \dots, m$ .

This definition is a natural extension of global SOS-convexity on  $\mathbb{R}^d$  (introduced in [4], see also [7]) to local SOS-convexity on  $\Omega$ . Notice that (1.5) can be viewed as a Putinar-type *algebraic certificate* that  $f$  is convex on

$\Omega$ . In particular, if  $f$  is strongly convex on  $\Omega$ , then by the Putinar's matrix Positivstellensatz [8, Theorem 2.19], (1.5) holds and so  $f$  is SOS-convex on  $\Omega$ . In principle, checking whether (1.5) holds can be done efficiently by solving a semidefinite program of finite size determined by an à priori bound on the maximum degree appearing in entries of matrix polynomials  $\mathbf{L}_j(\mathbf{x})$ .

The scope of this work is determined by the class of POPs for which:

- the  $g_j$ 's are all SOS-concave and so  $\Omega$  is convex. Typical examples of such feasible sets include the simplex, the Euclidean unit ball, convex polytopes, intersection of ellipsoids, etc.
- $f$  is *not* convex on  $\mathbb{R}^d$  but is SOS-convex on  $\Omega$ , i.e., (1.5) holds.

Hence, the resulting POPs define convex optimization problems (but not convex programs in the standard sense, since  $f$  is not convex on the whole  $\mathbb{R}^d$ ).

Our main results are as follows.

- (i) If  $f$  is strongly convex on  $\Omega^1$ , then the Moment-SOS hierarchy converges in finitely many steps.
- (ii) If, in addition,  $\deg(g_j) \leq 2$  for all  $j = 1, \dots, m$ , then finite convergence also holds under the weaker assumption that  $f$  is SOS-convex on  $\Omega$  (i.e., (1.5) holds). Moreover, an explicit lower bound on the order  $n$  of an exact relaxation is given by

$$1 + \max \left\{ \max_{j=1, \dots, m+1} \{a_j + 1\}, a_0 \right\}, \text{ with } a_j := \max_{(k,l) \in \{1, \dots, d\} \times \{1, \dots, r_j\}} \deg(\mathbf{L}_j(k,l)), \quad (1.6)$$

where  $\mathbf{L}_j(k,l)$  denotes the polynomial entry  $(k,l)$  of the matrix  $\mathbf{L}_j$  from (1.5).

A key feature is that neither result requires à priori verification of the corresponding convexity assumption: one need not check that  $f$  is strongly convex in (i), nor that (1.5) holds in (ii). In other words, the convexity may be hidden. Nevertheless, for the standard Moment-SOS relaxations (1.4) (with no ad hoc modification), finite convergence still occurs.

Crucial for our proofs and of independent interest, we provide a Jensen-type inequality for polynomials that are SOS-convex on  $\Omega$ , and linear functionals  $\phi$  whose associated localizing matrices  $\mathbf{M}_{n-d_j}(g_j \phi)$ ,  $j = 0, \dots, m$ , are positive semidefinite (thus an extension beyond probability measures on  $\Omega$ , but for a restricted class of convex functions).

We emphasize once again that, *if one knows that  $f$  is convex on  $\Omega$* , then running a local algorithm that converges to a KKT point  $\mathbf{x}^* \in \Omega$  might be enough, because (1.2) provides  $\mathbf{x}^*$  with a certificate of global optimality on  $\Omega$ , and there is no need to run the Moment-SOS hierarchy. But if one is unable to check beforehand whether  $f$  is convex on  $\Omega$  (an NP-hard problem), then the local KKT-optimality conditions are not sufficient to ensure that  $\mathbf{x}^*$  is a global minimizer.

## 2 Main result

### 2.1 Notation, definitions and preliminary result

Let  $\mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1, \dots, x_d]$  be the ring of  $d$ -variate polynomials in the variables  $\mathbf{x} = (x_1, \dots, x_d)$ , and let  $\mathbb{R}[\mathbf{x}]_n \subset \mathbb{R}[\mathbf{x}]$  be the vector space of polynomials of total degree at most  $n$ . Let  $\Sigma[\mathbf{x}]$  be the cone of sum of squares (SOS) polynomials and let  $\Sigma[\mathbf{x}]_n := \Sigma[\mathbf{x}] \cap \mathbb{R}[\mathbf{x}]_{2n}$ . For  $\boldsymbol{\alpha} \in \mathbb{N}^d$ , let  $|\boldsymbol{\alpha}| := \sum_{i=1}^d \alpha_i$ , and denote by

$\mathbb{N}_n^d = \{\boldsymbol{\alpha} \in \mathbb{N}^d : |\boldsymbol{\alpha}| \leq n\}$  the finite set of multi-indices of size  $s(n) := \binom{d+n}{n}$ . Let  $\mathbf{v}_n(\mathbf{x}) = (\mathbf{x}^\alpha)_{\boldsymbol{\alpha} \in \mathbb{N}_n^d}$  be the vector of all monomials with total degree bounded by  $n$ . For a real symmetric matrix  $\mathbf{A}$ , the notation  $\mathbf{A} \succeq 0$  (resp.  $\mathbf{A} \succ 0$ ) means that  $\mathbf{A}$  is positive semidefinite (resp. positive definite). A polynomial  $f \in \mathbb{R}[\mathbf{x}]_n$  is written  $\mathbf{x} \mapsto f(\mathbf{x}) = \sum_{\boldsymbol{\alpha} \in \mathbb{N}_n^d} f_\alpha \mathbf{x}^\alpha = \langle \mathbf{f}, \mathbf{v}_n(\mathbf{x}) \rangle$ , with vector of coefficients  $\mathbf{f} = (f_\alpha)_{\boldsymbol{\alpha} \in \mathbb{N}_n^d} \in \mathbb{R}^{s(n)}$  in the basis  $\mathbf{v}_n(\mathbf{x})$ . A

$d$ -dimensional unit sphere is denoted by  $\mathbb{S}^{d-1}$ . Euclidean ball centered at origin and having radius  $R > 0$  is denoted by  $\mathbf{B}(0, R)$ .

<sup>1</sup> Recall that a polynomial  $f$  is strongly convex on  $\Omega$  if and only if there exists  $\mu > 0$  such that  $\nabla^2 f(\mathbf{x}) - \mu \mathbf{I} \succeq 0$  for all  $\mathbf{x} \in \Omega$ .

### Riesz linear functional.

Given a sequence  $\boldsymbol{\phi} = (\phi_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in \mathbb{N}^d}$  (in bold), its associated Riesz functional is the linear mapping  $\phi : \mathbb{R}[\mathbf{x}] \rightarrow \mathbb{R}$  (not in bold) defined by:

$$f (= \sum_{\boldsymbol{\alpha}} f_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}) \mapsto \phi(f) = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^d} f_{\boldsymbol{\alpha}} \phi_{\boldsymbol{\alpha}} = \langle \mathbf{f}, \boldsymbol{\phi} \rangle. \quad (2.1)$$

A sequence  $\boldsymbol{\phi}$  has a *representing* measure if its associated Riesz linear functional  $\phi$  is a (positive) Borel measure on  $\mathbb{R}^d$ , in which case,

$$\phi_{\boldsymbol{\alpha}} = \int_{\mathbb{R}^d} \mathbf{x}^{\boldsymbol{\alpha}} d\phi, \quad \forall \boldsymbol{\alpha} \in \mathbb{N}^d. \quad (2.2)$$

Given a sequence  $\boldsymbol{\phi} = (\phi_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in \mathbb{N}^d}$  and a polynomial  $\mathbb{R}[\mathbf{x}] \ni g : \mathbf{x} \mapsto g(\mathbf{x}) = \sum_{\boldsymbol{\gamma}} g_{\boldsymbol{\gamma}} \mathbf{x}^{\boldsymbol{\gamma}}$ , we define the new sequence  $g \cdot \boldsymbol{\phi}$  via

$$(g \cdot \boldsymbol{\phi})_{\boldsymbol{\alpha}} := \phi(\mathbf{x}^{\boldsymbol{\alpha}} g) = \sum_{\boldsymbol{\gamma} \in \mathbb{N}^d} g_{\boldsymbol{\gamma}} \phi_{\boldsymbol{\alpha} + \boldsymbol{\gamma}}, \quad \forall \boldsymbol{\alpha} \in \mathbb{N}^d, \quad (2.3)$$

so that its associated Riesz linear functional, denoted by  $g \cdot \phi$ , satisfies

$$g \cdot \phi(f) = \phi(gf), \quad \forall f \in \mathbb{R}[\mathbf{x}]. \quad (2.4)$$

### Moment matrix.

The *moment* matrix of order  $n \in \mathbb{N}$  associated with a sequence  $\boldsymbol{\phi} = (\phi_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in \mathbb{N}^d}$  (or, equivalently, with the Riesz linear functional  $\phi$ ), is the real symmetric matrix of size  $s(n) \times s(n)$  denoted by  $\mathbf{M}_n(\boldsymbol{\phi})$  (or  $\mathbf{M}_n(\phi)$ ) with rows and columns indexed by  $\mathbb{N}_n^d$ , and whose entry  $(\boldsymbol{\alpha}, \boldsymbol{\beta})$  corresponds to  $\phi(\mathbf{x}^{\boldsymbol{\alpha} + \boldsymbol{\beta}}) = \phi_{\boldsymbol{\alpha} + \boldsymbol{\beta}}$ , for every  $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}_n^d$ . Therefore,  $\mathbf{M}_n(\boldsymbol{\phi})$  depends only on moments  $\phi_{\boldsymbol{\alpha}}$  of degree at most  $2n$ . Moreover, if  $\boldsymbol{\phi}$  has a representing measure  $\phi$ , then  $\mathbf{M}_n(\boldsymbol{\phi}) \succeq 0$  because  $\langle \mathbf{f}, \mathbf{M}_n(\boldsymbol{\phi}) \mathbf{f} \rangle = \int f^2 d\phi \geq 0$ , for all  $f \in \mathbb{R}[\mathbf{x}]_n$ .

### Localizing matrix.

With  $\boldsymbol{\phi}$  as above and  $g \in \mathbb{R}[\mathbf{x}]$  (with  $g(\mathbf{x}) = \sum_{\boldsymbol{\gamma}} g_{\boldsymbol{\gamma}} \mathbf{x}^{\boldsymbol{\gamma}}$ ), the *localizing* matrix associated with  $\boldsymbol{\phi}$  and  $g$  is the moment matrix  $\mathbf{M}_n(g \cdot \boldsymbol{\phi})$  associated with the sequence  $g \cdot \boldsymbol{\phi}$ . That is,  $\mathbf{M}_n(g \cdot \boldsymbol{\phi})$  is the real symmetric matrix with rows and columns indexed by  $\mathbb{N}_n^d$ , and whose entry  $(\boldsymbol{\alpha}, \boldsymbol{\beta})$  is just  $(g \cdot \boldsymbol{\phi})_{\boldsymbol{\alpha} + \boldsymbol{\beta}} = \phi(\mathbf{x}^{\boldsymbol{\alpha} + \boldsymbol{\beta}} g(\mathbf{x}))$ , that is,  $\mathbf{M}_n(g \cdot \boldsymbol{\phi})(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_{\boldsymbol{\gamma}} g_{\boldsymbol{\gamma}} \phi_{\boldsymbol{\alpha} + \boldsymbol{\beta} + \boldsymbol{\gamma}}$ , for every  $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}_n^d$ . If  $\boldsymbol{\phi}$  has a representing measure  $\phi$  whose support is contained in the set  $\{\mathbf{x} \in \mathbb{R}^d : g(\mathbf{x}) \geq 0\}$  then  $\mathbf{M}_n(g \cdot \boldsymbol{\phi}) \succeq 0$  for all  $n$ , because

$$\langle \mathbf{f}, \mathbf{M}_n(g \cdot \boldsymbol{\phi}) \mathbf{f} \rangle = g \cdot \phi(f^2) = \phi(f^2 g) = \int f^2 g d\phi \geq 0, \quad \forall f \in \mathbb{R}[\mathbf{x}]_n. \quad (2.5)$$

### SOS-convex polynomials.

▷ Definition 2.1. A polynomial  $p \in \mathbb{R}[\mathbf{x}]$  is (*globally*) *SOS-convex* if its Hessian  $\nabla^2 p$  is a matrix-SOS. That is, there exists a matrix polynomial  $\mathbf{L} \in \mathbb{R}[\mathbf{x}]^{d \times r}$  such that  $\nabla^2 p(\mathbf{x}) = \mathbf{L}(\mathbf{x}) \mathbf{L}(\mathbf{x})^T$ , for all  $\mathbf{x} \in \mathbb{R}^d$ . A polynomial  $p$  is SOS-concave if  $-p$  is SOS-convex.

▷ Definition 2.2. With  $\boldsymbol{\Omega} \subset \mathbb{R}^d$  as in (1.1), a polynomial  $f \in \mathbb{R}[\mathbf{x}]$  is *SOS-convex on  $\boldsymbol{\Omega}$*  if its Hessian  $\nabla^2 f$  satisfies

$$\nabla^2 f(\mathbf{x}) = \sum_{j=0}^m \mathbf{L}_j(\mathbf{x}) \mathbf{L}_j(\mathbf{x})^T g_j(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^d, \quad (2.6)$$

for some matrix polynomials  $\mathbf{L}_j \in \mathbb{R}[\mathbf{x}]^{d \times r_j}$ , with  $r_j \in \mathbb{N}$  for  $j = 0, \dots, m$ .

### Quadratic module

▷ Definition 2.3. Let  $\mathbf{g} = (g_0, g_1, \dots, g_m) \in \mathbb{R}[\mathbf{x}]^{m+1}$ . The set  $Q(\mathbf{g}) \subset \mathbb{R}[\mathbf{x}]$  defined by

$$Q(\mathbf{g}) := \left\{ \sum_{j=0}^m \sigma_j g_j : \sigma_j \in \Sigma[\mathbf{x}], j = 0, \dots, m \right\}. \quad (2.7)$$

is called a quadratic module generated by  $g_1, \dots, g_m$ . Its truncated version of order  $n \in \mathbb{N}$ , namely  $Q_n(\mathbf{g}) \subset \mathbb{R}[\mathbf{x}]_n$ , reads

$$Q_n(\mathbf{g}) := \left\{ \sum_{j=0}^m \sigma_j g_j : \sigma_j \in \Sigma[\mathbf{x}], \deg(\sigma_j g_j) \leq n, j = 0, \dots, m \right\}. \quad (2.8)$$

When both  $f$  is strongly convex on  $\Omega$  and the quadratic module  $Q(\mathbf{g})$  is Archimedean, then by Putinar's matrix Positivstellensatz [8, Theorem 2.22], equation (2.6) holds, that is,  $f$  is SOS-convex on  $\Omega$ . Recall that  $Q(\mathbf{g})$  is said to be Archimedean if there exists  $R > 0$  such that  $R^2 - \|\mathbf{x}\|^2 \in Q(\mathbf{g})$ .

## 2.2 Main result

Since  $\Omega$  is compact, we may and will assume that  $\Omega \subset \mathbf{B}(0, 1)$  possibly after re-scaling. Hence, we can include in the definition of  $\Omega$  the redundant quadratic constraint  $g_{m+1}(\mathbf{x}) \geq 0$ , with  $g_{m+1} : \mathbf{x} \mapsto 1 - \|\mathbf{x}\|^2$  (which is SOS-concave). This ensures that the quadratic module  $Q(\mathbf{g})$  in (2.7) is Archimedean. Therefore, throughout the paper, we assume the following:

- ▷ Assumption 2.4. (i) The set  $\Omega \subset \mathbb{R}^d$  in (1.1) is compact, and  $g_{m+1}$  is one of the defining constraints of  $\Omega$ ;
- (ii) For every  $j = 1, \dots, m+1$ , the constraint  $g_j$  is SOS-concave (hence  $\Omega$  is convex);
- (iii) Objective function  $f$  not convex on  $\mathbb{R}^d$ , but satisfies (1.5), i.e., it is SOS-convex on  $\Omega$ .

## A Jensen inequality for linear functionals

We may now state the main result underpinning our analysis of finite convergence of the Moment–SOS hierarchy under Assumption 2.4.

▷ Lemma 2.5. Let Assumption 2.4 hold, and let  $\phi = (\phi_\alpha)_{\alpha \in \mathbb{N}_{2n}^d}$  be such that

$$\phi(1) = 1 \text{ and } \mathbf{M}_{n-d_j}(g_j \phi) \succeq 0, \quad j = 0, \dots, m+1. \quad (2.9)$$

- (i) If  $f$  is strongly convex on  $\Omega$ , and if  $n$  is large enough, then

$$\phi(f) \geq f(\phi(\mathbf{x})), \quad \text{where } \phi(\mathbf{x}) := (\phi(x_1), \dots, \phi(x_d)) \in \Omega \quad (2.10)$$

- (ii) If  $\deg(g_j) \leq 2$  for all  $j = 1, \dots, m+1$ , then (2.10) holds as soon as  $2n \geq 2 + \max_{j=1, \dots, m+1} \{2a_j + 2\}$  and

$$2n \geq \max_{j=0, \dots, m+1} \{2a_j + 2\}, \text{ or equivalently, } n \geq 1 + \max \left\{ \max_{j=1, \dots, m+1} \{a_j + 1\}, a_0 \right\}.$$

**Proof.** (i) Observe that  $\mathbf{M}_{n-d_j}(g_j \phi) \succeq 0 \Rightarrow \phi(g_j) \geq 0$ , and since  $g_j$  is SOS-concave, by [8, Theorem 5.13],  $0 \leq \phi(g_j) \leq g_j(\phi(\mathbf{x}))$ , which turn implies that  $\phi(\mathbf{x}) \in \Omega$ .

Next, let  $\mathbf{z} \in \Omega$  be fixed, arbitrary. Then, as  $\phi$  is linear and satisfies  $\phi(1) = 1$ , we get  $\phi(f - f(\mathbf{z})) = \phi(f) - f(\mathbf{z})$ . We will prove that

$$\phi(f - f(\mathbf{z})) \geq \phi(\langle \nabla f(\mathbf{z}), \mathbf{x} - \mathbf{z} \rangle) = \langle \nabla f(\mathbf{z}), \phi(\mathbf{x}) - \phi(\mathbf{z}) \rangle, \quad (2.11)$$

and therefore substituting  $\mathbf{z}$  with  $\phi(\mathbf{x}) \in \Omega$ , one obtains the desired result (2.10).

Observe that

$$f(\mathbf{x}) - f(\mathbf{z}) = \langle \nabla f(\mathbf{z}), \mathbf{x} - \mathbf{z} \rangle + \left\langle (\mathbf{x} - \mathbf{z}), \left( \int_0^1 (1-t) \nabla^2 f(\mathbf{z} + t(\mathbf{x} - \mathbf{z})) dt \right) (\mathbf{x} - \mathbf{z}) \right\rangle. \quad (2.12)$$

As the maximum degree (in  $t$ ) of the entries of the integrand  $(1-t)\nabla^2 f(\mathbf{z} + t(\mathbf{x} - \mathbf{z}))$  is  $2d_f - 1$ , the entire integral from (2.12) may be computed using a cubature formula for Lebesgue measure on  $[0, 1]$ , supported on  $s$  points  $t_1, \dots, t_s \in [0, 1]$ , with weights  $0 < \gamma$ . The formula being exact for polynomials of degree  $2d_f - 1$ , we obtain

$$f(\mathbf{x}) - f(\mathbf{z}) = \langle \nabla f(\mathbf{z}), \mathbf{x} - \mathbf{z} \rangle + \sum_{k=1}^s \gamma_k \left\langle (\mathbf{x} - \mathbf{z}), ((1-t_k) \nabla^2 f(\mathbf{z} + t_k(\mathbf{x} - \mathbf{z}))) (\mathbf{x} - \mathbf{z}) \right\rangle. \quad (2.13)$$

Next for  $\mathbf{x}, \mathbf{z} \in \Omega$  and each  $k = 1, \dots, s$ ,  $\mathbf{z} + t_k(\mathbf{x} - \mathbf{z}) \in \Omega$ , and so

$$\nabla^2 f(\mathbf{z} + t_k(\mathbf{x} - \mathbf{z})) = \sum_{j=0}^{m+1} \mathbf{L}_j(\mathbf{z} + t_k(\mathbf{x} - \mathbf{z})) \mathbf{L}_j(\mathbf{z} + t_k(\mathbf{x} - \mathbf{z}))^\top g_j(\mathbf{z} + t_k(\mathbf{x} - \mathbf{z})) \quad (2.14)$$

$$:= \mathbf{H}_k(\mathbf{x}, \mathbf{z}) \in \mathbb{R}[\mathbf{x}, \mathbf{z}]^{d \times d}. \quad (2.15)$$

Observe that each entry of  $\mathbf{H}_k(\mathbf{x}, \mathbf{z})$  is a polynomial in  $(\mathbf{x}, \mathbf{z})$  of degree at most  $\max_j [2a_j + 2d_j]$ . Moreover, since, by assumption,  $f$  is strongly convex on  $\Omega$ , we have

$$\langle \mathbf{u}, \mathbf{H}_k(\mathbf{x}, \mathbf{z}) \mathbf{u} \rangle > 0, \quad \forall \mathbf{u} \in \mathbb{S}^{d-1}, \forall \mathbf{x}, \mathbf{z} \in \Omega. \quad (2.16)$$

Moreover, as the smallest eigenvalue  $\lambda_{\min}(\mathbf{H}_k(\mathbf{x}, \mathbf{z}))$  is a continuous function of its entries, it is also a continuous function of  $\mathbf{x}, \mathbf{z} \in \Omega$ , and as  $\Omega$  is compact, there exists  $\delta > 0$  such that  $\mathbf{H}_k(\mathbf{x}, \mathbf{z}) \succeq \delta \mathbf{I}$  for all  $\mathbf{x}, \mathbf{z} \in \Omega$ . Therefore by Putinar's matrix Positivstellensatz [8, Theorem 2.22],

$$\mathbf{H}_k(\mathbf{x}, \mathbf{z}) = \sum_{j=0}^{m+1} \mathbf{V}_{kj}(\mathbf{x}, \mathbf{z}) \mathbf{V}_{kj}(\mathbf{x}, \mathbf{z})^\top g_j(\mathbf{x}) + \sum_{j=0}^{m+1} \mathbf{W}_{kj}(\mathbf{x}, \mathbf{z}) \mathbf{W}_{kj}(\mathbf{x}, \mathbf{z})^\top g_j(\mathbf{z}), \quad (2.17)$$

for some matrix polynomials  $\mathbf{V}_{kj}(\mathbf{x}, \mathbf{z}), \mathbf{W}_{kj}(\mathbf{x}, \mathbf{z}) \in \mathbb{R}[\mathbf{x}, \mathbf{z}]^{d \times r_{kj}}$  with  $r_{kj} \in \mathbb{N}$ , and whose entries themselves are polynomials of degree at most  $\delta_{kj}$ . Introduce

$$S_{kj}(\mathbf{x}, \mathbf{z}) := (1 - t_k) \langle (\mathbf{x} - \mathbf{z}), \mathbf{V}_{kj}(\mathbf{x}, \mathbf{z}) \mathbf{V}_{kj}(\mathbf{x}, \mathbf{z})^\top (\mathbf{x} - \mathbf{z}) \rangle, \quad (2.18)$$

$$T_{kj}(\mathbf{x}, \mathbf{z}) := (1 - t_k) \langle (\mathbf{x} - \mathbf{z}), \mathbf{W}_{kj}(\mathbf{x}, \mathbf{z}) \mathbf{W}_{kj}(\mathbf{x}, \mathbf{z})^\top (\mathbf{x} - \mathbf{z}) \rangle. \quad (2.19)$$

Then, we can write

$$f(\mathbf{x}) - f(\mathbf{z}) = \langle \nabla f(\mathbf{z}), \mathbf{x} - \mathbf{z} \rangle + \sum_{j=0}^{m+1} \left( \sum_{k=1}^s \gamma_k S_{kj}(\mathbf{x}, \mathbf{z}) \right) g_j(\mathbf{x}) + \sum_{j=0}^{m+1} \left( \sum_{k=1}^s \gamma_k T_{kj}(\mathbf{x}, \mathbf{z}) \right) g_j(\mathbf{z}). \quad (2.20)$$

Observe that  $S_{kj}, T_{kj} \in \Sigma[\mathbf{x}, \mathbf{z}]$  with degree bounded by  $2\ell^* := 2 + \max_{j,k} \delta_{kj}$ , and that the same is true for  $\sigma_j : \mathbf{x} \mapsto \sum_{k=1}^s \gamma_k S_{kj}(\mathbf{x}, \mathbf{z})$  and  $\psi_j : \mathbf{x} \mapsto \sum_{k=1}^s \gamma_k T_{kj}(\mathbf{x}, \mathbf{z})$ , i.e.,  $\sigma_j, \psi_j \in \Sigma[\mathbf{x}]^{\ell^*}$ . So let  $\phi$  be as in (2.9), with  $2(n - d_j) \geq 2\ell^*$ , and let  $\mathbf{z} := \phi(\mathbf{x})$ . Recall that  $\phi(\mathbf{x}) \in \Omega$  and so  $g_j(\phi(\mathbf{x})) \geq 0$  for  $j = 0, \dots, m+1$ . Next, substituting  $\mathbf{z} := \phi(\mathbf{x})$  (a constant vector) into (2.20) yields,

$$\phi(f) - f(\phi(\mathbf{x})) = \langle \nabla f(\mathbf{z}), \phi(\mathbf{x}) - \phi(\mathbf{x}) \rangle + \sum_{j=0}^{m+1} \phi(\sigma_j g_j) + \sum_{j=0}^{m+1} \phi(\psi_j) g_j(\phi(\mathbf{x})). \quad (2.21)$$

Since for all  $j = 0, \dots, m+1$ ,  $\sigma_j, \psi_j \in \Sigma[\mathbf{x}]^{\ell^*}$  and since  $\deg(\sigma_j g_j) \leq 2n$ , we have  $\phi(\sigma_j g_j) \geq 0$  and  $\phi(\psi_j) \geq 0$ . Therefore, we obtain  $\phi(f) \geq f(\phi(\mathbf{x}))$ , which is the desired result.

(ii) If  $g_j \in \mathbb{R}[\mathbf{x}]_2$  and  $g_j$  is concave, then for every  $k = 1, \dots, s$ ,

$$0 \leq g_j(\mathbf{z} + t_k(\mathbf{x} - \mathbf{z})) - (t_k g_j(\mathbf{x}) + (1 - t_k) g_j(\mathbf{z})) =: q_{kj}(\mathbf{x}, \mathbf{z}) \in \Sigma[\mathbf{x}, \mathbf{z}]_1, \quad (2.22)$$

because all nonnegative quadratic polynomials are SOS. Therefore, in (2.15), we can replace  $g_j(\mathbf{z} + t_k(\mathbf{x} - \mathbf{z}))$  with  $t_k g_j(\mathbf{x}) + (1 - t_k) g_j(\mathbf{z}) + q_{kj}(\mathbf{x}, \mathbf{z})$ , and let

$$\mathbf{L}_j(\mathbf{z} + t_k(\mathbf{x} - \mathbf{z})) \mathbf{L}_j(\mathbf{z} + t_k(\mathbf{x} - \mathbf{z}))^\top =: \Delta_{kj}(\mathbf{x}, \mathbf{z}) \in \mathbb{R}[\mathbf{x}, \mathbf{z}]^{d \times d}. \quad (2.23)$$

We then obtain

$$\nabla^2 f(\mathbf{z} + t_k(\mathbf{x} - \mathbf{z})) = \sum_{j=0}^{m+1} \Delta_{kj}(\mathbf{x}, \mathbf{z}) (t_k g_j(\mathbf{x}) + (1 - t_k) g_j(\mathbf{z}) + q_{kj}(\mathbf{x}, \mathbf{z})), \quad (2.24)$$

for every  $k = 1, \dots, s$ . Hence, letting  $\sigma_{kj}(\mathbf{x}, \mathbf{z}) := \langle (\mathbf{x} - \mathbf{z}), \Delta_{kj}(\mathbf{x}, \mathbf{z}) (\mathbf{x} - \mathbf{z}) \rangle$ , we observe that  $\sigma_{kj}$  is SOS with  $\deg(\sigma_{kj}) \leq 2(a_j + 1)$ . Next,

$$\begin{aligned} \langle (\mathbf{x} - \mathbf{z}), \nabla^2 f(\mathbf{z} + t_k(\mathbf{x} - \mathbf{z})) (\mathbf{x} - \mathbf{z}) \rangle &= t_k \sum_{j=0}^{m+1} \sigma_{kj}(\mathbf{x}, \mathbf{z}) g_j(\mathbf{x}) \\ &\quad + (1 - t_k) \sum_{j=0}^{m+1} \sigma_{kj}(\mathbf{x}, \mathbf{z}) g_j(\mathbf{z}) + \sum_{j=0}^{m+1} \sigma_{kj}(\mathbf{x}, \mathbf{z}) q_{kj}(\mathbf{x}, \mathbf{z}), \end{aligned} \quad (2.25)$$

for every  $k = 1, \dots, s$ .

Let  $2n-2 \geq \max_{j=1, \dots, m+1} \{2a_j+2\}$  and  $2n \geq \max_{j=0, \dots, m+1} \{2a_j+2\}$ . Then with  $\mathbf{z} \in \Omega$  fixed, and as  $\mathbf{M}_{n-1}(g_j \phi) \succeq 0$  (recall that  $d_j = 1$  for quadratic  $g_j$ ),  $\phi(\theta g_j) \geq 0$  for all  $\theta \in \Sigma[\mathbf{x}]_{n-1}$ . Therefore, for every  $k = 1, \dots, s$ ,

$$\phi \left( \sum_{j=0}^{m+1} \sigma_{kj}(\mathbf{x}, \mathbf{z}) g_j(\mathbf{x}) \right) = \phi(\sigma_{k0}(\mathbf{x}, \mathbf{z})) + \sum_{j=1}^{m+1} \phi(\sigma_{kj}(\mathbf{x}, \mathbf{z}) g_j(\mathbf{x})) \geq 0. \quad (2.26)$$

Similarly, as  $g_j(\mathbf{z}) \geq 0$  is a constant, and  $\mathbf{M}_n(\phi) \succeq 0$ , we obtain

$$\phi \left( \sum_{j=0}^{m+1} \sigma_{kj}(\mathbf{x}, \mathbf{z}) g_j(\mathbf{z}) \right) = \sum_{j=0}^{m+1} \phi(\sigma_{kj}(\mathbf{x}, \mathbf{z}) g_j(\mathbf{z})) \geq 0. \quad (2.27)$$

Finally, as  $q_{kj}(\mathbf{x}, \mathbf{z}) \in \Sigma[\mathbf{x}, \mathbf{z}]_1$ , we also have  $\sigma_{kj}(\mathbf{x}, \mathbf{z}) q_{kj}(\mathbf{x}, \mathbf{z}) \in \Sigma[\mathbf{x}, \mathbf{z}]_{a_j+2}$ . Hence, recalling that, by assumption,  $\mathbf{M}_n(\phi) \succeq 0$ , and since  $q_{k0} = 0$ , we conclude that

$$\phi \left( \sum_{j=0}^{m+1} \sigma_{kj}(\mathbf{x}, \mathbf{z}) q_{kj}(\mathbf{x}, \mathbf{z}) \right) = \sum_{j=1}^{m+1} \phi(\sigma_{kj}(\mathbf{x}, \mathbf{z}) q_{kj}(\mathbf{x}, \mathbf{z})) \geq 0. \quad (2.28)$$

As the above is valid for all  $k$ , using  $\mathbf{z} := \phi(\mathbf{x})$ , one obtains the desired result

$$\phi(f) - f(\phi(\mathbf{x})) \geq \sum_{k=1}^s \gamma_k t_k \underbrace{\phi \left( \sum_{j=0}^{m+1} \sigma_{kj}(\mathbf{x}, \mathbf{z}) g_j(\mathbf{x}) \right)}_{\geq 0} + \sum_{k=1}^s \gamma_k (1-t_k) \underbrace{\phi \left( \sum_{j=0}^{m+1} \sigma_{kj}(\mathbf{x}, \mathbf{z}) g_j(\mathbf{z}) \right)}_{\geq 0} \quad (2.29)$$

$$+ \sum_{k=1}^s \gamma_k \underbrace{\phi \left( \sum_{j=0}^{m+1} \sigma_{kj}(\mathbf{x}, \mathbf{z}) q_{kj}(\mathbf{x}, \mathbf{z}) \right)}_{\geq 0} \geq 0. \quad (2.30)$$

◀

### 2.3 Finite convergence of the Moment-SOS hierarchy

Recall that, under Assumption 2.4, the quadratic module (2.7) associated with  $\mathbf{g}$  is Archimedean, and that the  $n$ -th moment semidefinite relaxation of POP (1.1) has the following form:

$$\rho_n = \min_{\phi} \{ \phi(f) : \phi(1) = 1, \mathbf{M}_{n-d_j}(g_j \phi) \succeq 0, j = 0, \dots, m+1 \}. \quad (2.31)$$

▷ Theorem 2.6. Let Assumption 2.4 hold. Then, for every  $n \geq n_{\min} := \max \left\{ \max_{j=0, \dots, m+1} d_j, d_f \right\}$ , the semidefinite relaxation (2.31) has an optimal solution  $\phi^{(n)}$ . In addition:

- (i) If  $f$  is strongly convex on  $\Omega$ , then there exists  $n \geq n_{\min}$  such that  $\rho_n = f^* = f(\phi^{(n)}(\mathbf{x}))$ .
- (ii) If  $\deg(g_j) \leq 2$  for every  $j = 1, \dots, m+1$ , then, as soon as  $n \geq 1 + \max \left\{ \max_{j=1, \dots, m+1} \{a_j + 1\}, a_0 \right\}$ , we have  $\rho_n = f^* = f(\phi^{(n)}(\mathbf{x}))$ , i.e., the point  $\mathbf{x}^* := \phi^{(n)}(\mathbf{x}) \in \Omega$  is globally optimal in (1.1).

**Proof.** (i) Recall that  $g_{m+1}(\mathbf{x}) = 1 - \|\mathbf{x}\|^2$  and that  $\mathbf{M}_{n-d_{m+1}}(g_{m+1} \phi) \succeq 0$ . This implies that  $\phi(x_i^{2n}) \leq 1$  for every  $i = 1, \dots, d$ . Hence, by [8, Proposition 2.38],

$$|\phi_{\alpha}| \leq \max \{ 1, \max_{i=1, \dots, d} \phi(x_i^{2n}) \} \leq 1, \quad \forall \alpha \in \mathbb{N}_{2n}^d. \quad (2.32)$$

Therefore, the feasible set of (2.31) is compact, implying that (2.31) has an optimal solution  $\phi^{(n)}$ . By construction,  $\rho_n \leq f^*$  for all  $n \geq n_{\min}$ .

Next let  $\phi^{(n)}$  be an optimal solution, hence satisfying  $\phi^{(n)}(f) = \rho_n \leq f^*$ . Observe that  $\mathbf{M}_{n-d_j}(g_j \phi) \succeq 0 \Rightarrow \phi^{(n)}(g_j) \geq 0$ , and since  $g_j$  is SOS-concave,  $0 \leq \phi^{(n)}(g_j) \Rightarrow g_j(\phi^{(n)}(\mathbf{x})) \geq 0$ . Therefore  $\mathbf{x}^* := \phi^{(n)}(\mathbf{x}) \in \Omega$ ,

which in turn implies  $f(\phi^{(n)}(\mathbf{x})) \geq f^*$ .

Finally, for  $n$  large enough, Lemma 2.5(i) guarantees that

$$f^* \leq f(\phi^{(n)}(\mathbf{x})) \leq \phi^{(n)}(f) = \rho_n \leq f^*, \quad (2.33)$$

which yields the desired result.

(ii) As in the proof of (i), but one now invokes Lemma 2.5(ii). ◀

When solving (2.31) at step  $n \geq n_{\min}$  of the Moment-SOS hierarchy, it is always recommended to check whether  $f(\phi^{(n)}(\mathbf{x})) \leq \rho_n$  at an optimal solution  $\phi^{(n)}$ . Indeed if the  $g_j$ 's are SOS-concave then  $\mathbf{x}^* := \phi^{(n)}(\mathbf{x}) \in \Omega$  is a global minimizer as  $f(\mathbf{x}^*) \leq \rho_n \leq f^*$ . Theorem 2.6 states that under the hidden convexity of Assumption 2.4, it is guaranteed to happen at some  $n \geq n_{\min}$ .

Finally, we give a toy example that illustrates the main notions introduced above, as well as the properties of the associated hierarchy of semidefinite relaxations.

▷ Example 1 (Illustration when  $d = 2$ ). Consider the following optimization problem

$$f^* = \min_{\mathbf{x} \in \Omega := \{\mathbf{x} \in \mathbb{R}^2 : \underbrace{1 - x_1^2 - x_2^2}_{:=g_1(\mathbf{x})} \geq 0\}} f(\mathbf{x}) := \frac{1}{6}x_1^3 + \frac{1}{2}x_1^2 + x_1 + \frac{1}{2}x_2^2. \quad (2.34)$$

Observe that the quadratic constraint  $g_1$  is SOS-concave, therefore,  $\Omega$  is convex, but also compact. In addition,  $Q(\mathbf{g})$  is Archimedean by construction. A simple calculation shows that

$$\nabla^2 f(\mathbf{x}) = \begin{bmatrix} x_1 + 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \forall \mathbf{x} \in \mathbb{R}^2. \quad (2.35)$$

The function  $f$  is clearly not convex on  $\mathbb{R}^2$ . Next, define the SOS polynomials  $\sigma_0 : \mathbf{x} \mapsto \frac{1}{2}(x_1 + 1)^2 + \frac{1}{2}x_2^2 \in \Sigma[\mathbf{x}]_1$  and  $\sigma_1 : \mathbf{x} \mapsto \frac{1}{2} \in \Sigma[\mathbf{x}]_0$ . Then the following identity holds *exactly*:

$$x_1 + 1 = \sigma_0(\mathbf{x}) + g_1(\mathbf{x})\sigma_1(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^2. \quad (2.36)$$

Consequently,

$$\nabla^2 f(\mathbf{x}) = \mathbf{S}_0(\mathbf{x}) \underbrace{g_0(\mathbf{x})}_{:=1} + \mathbf{S}_1(\mathbf{x}) g_1(\mathbf{x}) = \begin{bmatrix} \sigma_0(\mathbf{x}) & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \sigma_1(\mathbf{x}) & 0 \\ 0 & 0 \end{bmatrix} g_1(\mathbf{x}) \quad (2.37)$$

$$= \mathbf{L}_0 \mathbf{L}_0^\top + \mathbf{L}_1 \mathbf{L}_1^\top g_1(\mathbf{x}), \quad \text{with } \mathbf{L}_0(\mathbf{x}) := \begin{bmatrix} \frac{1+x_1}{\sqrt{2}} & \frac{x_2}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{L}_1(\mathbf{x}) := \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \quad (2.38)$$

which certifies that  $f$  is SOS-convex on  $\Omega$ . Finally, since  $\deg(g_1) = 2$ ,  $a_0 = 1$ , and  $a_1 = 0$ , we deduce that the moment semidefinite relaxation for the problem (2.34) will be exact as soon as  $n \geq 1 + \max\{a_1 + 1, a_0\} \iff n \geq 2$ . Indeed, we have,  $-\frac{2}{3} = \rho_2 = f^* = f(\phi^{(2)}(\mathbf{x}))$  with  $\mathbf{x}^* = [-1, 0]^\top \in \Omega$ .

### 3 Conclusion

We have shown that for a class of POPs with some hidden convexity, the Moment-SOS hierarchy has finite convergence with a guarantee to provide a global minimizer. Importantly, while the feasible set  $\Omega$  is identified as convex (because its defining polynomials are SOS-concave), the objective  $f$  is not assumed to be convex on  $\mathbb{R}^d$ , but SOS-convex on  $\Omega$ , without the knowledge of this SOS-convexity being explicit. As this hidden convexity is hard to check, any minimizer on  $\Omega$  (even global) provided by a (local) algorithm is lacking a guarantee to be a global minimizer. Our results complement existing work on convex POPs and highlight a remarkable aspect of the Moment-SOS hierarchy: it can blindly adapt to convex structure in the instance at hand, even when this structure is neither known nor identified in advance. Future work could investigate whether this phenomenon persists for structured variants of the Moment-SOS hierarchy designed to exploit structural properties of considered problems, such as symmetries or different sparsity types.

---

**References**

---

- 1 A.A Ahmadi, A. Olshevsky, P.A. Parrilo, J.N. Tsitsiklis. NP-hardness of deciding convexity of quartic polynomials and related problems, *Math. Program.* 137, pp. 453–476, 2013
- 2 M. Anjos, J.B. Lasserre. *Handbook of Semidefinite, Conic and Polynomial Optimization*, Editors, Springer, New York, 2011.
- 3 D.P. Bertsekas. *Nonlinear Programming*, 2nd ed., Athena Scientific, Nashua, NH, 2004
- 4 J.W. Helton, J. Nie. Sufficient and Necessary Conditions for Semidefinite Representability of Convex Hulls and Sets, *SIAM J. Optim.* 20(2), pp. 759–791, 2009.
- 5 E. de Klerk, M. Laurent. On the Lasserre hierarchy of semidefinite programming relaxations of convex polynomial optimization problems, *SIAM J. Optim.* 21(3), pp. 824–832, 2011.
- 6 J.B Lasserre. On representations of the feasible set in convex optimization, *Optim. Letters* 4 (1), pp. 1-5, 2010.
- 7 J.B. Lasserre. Convexity in semialgebraic geometry and polynomial optimization, *SIAM J. Optim.* 19, pp. 1995–2014, 2009
- 8 J.B. Lasserre. *An Introduction to Polynomial and Semi-Algebraic Optimization*, Cambridge University Press, Cambridge, UK, 2015
- 9 J.B. Lasserre. The Moment-SoS hierarchy: Applications and related topics, *Acta Numerica* 33, pp. 841–908, 2024.
- 10 J. Nie. Optimality Conditions and Finite Convergence of Lasserre’s Hierarchy, *Math. Program.* 146(1-2), pp. 97–121, 2014.
- 11 C. Scheiderer. Sums of squares on real algebraic curves, *Math. Z.* 245, pp. 725–760, 2003.