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On Hyperexponential Stabilization of a Class of Nonlinear Systems in Continuous and Discrete Time

Moussa Labbadi and Denis Efimov

Abstract—In this paper, we address the problem of accelerated (hyperexponential) stabilization for a class of nonlinear systems in both continuous- and discrete-time settings. The considered systems are subject to matched disturbances and unmatched state perturbations, where the nonlinearities are unknown but satisfy a linear growth condition. In the continuous-time case, a time-varying control law based on parametric Lyapunov equations is proposed. By employing a suitable time transformation, we establish hyperexponential stability of the closed-loop system using a quadratic Lyapunov function. For the discrete-time case, a semi-implicit Euler discretization is applied to the continuous-time closed-loop dynamics. A rigorous stability analysis is provided, showing that the discrete-time implementation preserves the stability properties of the continuous-time system.

Index Terms—Nonlinear systems, Hyperexponential stability, robust stabilization, semi-implicit Euler discretization

I. INTRODUCTION

Accelerated stability/convergence to equilibrium is essential in many applications [1] and is commonly obtained using nonlinear sliding mode or finite- and fixed-time/prescribed-time control laws. However, most existing results focus on linear systems in canonical forms. When the system includes unknown nonlinear functions, ensuring accelerated convergence becomes significantly more difficult than achieving standard exponential, finite-time, or fixed-time stabilization for controlled linear systems [2]. Moreover, unknown nonlinearities acting outside the control channel frequently arise in practice applications, for example in sampled-data systems [3], time-delay systems [4], multi-agent systems [5], and systems with input saturation and delays [6].

In recent years, accelerated stabilization problems have attracted significant attention. Several approaches have been developed, including sliding mode control [7], [8], Lyapunov differential inequality-based methods [9], [10], and homogeneous control techniques [11], [12]. More recently, prescribed-time feedback control has emerged as an effective framework for both linear and nonlinear systems, where time-varying gains grow unbounded within a prescribed time interval [13], [14], [15], [16], [17]. This approach ensures uniform convergence in the presence of matched perturbations, similarly to sliding mode control.

Alternative concepts, such as hyperexponential convergence [18], [19], provide convergence rates faster than any exponential dynamics while avoiding the finite-time blow-up of control gains. Hyperexponential convergence guarantees asymptotic

stabilization with a rate exceeding that of all linear systems. As shown in [20], output-based accelerated stabilization can be achieved for a chain of integrators. Hyperexponential and fixed-time stabilization of time-delay systems are further investigated in [21], and a broader discussion of hyperexponential stability and stabilization can be found in [22].

Hyperexponential stabilization has recently emerged as an effective framework for achieving enhanced robustness and convergence performance in complex systems subject to perturbations [23], [24], [25]. In particular, hyperexponential convergence of the double integrator was established in [26] and subsequently extended to n th-order integrator chains with matched perturbations in [2]. However, in the presence of unknown nonlinear dynamics, the stability guarantees of the resulting closed-loop system remain unresolved. Moreover, the complexity of the feedback design and the associated stability analysis hinders a precise quantitative characterization of the achievable performance and complicates extensions to broader classes of nonlinear systems. Furthermore, the implicit Euler discretization employed in [2] is not directly applicable to nonlinear systems, since, in contrast to the linear case where explicit solutions can be derived, the nonlinear setting generally leads to implicit equations that are difficult to solve. These limitations motivate the development of new hyperexponential stabilization frameworks capable of handling nonlinear systems and more general classes of state uncertainties.

Motivated by these challenges, this paper extends the study of stabilizing feedback laws for perturbed linear systems proposed in [2], where convergence is achieved at a rate faster than any exponential. While the results in [2] establish uniform convergence with respect to matched disturbances, they do not address the presence of uncertain Lipschitz nonlinearities in either continuous- or discrete-time settings. In this paper, we generalize the results of [2] to a class of nonlinear systems. Specifically, we adopt the approach of [27], which relies on time-varying parametric Lyapunov equations (PLEs) combined with a suitable time transformation, to address the hyperexponential stabilization problem for nonlinear dynamics. In the continuous-time case, the proposed time transformation allows the closed-loop system to be analyzed in a transformed time scale, under which a quadratic Lyapunov function decays exponentially. This result is then mapped back to the original time variable to establish uniform hyperexponential convergence of the closed-loop system. For the discrete-time setting, a semi-implicit Euler discretization scheme is applied to the continuous-time closed-loop dynamics. We show that the discrete-time system preserves the key stability properties of the continuous-time system, ensuring boundedness of the entire state vector while achieving hyperexponential convergence of the state variables.

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The early work of this paper was presented in [23] as a conference paper. This version of the paper provides more details compared to [23], including the full proofs of the results, as well as the complete discretization procedure and its stability analysis in Theorem 9. However, the simulation example is given only in [23].

The paper is organized as follows. Section III presents the preliminaries, including uniform hyperexponential stability, auxiliary properties, and properties of parametric Lyapunov equations. Section IV introduces the continuous-time control design. Section V discusses the semi-implicit Euler discretization of the closed-loop system. Section VI provides proofs of technical results. Finally, Section VII concludes the paper with a discussion of the main contributions and potential future work.

NOTATION

- The set of nonnegative real numbers is denoted as $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geq 0\}$, where \mathbb{R} represents the set of real numbers, \mathbb{Z} denotes the set of integers, and $\mathbb{Z}^+ = \mathbb{Z} \cap \mathbb{R}^+$ represents the set of non-negative integers.
- The absolute value of a scalar $x \in \mathbb{R}$ is denoted by $|x|$, while $\|\cdot\|$ represents the Euclidean norm in \mathbb{R}^n .
- For a Lebesgue-measurable function $d : \mathbb{R}^+ \rightarrow \mathbb{R}^m$, the essential supremum norm is defined as $\|d\|_\infty = \text{ess sup}_{t \geq 0} \|d(t)\|$. The space of functions d satisfying $\|d\|_\infty < +\infty$ is denoted by L_m^∞ .
- Given a sequence $d_k \in \mathbb{R}^m$, $k \in \mathbb{Z}^+$, we define its supremum norm as $|d|_\infty = \sup_{k \in \mathbb{Z}^+} \|d_k\|$. The set of sequences satisfying $|d|_\infty < +\infty$ is denoted as ℓ_m^∞ .
- A continuous function $\alpha : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ belongs to the class \mathcal{K} if it is strictly increasing and satisfies $\alpha(0) = 0$. A function $\alpha : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is in the class \mathcal{K}_∞ if $\alpha \in \mathcal{K}$ and $\alpha(s) \rightarrow \infty$ as $s \rightarrow \infty$. A continuous function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ belongs to the class \mathcal{KL} if $\beta(\cdot, t) \in \mathcal{K}$ for each fixed $t \in \mathbb{R}^+$ and $\beta(s, \cdot)$ is strictly decreasing and satisfies $\lim_{t \rightarrow \infty} \beta(s, t) = 0$ for all $s > 0$.
- For any matrix X , we use X^\top to denote its transpose, and $\lambda_i(X)$ to represent its i th eigenvalue (if X is square). For a symmetric matrix P , the largest and smallest eigenvalues are denoted by $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$, respectively. Moreover, $P > 0$ (resp. $P \geq 0$) indicates that P is positive definite (resp. positive semidefinite).
- The identity matrix of size $n \times n$ is denoted as I_n .
- $\exp(1) = e$.

II. PROBLEM STATEMENT

In this paper, we consider the following single-input nonlinear time-varying system in the presence of matched perturbation:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) + g_1(t, x, u), \\ \dot{x}_2(t) &= x_3(t) + g_2(t, x, u), \\ &\vdots \\ \dot{x}_{n-1}(t) &= x_n(t) + g_{n-1}(t, x, u), \\ \dot{x}_n(t) &= u(t) + g_n(t, x, u) + d(t), \end{aligned} \quad (1)$$

where $x(t) = [x_1(t) \ x_2(t) \ \cdots \ x_n(t)]^\top \in \mathbb{R}^n$ is the state vector, and $u(t) \in \mathbb{R}$ is the input. The functions $g_i(t, x, u)$ are unknown and continuous for $i = 1, \dots, n$, with $t \geq 0$. The function $d(t) \in \mathbb{R}$ represents an external disturbance acting on the system. The stabilization of the nonlinear system (1) with the nonlinearities g_i satisfying for the linear growth rate condition in the state was first investigated in [28], where it was demonstrated that exponential global stabilization can be achieved via linear state feedback. Additionally, observer-based output feedback stabilization was addressed. More recently, global prescribed-time stabilization was studied in [29].

Similar to [28], we impose the following assumption:

Assumption 1: There exist known constants $c_{ij} \geq 0$ for $j = 1, \dots, i$ and $i = 1, \dots, n$ such that

$$|g_i(t, x, u)| \leq \sum_{j=1}^i c_{ij} |x_j|, \quad i = 1, \dots, n.$$

for all $t \geq 0$, $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$.

Additionally, we assume:

Assumption 2: The disturbance d is bounded, i.e., $\|d\|_\infty < +\infty$.

We first address the control design problem in continuous time before extending the approach to discrete time.

For future reference, we rewrite the system as:

$$\dot{x}(t) = Ax(t) + b(u(t) + d(t)) + g(t, x, u),$$

where $g(t, x, u) = [g_1(t, x, u), \dots, g_n(t, x, u)]^\top$, and

$$b = \begin{bmatrix} 0_{(n-1) \times 1} \\ 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0_{(n-1) \times 1} & I_{n-1} \\ 0 & 0_{1 \times (n-1)} \end{bmatrix}.$$

It is required to design a feedback control $u(t) = u(x(t), t)$ that ensures stabilization of $x(t)$ at the origin with a hyperexponential convergence rate, uniformly for all $d \in L_\infty^1$ and the functions $g_i(t, x, u)$ satisfying Assumption 1. Note that any controllable nonlinear single-input, single-output system can be expressed in this canonical form [28], [30], where controllability is essential for achieving accelerated stabilization.

III. PRELIMINARIES

Standard stability concepts for non-autonomous nonlinear systems, such as asymptotic stability and input-to-state stability, are detailed in [31]. Below, we recall less conventional notions, an auxiliary lemma used in this paper, and key properties of parametric Lyapunov equations.

A. Uniform hyperexponential stability

Consider the following non-autonomous system:

$$\frac{dx(t)}{dt} = f(t, x(t), d(t)), \quad t \geq t_0, \quad t_0 \in \mathbb{R}^+ \quad (2)$$

where $x(t) \in \mathbb{R}^n$ is the state, $d(t) \in \mathbb{R}^m$ represents the external disturbances, $d \in L_m^\infty$, and $f : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a continuous function with respect to x and d , and piecewise continuous with respect to t . Additionally, $f(t, 0, 0) = 0$ for all $t \in \mathbb{R}^+$. The solution of system (2) with an initial condition $x_0 \in \mathbb{R}^n$ at time $t_0 \in \mathbb{R}^+$ and a disturbance $d \in L_m^\infty$ is denoted as $\varphi(t, t_0, x_0, d)$. It is assumed that f ensures the existence of

solutions $\varphi(t, t_0, x_0, d)$, at least locally in forward time. The following definition introduces the uniform stability property for (2) with a hyperexponential rate of convergence.

Definition 1: [19], [2] Given a set $D \subset L_m^\infty$, the system (2) is termed uniformly hyperexponentially stable if there exist constant $\kappa_0 \in \mathbb{R}^+$, functions $\rho, \kappa \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ such that:

$$\|\varphi(t, t_0, x_0, d)\| \leq e^{-(\kappa(t-t_0)+\kappa_0)(t-t_0)} \rho(\|x_0\|) + \beta(\|d\|_\infty, t-t_0), \quad \forall t \geq t_0 \quad (3)$$

for all $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}^+$, and $d \in D$.

An illustrative example is:

$$\dot{x}(t) = -(1+t)x(t) + d(t), \quad t \geq 0 \quad (4)$$

where $x(t), d(t) \in \mathbb{R}$. Let $t_0 = 0$, the solutions to (4) can be bounded by:

$$|x(t)| \leq e^{-(\frac{t^2}{2}+t)} |x(0)| + \frac{2\|d\|_\infty}{1+(\frac{t^2}{2}+t)} \quad (5)$$

for all $t \geq 0$, any $x(0) \in \mathbb{R}$, and $d \in L_1^\infty$. Thus, we can define $\rho(s) = s$, $\kappa(s) = \frac{s}{2}$, $\kappa_0 = 1$, and $\beta(s, t) = \frac{2s}{1+(\frac{t^2}{2}+1)t}$.

In this definition, the hyperexponential rate of convergence is required only for the initial conditions, whereas uniformity is understood as independence from the exogenous input $d \in D$ and the initial time t_0 . Although $D \subset L_m^\infty$ is assumed, any other suitable class of inputs can be considered. Similarly, the uniform hyperexponential stability can be defined with respect to an output or a state component by replacing in the left-hand side of (3) the norm of the state trajectory with the corresponding quantity.

B. Auxiliary properties

In this paper, the following lemma will be used.

Lemma 2: [32] For all $\tau \geq 0$, and $a, \alpha > 0$ such that $\alpha a > 1$, the following holds:

$$\int_0^\tau e^{s-\tau} \frac{ds}{(as+1)^\alpha} \leq \frac{r_{a,\alpha}}{(a\tau+1)^\alpha},$$

where

$$r_{a,\alpha} = (a\alpha)^\alpha \int_0^{\frac{\alpha\alpha-1}{a}} e^{s-\frac{\alpha\alpha-1}{a}} \frac{ds}{(as+1)^\alpha} + (a\alpha+1) + \left(\frac{a\alpha+1}{a\alpha}\right)^\alpha \left(1 - e^{-\frac{1}{a}}\right).$$

The following block matrix inversion formulas are used in the sequel:

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathcal{A}^{-1} + \mathcal{A}^{-1}\mathcal{B}M_1\mathcal{C}\mathcal{A}^{-1} & -\mathcal{A}^{-1}\mathcal{B}M_1 \\ -M_1\mathcal{C}\mathcal{A}^{-1} & M_1 \end{bmatrix} \\ = \begin{bmatrix} M_2 & -M_2\mathcal{B}\mathcal{D}^{-1} \\ -\mathcal{D}^{-1}\mathcal{C}M_2 & \mathcal{D}^{-1} + \mathcal{D}^{-1}\mathcal{C}M_2\mathcal{B}\mathcal{D}^{-1} \end{bmatrix},$$

where $M_1 = (\mathcal{D} - \mathcal{C}\mathcal{A}^{-1}\mathcal{B})^{-1}$, $M_2 = (\mathcal{A} - \mathcal{B}\mathcal{D}^{-1}\mathcal{C})^{-1}$, and $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are matrices of appropriate dimensions, with $\mathcal{A}, \mathcal{D}, M_1$, and M_2 assumed to be nonsingular.

C. Properties of parametric Lyapunov equations

Consider the following parametric Lyapunov equation (PLE) [33], [29], [27], [34]:

$$A^\top P_\psi + P_\psi A - P_\psi b b^\top P_\psi = -\psi P_\psi, \quad (6)$$

where ψ is a (possibly time-varying) parameter to be determined later.

Lemma 3: [33], [29], [27], [34] Let $(A, b) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$ be controllable. Then, the following statements hold:

i) The PLE (6) admits a unique positive definite solution P_ψ if and only if

$$\psi > -2 \min_{i=1,2,\dots,n} \{\operatorname{Re}\{\lambda_i(A)\}\}. \quad (7)$$

When this condition is satisfied, the solution is explicitly given by $P_\psi = \mathcal{W}_\psi^{-1}$, which is a polynomial function of ψ verifying the Lyapunov equation

$$\left(A + \frac{\psi}{2} I_n\right) \mathcal{W}_\psi + \mathcal{W}_\psi \left(A + \frac{\psi}{2} I_n\right)^\top = b b^\top.$$

ii) If ψ satisfies (7), then P_ψ exhibits the following properties:

$$\frac{dP_\psi}{d\psi} > 0, \quad (8)$$

ensuring that P_ψ is a strictly increasing function of ψ . Additionally, its trace satisfies

$$\operatorname{tr}(b^\top P_\psi b) = 2\operatorname{tr}(A) + n\psi.$$

iii) If all eigenvalues of A lie on the imaginary axis, then

$$\lim_{\psi \rightarrow 0^+} P_\psi = 0.$$

iv) When all eigenvalues of A are zero, there exists a constant $d_c \geq 1$, independent of ψ , such that

$$\frac{P_\psi}{n\psi} \leq \frac{dP_\psi}{d\psi} \leq d_c \frac{P_\psi}{n\psi}, \quad \forall \psi > 0. \quad (9)$$

v) Under these conditions, the unique solution of (6) is expressed as

$$P_\psi = \psi \Gamma(\psi) P_1 \Gamma(\psi), \quad (10)$$

where P_1 is a positive matrix defined for $\psi = 1$, and the matrix $\Gamma(\psi)$ is defined as

$$\Gamma(\psi) = \operatorname{diag}(\psi^{n-1}, \psi^{n-2}, \dots, 1). \quad (11)$$

Moreover, it satisfies the differential relation

$$\frac{dP_\psi}{d\psi} = \frac{P_\psi}{\psi} + \Gamma(\psi)(H P + P H)\Gamma(\psi), \quad (12)$$

where $H = \operatorname{diag}\{n-1, n-2, \dots, 0\}$.

Lemma 4: Let Assumption 1 be satisfied and $\Gamma(\psi)$ be defined in (11). Then, for any $\psi > 1$, any $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, and any $t \geq 0$, we have:

$$G(x) := (\Gamma(\psi)g(t, x, u))^\top (\Gamma(\psi)g(t, x, u)) \\ \leq \delta (\Gamma(\psi)x)^\top \Gamma(\psi)x$$

where $\delta \geq 0$ is defined by

$$\delta = n \max_{1 \leq i \leq n} \left(\max_{1 \leq j \leq i} c_{ij}^2 \right). \quad (13)$$

The proof of this result is given in Section VI.

IV. CONTROL IN CONTINUOUS TIME

Consider a control law for (1) chosen in the following form (the same control but written differently was used in [2]):

$$u(t) = -\frac{1}{2}b^\top P_\psi(t)x(t), \quad (14)$$

where $\psi(t) = \lambda + \alpha t$, $\lambda \geq 1$, $\alpha > 0$ are tuning parameters and $P_\psi(t) = P_{\psi(t)}$ is the unique positive definite solution to the PLE (6) (it is given in (10) with the matrix P).

Remark 5: Any continuous, strictly increasing function $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\psi(0) \geq 1$ and an unbounded integral can be utilized in (14). For instance, $\psi(t) = ae^{\alpha t}$ or $\psi(t) = at^t$, where $a > 0$ and $\alpha > 0$.

The closed-loop system of (1) and (14) is given by

$$\dot{x}(t) = \left(A - \frac{1}{2}bb^\top P_\psi(t)\right)x(t) + bd(t) + g(t, x, u). \quad (15)$$

The following result is presented with the proof in Section VI.

Theorem 6: Under assumptions in 1 and 2, let d_c from (9), δ from (13), λ and α satisfy the following condition:

$$\frac{1}{2} > \frac{\alpha d_c}{n\lambda^2} + \frac{2\sqrt{\delta\Lambda}}{\lambda} \quad \text{for} \quad \Lambda = \frac{\lambda_{\max}(P_1)}{\lambda_{\min}(P_1)}. \quad (16)$$

Then, the linear time-varying state feedback (14) renders the closed-loop system (1) uniformly hyperexponential stable.

Remark 7: The results of the proposed approach (Theorem 6) for SISO systems can be extended to MIMO nonlinear systems using the concept of left and right clustered matrices [16].

Remark 8: The proposed control approach can be extended to design an observer with hyperexponential convergence of the output states, as in [29]. This can be achieved by using a similar PLE (6), with the input matrix replaced by the output matrix.

V. DISCRETIZATION OF THE CONTROLLED SYSTEM (15)

In this section, we investigate a consistent discretization of (15). The system (15) is a nonlinear time-varying system with external input $d(t)$. Since the control gain P_ψ grows unbounded over time and nonlinear function g appears in the right-hand side, the implicit or explicit Euler methods are not suitable for all $t \geq 0$. However, the semi-implicit Euler method can provide a stable discretization approach for such a class of systems [35], [36].

Let $h > 0$ be the time step, and define the discretization time instants as $t_k = hk$, $k \in \mathbb{Z}^+$. Applying the semi-implicit Euler method to (15) results in the discrete-time system:

$$\xi_{k+1} = Z^{-1}(t_{k+1})(\xi_k + hbd_{k+1} + hg_{k+1}), \quad (17)$$

where the matrix $Z(t)$ is given by:

$$Z(t) = I_n - h\left(A - \frac{1}{2}bb^\top P_\psi(t)\right). \quad (18)$$

and $\xi_k \in \mathbb{R}^n$ is a discrete approximation of $x_k = x(t_k)$, ensuring that $\xi_k \rightarrow x_k$ as $h \rightarrow 0$. The inputs are defined as: $g_{k+1} = g(t_{k+1}, x_k, u_k)$, $d_k = d(t_k)$. We assume that perturbation signal remain bounded, i.e., $d \in \ell_1^\infty$.

The discrete-time matrix $Z(t)$ can be rewritten as follows

$$Z(t) = I_n - h\left(A - \frac{1}{2}bb^\top P_\psi(t)\right) = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C}(t) & \mathcal{D}(t) \end{bmatrix},$$

where the blocks are

$$\mathcal{A} = I_{n-1} - h \begin{bmatrix} 0_{n-2 \times 1} & I_{n-2} \\ 0 & 0_{1 \times n-2} \end{bmatrix}, \mathcal{B} = \begin{bmatrix} 0_{n-2 \times 1} \\ -h \end{bmatrix},$$

$$\mathcal{C}(t) = \frac{h}{2}\psi(t) [P_{n1}\psi^{n-1}(t) \quad \cdots \quad P_{n,n-1}\psi(t)],$$

$$\mathcal{D}(t) = 1 + \frac{h}{2}P_{nn}\psi(t),$$

the coefficients $P_{n,j}$, $j = 1, \dots, n$, denote the entries of the last row P_n of P , i.e., $P_n := [P_{n1}, P_{n2}, \dots, P_{nn}]$. Accordingly, the terms $P_{n,j}\psi^{n-j+1}(t)$ appearing in $\mathcal{C}(t)$ and $\mathcal{D}(t)$ arise from the scaling induced by $P_\psi(t) = \psi(t)\Gamma(\psi(t))P_1\Gamma(\psi(t))$ and reflect the contribution of the last row of the funnel gain matrix in the discretized dynamics.

Using the standard block inversion formula, the inverse is

$$Z^{-1}(t) = \frac{\Omega(t)}{O(t)},$$

where

$$O(t) = \mathcal{D}(t) - \mathcal{C}(t)\mathcal{A}^{-1}\mathcal{B} > 0,$$

and

$$\Omega(t) = \begin{bmatrix} \mathcal{A}^{-1}(O(t)I_{n-1} + \mathcal{B}\mathcal{C}(t)\mathcal{A}^{-1}) & -\mathcal{A}^{-1}\mathcal{B} \\ -\mathcal{C}(t)\mathcal{A}^{-1} & 1 \end{bmatrix}.$$

The denominator $O(t)$ grows like $\psi^n(t)$, ensuring non-singularity and boundedness of $Z^{-1}(t)$ for all $t \geq 0$. As $\psi(t) \rightarrow \infty$, the Schur complement dominates, yielding the asymptotic limit

$$\lim_{t \rightarrow +\infty} Z^{-1}(t) = - \begin{bmatrix} 0 & 0 & \cdots & 0 \\ h^{-1} & 0 & \cdots & 0 \\ h^{-2} & h^{-1} & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ h^{1-n} & \cdots & h^{-1} & 0 \end{bmatrix},$$

which implies that asymptotically the linear dynamics of (17) approaches a static linear system with a nilpotent matrix of index n . Obviously, the effect of the disturbance d is still asymptotically annihilated by the control, since

$$hZ^{-1}(t)b = \frac{1}{O(t)} \begin{bmatrix} -h^n \\ -h^{n-1} \\ \vdots \\ -h^2 \\ h \end{bmatrix} \rightarrow 0 \quad \text{as } t \rightarrow +\infty.$$

It remains to demonstrate that the hyperexponential rate of convergence is also preserved in (17). Moreover, the effect of the nonlinearities g is still asymptotically annihilated by the control, which follows from the structure of the corresponding

discrete-time term. Indeed, using the representation $Z^{-1}(t) = \Omega(t)/O(t)$, one obtains

$$hZ^{-1}(t)g(t, \xi, u) = \begin{pmatrix} \sum_{i=1}^n \frac{h g_i(t, \xi, u) \mathcal{P}_{i1}(\psi(t), h)}{O(t)} \\ \sum_{i=1}^n \frac{h g_i(t, \xi, u) \mathcal{P}_{i2}(\psi(t), h)}{O(t)} \\ \vdots \\ \sum_{i=1}^n \frac{h g_i(t, \xi, u) \mathcal{P}_{in}(\psi(t), h)}{O(t)} \end{pmatrix},$$

where $\mathcal{P}_{ij}(\psi, h)$ are polynomial functions in ψ and h of degree at most $n-1$, while $O(t)$ is a polynomial of degree n in $\psi(t)$. Using the growth condition from Assumption 1:

$$|g_i(t, \xi, u)| \leq \sum_{j=1}^i c_{ij} |\xi_j|, \quad i = 1, \dots, n,$$

and the fact that $O(t)$ is a polynomial of degree n in $\psi(t)$, it follows that each component of $hZ^{-1}(t)g$ satisfies

$$|[hZ^{-1}(t)g(t, \xi, u)]_k| \leq \frac{c}{\psi(t)} \|\xi\|, \quad k = 1, \dots, n,$$

for some constant $c > 0$. Consequently if $\|\xi\|_\infty < +\infty$ then,

$$\lim_{t \rightarrow +\infty} hZ^{-1}(t)g(t, \xi, u) = 0.$$

The following result is presented with the proof in Section VI.

Theorem 9: Let $\alpha > 0$, $\gamma > 0$, and $\beta > 0$, and suppose there exists $P_1 = P_1^\top \in \mathbb{R}^{n \times n}$, $P_1 \succ 0$, such that the following inequalities hold:

$$(A - \frac{1}{2}bb^\top P_1)P_1^{-1} + P_1^{-1}(A - \frac{1}{2}bb^\top P_1)^\top \preceq 0, \quad (19a)$$

$$(A - \frac{1}{2}bb^\top P_1)P_1^{-1}(A - \frac{1}{2}bb^\top P_1)^\top - \beta^{-1}I_n - \gamma^{-1}bb^\top - \alpha^{-1}P_1^{-1} \succeq 0. \quad (19b)$$

Then, for any initial state $\xi_0 \in \mathbb{R}^n$, the solution ξ_k satisfies

$$\|\xi_k\|^2 \leq \psi^{2(n-1)}(t_k) \frac{\lambda_{\max}(P_1)}{\lambda_{\min}(P_1)} \left(\prod_{i=0}^{k-1} \bar{\alpha}_i \right) \|\xi_0\|^2 + \frac{\gamma \|d\|_\infty^2}{\lambda_{\min}(P_1)} \psi^{2(n-1)}(t_k) \sum_{j=0}^{k-1} \left(\prod_{i=j+1}^{k-1} \bar{\alpha}_i \right) \psi^{-2n}(t_{j+1}),$$

where

$$\bar{\alpha}_i := (h^{-2}\alpha + \beta \delta \lambda_{\max}(P_1^{-1})) \psi^{-2}(t_{i+1}).$$

VI. PROOFS OF TECHNICAL RESULTS

Proof: (Lemma 4) To establish the inequality for $G(x)$ and determine δ , we begin by deriving the necessary terms as:

$$\Gamma(\psi)g(t, x, u) = \begin{pmatrix} \psi^{n-1}g_1(t, x, u) \\ \vdots \\ \psi^0g_n(t, x, u) \end{pmatrix},$$

then recalling Assumption 1, we have

$$\begin{aligned} (\Gamma(\psi)g(t, x, u))^\top \Gamma(\psi)g(t, x, u) &= \sum_{i=1}^n (\psi^{n-i}g_i(t, x, u))^2 \\ &\leq \sum_{i=1}^n \left[\psi^{n-i} \sum_{j=1}^i \frac{c_{ij}}{\psi^{n-j}} |\psi^{n-j}x_j| \right]^2 \\ &= \sum_{i=1}^n \left[\sum_{j=1}^i \frac{c_{ij}}{\psi^{i-j}} |\psi^{n-j}x_j| \right]^2. \end{aligned}$$

Consequently, under Assumption 1, we have the following inequality for $G(x)$:

$$\begin{aligned} G(x) &\leq \max_{1 \leq i \leq n} \max_{1 \leq j \leq i} c_{ij}^2 \sum_{i=1}^n \left[\sum_{j=1}^i (|\psi^{n-j}x_j|)^2 \right]^2 \\ &\leq n \max_{1 \leq i \leq n} \max_{1 \leq j \leq i} c_{ij}^2 \left[\sum_{j=1}^n (|\psi^{n-j}x_j|)^2 \right]^2 \\ &= \delta (|\Gamma(\psi)x|^2), \end{aligned}$$

the proof is thus complete. \blacksquare

Proof: (Theorem 6) We note that when $\delta = 0$ (i.e., $c_{ij} = 0$), system (1) simplifies to a chain of integrators with matched perturbations, and Theorem 6 corresponds to Theorem 1 in [2]. Therefore, we will assume $\delta > 0$ in the following proof.

Consider the Lyapunov function

$$V(t, x) = x^\top P_\psi(t)x.$$

The time derivative of $V(t, x)$ is

$$\begin{aligned} \dot{V}(t, x) &= \dot{\psi}(t)x(t)^\top \frac{dP_\psi}{d\psi} x(t) + 2x(t)^\top P_\psi(t)g(t, x, u) \\ &\quad + x(t)^\top (A^\top P_\psi(t) + P_\psi(t)A - P_\psi(t)bb^\top P_\psi(t))x(t) \\ &\quad + 2x(t)^\top P_\psi(t)bd(t) \\ &\leq \frac{\alpha d_c}{n\psi(t)} x(t)^\top P_\psi(t)x(t) - \psi(t)x(t)^\top P_\psi(t)x(t) \\ &\quad + 2x(t)^\top P_\psi(t)g(t, x, u) + 2x(t)^\top P_\psi(t)bd(t) \\ &\leq \left(\frac{\alpha d_c}{n\psi(t)} - \psi(t) \right) x(t)^\top P_\psi(t)x(t) + k_0 x(t)^\top P_\psi(t)x(t) \\ &\quad + \frac{g(t, x, u)^\top P_\psi(t)g(t, x, u)}{k_0} + 2x(t)^\top P_\psi(t)bd(t) \end{aligned}$$

where $k_0 > 0$ is a scalar to be determined, and we have noted that $\dot{\psi} = \alpha$. Under Assumption 1 and taking (11) and (13) into account, we have

$$\begin{aligned} &g(t, x, u)^\top P_\psi(t)g(t, x, u) \\ &= g(t, x, u)^\top \psi(t) \Gamma(\psi(t)) P \Gamma(\psi(t)) g(t, x, u) \\ &\leq \psi(t) \lambda_{\max}(P_1) (\Gamma(\psi(t))g(t, x, u))^\top (\Gamma(\psi(t))g(t, x, u)) \\ &\leq \psi(t) \delta \Lambda (\Gamma(\psi(t))x(t))^\top \lambda_{\min}(P_1) (\Gamma(\psi(t))x(t)) \\ &\leq \psi(t) \delta \Lambda (\Gamma(\psi(t))x(t))^\top P (\Gamma(\psi(t))x(t)) \\ &= \delta \Lambda x(t)^\top P_\psi(t)x(t) \end{aligned}$$

where $\Lambda = \frac{\lambda_{\max}(P_1)}{\lambda_{\min}(P_1)}$ and

$$2x(t)^\top P_\psi(t)bd(t) \leq \frac{1}{2}\psi(t)x(t)^\top P_\psi(t)x(t) + 2\frac{d(t)b^\top P_\psi(t)bd(t)}{\psi(t)}.$$

Hence, $\dot{V}(t, x)$ can be continued as

$$\dot{V}(t, x) \leq -\mu(\psi(t))\psi(t)x(t)^\top P_\psi(t)x(t) + 2\frac{d(t)b^\top P_\psi(t)bd(t)}{\psi(t)}$$

where

$$\mu(\psi) = \left(\frac{1}{2} - \alpha \frac{d_c}{n\psi^2} - \frac{1}{\psi} \left(\frac{\delta\Lambda}{k_0} + k_0 \right) \right)$$

Defining $k_0 = \sqrt{\delta\Lambda}$ (we can always assume without losing generality that $\delta > 1$), we obtain:

$$\mu(\psi) = \frac{1}{2} - \frac{\alpha d_c}{n\psi^2} - \frac{2\sqrt{\delta\Lambda}}{\psi} > 0$$

due to the imposed restriction on α and λ . Calculating the derivative of $\mu(\psi)$ with respect to ψ :

$$\frac{d\mu}{d\psi} = \alpha \frac{2d_c}{n\psi^3} + \frac{1}{\psi^2} 2\sqrt{\delta\Lambda}$$

we obtain that $\mu(\psi)$ is monotonously growing due to the derivative is always positive. Then, we get

$$\mu(\psi) > \mu(\lambda)$$

Therefore, we have

$$\dot{V} \leq -\mu(\lambda)\psi V + 2P_{n,n}d^2,$$

where $P_{n,n}$ is a the corresponding element of the matrix P_1 .

We introduce a new time variable τ defined as:

$$\tau = \phi(t) = \mu_0 \frac{\alpha t(t + 2\frac{\lambda}{\alpha})}{2}, \quad \text{with its inverse}$$

$$t = \phi^{-1}(\tau) = \sqrt{\frac{2\tau}{\alpha\mu_0} + \frac{\lambda^2}{\alpha^2}} - \frac{\lambda}{\alpha},$$

where $\mu(\lambda) = \mu_0$ and $\phi^{-1}(\tau)$ represents the inverse function of $\phi(t)$. Differentiating τ with respect to t gives:

$$d\tau = \mu_0(\alpha t + \lambda)dt.$$

After applying the time transformation $t \rightarrow \tau$, the dynamics of the Lyapunov function can be simplified into a more convenient form, valid for $\tau \geq 0$:

$$\frac{dV(\tau, x)}{d\tau} \leq -V(\tau, x) + \frac{P_{n,n}d^2(\tau)}{\mu_0\psi(\tau)}$$

whose solution admits an upper bound:

$$V(\tau, x) \leq e^{-\tau}V(0, x_0) + e^{-\tau} \int_0^\tau e^s \frac{P_{n,n}\|d\|_\infty^2}{\mu_0\psi(s)} ds.$$

Applying Lemma 2, we have

$$V(\tau, x) \leq e^{-\tau}V(0, x_0) + r \frac{2\alpha}{\lambda^2\mu_0}, \frac{1}{2} \frac{P_{n,n}\|d\|_\infty^2}{\mu_0\psi(\tau)} ds,$$

and in original time,

$$V(t, x) \leq e^{-\mu_0 \frac{t(t+2\lambda)}{2}} V(0, x_0) + r \frac{2\alpha}{\lambda^2\mu_0}, \frac{1}{2} \frac{P_{n,n}\|d\|_\infty^2}{\mu_0\psi(t)}.$$

Moreover,

$$V(t, x) \geq x(t)^\top P_\lambda x(t) \geq \lambda_{\min}(P_\lambda)\|x(t)\|^2,$$

which in turn implies

$$\|x(t)\| \leq \sqrt{\frac{e^{-\mu_0 \frac{t(t+2\lambda)}{2}} \lambda_{\max}(P_\lambda)}{\lambda_{\min}(P_\lambda)} \|x_0\|^2 + \frac{r \frac{2\alpha}{\lambda^2\mu_0}, \frac{1}{2} P_{n,n}\|d\|_\infty^2}{\lambda_{\min}(P_\lambda) \mu_0\psi(t)}},$$

which indicates that the states exhibit a hyperexponential convergence in response to the initial conditions. Furthermore, the perturbation term vanishes asymptotically as $t \rightarrow \infty$, ensuring robust stability of the system in the presence of external disturbances. ■

Proof: (Theorem 9) To investigate the stability and the convergence rate of the discrete-time system (17), we consider the following time-varying Lyapunov function candidate:

$$V_k = \xi_k^\top \Theta(t_k) \xi_k, \quad \Theta(t_k) = \Upsilon(t_k) P_1 \Upsilon(t_k), \\ \Upsilon(t_k) = \psi^{1-n}(t_k) \Gamma(t_k),$$

where $P_1 = P_1^\top \succ 0$ is the matrix introduced in the hypotheses of the theorem.

For given positive constants $\alpha, \gamma, \beta > 0$, we define the following time-varying parameters: $\alpha_k = h^{-2}\alpha\psi^{-2}(t_{k+1})$, $\gamma_k = \gamma\psi^{-2n}(t_{k+1})$, $\beta_k = \beta\psi^{-2}(t_{k+1})\Upsilon^2(t_{k+1})$. we obtain for all $k \in \mathbb{Z}_+$:

$$V_{k+1} - \alpha_k V_k = \begin{bmatrix} \xi_k \\ d_{k+1} \\ g_{k+1} \end{bmatrix}^\top Q_k \begin{bmatrix} \xi_k \\ d_{k+1} \\ g_{k+1} \end{bmatrix} + \gamma_k d_{k+1}^2 + g_{k+1}^\top \beta_k g_{k+1},$$

where

$$Q_k = \begin{bmatrix} I_n \\ hb^\top \\ h \end{bmatrix} (Z^{-1}(t_{k+1}))^\top \Theta(t_{k+1}) Z^{-1}(t_{k+1}) \\ \times \begin{bmatrix} I_n \\ hb^\top \\ h \end{bmatrix}^\top - \begin{bmatrix} \alpha_k \Theta(t_k) & 0 & 0 \\ 0 & \gamma_k & 0 \\ 0 & 0 & \beta_k \end{bmatrix}.$$

We need to find the restrictions on P_1 and the gains α, γ, β such that $Q_k \leq 0$ for all $k \geq 0$. To this end, note that $-Q_k$ is the Schur complement of

$$\tilde{Q}_k = \begin{bmatrix} \alpha_k \Theta(t_k) & 0 & 0 & I_n \\ 0 & \gamma_k & 0 & hb^\top \\ 0 & 0 & \beta_k & h \\ I_n & hb & h & Z(t_{k+1})\Theta(t_{k+1})^{-1}Z^\top(t_{k+1}) \end{bmatrix},$$

then we seek for conditions guaranteeing $\tilde{Q}_k \succeq 0$, which calculating another Schur complement of \tilde{Q}_k are equivalent to

$$\begin{aligned} \hat{Q}_k &= \begin{bmatrix} \beta_k & h \\ h & Z(t_{k+1})\Theta(t_{k+1})^{-1}Z^\top(t_{k+1}) \end{bmatrix} \\ &- \begin{bmatrix} 0 & 0 \\ I_n & hb \end{bmatrix} \begin{bmatrix} \alpha_k^{-1}\Theta(t_{k+1})^{-1} & 0 \\ 0 & \gamma_k^{-1} \end{bmatrix} \begin{bmatrix} 0 & I_n \\ 0 & hb^\top \end{bmatrix} \\ &= \begin{bmatrix} \beta_k & h \\ h & L_k - h^2\gamma_k^{-1}bb^\top \end{bmatrix} \succeq 0, \end{aligned}$$

$$L_k = Z(t_{k+1})\Theta(t_{k+1})^{-1}Z^\top(t_{k+1}) - \alpha_k^{-1}\Theta(t_k)^{-1}.$$

Finally, computing the Schur complement of \hat{Q}_k we get an equivalent condition to check:

$$L_k - h^2\gamma_k^{-1}bb^\top - h^2\beta_k^{-1} \succeq 0.$$

By multiplying the inequality from both sides with $\Upsilon(t_{k+1})$ and applying (6), we obtain the following matrix inequality:

$$\begin{aligned} 0 &\preceq \Upsilon(t_{k+1})(L_k - h^2\gamma_k^{-1}bb^\top - h^2\beta_k^{-1})\Upsilon(t_{k+1}) \\ &= \Upsilon(t_{k+1})\left(Z(t_{k+1})\Theta^{-1}(t_{k+1})Z^\top(t_{k+1}) - \alpha_k^{-1}\Theta^{-1}(t_k) \right. \\ &\quad \left. - h^2\gamma_k^{-1}bb^\top - h^2\beta_k^{-1}\right)\Upsilon(t_{k+1}) \\ &= \underbrace{\Upsilon(t_{k+1})Z(t_{k+1})\Upsilon^{-1}(t_{k+1})}_{I_n - h\psi(t_{k+1})(A - \frac{1}{2}bb^\top P_1)} P_1^{-1} \\ &\quad \times \underbrace{\left(\Upsilon(t_{k+1})Z(t_{k+1})\Upsilon^{-1}(t_{k+1})\right)^\top}_{I_n - h\psi(t_{k+1})(A - \frac{1}{2}bb^\top P_1)^\top} \\ &\quad - \alpha_k^{-1}\Upsilon(t_{k+1})\Upsilon^{-1}(t_k)P_1^{-1}\Upsilon^{-1}(t_k)\Upsilon(t_{k+1}) \\ &\quad - h^2\beta_k^{-1}\Upsilon^2(t_{k+1}) - h^2\gamma_k^{-1}\psi^{2-2n}(t_{k+1})bb^\top \\ &= P_1^{-1} - \alpha_k^{-1}\Upsilon(t_{k+1})\Upsilon^{-1}(t_k)P_1^{-1}\Upsilon^{-1}(t_k)\Upsilon(t_{k+1}) \\ &\quad + h^2\psi^2(t_{k+1})(A - \frac{1}{2}bb^\top P_1)P_1^{-1}(A - \frac{1}{2}bb^\top P_1)^\top \\ &\quad - h\psi(t_{k+1})\left((A - \frac{1}{2}bb^\top P_1)P_1^{-1} + P_1^{-1}(A - \frac{1}{2}bb^\top P_1)^\top\right) \\ &\quad - h^2\beta_k^{-1}\Upsilon^2(t_{k+1}) - h^2\gamma_k^{-1}\psi^{2-2n}(t_{k+1})bb^\top \end{aligned}$$

Note that $P_1^{-1} \succeq \Upsilon(t_{k+1})\Upsilon^{-1}(t_k)P_1^{-1}\Upsilon^{-1}(t_k)\Upsilon(t_{k+1})$ [2]. Therefore, the property $Q_k \prec 0$ can be obtained using the following conditions:

$$\begin{aligned} 0 &\preceq h^2\psi^2(t_{k+1})(A - \frac{1}{2}bb^\top P_1)P_1^{-1}(A - \frac{1}{2}bb^\top P_1)^\top \\ &- h\psi(t_{k+1})\left((A - \frac{1}{2}bb^\top P_1)P_1^{-1} + P_1^{-1}(A - \frac{1}{2}bb^\top P_1)^\top\right) \\ &\quad - h^2\beta_k^{-1}\Upsilon^2(t_{k+1}) \\ &\quad - h^2\gamma_k^{-1}\psi^{2-2n}(t_{k+1})bb^\top + P_1^{-1} - \alpha_k^{-1}P_1^{-1}. \end{aligned}$$

Substituting the expressions of α_k, γ_k and β_k in the above formula, we get

$$\begin{aligned} 0 &\preceq -h\psi(t_{k+1})\left((A - \frac{1}{2}bb^\top P_1)P_1^{-1} + P_1^{-1}(A - \frac{1}{2}bb^\top P_1)^\top\right) \\ &\quad + h^2\psi^2(t_{k+1})\left((A - \frac{1}{2}bb^\top P_1)P_1^{-1}(A - \frac{1}{2}bb^\top P_1)^\top \right. \\ &\quad \left. - \beta^{-1}I_n - \gamma^{-1}bb^\top - \alpha^{-1}P_1^{-1}\right) + P_1^{-1}. \end{aligned}$$

If (19) is satisfied, then the matrix inequality above is true.

Given that the conditions of theorem explicitly includes all required matrix inequalities, the preceding derivation demonstrates that

$$\begin{aligned} V_{k+1} &\leq \alpha_k V_k + \gamma_k d_{k+1}^2 + g_{k+1}^\top \beta_k g_{k+1} \\ &= h^{-2}\alpha\psi^{-2}(t_{k+1})V_k + \gamma\psi^{-2n}(t_{k+1})d_{k+1}^2 \\ &\quad + g_{k+1}^\top \beta\psi^{-2}(t_{k+1})\Upsilon^2(t_{k+1})g_{k+1}, \end{aligned}$$

Applying Lemma 4 and using the relation $\Gamma(\psi(t_{k+1})) = \psi^{n-1}(t_{k+1})\Upsilon(t_{k+1})$, we obtain

$$\begin{aligned} g_{k+1}^\top \beta_k g_{k+1} &= \beta\psi^{-2}(t_{k+1})(\Upsilon(t_{k+1})g_{k+1})^\top(\Upsilon(t_{k+1})g_{k+1}) \\ &\leq \beta\delta\psi^{-2}(t_{k+1})(\Upsilon(t_{k+1})\xi_k)^\top(\Upsilon(t_{k+1})\xi_k). \end{aligned}$$

Since the Lyapunov function is defined as

$$V_k = (\Upsilon(t_k)\xi_k)^\top P_1(\Upsilon(t_k)\xi_k),$$

and $P_1 > 0$, it follows that

$$\begin{aligned} (\Upsilon(t_{k+1})\xi_k)^\top(\Upsilon(t_{k+1})\xi_k) &\leq (\Upsilon(t_k)\xi_k)^\top(\Upsilon(t_k)\xi_k) \\ &\leq \lambda_{\max}(P_1^{-1})V_k. \end{aligned}$$

Therefore,

$$g_{k+1}^\top \beta_k g_{k+1} \leq \beta\delta\lambda_{\max}(P_1^{-1})\psi^{-2}(t_{k+1})V_k.$$

The Lyapunov inequality becomes

$$\begin{aligned} V_{k+1} &\leq (h^{-2}\alpha + \beta\delta\lambda_{\max}(P_1^{-1}))\psi^{-2}(t_{k+1})V_k \\ &\quad + \gamma\psi^{-2n}(t_{k+1})d_{k+1}^2. \end{aligned}$$

Defining

$$\bar{\alpha}_k := (h^{-2}\alpha + \beta\delta\lambda_{\max}(P_1^{-1}))\psi^{-2}(t_{k+1}),$$

we obtain

$$V_{k+1} \leq \bar{\alpha}_k V_k + \gamma\psi^{-2n}(t_{k+1})d_{k+1}^2.$$

Iterating gives

$$V_k \leq \left(\prod_{i=0}^{k-1} \bar{\alpha}_i\right) V_0 + \sum_{j=0}^{k-1} \left(\prod_{i=j+1}^{k-1} \bar{\alpha}_i\right) \gamma\psi^{-2n}(t_{j+1})d_{j+1}^2.$$

which gives the required estimate. Since $\psi(t) \rightarrow \infty$, we have $\bar{\alpha}_k \rightarrow 0$, and therefore the product $\prod_{i=0}^{k-1} \bar{\alpha}_i$ converges to zero faster than any exponential rate.

This proves hyperexponential convergence in the initial error and there is no asymptotic dependence in d , which completes the proof. \blacksquare

VII. CONCLUSIONS

We have addressed hyperexponential stabilization of nonlinear systems with matched disturbances and unmatched state perturbations under a linear growth condition. A time-varying control law ensures hyperexponential stability in continuous time, while a semi-implicit Euler discretization preserves these properties in discrete time. The results provide a systematic framework for robust, accelerated stabilization compatible with practical implementation. For future work, the proposed approach will be extended to more complex systems, including bilinear dynamics.

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