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# Lacunary matrices

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## Abstract

We study unconditional subsequences of the canonical basis  $(e_{rc})$  of elementary matrices in the Schatten class  $S^p$ . They form the matrix counterpart to Rudin's  $\Lambda(p)$  sets of integers in Fourier analysis. In the case of  $p$  an even integer, we find a sufficient condition in terms of trails on a bipartite graph. We also establish an optimal density condition and present a random construction of bipartite graphs. As a byproduct, we get a new proof for a theorem of Erdős on circuits in graphs.

## 1 Introduction

We study the following question on the Schatten class  $S^p$ .

- (†) How many matrix coefficients of an operator  $x \in S^p$  must vanish so that the norm of  $x$  has a bounded variation if we change the sign of the remaining nonzero matrix coefficients?

Let  $C$  be the set of columns and  $R$  be the set of rows for coordinates in the matrix, in general two copies of  $\mathbb{N}$ . Let  $I \subseteq R \times C$  be the set of matrix coordinates of the remaining nonzero matrix coefficients of  $x$ . Property (†) means that the subsequence  $(e_{rc})_{(r,c) \in I}$  of the canonical basis of elementary matrices is an unconditional basic sequence in  $S^p$ :  $I$  forms a  $\sigma(p)$  set in the terminology of [5, §4].

It is natural to wonder about the operator valued case, where the matrix coefficients are themselves operators in  $S^p$ . As the proof of our main result carries over to that case, we shall state it in the more general terms of complete  $\sigma(p)$  sets.

We show that for our purpose, a set of matrix entries  $I \subseteq R \times C$  is best understood as a bipartite graph. Its two vertex classes are  $C$  and  $R$ , whose elements will respectively be termed “column vertices” and “row vertices”. Its edges join only row vertices  $r \in R$  with column vertices  $c \in C$ , this occurring exactly if  $(r, c) \in I$ .

We obtain a generic condition for  $\sigma(p)$  sets in the case of even  $p$  (Theorem 3.2) that generalises [5, Prop. 6.5]. These sets reveal in fact as a matrix counterpart to Rudin's  $\Lambda(p)$  sets and we are able to transfer Rudin's proof of [9, Theorem 4.5(b)] to a non-commutative context: his number  $r_s(E, n)$  is replaced by the numbers of Def. 2.4(b) and we count trails between given vertices instead of representations of an integer.

We also establish an upper bound for the intersection of a  $\sigma(p)$  set with a finite product set  $R' \times C'$  (Theorem 4.2): this is a matrix counterpart to Rudin's [9, Theorem 3.5]. In terms of bipartite graphs, this intersection is the subgraph induced by the vertex subclasses  $C' \subseteq C$  and  $R' \subseteq R$ .

The bound of Theorem 4.2 provides together with Theorem 3.2 a generalisation of a theorem by Erdős [4, p. 33] on graphs without circuits of a given even length. In the last part of this article, we present a random construction of maximal  $\sigma(p)$  sets for even integers  $p$ .

**Terminology.**  $C$  is the set of columns and  $R$  is the set of rows, in general both indexed by  $\mathbb{N}$ . The set  $V$  of all vertices is their disjoint union  $R \amalg C$ . An edge on  $V$  is a pair  $\{v, w\} \subseteq V$ . A graph on  $V$  is given by its set of edges  $E$ . A bipartite graph on  $V$  with vertex classes  $C$  and  $R$  has only edges  $\{r, c\}$  such that  $c \in C$  and  $r \in R$  and may therefore be described alternatively by the set  $I = \{(r, c) \in R \times C : \{r, c\} \in E\}$ . A trail of length  $s$  in a graph is a sequence  $(v_0, \dots, v_s)$  of  $s + 1$  vertices such that  $\{v_0, v_1\}, \dots, \{v_{s-1}, v_s\}$  are pairwise distinct edges of the graph. A trail is a path

if its vertices are pairwise distinct. A circuit of length  $p$  in a graph is a sequence  $(v_1, \dots, v_p)$  of  $p$  vertices such that  $\{v_1, v_2\}, \dots, \{v_{p-1}, v_p\}, \{v_p, v_1\}$  are pairwise distinct edges of the graph. A circuit is a cycle if its vertices are pairwise distinct.

**Notation.**  $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ . Let  $q = (r, c) \in R \times C$ . The transpose of  $q$  is  $q^* = (c, r)$ . The entry (elementary matrix)  $e_q = e_{rc}$  is the operator on  $\ell_2$  that maps the  $c$ th basis vector on the  $r$ th basis vector and all other basis vectors on 0. The matrix coefficient at coordinate  $q$  of an operator  $x$  on  $\ell_2$  is  $x_q = \text{tr } e_q^* x$  and its matrix representation is  $(x_q)_{q \in R \times C} = \sum_{q \in R \times C} x_q e_q$ . The Schatten class  $S^p$ ,  $1 \leq p < \infty$ , is the space of those compact operators  $x$  on  $\ell_2$  such that  $\|x\|_p^p = \text{tr } |x|^p = \text{tr}(x^* x)^{p/2} < \infty$ . For  $I \subseteq R \times C$ , the entry space  $S_I^p$  is the space of those  $x \in S^p$  whose matrix representation is supported by  $I$ :  $x_q = 0$  if  $q \notin I$ .  $S_I^p$  is also the closed subspace of  $S^p$  spanned by  $(e_q)_{q \in I}$ . The  $S^p$ -valued Schatten class  $S^p(S^p)$  is the space of those operators  $x$  from  $\ell_2$  to  $S^p$  such that  $\|x\|_p^p = \text{tr}(\text{tr } |x|^p) < \infty$ , where the inner trace is the  $S^p$ -valued analogue of the usual trace. The  $S^p$ -valued entry space  $S_I^p(S^p)$  is the closed subspace spanned by the  $x_q e_q$  with  $x_q \in S^p$  and  $q \in I$ :  $x_q = \text{tr } e_q^* x$  is the operator coefficient of  $x$  at matrix coordinate  $q$ . Thus, for even integers  $p$  and  $x = (x_q)_{q \in I} = \sum_{q \in I} x_q e_q$  with  $x_q \in S^p$  and  $I$  finite,

$$\|x\|_p^p = \sum_{q_1, \dots, q_p \in I} \text{tr } x_{q_1}^* x_{q_2} \dots x_{q_{p-1}}^* x_{q_p} \text{tr } e_{q_1}^* e_{q_2} \dots e_{q_{p-1}}^* e_{q_p}.$$

A Schur multiplier  $T$  on  $S_I^p$  associated to  $(\mu_q)_{q \in I} \in \mathbb{C}^I$  is a bounded operator on  $S_I^p$  such that  $T e_q = \mu_q e_q$  for  $q \in I$ .  $T$  is furthermore completely bounded (c.b. for short) if  $T$  is bounded as the operator on  $S_I^p(S^p)$  defined by  $T(x_q e_q) = \mu_q x_q e_q$  for  $x_q \in S^p$  and  $q \in I$ .

We shall stick to this harmonic analysis type notation; let us nevertheless show how these objects are termed with tensor products:  $S^p(S^p)$  is also  $S^p(\ell_2 \otimes_2 \ell_2)$  endowed with  $\|x\|_p^p = \text{tr} \otimes \text{tr } |x|^p$ ; one should write  $x_q \otimes e_q$  instead of  $x_q e_q$ ; here  $x_q = \text{Id}_{S^p} \otimes \text{tr}((\text{Id}_{\ell_2} \otimes e_q^*)x)$ ;  $T$  is c.b. if  $\text{Id}_{S^p} \otimes T$  is bounded on  $S^p(\ell_2 \otimes_2 \ell_2)$ .

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## 2 Definitions

We use the notion of unconditionality in order to define the matrix analogue of Rudin's "commutative"  $\Lambda(p)$  sets.

**Definition 2.1.** Let  $X$  be a Banach space. The sequence  $(y_n) \subseteq X$  is an unconditional basic sequence in  $X$  if there is a constant  $D$  such that

$$\left\| \sum \vartheta_n c_n y_n \right\|_X \leq D \left\| \sum c_n y_n \right\|_X$$

for every real (vs. complex) choice of signs  $\vartheta_n \in \{-1, 1\}$  (vs.  $\vartheta_n \in \mathbb{T}$ ) and every finitely supported sequence of scalar coefficients  $(c_n)$ . The optimal  $D$  is the real (vs. complex) unconditionality constant of  $(y_n)$  in  $X$ .

Real and complex unconditionality are isomorphically equivalent: the complex unconditionality constant is at most  $\pi/2$  times the real one. The notions of unconditionality and multipliers are intimately connected: we have

**Proposition 2.2.** *Let  $(y_n) \subseteq X$  be an unconditional basic sequence in  $X$  and let  $Y$  be the closed subspace of  $X$  spanned by  $(y_n)$ . The real (vs. complex) unconditionality constant of  $(y_n)$  in  $X$  is exactly the least upper bound for the norms  $\|T\|_{\mathcal{L}(Y)}$ , where  $T$  is the multiplication operator defined by  $T y_n = \mu_n y_n$ , and the  $\mu_n$  range over all real (vs. complex) numbers with  $|\mu_n| \leq 1$ .*

Let us encompass the notions proposed in Question (†).

**Definition 2.3.** Let  $I \subseteq R \times C$  and  $p > 2$ .

- (a) [5, Def. 4.1]  $I$  is a  $\sigma(p)$  set if  $(e_q)_{q \in I}$  is an unconditional basic sequence in  $S^p$ . This amounts to the uniform boundedness of the family of all relative Schur multipliers by signs

$$T_\vartheta: S_I^p \rightarrow S_I^p, x = (x_q)_{q \in I} \mapsto T_\vartheta x = (\vartheta_q x_q)_{q \in I} \text{ with } \vartheta_q \in \{-1, 1\}. \quad (1)$$

By [5, Lemma 0.5], this means that there is a constant  $D$  such that for every finitely supported operator  $x = (x_q)_{q \in I} = \sum_{q \in I} x_q e_q$  with  $x_q \in \mathbb{C}$

$$D^{-1} \|x\|_p \leq \|x\|_p \leq \|x\|_p, \quad (2)$$

where the second inequality is a convexity inequality that is always satisfied (see [10, Theorem 8.9]) and

$$\|x\|_p^p = \sum_c \left( \sum_r |x_{rc}|^2 \right)^{p/2} \vee \sum_r \left( \sum_c |x_{rc}|^2 \right)^{p/2}. \quad (3)$$

- (b) [5, Def. 4.4]  $I$  is a complete  $\sigma(p)$  set if the family of all relative Schur multipliers by signs (1) is uniformly c.b. By [5, Lemma 0.5],  $I$  is completely  $\sigma(p)$  if and only if there is a constant  $D$  such that for every finitely supported operator valued operator  $x = (x_q)_{q \in I} = \sum_{q \in I} x_q e_q$  with  $x_q \in S^p$

$$D^{-1} \|x\|_p \leq \|x\|_p \leq \|x\|_p, \quad (4)$$

where the second inequality is a convexity inequality that is always satisfied and

$$\|x\|_p^p = \sum_c \left\| \left( \sum_r x_{rc}^* x_{rc} \right)^{1/2} \right\|_p^p \vee \sum_r \left\| \left( \sum_c x_{rc} x_{rc}^* \right)^{1/2} \right\|_p^p.$$

The notion of a complete  $\sigma(p)$  set is stronger than that of a  $\sigma(p)$  set: Inequality (2) amounts to Inequality (4) tested on operators of the type  $x = \sum_{q \in I} x_q e_q$  with each  $x_q$  acting on the same one-dimensional subspace of  $\ell_2$ . It is an important open problem to decide whether the notions differ. An affirmative answer would solve Pisier's conjecture about completely bounded Schur multipliers [8, p. 113].

Notorious examples of 1-unconditional basic sequences in all Schatten classes  $S^p$  are single columns, single rows, single diagonals and single anti-diagonals — and more generally “column sets” (vs. “row sets”)  $I$  such that for each  $(r, c) \in I$ , no other element of  $I$  is in the column  $c$  (vs. row  $r$ ). These sets are called sections in [11, Def. 4.3]

We shall try to express these notions in terms of trails on bipartite graphs. We proceed as announced in the Introduction: then each example above is a union of disjoint star graphs in which one vertex of one class is connected to some vertices of the other class: trails in a star graph have at most length 2.

**Definition 2.4.** Let  $I \subseteq R \times C$  and  $s \geq 1$  an integer. We consider  $I$  as a bipartite graph: its vertex set is  $V = R \amalg C$  and its edge set is  $E = \{ \{r, c\} \subseteq V : (r, c) \in I \}$ .

(a) The sets of trails of length  $s$  on the graph  $I$  from the column (vs. row) vertex  $v_0$  to the vertex  $v_s$  are respectively

$$\mathcal{C}^s(I; v_0, v_s) = \{ (v_0, \dots, v_s) \in V^{s+1} : v_0 \in C \ \& \ \text{all } \{v_i, v_{i+1}\} \in E \text{ are distinct} \},$$

$$\mathcal{R}^s(I; v_0, v_s) = \{ (v_0, \dots, v_s) \in V^{s+1} : v_0 \in R \ \& \ \text{all } \{v_i, v_{i+1}\} \in E \text{ are distinct} \}.$$

(b) We define the Rudin numbers of trails starting respectively with a column vertex and a row vertex by  $c_s(I; v_0, v_s) = \#\mathcal{C}^s(I; v_0, v_s)$  and  $r_s(I; v_0, v_s) = \#\mathcal{R}^s(I; v_0, v_s)$ .

*Remark 2.5.* In other words, for an integer  $l \geq 1$ ,

$$\begin{aligned} c_{2l-1}(I; v_0, v_{2l-1}) &= \# \left[ \begin{array}{l} (r_1, c_1), (r_1, c_2), (r_2, c_2), (r_2, c_3), \dots, (r_l, c_l) \\ \text{pairwise distinct in } I : c_1 = v_0, r_l = v_{2l-1} \end{array} \right] \\ c_{2l}(I; v_0, v_{2l}) &= \# \left[ \begin{array}{l} (r_1, c_1), (r_1, c_2), \dots, (r_l, c_l), (r_l, c_{l+1}) \\ \text{pairwise distinct in } I : c_1 = v_0, c_{l+1} = v_{2l} \end{array} \right] \end{aligned}$$

and similarly for  $r_s(I; v_0, v_s)$ . If  $s$  is odd, then  $c_s(I; v_0, v_s) = r_s(I; v_s, v_0)$  for all  $(v_0, v_s) \in C \times R$ . But if  $s$  is even, one Rudin number may be bounded while the other is infinite: see [5, Rem. 6.4(ii)].

### 3 $\sigma(p)$ sets as matrix $\Lambda(p)$ sets

We claim the following result.

**Theorem 3.1.** *Let  $I \subseteq R \times C$  and  $p = 2s$  be an even integer. If  $I$  is a union of sets  $I_1, \dots, I_l$  such that one of the Rudin numbers  $c_s(I_j; v_0, v_s)$  or  $r_s(I_j; v_0, v_s)$  is a bounded function of  $(v_0, v_s)$ , for each  $j$ , then  $I$  is a complete  $\sigma(p)$  set.*

This follows from Theorem 3.2 below: the union of two complete  $\sigma(p)$  sets is a complete  $\sigma(p)$  set by [5, Rem. after Def. 4.4]; furthermore the transposed set  $I^* = \{q^* : q \in I\} \subseteq C \times R$  is a complete  $\sigma(p)$  set provided  $I$  is. Note that the case of  $\sigma(\infty)$  sets (see [5, Rem. 4.6(iii)]) provides evidence that Theorem 3.1 might be a characterisation of complete  $\sigma(p)$  sets for even  $p$ .

**Theorem 3.2.** *Let  $I \subseteq R \times C$  and  $p = 2s$  be an even integer. If the Rudin number  $c_s(I; v_0, v_s)$  is a bounded function of  $(v_0, v_s)$ , then  $I$  is a complete  $\sigma(p)$  set.*

This is proved for  $p = 4$  in [5, Prop. 6.5]. We wish to emphasise that the proof below follows the scheme of the proof of [5, Theorem 1.13]. In particular, we make crucial use of Pisier's idea to express repetitions by dependent Rademacher variables ([5, Prop. 1.14]).

*Proof.* Let  $x = \sum_{q \in I} x_q e_q$  with  $x_q \in S^p$ . We have the following expression for  $\|x\|_p$ .

$$\|x\|_p^p = \text{tr} \otimes \text{tr}(x^* x)^s = \|y\|_2^2 \quad \text{with} \quad y = \overbrace{x^* x x^* \dots x^{(*)}}^{s \text{ terms}},$$

i.e.,  $y$  is the product of  $s$  terms which are alternatively  $x^*$  and  $x$ , and we set  $x^{(*)} = x$  for even  $s$ ,  $x^{(*)} = x^*$  for odd  $s$ . Set  $C^{(*)} = C$  for even  $s$  and  $C^{(*)} = R$  for odd  $s$ . Let  $(v_0, v_s) \in C \times C^{(*)}$  and  $y_{v_0 v_s} = \text{tr} e_{v_0 v_s}^* y$  be the matrix coefficient of  $y$  at coordinate  $(v_0, v_s)$ . Then we obtain by the rule of matrix multiplication

$$\begin{aligned} y &= \sum_{q_1, \dots, q_s \in I} (x_{q_1}^* e_{q_1}^*) (x_{q_2} e_{q_2}) \dots (x_{q_s}^{(*)} e_{q_s}^{(*)}) \\ y_{v_0 v_s} &= \sum_{(v_1, v_0), (v_1, v_2), \dots \in I} x_{v_1 v_0}^* x_{v_1 v_2} x_{v_3 v_2}^* \dots x_{(v_{s-1}, v_s)^{(*)}}^{(*)}. \end{aligned} \quad (5)$$

Let  $\mathcal{E}$  be the set of equivalence relations on  $\{1, \dots, s\}$ . Then

$$y = \sum_{\sim \in \mathcal{E}} \sum_{i \sim j \Leftrightarrow q_i = q_j} (x_{q_1}^* e_{q_1}^*) (x_{q_2} e_{q_2}) \dots (x_{q_s}^{(*)} e_{q_s}^{(*)}). \quad (6)$$

We shall bound the sum above in two steps.

(a) Let  $\sim$  be equality and consider the corresponding term in the sum (6). The number of terms in the sum (5) such that  $\{v_{i-1}, v_i\} \neq \{v_{j-1}, v_j\}$  if  $i \neq j$  is  $c_s(I; v_0, v_s)$ . If  $c$  is an upper bound for  $c_s(I; v_0, v_s)$ , we have by the expression of the Hilbert–Schmidt norm and the Arithmetic-Quadratic Mean Inequality

$$\begin{aligned} & \left\| \sum_{\substack{q_1, \dots, q_s \\ \text{pairwise distinct}}} (x_{q_1}^* e_{q_1}^*) (x_{q_2} e_{q_2}) \dots (x_{q_s}^{(*)} e_{q_s}^{(*)}) \right\|_2^2 \\ &= \sum_{(v_0, v_s) \in C \times C^{(*)}} \left\| \sum_{v \in \mathcal{C}^s(I; v_0, v_s)} x_{v_1 v_0}^* x_{v_1 v_2} x_{v_3 v_2}^* \dots x_{(v_{s-1}, v_s)^{(*)}}^{(*)} \right\|_2^2 \\ &\leq c \sum_{(v_0, v_s) \in C \times C^{(*)}} \sum_{v \in \mathcal{C}^s(I; v_0, v_s)} \left\| x_{v_1 v_0}^* x_{v_1 v_2} x_{v_3 v_2}^* \dots x_{(v_{s-1}, v_s)^{(*)}}^{(*)} \right\|_2^2 \\ &= c \sum_{\substack{q_1, \dots, q_s \\ \text{pairwise distinct}}} \left\| (x_{q_1}^* e_{q_1}^*) (x_{q_2} e_{q_2}) \dots (x_{q_s}^{(*)} e_{q_s}^{(*)}) \right\|_2^2 \\ &\leq c \sum_{q_1, \dots, q_s} \left\| (x_{q_1}^* e_{q_1}^*) (x_{q_2} e_{q_2}) \dots (x_{q_s}^{(*)} e_{q_s}^{(*)}) \right\|_2^2 \\ &= c \left\| \sum_{q_1, \dots, q_s} |(x_{q_1}^* e_{q_1}^*) (x_{q_2} e_{q_2}) \dots (x_{q_s}^{(*)} e_{q_s}^{(*)})|^2 \right\|_1 \end{aligned}$$

Now this last expression may be bounded accordingly to [5, Cor. 0.9] by

$$c \left( \left\| \sum (x_q^* e_q^*) (x_q e_q) \right\|_s \vee \left\| \sum (x_q e_q) (x_q^* e_q^*) \right\|_s \right)^s = c \|x\|_p^p : \quad (7)$$

see [5, Lemma 0.5] for the last equality.

(b) Let  $\sim$  be distinct from equality. The corresponding term in the sum (6) cannot be bounded directly. Consider instead

$$\Psi(\sim) = \left\| \sum_{i \sim j \Rightarrow q_i = q_j} (x_{q_1}^* e_{q_1}^*) (x_{q_2} e_{q_2}) \cdots (x_{q_s}^{(*)} e_{q_s}^{(*)}) \right\|_2 = \left\| \sum_{i \sim j \Rightarrow q_i = q_j} \prod_{i=1}^s f_i(q_i) \right\|_2$$

with  $f_i(q) = x_q e_q$  for even  $i$  and  $f_i(q) = x_q^* e_q^*$  for odd  $i$ . We may now apply Pisier's Lemma [5, Prop. 1.14]: let  $0 \leq r \leq s - 2$  be the number of one element equivalence classes modulo  $\sim$ ; then

$$\Psi(\sim) \leq \|x\|_p^r (B \|x\|_p)^{s-r}, \quad (8)$$

where  $B$  is the constant arising in Lust-Piquard's non-commutative Khinchin inequality. In order to finish the proof, one does an induction on the number of atoms of the partition induced by  $\sim$ , along the lines of step 2 of the proof of [5, Theorem 1.13].  $\square$

The Moebius inversion formula for partitions enabled Pisier [7] to obtain the following explicit bounds in the computation above:

$$\begin{aligned} \|y\|_2 &\leq c^{1/2} \|x\|_p^s + \sum_{0 \leq r \leq s-2} \binom{s}{r} (s-r)! \|x\|_p^r ((3\pi/4) \|x\|_p)^{s-r} \\ \|x\|_p &\leq ((4c)^{1/p} \vee 9\pi p/8) \|x\|_p. \end{aligned} \quad (9)$$

Let us also record the following consequence of his study of  $p$ -orthogonal sums. The family  $(x_q e_q)_{q \in I}$  is  $p$ -orthogonal in the sense of [7] if and only if the graph associated to  $I$  does not contain any circuit of length  $p$ , so that we have by [7, Theorem 3.1]:

**Theorem 3.3.** *Let  $p \geq 4$  be an even integer. If  $I$  does not contain any circuit of length  $p$ , then  $I$  is a complete  $\sigma(p)$  set with constant at most  $3\pi p/2$ .*

*Remark 3.4.* Pisier proposed to us the following argument to deduce a weaker version of Theorem 3.2 from [5, Theorem 1.13]. Let  $\Gamma = \mathbb{T}^V$  and  $z_v$  denote the  $v$ th coordinate function on  $\Gamma$ . Associate to  $I$  the set  $\Lambda = \{z_r z_c : (r, c) \in I\}$ . Let still  $p = 2s$  be an even integer. Then  $I$  is a complete  $\sigma(p)$  set if  $\Lambda$  is a complete  $\Lambda(p)$  set as defined in [5, Def. 1.5], which in turn holds if  $\Lambda$  has property  $Z(s)$  as given in [5, Def. 1.11]. It turns out that this condition implies the uniform boundedness of

$$c_t(I; v_0, v_t) \vee r_t(I; v_0, v_t) \quad \text{for } t \leq s, v_0, v_t \in V.$$

For  $p \geq 8$ , this implication is strict: in fact, the countable union of disjoint cycles of length 4 ("quadrilaterals")

$$I = \bigcup_{i \geq 0} \{(2i, 2i), (2i, 2i+1), (2i+1, 2i+1), (2i+1, 2i)\}$$

satisfies  $c_t(I; v_0, v_t) \vee r_t(I; v_0, v_t) \leq 2$  whereas  $\Lambda$  does not satisfy  $Z(s)$  for any  $s \geq 4$ .

*Remark 3.5.* Theorem 3.1 is especially useful to construct c.b. Schur multipliers: by [5, Rem. 4.6(ii)], if  $I$  is a complete  $\sigma(p)$  set, there is a constant  $D$  (the constant  $D$  in (4)) such that for every sequence  $(\mu_q) \in \mathbb{C}^{R \times C}$  supported by  $I$  and every operator  $T_\mu : (x_q) \mapsto (\mu_q x_q)$  we have

$$\|T_\mu\|_{\mathcal{L}(S^p(S^p))} \leq D \sup_{q \in I} |\mu_q|.$$

## 4 The intersection of a $\sigma(p)$ set with a finite product set

Let  $I \subseteq R \times C$  considered as a bipartite graph as in the Introduction and let  $I' \subseteq I$  be the subgraph induced by the vertex set  $C' \amalg R'$ , with  $C' \subseteq C$  a set of  $m$  column vertices and  $R' \subseteq R$  a set of  $n$  row vertices. In other words,  $I' = I \cap R' \times C'$ . Let  $d(v)$  be the degree of the vertex  $v \in C' \amalg R'$  in  $I'$ : in other words,

$$\begin{aligned} \forall c \in C' \quad d(c) &= \#[I' \cap R' \times \{c\}], \\ \forall r \in R' \quad d(r) &= \#[I' \cap \{r\} \times C']. \end{aligned}$$

Let us recall that the dual norm of (3) is

$$\|x\|_{p'} = \inf_{\substack{\alpha, \beta \in S^{p'} \\ \alpha + \beta = x}} \left( \sum_c \left( \sum_r |\alpha_{rc}|^2 \right)^{p'/2} \right)^{1/p'} + \left( \sum_r \left( \sum_c |\beta_{rc}|^2 \right)^{p'/2} \right)^{1/p'},$$

where  $p \geq 2$  and  $1/p + 1/p' = 1$  (see [5, Rem. after Lemma 0.5]).

**Lemma 4.1.** *Let  $1 \leq p' \leq 2$  and  $x = \sum_{q \in I'} x_q$ . Then*

$$\|x\|_{p'}^{p'} \geq \sum_{(r,c) \in I'} \sum_{(r,c) \in I'} \left( \max(d(c), d(r))^{1/2-1/p'} |x_{rc}| \right)^{p'}.$$

*Proof.* By the  $p'$ -Quadratic Mean Inequality and by Minkowski's Inequality,

$$\begin{aligned} & \left( \sum_{c \in C'} \left( \sum_{(r,c) \in I'} |\alpha_{rc}|^2 \right)^{p'/2} \right)^{1/p'} + \left( \sum_{r \in R'} \left( \sum_{(r,c) \in I'} |\beta_{rc}|^2 \right)^{p'/2} \right)^{1/p'} \\ & \geq \left( \sum_{c \in C'} d(c)^{p'/2-1} \sum_{(r,c) \in I'} |\alpha_{rc}|^{p'} \right)^{1/p'} + \left( \sum_{r \in R'} d(r)^{p'/2-1} \sum_{(r,c) \in I'} |\beta_{rc}|^{p'} \right)^{1/p'} \\ & \geq \left( \sum_{(r,c) \in I'} \left( d(c)^{1/2-1/p'} |\alpha_{rc}| + d(r)^{1/2-1/p'} |\beta_{rc}| \right)^{p'} \right)^{1/p'} \end{aligned}$$

The lemma follows by taking the infimum over all  $\alpha, \beta$  with  $\alpha_q + \beta_q = x_q$  for  $q \in I'$  as one can suppose that  $\alpha_q = \beta_q = 0$  if  $q \notin I'$ ; note further that  $1/2 - 1/p' \leq 0$ .  $\square$

**Theorem 4.2.** *If  $I$  is a  $\sigma(p)$  set with constant  $D$  as in (2), then the size  $\#I'$  of any subgraph  $I'$  induced by  $m$  column vertices and  $n$  row vertices, in other words the cardinal of any subset  $I' = I \cap R' \times C'$  with  $\#C' = m$  and  $\#R' = n$ , satisfies*

$$\begin{aligned} \#I' &\leq D^2 (m^{1/p} n^{1/2} + m^{1/2} n^{1/p})^2 \\ &\leq 4D^2 \min(m, n)^{2/p} \max(m, n). \end{aligned} \tag{10}$$

*The exponents in this inequality are optimal even for a complete  $\sigma(p)$  set  $I$  in the following cases:*

- (a) if  $m$  or  $n$  is fixed;
- (b) if  $p$  is an even integer and  $m = n$ .

Bound (10) holds *a fortiori* if  $I$  is a complete  $\sigma(p)$  set. Density conditions thus do not so far permit to distinguish  $\sigma(p)$  sets and complete  $\sigma(p)$  sets. One may conjecture that Inequality (10) is also optimal for  $p$  not an even integer and  $m = n$ : this would be a matrix counterpart to Bourgain's theorem [3] on maximal  $\Lambda(p)$  sets.

*Proof.* If (2) holds, then  $\|x|_{I'}\|_p \leq D \|x\|_p$  for all  $x \in S^p$  by Remark 3.5 applied to  $(\mu_q)$  the indicator function of  $I'$ , and by duality  $\|x|_{I'}\|_{p'} \leq D \|x\|_{p'}$  for all  $x \in S^{p'}$  (compare with [5, Rem. 4.6(iv)]). Let

$$\begin{aligned} y &= \sum_{(r,c) \in I'} d(c)^{1/p'-1/2} e_{rc}, \\ z &= \sum_{(r,c) \in I'} d(r)^{1/p'-1/2} e_{rc}, \end{aligned}$$

Then the  $n$  rows of  $y$  are all equal, as well as the  $m$  columns of  $z$ :  $y$  and  $z$  have rank 1 and a single singular value. By the norm inequality followed by the  $(2/p' - 1)$ -Arithmetic Mean Inequality,

$$\begin{aligned} \|y + z\|_{p'} &\leq \|y\|_{p'} + \|z\|_{p'} \\ &= n^{1/2} \left( \sum_{c \in C'} d(c)^{2/p'-1} \right)^{1/2} + m^{1/2} \left( \sum_{r \in R'} d(r)^{2/p'-1} \right)^{1/2} \\ &\leq n^{1/2} m^{1-1/p'} (\#I')^{1/p'-1/2} + m^{1/2} n^{1-1/p'} (\#I')^{1/p'-1/2}. \end{aligned}$$

We used that  $\sum_{c \in C'} d(c) = \sum_{r \in R'} d(r) = \#I'$ . By Lemma 4.1 applied to  $x = y + z$ ,

$$(\#I')^{1/p'} \leq D(n^{1/2} m^{1-1/p'} + m^{1/2} n^{1-1/p'}) (\#I')^{1/p'-1/2},$$

and we get therefore the first part of the theorem.

Let us show optimality in the given cases.

(a) Suppose that  $n$  is fixed and  $C' = C$ :  $I' = R' \times C$  is a complete  $\sigma(p)$  set for any  $p$  as a union of  $n$  rows and  $\#I' = n \cdot m$ .

(b) is proved in [5, Theorem 4.8]. □

*Remark 4.3.* If  $n \approx m$ , the method used in [5, Theorem 4.8] does not provide optimal  $\sigma(p)$  sets but the following lower bound. Let  $p = 2s$  with  $s \geq 2$  an integer. Consider a prime  $q$  and let  $k = s^{s-1} q^s$ . By [9, 4.7] and [5, Theorem 2.5], there is a subset  $F \subseteq \{0, \dots, k-1\}$  with  $q$  elements whose complete  $\Lambda(2s)$  constant is independent of  $q$ . Let  $m \geq k$  and  $0 \leq n \leq m$  and consider the Hankel set

$$I = \{ (r, c) \in \{0, \dots, n-1\} \times \{0, \dots, m-1\} : r + c \in F + m - k \}.$$

Then the complete  $\sigma(p)$  constant of  $I$  is independent of  $q$  by [5, Prop. 4.7] and

$$\#I \geq \begin{cases} nq & \text{if } n \leq m - k + 1 \\ (m - k + 1)q & \text{if } n \geq m - k + 1. \end{cases}$$

If we choose  $m = (s+1)k - 1$ , this yields

$$\#I \geq \frac{s^{1/s}}{(s+1)^{1+1/s}} \min(n, m) \max(m, n)^{1/s}.$$

Random construction 6.1 provides bigger sets than this deterministic construction; however, it also does not provide sets that would show the optimality of Inequality (10) unless  $s = 2$ .

## 5 Circuits in graphs

Non-commutative methods yield a new proof to a theorem of Erdős [4, p. 33]. Note that its generalisation by Bondy and Simonovits [2] is stronger than Theorem 5.1 below as it deals with cycles instead of circuits. By Theorem 3.3 and (10)

**Theorem 5.1.** *Let  $p \geq 4$  be an even integer. If  $G$  is a nonempty graph on  $v$  vertices with  $e$  edges without circuit of length  $p$ , then*

$$e \leq 18\pi^2 p^2 v^{1+2/p}.$$

*If  $G$  is furthermore a bipartite graph whose two vertex classes have respectively  $m$  and  $n$  elements, then*

$$e \leq 9\pi^2 p^2 \min(m, n)^{2/p} \max(m, n). \tag{11}$$

*Proof.* For the first assertion, recall that a graph  $G$  with  $e$  edges contains a bipartite subgraph with more than  $e/2$  edges (see [1, p. xviii]). □

*Remark 5.2.* Łuczak showed to us that (11) cannot be optimal if  $m$  and  $n$  are of very different order of magnitude. In particular, let  $p$  be a multiple of 4. Let  $e'$  be the maximal number of edges of a graph on  $n$  vertices without circuit of length  $p/2$ . If  $m > pe'$ , he shows that (11) may be replaced by  $e < 3m$ .

We also get the following result, which enables us to conjecture a generalisation of the theorems of Erdős and Bondy and Simonovits.

**Theorem 5.3.** *Let  $G$  be a nonempty graph on  $v$  vertices with  $e$  edges. Let  $s \geq 2$  be an integer.*

(i) *If*

$$e > 8D^2 v^{1+1/s} \quad \text{with } D > 9\pi s/4,$$

*then one may choose two vertices  $v_0$  and  $v_s$  such that  $G$  contains more than  $D^{2s}/4$  pairwise distinct trails from  $v_0$  to  $v_s$ , each of length  $s$  and with pairwise distinct edges.*

(ii) *One may draw the same conclusion if  $G$  is a bipartite graph whose two vertex classes have respectively  $m$  and  $n$  elements and*

$$e > 4D^2 \min(m, n)^{1/s} \max(m, n) \quad \text{with } D > 9\pi s/4.$$

*Proof.* (i) According to [1, p. xvii], the graph  $G$  contains a bipartite subgraph with more than  $e/2$  edges, so that we may apply (ii).

(ii) Combining inequalities (9) and (10), if  $D > 9\pi s/4$ , then there are vertices  $v_0$  and  $v_s$  such that the number  $c$  of pairwise distinct trails from  $v_0$  to  $v_s$ , each of length  $s$  and with pairwise distinct edges, satisfies  $(4c)^{1/2s} > D$ .  $\square$

Two paths with equal endvertices are called independent if they have only their endvertices in common.

*Question 5.4.* Let  $G$  be a graph on  $v$  vertices with  $e$  edges. Let  $s, l \geq 2$  be integers. Is it so that there is a constant  $D$  such that if  $e > Dv^{1+1/s}$ , then  $G$  contains  $l$  pairwise independent paths of length  $s$  with equal endvertices?

*Remark 5.5.* Note that by Theorem 4.2, the exponent  $1 + 1/s$  is optimal in Theorem 5.3(i), whereas optimality of the exponent  $1 + 2/p$  in Theorem 5.1 is an important open question in Graph Theory (see [6]).

One may also formulate Theorem 5.3(ii) in the following way.

**Theorem 5.6.** *If a bipartite graph  $G_2(n, m)$  with  $n$  and  $m$  vertices in its two classes avoids any union of  $c$  pairwise distinct trails along  $s$  pairwise distinct edges between two given vertices as a subgraph, where the class of the first vertex is fixed, then the size  $e$  of the graph satisfies*

$$e \leq 4 \max((4c)^{1/2s}, 9\pi s/4) \min(m, n)^{1/s} \max(m, n).$$

## 6 A random construction of graphs

Let us precise our construction of a random graph.

**Random construction 6.1.** *Let  $C, R$  be two sets such that  $\#C = m$  and  $\#R = n$ . Let  $0 \leq \alpha \leq 1$ . A random bipartite graph on  $V = C \amalg R$  is defined by selecting independently each edge in  $E = \{ \{r, c\} \subseteq V : (r, c) \in R \times C \}$  with the same probability  $\alpha$ . The resulting random edge set is denoted by  $E' \subseteq E$  and  $I' \subseteq R \times C$  denotes the associated random subset.*

Our aim is to construct large sets while keeping down the Rudin number  $c_s$ .

**Theorem 6.2.** *For each  $\varepsilon > 0$  and for each integer  $s \geq 2$ , there is an  $\alpha$  such that Random construction 6.1 yields subsets  $I' \subseteq R \times C$  with size*

$$\#I' \sim \min(m, n)^{1/2+1/s} \max(m, n)^{1/2-\varepsilon}$$

*and with  $\sigma(2s)$  constant independent of  $m$  and  $n$  for  $mn \rightarrow \infty$ .*

*Proof.* Let us suppose without loss of generality that  $m \geq n$ . We want to estimate the Rudin number of trails in  $I'$ . Set  $C^{(*)} = C$  for even  $s$ ,  $C^{(*)} = R$  for odd  $s$  and let  $(v_0, v_s) \in C \times C^{(*)}$ . Let  $l \geq 1$  be a fixed integer. Then

$$\begin{aligned} \mathbb{P}[c_s(I'; v_0, v_s) \geq l] &= \mathbb{P}[\exists l \text{ distinct trails } (v_0^j, \dots, v_s^j) \text{ in } \mathcal{C}^s(I'; v_0, v_s)] \\ &= \mathbb{P}[E' \supseteq \{\{v_{i-1}^j, v_i^j\}_{i,j} : \{(v_0^j, \dots, v_s^j)\}_{j=1}^l \subseteq \mathcal{C}^s(R \times C; v_0, v_s)\}] \\ &\leq \sum_{k=\lceil l^{1/s} \rceil}^{ls} \#A_k \cdot \alpha^k, \end{aligned}$$

where  $A_k$  is the following set of  $l$ -element subsets of trails in  $\mathcal{C}^s(R \times C; v_0, v_s)$  built with  $k$  pairwise distinct edges

$$A_k = \left\{ \{(v_0^j, \dots, v_s^j)\}_{j=1}^l \subseteq \mathcal{C}^s(R \times C; v_0, v_s) : \#\{v_{i-1}^j, v_i^j\}_{i,j} = k \right\};$$

the lower limit of summation is  $\lceil l^{1/s} \rceil$  because one can build at most  $k^s$  pairwise distinct trails of length  $s$  with  $k$  pairwise distinct edges.

In order to estimate  $\#A_k$ , we now have to bound the number of pairwise distinct vertices and the number of pairwise distinct column vertices in each set of  $l$  trails  $\{(v_0^j, \dots, v_s^j)\}_{j=1}^l \in A_k$ . We claim that

$$\#\{v_i^j : 1 \leq i \leq s-1, 1 \leq j \leq l\} \leq k(s-1)/s, \quad (12)$$

$$\#\{v_{2i}^j : 1 \leq i \leq \lceil s/2 \rceil - 1, 1 \leq j \leq l\} \leq k/2. \quad (13)$$

The second estimate is trivial, because each column vertex  $v_{2i}^j$  accounts for two distinct edges  $\{v_{2i-1}^j, v_{2i}^j\}$  and  $\{v_{2i}^j, v_{2i+1}^j\}$ . For the first estimate, note that each maximal sequence of  $h$  consecutive pairwise distinct vertices  $(v_{a+1}^j, \dots, v_{a+h}^j)$  accounts for  $h+1$  pairwise distinct edges

$$\{v_a^j, v_{a+1}^j\}, \{v_{a+1}^j, v_{a+2}^j\}, \dots, \{v_{a+h}^j, v_{a+h+1}^j\};$$

as  $h \leq s-1$ ,  $h+1 \geq hs/(s-1)$ . By (12) and (13),

$$\#A_k \leq m^{k/2} n^{k/2 - k/s} (k - k/s)^{ls-l} \leq (ls)^{ls} m^{k/2} n^{k/2 - k/s};$$

each element of  $A_k$  is obtained by a choice of at most  $k - k/s$  vertices, of which at most  $k/2$  are column vertices, and the choice of an arrangement with repetitions of  $ls - l$  out of at most  $k - k/s$  vertices.

Put  $\alpha = m^{-1/2} n^{-1/2 + 1/s} (\#C \cdot \#C^{(*)})^{-\varepsilon}$ . Then

$$\begin{aligned} \mathbb{P}\left[\sup_{(v_0, v_s)} c_s(I'; v_0, v_s) \geq l\right] &\leq \#C \cdot \#C^{(*)} \cdot (ls)^{ls} \sum_{k=\lceil l^{1/s} \rceil}^{ls} (\#C \cdot \#C^{(*)})^{-k\varepsilon} \\ &\leq (ls)^{ls} \frac{(\#C \cdot \#C^{(*)})^{1 - \lceil l^{1/s} \rceil \varepsilon}}{1 - (\#C \cdot \#C^{(*)})^{-\varepsilon}}. \end{aligned}$$

Choose  $l$  such that  $\lceil l^{1/s} \rceil \varepsilon > 1$ . Then this probability is little for  $mn$  large. On the other hand,  $\#I'$  is of order  $mn\alpha$  with probability close to 1.  $\square$

*Remark 6.3.* This construction yields much better results for  $s = 2$ . Keeping the notation of the proof above and  $m \geq n$ , we get  $k = 2l$ ,  $A_k = \binom{n}{l}$  and

$$\mathbb{P}\left[\sup_{(v_0, v_2) \in C \times C} c_2(I'; v_0, v_2) \geq l\right] \leq m^2 \binom{n}{l} \alpha^{2l}.$$

Let  $l \geq 2$  and  $\alpha = m^{-1/l} n^{-1/2}$ . This yields sets  $I' \subseteq R \times C$  with size

$$\#I' \sim n^{1/2} m^{1-1/l}$$

and with  $\sigma(4)$  constant independent of  $m$  and  $n$ . This case has been extensively studied in Graph theory as the “Zarankiewicz problem:” if  $c_2(I'; v_0, v_2) \leq l$  for all  $v_0, v_2 \in C$ , then the graph  $I'$  does not contain a complete bipartite subgraph on any two column vertices  $v_0, v_2$  and  $l + 1$  row vertices. Reiman (see [1, Theorem VI.2.6]) showed that then

$$\#I' \leq (lnm(m-1) + n^2/4)^{1/2} + n/2 \sim l^{1/2}n^{1/2}m.$$

With use of finite projective geometries, he also showed that this bound is optimal for

$$n = l \frac{q^{r+1} - 1}{q^2 - 1} \frac{q^r - 1}{q - 1}, \quad m = \frac{q^{r+1} - 1}{q - 1}$$

with  $q$  a prime power and  $r \geq 2$  an integer, and thus with  $m \leq n$ : there seems to be no constructive example of extremal graphs with  $c_2(I'; v_0, v_2) \leq l$  and  $m > n$  besides the trivial case of complete bipartite graphs with  $m > n = l - 1$ .

*Remark 6.4.* In the case  $s = 3$ , our result cannot be improved just by refining the estimation of  $\#A_k$ . If we consider first  $l$  distinct paths that have their second vertex in common and then  $l$  independent paths, we get

$$\#A_{2l+1} \geq \binom{m}{l} n, \quad \#A_{3l} \geq \binom{m}{l} \binom{n}{l}.$$

Therefore any choice of  $\alpha$  as a monomial  $m^{-t}n^{-u}$  in the proof above must satisfy  $t \geq (l+1)/(2l+1)$ ,  $t+u \geq (2l+2)/(3l)$  and this yields sets with

$$\#I' \preceq m^{1/2-1/2(4l+2)} n^{5/6-(7l+6)/(12l^2+6l)}.$$

## Bibliography

- [1] Béla Bollobás. *Extremal graph theory*. Academic Press, 1978. (pp. 7, 8, and 10).
- [2] J. A. Bondy and M. Simonovits. **Cycles of even length in graphs**. *J. Combin. Theory Ser. B*, 16:97–105, 1974. (p. 7).
- [3] J. Bourgain. **Bounded orthogonal systems and the  $\Lambda(p)$ -set problem**. *Acta Math.*, 162(3-4):227–245, 1989. (p. 6).
- [4] Paul Erdős. Extremal problems in graph theory. In *Theory of Graphs and its Applications (Smolenice, 1963)*, pages 29–36. Publ. House Czechoslovak Acad. Sci., 1964. (pp. 1 and 7).
- [5] Asma Harcharras. **Fourier analysis, Schur multipliers on  $S^p$  and non-commutative  $\Lambda(p)$ -sets**. *Studia Math.*, 137(3):203–260, 1999. (pp. 1, 3, 4, 5, 6, and 7).
- [6] Felix Lazebnik, Vasilii A. Ustimenko, and Andrew J. Woldar. **Polarities and  $2k$ -cycle-free graphs**. *Discrete Math.*, 197/198:503–513, 1999. (p. 8).
- [7] Gilles Pisier. An inequality for  $p$ -orthogonal sums in non-commutative  $L_p$ . *Illinois J. Math.*, 44(4):901–923, 2000. (p. 5).
- [8] Gilles Pisier. *Similarity problems and completely bounded maps*, volume 1618 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, expanded edition, 2001. Includes the solution to “The Halmos problem”. (p. 3).
- [9] Walter Rudin. **Trigonometric series with gaps**. *J. Math. Mech.*, 9:203–228, 1960. (pp. 1 and 7).
- [10] Barry Simon. *Trace ideals and their applications*. Cambridge University Press, 1979. (p. 3).
- [11] Nicholas Varopoulos. **Tensor algebras over discrete spaces**. *J. Functional Analysis*, 3:321–335, 1969. (p. 3).