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Approximation theory for distant Bang calculus

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Abstract

Approximation semantics capture the observable behaviour of λ -terms. Böhm Trees and Taylor Expansion are its two central paradigms, related by the Commutation Theorem. While these notions are well understood in Call-by-Name (CbN), they have only recently been developed for Call-by-Value (CbV), which motivate the search for a unified approximation framework. The Bang-calculus provides such a framework: it subsumes both CbN and CbV through linear-logic translations and enjoys robust rewriting properties. We develop the approximation semantics of dBang (the Bang-calculus with explicit substitutions and distant reductions) by introducing approximation trees in the Böhm tradition together with Taylor expansion. We establish their fundamental properties, including a commutation theorem. Via translations, our results recover the CbN and CbV cases within a single unifying framework capturing infinitary and resource-sensitive semantics.

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1 Introduction

A central approach to λ -calculus semantics is program approximation theory, which captures program behaviour, finitarily or infinitarily, offering a characterization of “meaningful” terms.

In *Call-by-Name* (CbN), meaningful terms are the *solvable* ones, i.e. those reducing to identity under some “testing context”. In *Call-by-Value* (CbV), meaningfulness is given by *scrutability*¹, which only requires reduction to a value and is strictly finer. Among approximation techniques, *evaluation trees* and *Taylor Expansions* are the most influential. *Böhm trees* introduced by Barendregt [14] are the seminal evaluation trees. They associate to each λ -term a (possibly infinite) tree whose nodes describe successive approximations of the term’s *head normal form*, or \perp — canonical term meaning nonsense, or divergence — when none exists, thus capturing its asymptotic CbN behaviour. Böhm trees were related to the notion of *solvability* of CbN terms by the fact that a term is solvable if and only if its Böhm tree is not \perp .

Ehrhard and Regnier later introduced Taylor expansion [26], inspired by the differential λ -calculus [25] and relational semantics [32]. Taylor expansion unfolds a λ -term into an (infinite)

¹ Also called potential valuability

formal sum of resource terms underlying the linear-use of resources during computations, rather than progressively revealing their shape as Böhm trees do.

The Commutation Theorem [26, Corollary 35] states that normalizing the Taylor expansion of a term yields exactly the Taylor expansion of its Böhm tree. Taylor expansions can then be understood as a *resource-sensitive* version of Böhm trees. Originally proved for the CbN λ -calculus, this result provides a deep bridge between infinitary semantics and differential/resource semantics.

While Taylor expansions for the CbV calculus were already studied in [18, 22], the development of CbV Böhm trees remained open until Kerinec's PhD work [30, 29]. Those Böhm trees have the same commutation theorem with Taylor expansion as in the CbN case. They also respect the same relation with scrutability as CbN Böhm trees do with solvability. This emphasizes that scrutability is the appropriate notion of meaningfulness in CbV. However, this result is obtained in an alternative version of the original Plotkin CbV, which is known to have issues due to the β_v -reduction being "too weak". Concretely, unlike in CbN, CbV reduction may get stuck: redexes can be blocked since their argument is in normal form but not a value. This phenomenon prevents a straightforward infinitary unfolding analogous to the CbN case.

The aforementioned CbV Böhm trees are defined in the λ_v^σ -calculus from [17], where the β_v -reduction is extended with permutation rules, the so-called σ -rules, originating from the translation of λ -terms to proof-nets [1]. Another way to solve the CbV issue, also coming from proof-nets, is using a distance-based CbV calculus: *distant CbV* (dCbV)² [3, 2, 4]. In this system, substitutions may be frozen thanks to *explicit substitutions*, written $M[N/x]$, which do *not* correspond to (effective) substitutions, but instead represent substitutions that are yet to be evaluated. The rewriting rules then act *at a distance* with respect to the explicit substitutions.

The divergence between CbN and CbV has historically required separate developments of most semantic notions. Such a duplication naturally called for a unifying perspective. *Call-by-Push-Value* (CbPV) [33] provides precisely this, reconciling typed CbN and CbV within a single calculus structured around a clear distinction between values and computations. CbPV later gave rise to its untyped analogue *Bang-calculus* (Bang) [24], by use of Linear Logic [23]. Both CbN and CbV arise within the Bang-calculus via Girard's translations of the intuitionistic arrow into Linear Logic [28], making it a natural setting in which to seek a uniform approximation theory. The Bang-calculus is then an extension of the λ -calculus with two new constructs: $!M$ (pronounced "bang M ") which *freezes* the computation of M , and $\mathbf{der}(M)$ which *unfreezes* it. Both CbN and CbV can then be translated into the Bang-calculus, which simulates their rewriting strategies within a single rewriting system.

However, the Bang-calculus exhibits the same issue as CbV: ill-formed redexes may block evaluation. One might expect that adapting the σ -rules to the Bang-calculus would solve this issue; however, while the CbV calculus with σ -rules is confluent, this is not the case for the Bang calculus [24, Sec.2.3]. Confluence can be recovered modulo an equivalence relation contained in the σ -equivalence generated by the σ -rules, but this requires working modulo said equivalence. Another solution to this problem is to consider the distance variant (dBang) [16], similar to the distant variant of CbV. This allows us to work without the σ -rules and the aforementioned equivalences. This new system has been shown to be confluent [16]. It has also been used in unifying multiple results for dCbN and dCbV. For instance, in [9]

² Also called Value Substitution Calculus

the authors show that the rewriting results (confluence, factorization) of **dBang** carry over to the **dCbN** and **dCbV** setting using the translations into **dBang**³. Similarly, the notion of solvability in both **dCbN** and **dCbV** has been captured through the notion of *meaningfulness* in **dBang** [31, 11]. However, despite significant progress, the approximation theory of the **dBang**-calculus remains largely underdeveloped.

A recent work by Mazza & Dufour tried to close that gap [21]: they developed a generic notion of Böhm tree and Taylor expansion for a language called **Proc**, representing untyped proof structures. They showed how any language that can be embedded into **Proc** in a “nice way” inherits the notions of Böhm tree and Taylor expansion from **Proc**, and their commutation Theorem. In particular, **dBang** admit such an embedding. However, no notion of meaningfulness exists in **Proc** and it is not clear how to relate the results from **CbN** and **CbV** with the notion developed in [21].

1.1 Contributions

We develop a theory of approximation of the distance **Bang**-calculus (**dBang**). To this end we recall the definitions and main results of **dBang** from [31, 10, 16] in Section 2. We start by developing the Taylor expansion of **dBang** (Section 2.2) where we introduce the resource calculus (Section 2.2.1) and define the approximation relation (Section 2.2.2) and establish a simulation result between **dBang** and its approximants in the Taylor expansion (Theorem 21). We next develop the Böhm approximants of **dBang** (Section 2.3)⁴ and prove a commutation theorem between the Taylor Expansion of Böhm trees and the Taylor normal form (Theorem 46).

We establish the soundness of our definitions with regard to the standard notion of Böhm trees and Taylor Expansions in the distance call-by-name (**dCbN**) and distance call-by-value (**dCbV**) λ -calculus by translating these systems into **dBang** (Section 3), in particular, we show that the Böhm trees of a term M in **dCbN** (respectively **dCbV**) are the same as its translation into **dBang** (Theorem 59). We show a similar result for Taylor expansion (Lemmas 54 and Corollaries 55).

Finally, we establish the commutation result for both **dCbN** and **dCbV** as a consequence of the **dBang** commutation theorem 60.

Proofs are available in the Appendix.

2 The (distance) Bang-calculus

We begin by recalling the theory of **dBang**, first with some results from previous studies [31, 10, 16], before developing its approximation theory via Taylor expansion and approximation trees.

► **Definition 1.** (*dBang: terms and contexts*)

$$\begin{array}{ll}
 (\text{Terms}) & M, N := x \mid MN \mid \lambda x M \mid !M \mid \mathbf{der}(M) \mid M[N/x] \\
 (\text{List contexts}) & L := \square \mid L[M/x] \\
 (\text{Surface contexts}) & S := \square \mid SM \mid MS \mid \lambda x S \mid \mathbf{der}(S) \mid S[M/x] \mid M[S/x] \\
 (\text{Full contexts}) & F := \square \mid FM \mid MF \mid \lambda x F \mid \mathbf{der}(F) \mid F[M/x] \mid M[F/x] \mid !F
 \end{array}$$

³ Note that while the translation of **dCbN** into **dBang** is the usual one mentioned before, the authors use a new translations for **CbV** which will be discussed in Section 3

⁴ In the rest of the paper, we will call them *approximation trees* instead, due to subtle difference compared to standard Böhm trees. These differences will be explained in Section 2.3.

Terms include the standard λ -calculus constructs: variable (ranging over a countably infinite set), application and abstraction. Two additional ones are used to define the **Bang**-calculus: the *bang* (or *exponential*) $!M$, representing delayed evaluation of the subterm M , and *dereliction* $\mathbf{der}(M)$, which reactivates it. Finally, *explicit substitutions* $M[N/x]$ comes from the "at distance" mechanism, and represents pending substitutions. The lambda-abstraction (as usual) and the explicit substitution bind the variable x in M . We use *contexts*, *i.e.* terms with a subterm hole (\square) and we denote by $C\langle M \rangle$ the term obtained by filling the hole in C by M . We distinguish *list*, *surface* and *full* contexts. List contexts are sequences of explicit substitutions and will be used for the reduction at a distance. Surface and full contexts determine whether reduction under a $!$ is allowed (in particular it is forbidden in *weak* calculi (CbN or CbV)).

The **dBang** calculus has three reduction rules:

$$L\langle \lambda x M \rangle N \mapsto! L\langle M[N/x] \rangle \quad M[L\langle !N \rangle/x] \mapsto! L\langle M\{N/x\} \rangle \quad \mathbf{der}(L\langle !M \rangle) \mapsto! L\langle M \rangle$$

Notice that requiring subterms of the form $!M$ assigns them *value* status in the CbV sense.

We write $\rightarrow_{!s}$ and $\rightarrow_{!f}$ for the closures under surface and full contexts, respectively; both are confluent [31, Theorem 1]. We also denote by \rightarrow_i the internal reductions (*i.e.* $\rightarrow_i = \rightarrow_{!f} \setminus \rightarrow_{!s}$, these are the reductions occurring under a $!$ -construct).

► **Example 2.**

$$\begin{aligned} & (\lambda x xy)[!z/y]!!((\lambda ww)!N) \\ & \rightarrow_{!s} (xy)[!!((\lambda ww)!N)/x][!z/y] \\ & \rightarrow_{!s} !((\lambda ww)!N)y[!z/y] \\ & \rightarrow_{!s} !((\lambda ww)!N)z \\ & \rightarrow_i !!Nz \end{aligned}$$

► **Example 3.**

- $\Delta = \lambda x(x!x), \Omega = \Delta!\Delta$. We have $\Omega \rightarrow_{!s}^2 \Omega$.
- $Y_x^n = (\lambda yx!(y!y))!(\lambda yx!(y!y))$. We have $Y_x^n \rightarrow_{!s}^+ x!Y_x^n$
- $Y_x^v = (\lambda yx(y!y))!(\lambda yx(y!y))$. We have $Y_x^v \rightarrow_{!s}^+ xY_x^v$

The upper scripts in the two last items, as we shall see further, represent the fact that Y_x^n and Y_x^v correspond respectively to the CbN and CbV translations from a fixpoint combinator Y of standard λ -calculus⁵.

2.1 Meaningfulness in dBang

This section focus on *meaningfulness*, later related to approximation theory in Section 4: intuitively, a term is meaningful if it reduces to a desired result under some *testing context*. It generalizes the notions of *solvability* CbN and *scrutability* CbV, and has been studied in detail for **dBang** in [31].

► **Definition 4.** A term M of **dBang** is said meaningful if there exists a testing context $T := \square \mid TM \mid (\lambda xT)M$ and a term P such that $T\langle M \rangle \rightarrow_{!s}^* !P$.

⁵ Note that Y_x^v lacks some good properties if one would want to use it for computations in CbV, and CbV fixpoints are usually defined differently. We nevertheless keep Y_x^v as a running example, so as to illustrate an infinite computation, with a fixpoint behaviour, but with empty semantics.

Notice that the surface reduction involved in meaningfulness is not restrictive: for any N , $N \rightarrow_!^* !P$, then there is some P' such that $N \rightarrow_{!s}^* !P'$ (this follows from a standardisation property, see Corollary 6 below). It has been shown in [31] that the smallest theory that identifies all meaningless terms is consistent; that meaningful terms and surface-normalizing terms can be characterized by an intersection type systems and finally that meaningless subterms do not affect the operational meaning of a given term [31, Proposition 8, Theorem 24 and Corollary 11].

A natural property of surface reduction is that it determines the external shape of a term. In other words, allowing full reduction does not unlock external redexes. Indeed, with terms like *e.g.* $\lambda x!M$, $y!M$, $!M!M$, reductions occurring contextually inside M cannot make the full term reducible at the surface (the syntax of such term will be precisely characterized further). This is expressed as the following factorization proposition:

► **Proposition 5.** ([10], Corollary 21) *Let $M \rightarrow_!^* N$. There is some P such that $M \rightarrow_{!s}^* P \rightarrow_i^* N$.*

Conversely, internal reductions preserve the *external shape*. We express this notion with multi-holes surface contexts, that let us reformulate the factorization property as follows:

► **Corollary 6 (Standardization).** *Let S^+ denote multi-holes surface contexts and given by the syntax: $S^+ := \square \mid M \mid S^+S^+ \mid S^+[S^+/x] \mid \mathbf{der}(S^+) \mid \lambda x.S^+$. For any reduction $M \rightarrow_!^* P$, there are some terms N_i and a multi-hole surface context S such that $P = S(!N_1, \dots, !N_k)$ and $M \rightarrow_{!s}^* S(!N'_1, \dots, !N'_k) \rightarrow_i^* S(!N_1, \dots, !N_k)$*

If $k = 0$, the context has no hole and the reduction occurs only at surface level.

2.2 Taylor expansion

We define the Taylor expansion of **dBang**, starting with the associated resource calculus $\delta\mathbf{Bang}$. This calculus will also serve as the resource language for **dCbN** and **dCbV** in Section 3. There is no necessity to define specific resource calculi, as $\delta\mathbf{Bang}$ fits well as a target of usual Taylor expansion (**CbN** [27] and **cbv** [22]), with a straightforward adaptation to distant setting.

2.2.1 dBang: resources

► **Definition 7 (Resource calculus $\delta\mathbf{Bang}$).**

$$\begin{aligned}
(\text{terms}) \quad m, n &:= x \mid mn \mid \lambda xm \mid \mathbf{der}(m) \mid m[n/x] \mid [m_1, \dots, m_k] \\
(\text{lists}) \quad l &:= \square \mid l[m/x] \\
(\text{surface}) \quad s &:= \square \mid sm \mid ms \mid \lambda xs \mid \mathbf{der}(s) \mid s[m/x] \mid m[s/x] \\
(\text{full}) \quad f &:= \square \mid fm \mid mf \mid \lambda xf \mid \mathbf{der}(f) \mid f[m/x] \mid m[f/x] \mid [f, m_1, \dots, m_k] \\
(\text{tests}) \quad t &:= \square \mid tm \mid (\lambda xt)m
\end{aligned}$$

Notations (resource terms, multisets/bags, multilinear substitutions)

Terms $[m_1, \dots, m_k]$ for $k \in \mathbb{N}$, called *bags*, denote finite multisets of resource terms, with $[\]$ the empty bag. When necessary, we use a subscript as $[m]_k$ or $[m, \dots, m]_k$ in order to make explicit the number of occurrences.

Contexts are exactly as in **dBang** (Definition 1), with bags replacing exponentials. We write P_k for the sets of permutations of the set $\{1, \dots, k\}$, and we denote as $d_x(m)$ the number of free occurrences of the variable x in m . Resource substitution is (multi-)linear,

$$\begin{array}{c}
\frac{}{x \triangleleft_! x} \quad \frac{m \triangleleft_! M}{\lambda x m \triangleleft_! \lambda x M} \quad \frac{m \triangleleft_! M}{\mathbf{der}(m) \triangleleft_! \mathbf{der}(M)} \quad \frac{m \triangleleft_! M \quad n \triangleleft_! N}{mn \triangleleft_! MN} \\
\frac{m \triangleleft_! M \quad n \triangleleft_! N}{m[n/x] \triangleleft_! M[N/x]} \quad \frac{m_1 \triangleleft_! M \quad \dots \quad m_k \triangleleft_! M}{[m_1, \dots, m_k] \triangleleft_! !M} \quad k \in \mathbb{N}
\end{array}$$

■ **Figure 1** Resource approximation for dBang

and ordered: when we write $m\{n_1/x_1, \dots, n_k/x_k\}$, it is always intended (in the resource setting) that x_i represents the i -th free occurrence of x in m . In that way, each term n_i is substituted exactly once in m .

We are now ready to define the reduction relation.

► **Definition 8.**

The reduction relation $\Rightarrow_\delta \subseteq \delta\mathbf{Bang} \times \wp(\delta\mathbf{Bang})$ is then defined as follows:

- $l\langle \lambda x m \rangle n \Rightarrow_\delta \{l\langle m[n/x] \rangle\}$
- $m[l\langle [n_1, \dots, n_k] \rangle / x] \Rightarrow_\delta \begin{cases} \bigcup_{\sigma \in P_k} l\langle m\{n_{\sigma(1)}/x_1, \dots, n_{\sigma(k)}/x_k\} \rangle & \text{if } k = d_x(m) \\ \emptyset & \text{otherwise} \end{cases}$
- $\mathbf{der}(l\langle [m_1, \dots, m_k] \rangle) \Rightarrow_\delta \{l\langle m_1 \rangle\}$ if $k = 1$ and \emptyset otherwise

We write $m \rightarrow_\delta n$ as soon as $m \Rightarrow_\delta X$ and $n \in X$ for some n (if $m \Rightarrow_\delta \emptyset$, we also abusively write $m \rightarrow_\delta \emptyset$, and we add the following equation: if $f\langle \emptyset \rangle = \emptyset$ for any full context f). We also define \rightarrow_{δ_s} and \rightarrow_δ the contextual closures of \rightarrow_δ under surface and full contexts, respectively. Notice that none of these reductions is deterministic. Both \rightarrow_{δ_s} and \rightarrow_δ are strongly normalizing, which is an immediate consequence of linearity: the size of bags of resource terms is decreasing. Confluence of \rightarrow_{δ_s} and \rightarrow_δ follows from standard resource-calculus arguments and from the proofs for $\rightarrow_{!_s}$ and $\rightarrow_{!}$.

2.2.2 Approximation

In Figure 1, defines the relation $\triangleleft_! \subseteq \delta\mathbf{Bang} \times \mathbf{dBang}$, where $m \triangleleft_! M$ means that m is a multilinear resource approximation on M .

We extend this definition to list contexts as follows: $\square \triangleleft_! \square$, $l[m/x] \triangleleft_! L[M/x]$ if $l \triangleleft_! L$ and $m \triangleleft_! M$. The extension to surface and full contexts follows analogously.

The expected behaviour of context approximation is given by the following lemma (proved by induction on contexts):

► **Lemma 9.**

- If $m \triangleleft_! L\langle N \rangle$, then there exist $l \triangleleft_! L$ and $n \triangleleft_! N$ such that $m = l\langle n \rangle$.
- If $m \triangleleft_! S\langle N \rangle$, then there exist $s \triangleleft_! S$ and $n \triangleleft_! N$ such that $m = s\langle n \rangle$.

Notice that this property fails for full contexts. Indeed, consider $t = [] \triangleleft_! !M = F\langle M \rangle$, there is no $m \triangleleft_! M$ or $f \triangleleft_! F$ such that $m \triangleleft_! M$ and $t = f\langle m \rangle$. Consequently, our definitions do not provide a convenient notion of approximation between full contexts. This is due to a need for parallel treatment of bags, which - as we shall see later - is incompatible with single-hole contexts.

► **Definition 10.** (Taylor expansion) For any $M \in \mathbf{dBang}$, we define its Taylor expansion as the set of its resource approximants:

$$\mathcal{T}(M) = \{m \in \delta\mathbf{Bang} \mid m \triangleleft_! M\}$$

By strong normalization of $\delta\mathbf{Bang}$, we can define the normal form $\mathbf{nf}(m)$ of a resource term m as the finite set made of its full reducts. Remember that the resource reduction we consider is not deterministic nor confluent (\Rightarrow_δ is, but not \rightarrow_δ), hence this set can contain multiple terms, and can be empty. We then define the *Taylor normal form* of \mathbf{dBang} terms as $\text{TNF}(M) = \bigcup_{m \triangleleft_1 M} \mathbf{nf}(m)$. Notice that Taylor normal form is made of full normal terms, not only surface normal forms.

► **Remark 11.** If $\mathcal{T}(M) \Rightarrow_\delta^* X$ and X is a set of full normal terms, then $X = \text{TNF}(M)$. But the converse is not true: we can have $\text{TNF}(M)$ non empty, but no k such that $M \Rightarrow_\delta^k \text{TNF}(M)$. This is due to the fact that terms in $\mathcal{T}(M)$ can require different number of reduction steps, as with Taylor expansion of fixpoint combinators where this number is unbounded, as illustrated in Example 12.

► **Example 12.** Consider the terms given in Example 3.

- An approximant of $m \triangleleft_1 \Omega$ must be of shape $(\lambda xx[x]_k)[\lambda xx[x]_{k_1}, \dots, \lambda xx[x]_{k_l}]$. Now, if $k = l - 1$ (otherwise $m \mapsto_1 \emptyset$) then $m \rightarrow_{\delta_s} (\lambda xx[x]_{k_1})[\lambda xx[x]_{k_2}, \dots, \lambda xx[x]_{k_l}]$ ⁶, which is again an approximant of Ω . But we can observe that the cardinality of the bag reduces during this reduction; hence if we iterate this reduction, we eventually reach a term like $(\lambda x[x]_k)[] \rightarrow_{\delta_s} \emptyset$ (if an empty reduction has not occurred before). So, $\text{TNF}(\Omega) = \emptyset$.
- Similarly, if $m \triangleleft_1 Y_x^n$, we verify easily that $m \rightarrow_{\delta_s} x[n_1, \dots, n_k]$, with $n_i \triangleleft_1 Y_x^n$. In particular, $x[] \triangleleft_1 Y_x^n$ and is in normal form. Actually, $\text{TNF}(Y_x^n)$ can be characterized inductively : $x[] \in \text{TNF}(Y_x^n)$, and if $n_1, \dots, n_k \in \text{TNF}(Y_x^n)$, then $x[n_1, \dots, n_k] \in \text{TNF}(Y_x^n)$, for any k .
- The other fixpoint term, Y_x^v , behaves slightly differently: if $m \triangleleft_1 Y_x^v$, then we verify $m \rightarrow_{\delta_s}^* xn$, for some $n \triangleleft_1 Y_x^v$. But here, because of the argument not being in a bag, all approximants reduce (if not \emptyset) to some term $xxx \dots xn$, but are not in normal form; since such a reduction terminates, we observe that $\text{TNF}(Y_x^v) = \emptyset$.

► **Remark 13.** [Clashes and normal forms] Note that we have not excluded so-called *clashes* from our calculus, as they cause no issue in our setting. Thus, terms such $\mathbf{der}(\lambda xx)$ are considered as regular normal forms. This approach is similar to the one of Dufour and Mazza [21]. However, giving an empty semantics to clashes could easily be done, for example, adding reductions such as $\mathbf{der}(\lambda xm) \rightarrow \emptyset$ to resource calculus.

2.2.3 Simulation

We now formalize the simulation of \mathbf{dBang} reduction in the approximants of $\delta\mathbf{Bang}$, as sketched in the Example 12.

► **Lemma 14** (Substitution Lemma for Taylor expansion). *For any M, N of \mathbf{dBang} , for any $m, n_1, \dots, n_{d_x(m)}$ of $\delta\mathbf{Bang}$, we have $m\{n_1/x_1 \dots n_{d_x(m)}/x_{d_x(m)}\} \triangleleft_1 M\{N/x\}$ if and only if $m \triangleleft_1 M$, and $n_i \triangleleft_1 N_i$ for all $i \leq d_x(m)$.*

We can show, by a straightforward induction, with the help of the Substitution Lemma stated above, that the surface reduction acts exactly the same way in a \mathbf{dBang} term and in its approximants:

► **Lemma 15.** *Let $m \triangleleft_1 M$ and $m \rightarrow_{\delta_s} n$. Then there is $N \in \mathbf{dBang}$ such that $M \rightarrow_{1_s} N$ and $n \triangleleft_1 N$.*

⁶ We consider here one possible reduction, any element of the bag could be substituted to the inner head variable x , not necessarily $\lambda x[x]_{k_1}$, the argument is valid for all the reduction paths.

However, this result is false when considering reduction in full contexts: take the following term $m = [(\lambda xx)y, (\lambda xx)y]$. We have $m \triangleleft_! !((\lambda xx)y)$ and $m \rightarrow_\delta n = [x[y/x], (\lambda xx)y]$, but n is not an approximant of any **dBang** term (it has lost its uniformity, in the sense of Ehrhard and Regnier [27]). We will need parallel reduction to achieve full simulation, this is done in Subsection 2.2.4.

► **Lemma 16.** *Considering the following configuration:*

$$\begin{array}{ccc} M & \rightarrow_{!s} & N \\ \nabla & & \nabla \\ m & \rightarrow_{\delta s} & n \end{array}$$

Given three of four terms, we can always obtain a convenient fourth that makes the square commute. (for a proof, see lemmas 74, 75 and 76 in Appendix A).

► **Definition 17.** *Let X and Y be sets, and $\mathcal{R} \subseteq X \times Y$ a relation. We write $X\overline{\mathcal{R}}Y$ whenever $\forall x \in X, \exists y \in Y$ such that $(x, y) \in \mathcal{R}$ and $\forall y \in Y, \exists x \in X$ such that $(x, y) \in \mathcal{R}$.*

Then, using Lemma 16 we obtain a simulation theorem for surface reduction:

► **Theorem 18 (Simulation).** *Let M, N in **dBang** such that $M \rightarrow_{!s} N$. We have $\mathcal{T}(M) \Rightarrow_{\delta s} \mathcal{T}(N)$.*

2.2.4 Parallel reduction and full contexts

Figure 2 defines a parallel resource reduction allowing us to handle full contexts. Intuitively, it needs to reduce at once every term occurring in a bag, in order to simulate internal reductions like $!M \rightarrow_! !M'$: in this case, approximants of the antecedent are of the form $[m_1, \dots, m_k]$ with $m_i \triangleleft_! M$, and the simulation needs to reduce all the m_i 's at once, in order to obtain an approximation of the reduct $[m'_1, \dots, m'_k] \triangleleft_! !M'$. In particular, the reduction needs to be reflexive because *e.g.* $[] \triangleleft_! !M$ and $[]$ does not reduce to \emptyset , we have to consider that $[] \triangleleft_! !M'$ is obtained from $[]$ by reduction. The resulting relation \Rightarrow_δ , is a well-known non-deterministic extension of standard reduction which can be used to prove confluence property. In general, for a reduction \rightarrow , we have $\rightarrow \subseteq \Rightarrow_\delta \rightarrow^*$. See *e.g.* Barendregt's proof of confluence for the λ -calculus [12], known as Tait—Martin-Löf technique.⁸ We abusively write $l \Rightarrow_\delta l'$ for contexts as soon as $l = \square[m_1/x_1] \dots [m_k/x_k], l' = \square[m'_1/x_1] \dots [m'_k/x_k]$ and $m_i \Rightarrow_\delta m'_i$ for any $i \leq k$.

Since \Rightarrow_δ is size-decreasing, it enjoys weak normalization, but obviously not strong, as it is reflexive. We can now state our simulation results for full contexts:

► **Lemma 19.** *Let $M, N \in \mathbf{dBang}$ with $M \rightarrow_! N$. For any $m \triangleleft_! M$, either $m \Rightarrow_\delta \emptyset$ or there is some $n \triangleleft_! N$ such that $m \Rightarrow_\delta n$*

The symmetric counterpart of this result is obtained with a similar reasoning:

► **Lemma 20.** *Let $M, N \in \mathbf{dBang}$ with $M \rightarrow_! N$. For any $n \triangleleft_! N$, there is some $m \triangleleft_! M$ such that $m \Rightarrow_\delta n$.*

⁷ Usually the parallel reduction enjoys the diamond property. It is not the case here because of the non-determinism induced by the permutations; in order to recover it, it is necessary to lift the reduction to set of terms. We mentioned it with non parallel reduction, which \Rightarrow_δ being confluent, but not $\rightarrow_{\delta s}$, for the same reason.

⁸ See Vaux-Auclair [35] for a detailed study of this reduction in λ -calculus and its interaction with quantitative Taylor expansion.

$$\begin{array}{c}
\frac{}{x \Rightarrow_{\delta} x} \quad \frac{m_1 \Rightarrow_{\delta} m'_1 \quad \dots \quad m_k \Rightarrow_{\delta} m'_k}{[m_1, \dots, m_k] \Rightarrow_{\delta} [m'_1, \dots, m'_k]} \quad k \in \mathbb{N} \\
\frac{m \Rightarrow_{\delta} m'}{s\langle m \rangle \Rightarrow_{\delta} s\langle m' \rangle} * \quad \frac{[m_1, \dots, m_k] \Rightarrow_{\delta} [m'_1, \dots, m'_k] \quad n \Rightarrow_{\delta} n' \quad l \Rightarrow_{\delta} l'}{n[l\langle [m_1, \dots, m_k] \rangle] \Rightarrow_{\delta} l'\langle n'\{m'_{\sigma(1)}/x_1, \dots, m'_{\sigma(k)}/x_k\} \rangle} ** \\
\frac{m \Rightarrow_{\delta} m' \quad n \Rightarrow_{\delta} n' \quad l \Rightarrow_{\delta} l'}{l\langle \lambda x m \rangle n \Rightarrow_{\delta} l'\langle m'[n'/x] \rangle} \quad \frac{m_1 \Rightarrow_{\delta} m'_1}{\mathbf{der}(m_1, \dots, m_k) \Rightarrow_{\delta} m'_1} ***
\end{array}$$

* s is any surface resource context

** $\sigma \in P_k$, and if $k = d_x(n')$ (otherwise the reduction gives \emptyset).

*** $k = 1$. Otherwise, the reduction gives \emptyset .

■ **Figure 2** Parallel reduction for $\delta\mathbf{Bang}$

The two previous lemmas give us the desired simulation result for full reduction:

► **Theorem 21.** *Let $M, N \in \mathbf{dBang}$ such that $M \rightarrow_{\dagger} N$, then we have $\mathcal{T}(M) \overline{\Rightarrow}_{\delta} \mathcal{T}(N)$*

We saw that surface reduction acts similarly in \mathbf{dBang} and in $\delta\mathbf{Bang}$, while parallel reduction is necessary to give a multilinear account to internal reductions (the definition of parallel reduction alone does not imply that they occur exclusively inside bags, but this is made mandatory through the use of invariant multi-hole surface contexts). The factorization properties established for \mathbf{dBang} (Corollary 6) can easily be translated in the resource setting:

► **Proposition 22 (Factorization).** *Let s^+ denote multi-holes surface contexts and given by the syntax $s^+ := \square \mid m \mid s^+s^+ \mid s^+[s^+/x] \mid \mathbf{der}(s^+) \mid \lambda xs^+$. For any reduction $m \rightarrow_{\delta}^* p$, there are some bags \bar{n}_i and some multi-hole context s such that $p = s\langle \bar{n}_1, \dots, \bar{n}_k \rangle$ and $m \rightarrow_{\delta_s}^* s\langle \bar{n}'_1, \dots, \bar{n}'_k \rangle \Rightarrow_{\delta}^* s\langle \bar{n}_1, \dots, \bar{n}_k \rangle$*

Following the reduction occurring on resource terms, we have defined previously Taylor normal form. We develop in this section some lemmas on those objects. They will be useful in order to prove the Commutation Theorem between approximation trees and Taylor expansions.

► **Lemma 23.** *Given $M \rightarrow_{\dagger}^* N$ then $TNF(M) = TNF(N)$.*

► **Lemma 24.** *Given $m \in TNF(M)$ then there exists M' such that $M \rightarrow_{\dagger}^* M'$ and $m \triangleleft_{\dagger} M'$.*

2.3 Approximation trees

A classical way to approximate λ -terms is through evaluation trees, which can be seen as infinitary normal forms. The evaluation tree of a λ -term corresponds to the limit of a sequence of increasingly precise approximants of this λ -term. Those approximants are terms in normal forms, but using potentially a specific subterm \perp , representing an undefined computation. When reducing the λ -term some subterms become stable under further reductions, these subterms are precisely those captured by approximants.

Different comparable evaluation trees coexist. The original and best-known ones are Böhm trees. They have been introduced in the \mathbf{CbN} paradigm where they intersect with numerous notions. Indeed, they give a precise classification of λ -terms in regard of their operational behaviour.

In CbN, using all partial normal forms (possibly containing \perp) as approximants, corresponds to Lévy-Longo trees [34]. Böhm trees use a restriction on this set of approximants, so that unsolvable terms have an empty Böhm tree. This restriction is based on restricting approximants to (partial) head normal forms, since being solvable is equivalent to having a head normal form. Lévy-Longo trees instead rely on weak head normal forms. In fact, it only leads to the suppression of "stupid" approximants such as $\lambda x \perp$. Böhm trees and Lévy-Longo trees differ essentially in their treatment of abstractions over undefined computations and therefore share similar properties. Among others, they both satisfy variants of the Commutation Theorem, which states that Taylor expansion of the Böhm tree of a term is equal to its Taylor normal form [27].

More recently, Böhm trees have been introduced in the CbV setting (in a variant with permutation rules) [30]. They are defined without restriction on approximants. Indeed, in this case, there is no known syntactic characterization of solvability in terms of a specific normal form. In that, they may be considered closer to the Lévy-Longo trees. However, the relation with scrutability — the notion of meaning — in CbV is respected: unscrutable λ -terms have an empty Böhm tree. An approximant such as $\lambda x \perp$ is meaningful in this setting, since it is a value.

In this work we define approximation trees for the distant version of CbV, CbN and Bang. As in the CbV case, we do not restrict our approximants for dCbN and dCbV. In order to be consistent between the different calculus, we adopt the same approach for dCbN. The resulting trees are called approximation trees here. They correspond to Lévy-Longo trees in the dCbN case, while in the dCbV case they are closer to Böhm trees in spirit, although the situation is more subtle. Since those two family of trees are very strongly related, we choose not to focus too much on whether our trees should be considered as Böhm or Lévy-Longo trees.

In this section we will introduce those trees for dBang and their relation to the Taylor expansion via the Commutation Theorem (Theorem 46).

► **Definition 25** (dBang $_{\perp}$). *Let dBang $_{\perp}$ be the set of dBang-terms extended with the symbol \perp . In the following we use subscripts as \perp_i in order to distinguish occurrences of \perp as a subterm. Similarly, we extend the different types of contexts; and we extend reductions in the obvious way.*

► **Definition 26.** *The set of approximants is a strict subset of dBang $_{\perp}$ generated by the gram-*

$$\begin{array}{ll} \text{mar:} & A := \perp \mid B \mid \lambda x A \mid !A \mid A[A_1/x] & A_! & := B \mid \lambda x A \mid A_1[A_1/x] \\ & B := x \mid A_{\lambda} A \mid \mathbf{der}(A_1) & A_{\lambda} & := B \mid !A \mid A_{\lambda}[A_1/x] \end{array}$$

► **Lemma 27.** *Approximants are the normal forms of dBang $_{\perp}$.*

Substituting \perp by any term preserves normal forms, as a consequence of the syntactic structure of approximants. First, approximants contain no redexes. Secondly, there are no approximants containing subterms of shape $\mathbf{der}(\perp)$, $\perp A$, $A[\perp/x]$ that could hide a potential redex. Formally:

► **Lemma 28.** *Let A be an approximant, M any term of dBang $_{\perp}$, and \perp_i some occurrence of \perp in A . If $A[M/\perp_i] \rightarrow_! N$, then $N = A[M'/\perp_i]$ with $M \rightarrow_! M'$. i.e. we cannot create a redex when replacing a \perp by a term M in an approximant: the only redexes obtained this way are those already present in M .*

► **Definition 29** (Order). *Let $\sqsubseteq \subseteq \text{dBang}_{\perp} \times \text{dBang}_{\perp}$ be the least contextual closed preorder on dBang $_{\perp}$ generated by setting: $\forall M \in \text{dBang}_{\perp}$ and for all full context F , $F[\perp] \subseteq F[M]$.*

In other words, $M \sqsubseteq N$ if and only if $M = N\overrightarrow{\{P/\perp\}}$. In the evaluation trees literature, we sometimes find approximation orders where only terms of shape $!P$ are replaced by a \perp (see *e.g.* Dufour and Mazza [21]). For technical reasons (Theorem 59 in particular) due to CbN and cbv embeddings into dBang, we need, for example, $\lambda x \perp \sqsubseteq \lambda x M$ to be a valid approximation.

The following lemma states that when a reduction occurs in a term, it cannot be *seen* by its approximations, that only represent a part of the skeleton of their normal form. In other words, when $A \sqsubseteq M$, then A represents a subtree of M which cannot be modified by some reduction.

► **Lemma 30.** $A \sqsubseteq M$ and $M \rightarrow_{\dagger}^* N$ then $A \sqsubseteq N$.

► **Definition 31** (Set of approximants). *Given $M \in \text{dBang}$, the set of approximants of M is defined as follows: $\mathcal{A}(M) = \{A \mid M \rightarrow_{\dagger}^* N, A \sqsubseteq N\}$.*

Note that $\mathcal{A}(M)$ is never empty, since it contains at least \perp .

► **Example 32.** Consider the terms given in Example 3.

- The only reducts of Ω being Ω itself, and since the syntax A of approximants contains no redex, we easily conclude that $\mathcal{A}(\Omega) = \{\perp\}$.
- The reducts of Y_x^n are of shape $x!x!x \dots !Y_x^n$, we conclude that $\mathcal{A}(Y_x^n)$ contains precisely the terms $\perp, x\perp, x!\perp, x!x\perp, x!x!\perp, \dots$
- The reducts of Y_x^v follow the same observation, minus the exponentials, and $\mathcal{A}(Y_x^v)$ contains precisely $\perp, x\perp, xx\perp, xxx\perp, \dots$

► **Lemma 33.** $M \rightarrow_{\dagger}^* N$ then $\mathcal{A}(M) = \mathcal{A}(N)$.

► **Remark 34.** If $A = F[\vec{\perp}] \in \mathcal{A}(M)$ (where F is a multi-hole context), then $M \rightarrow^* F[\vec{N}]$ for some \vec{N} . In particular, if A contains no \perp subterms, then M has a normal form, which is exactly A .

► **Definition 35** (Ideal). *Let $X \subseteq \text{dBang}_{\perp}$. We set X is an ideal when X is downwards closed for \sqsubseteq , and directed: for all $M, N \in X$ there exists some upper bound (with respect to \sqsubseteq) to $\{M, N\}$.*

► **Lemma 36.** *Let $M \in \text{dBang}$. $\mathcal{A}(M)$ is an ideal.*

In order to define approximation trees, we follow the long-established tradition for term rewriting systems, including CbN and CbV [6, 15, 13, 5, 7, 30], based on the ideal completion. This method constructs the set of ideals of approximants (ordered by a direct partial order). Evaluation trees, such as Böhm trees are then identified with such ideals. The finite and infinite ideals represent respectively the finite and infinite trees. For a λ -term M , its evaluation tree is the ideal generated by its set of approximants; equivalently, it can be seen as the supremum of these approximants in the associated directed-complete domain. This domain admits a concrete presentation as a coinductive grammar extending that of approximants, where constructors may be unfolded infinitely often. Given $M \in \text{dBang}$, $\mathcal{A}(M)$ has a supremum, noted $\cup \mathcal{A}(M)$, which is a potentially infinite tree.

► **Definition 37** (Approximation Tree). *The approximation tree of a term M in dBang, is given by $\cup \mathcal{A}(M)$, and denoted $\text{AT}_{\dagger}(M)$.*

Approximation trees satisfy the following immediate properties (we already used these facts on approximants for previous results, they lift immediately to approximation trees).

► **Proposition 38** (Some properties of approximation trees).

- If M is in normal form, then $\text{AT}_!(M) = M$.
- $\text{AT}_!(M) = \perp$ if and only if $\mathcal{A}(M) = \{\perp\}$ ⁹, if and only if any reduct of M is a redex.
- If $M \rightarrow^* N$, $\text{AT}_!(M) = \text{AT}_!(N)$, for example $\text{AT}_!(\mathbf{der}(!M)) = \text{AT}_!(M)$.
- $\text{AT}_!(\lambda x M) = \lambda x \text{AT}_!(M)$ and $\text{AT}_!(!M) = !(\text{AT}_!(M))$.
- If $\text{AT}_!(M) \neq L(\lambda x -)$, then $\text{AT}_!(MN) = \text{AT}_!(M)\text{AT}_!(N)$.
- If $\text{AT}_!(M) \neq L(!-)$, then $\text{AT}_!(\mathbf{der}(M)) = \mathbf{der}(\text{AT}_!(M))$ and $\text{AT}_!(N[M/x]) = \text{AT}_!(N)[\text{AT}_!(M)/x]$

In particular, we can infer from the facts above that if $\text{AT}_!(M)$ is an application, then it is equal to $(x)\text{AT}_!(N)$ for some N (hence $M \rightarrow^* L(x)N$).

► **Example 39.** From Example 32, we can infer that $\text{AT}_!(\Omega) = \perp$, $\text{AT}_!(Y_x^n)$ is the infinite application $x!x!x!x \dots$, and $\text{AT}_!(Y_x^v)$ is also an infinite application $xxxxxx \dots$. This is the intended behaviour of approximation trees, as in this category of terms they represent, at the limit, the amount of result produced by a computation, even if the term itself has no normal form.

2.4 The commutation between approximation tree and Taylor expansion

Combining the previous results, we state the Commutation Theorem. We start by defining the Taylor expansion of approximation trees.

► **Definition 40.** We extend the Taylor expansion to \mathbf{dBang}_\perp -terms by setting $\mathcal{T}(\perp) = \emptyset$ (recall that $f[\emptyset] = \emptyset$ for any full context f). In other words, there is no $\delta\mathbf{Bang}$ term m such that $m \triangleleft_\perp \perp$.

One immediate consequence of this definition is that, for M in \mathbf{dBang}_\perp , $\mathcal{T}(M) \neq \emptyset$ if M contains no \perp as a surface subterm. In other words, M expands to a non-empty set if and only if the \perp are under exponentials $!$ (in that case, these exponentials can be approximated by empty bags). Moreover, $M \sqsubseteq N$ implies $\mathcal{T}(M) \subseteq \mathcal{T}(N)$.

► **Definition 41** (Taylor expansion of approximation trees). Given $M \in \mathbf{dBang}$, $\mathcal{T}(\text{AT}_!(M)) = \bigcup_{a \in \mathcal{A}(M)} \mathcal{T}(a)$.

► **Example 42.** Following the terms of previous examples, we easily check that $\mathcal{T}(\text{AT}_!(\Omega)) = \emptyset$ (because $\mathcal{A}(\Omega) = \{\perp\}$). Recall that $\mathcal{A}(Y_x^n) = \{\perp, x\perp, x!\perp, \dots\}$. The first of these approximants having a non-empty expansion is $x!\perp$, approximated by $x[]$, $\mathcal{T}(\text{AT}_!(Y_x^n)) = \{x[], x[x[]], \dots, x[\dots]\}$. And $\mathcal{T}(\text{AT}_!(Y_x^v)) = \emptyset$. Indeed, all approximants of Y_x^v are of shape $xxx \dots x\perp$, and have a surface \perp that expands to \emptyset . With examples 32 and 12, we observe for these terms the identity of $\text{TNF}(M)$ and $\mathcal{T}(\text{AT}_!(M))$ which is proved globally in this section.

Remark that, since $\mathcal{A}(M)$ is an ideal, $\mathcal{T}(\text{AT}_!(M))$ is a directed union. The purpose of Taylor expansion is to approach a term by finitary resource terms only, so we do not consider any

⁹ This distinguishes our approximation trees from those of Mazza and Dufour [21], in which $\text{AT}_!(M) = \perp$ as soon as M has no surface normal form. Indeed, for technical reasons (relative to \mathbf{CbV} embeddings), if $\text{AT}_!(M) = \perp$, we need to have $\text{AT}_!(\lambda x M) = \lambda x \perp$, and not \perp , because $\lambda x \perp$ embeds to $!(\lambda x \perp)$, while \perp embeds to \perp , which loses the exponential and breaks commutation properties between approximation trees and \mathbf{CbV} embedding (Theorem 59). This distinction vanishes at the semantical level: as soon as we consider Taylor expansion of approximation trees, the trees having no surface normal form are given an empty expansion (Definition 40).

infinite supremum of this union, and keep a set of terms, this approximation being inductive; while Böhm trees are coinductive and consist of infinite objects.

The two following lemmas formalize a compatibility between approximation order and inclusion of Taylor expansion.

► **Lemma 43.** *Let $A \sqsubseteq M$, then $\mathcal{T}(A) \subseteq \mathcal{T}(M)$.*

► **Lemma 44.** *Let $A \in \mathcal{A}(M)$, then $\mathcal{T}(A) \subseteq TNF(M)$.*

Analogously, normalized resource approximants match the tree approximants.

► **Lemma 45.** *Let $m \triangleleft_! M$ in normal form, then there exists an approximant A such that $A \sqsubseteq M$ and $m \triangleleft_! A$ (remember that $m \triangleleft_! M$ and $m \in \mathcal{T}(M)$ are the same thing).*

From the previous lemmas we deduce the Commutation Theorem:

► **Theorem 46.** *Let $M \in \mathbf{dBang}$. $\mathcal{T}(\mathbf{AT}_!(M)) = TNF(M)$.*

3 Translations

This section is about the approximation theory of \mathbf{dCbN} and \mathbf{dCbV} , in particular about how the embeddings into \mathbf{dBang} can profit from the result of previous Section. We briefly recall the relevant definitions and results of those embeddings, that have been already well studied [16, 31, 8, 9, 10, 11] \mathbf{dCbV} and \mathbf{dCbN} share the same syntax: $M, N ::= x \mid \lambda x M \mid MN \mid M[N/x]$. Furthermore, the values V are defined as either a variable x or a λ -abstraction $\lambda x M$. List contexts are defined as in \mathbf{dBang} ; reduction rules are:

- In \mathbf{dCbN} : $L\langle \lambda x M \rangle N \rightarrow_n L\langle M[N/x] \rangle$ and $M[N/x] \rightarrow_n M\{N/x\}$
- In \mathbf{dCbV} : $L\langle \lambda x M \rangle N \rightarrow_v L\langle M[N/x] \rangle$ and $M[L\langle V \rangle/x] \rightarrow_v L\langle M\{N/x\} \rangle$

We are now ready to define the translations of \mathbf{dCbN} (noted $()^n$) and \mathbf{dCbV} (noted $()^v$) into \mathbf{dBang} . Several translations of \mathbf{dCbV} into \mathbf{dBang} exist (or the original \mathbf{Bang} -calculus without explicit substitutions). The first one [24], inspired by Girard *second translation*, does not preserve normal forms (xy translates to $\mathbf{der}(!x)!y$). Another translation was then proposed [16], which fixes this problem by simplifying the created redexes by the translations on the fly. This is the translation that we use here. It is worth noticing that this translation does not satisfy reverse simulation from \mathbf{dBang} to \mathbf{dCbV} [10, Figure 1] This issue has been addressed by a third translation [10]. However, reverse simulation is not necessary in our case, and we will keep with the translation proposed in [16], although we are convinced that the same developments could be carried out with the translation from [10]. This choice is motivated by the fact that the translation we consider fits better with the Linear Logic discipline from which stems Taylor expansion, and also because when considering the study of meaningfulness, we can rely on results that have been proved for this translation. Moreover, this translation enjoys a weaker property than strict reverse simulation, that we call embedding (see Lemma 58) and which is sufficient for our study (approximation trees and Taylor expansion concern mostly iterated reductions \rightarrow^* , and one-step, thus reverse simulation is then not mandatory for our results).

$$\begin{array}{ll}
 x^n = x & x^v = !x \\
 (\lambda x M)^n = \lambda x M^n & (\lambda x M)^v = !(\lambda x M^v) \\
 (M N)^n = M^n !N^n & (M N)^v = \begin{cases} L\langle P \rangle N^v & \text{if } M^v = L\langle !P \rangle \\ \mathbf{der}(M^v) N^v & \text{otherwise} \end{cases} \\
 (M[N/x])^n = M^n [!N^n/x] & (M[N/x])^v = M^v [N^v/x]
 \end{array}$$

We abusively extend the translations to list contexts: let $\circ \in \{n, v\}$; then given a list context $L = \square[M_1/x_1] \dots [M_k/x_k]$, we write $L^\circ = \square[M_1^\circ/x_1] \dots [M_k^\circ/x_k]$.

We are now ready to define meaningfulness in both systems:

► **Definition 47** (dCbV and dCbN meaningfulness [31]). *We say that a term $M \in \text{dCbV}$ is dCbV-meaningful (resp. $M \in \text{dCbN}$ and is dCbN-meaningful) if $T\langle M \rangle \rightarrow_v^* V$ for some value V (resp. $T\langle M \rangle \rightarrow_n^* \lambda x x$) where T is a testing context, defined by: $T := \square \mid T N \mid (\lambda x T) N$.*

¹⁰.

► **Example 48.** Consider the fixpoint $Y_x = (\lambda y x(y y))(\lambda y x(y y))$, its translations are:

- $(Y_x)^n = (\lambda y x!(y!y))!(\lambda y x!(y!y))$.
- $(Y_x)^v = (\lambda y x(y!y))!(\lambda y x(y!y))$.

Which are the terms Y_x^n and Y_x^v of Example 3.

It has been shown that dCbN dCbV can be simulated through their encoding in dBang, and that the meaningfulness of dCbV and dCbN coincides with the one of dBang (Definition 4):

► **Theorem 49.**

1. If $M \rightarrow_n N$ (resp. $M \rightarrow_v N$) then $M^n \mapsto_! N^n$ (resp. $M^v \mapsto_! N^v$) [16, Lemma 4.6].
2. M is dCbN-meaningful iff M^n is meaningful [31, Theorem 25].
3. M is dCbV-meaningful iff M^v is meaningful [31, Theorem 30].

We study approximation trees and Taylor expansion for dCbV and dCbN via their translations, in view of meaningfulness.

3.1 Taylor expansion and approximation trees for dCbN and dCbV

3.1.1 Taylor expansion for dCbN and dCbV

► **Definition 50** (Resource approximations and Taylor expansion of dCbN).

We define an approximation \triangleleft_n relation between δBang and dCbN. Notice that despite the approximants being defined in δBang , there is no dereliction needed in the case of dCbN. The relation is given by $x \triangleleft_n x$; if $m \triangleleft_n M$ then $\lambda x m \triangleleft_n \lambda x M$; and if $m \triangleleft_n M$ and $n_i \triangleleft_n N$ for any $i \leq k$ then we have $m[n_1, \dots, n_k] \triangleleft_n MN$ and $m[[n_1, \dots, n_k]/x] \triangleleft_n M[N/x]$.

Taylor expansion is again defined as sets of approximations: $\mathcal{T}^n(M) = \{m \in \delta\text{Bang} \mid m \triangleleft_n M\}$.

We do not need to define a specific resource calculus for dCbN nor dCbV, since δBang semantics precisely subsumes both approximation theories.

► **Remark 51.** The syntax of approximants of dCbN can be given by the following syntax: $m, n := x \mid m [n_1, \dots, n_k] \mid \lambda x m \mid m[[n_1, \dots, n_k]/x]$

► **Definition 52** (Resource approximation and Taylor expansion of dCbV). We define the relation $m \triangleleft_v M$ for $m \in \delta\text{Bang}$ and $M \in \text{dCbV}$.

- $[x, \dots, x]_k \triangleleft_v x$ for any $k \in \mathbb{N}$.

¹⁰Note that as for dBang, if such a reduction exists, then a weak reduction (analogous to surface reduction) is sufficient, i.e. it is not necessary to reduce under the λ s in dCbV or in the arguments of applications (or explicit substitutions) in dCbN.

- $[\lambda x m_1, \dots, \lambda x m_k] \triangleleft_v \lambda x M$ if $m_i \triangleleft_v M$ for any $i \leq k$.
- $\mathbf{der}(m)n \triangleleft_v MN$ if $m \triangleleft_v M, n \triangleleft_v N$ and $M \notin V$
- $mn \triangleleft_v VN$ if $[m] \triangleleft_v V$ and $n \triangleleft_v N$
- $m[n/x] \triangleleft_v M[N/x]$ if $m \triangleleft_v M$ and $n \triangleleft_v N$.

Taylor expansion is defined as $\mathcal{T}^v(M) = \{m \in \delta\mathbf{Bang} \mid m \triangleleft_v M\}$. Notice that, so as Taylor expansion commutes with this embedding, Taylor approximation also suppresses derelections redexes, so as the expansion also preserves normal forms.

► **Example 53.** Approximants can fit both for \mathbf{dCbN} and \mathbf{dCbV} terms, such as $x[x] \triangleleft_v xx$ and $x[x] \triangleleft_n xx$. But this is not the case in general, e.g. $[x] \triangleleft_v x$ but there is no \mathbf{dCbN} term M such that $[x] \triangleleft_v M$.

Both in \mathbf{dCbV} and \mathbf{dCbN} we also define $\mathbf{TNF}(M)$ as the set containing the normal forms of resource approximants of M . We get the following:

► **Lemma 54.** Let $M \in \mathbf{dCbN}$ (resp. \mathbf{dCbV}) then $\mathcal{T}^n(M) = \mathcal{T}(M^n)$ (resp. M^v)

In the \mathbf{dCbV} case, notice that the translation of application, as well as its Taylor expansion, is described by case on its first component such that the translation (resp. the expansion) of a term in normal form does not lead to any reducible pattern. From that we obtain the following properties:

► **Corollary 55.** Let $M \in \mathbf{dCbN}$ (resp. \mathbf{dCbV}) then $\mathbf{TNF}(M) = \emptyset \leftrightarrow \mathbf{TNF}(M^n) = \emptyset$ (resp. M^v).

3.1.2 Approximation trees for \mathbf{dCbN} and \mathbf{dCbV}

► **Definition 56.** The syntax of \mathbf{dCbN} approximants is given by:

$$A_n ::= \perp \mid N_\lambda \mid \lambda x A_n \quad N_\lambda ::= x \mid N_\lambda A_n$$

Notice that this syntax contains inductive weak head normal forms, as it is standard for Lévy-Longo trees [34]. Subterms such as $\lambda \vec{x} \perp$, which might correspond to non trivial approximants of non-solvable terms (e.g. $\lambda x \Omega$) are not included in Böhm trees, which correspond to inductive head normal forms. We could endow the approximants with equations identifying $\lambda \vec{x} \perp$ to \perp , but this is not necessary¹¹ since this trivialization is achieved by Taylor expansion.

► **Definition 57.** The syntax of \mathbf{dCbV} approximants is given by:

$$\begin{aligned} A_v & ::= \perp \mid A_\lambda \mid \lambda x A_v \mid A_v[A_{x\lambda}/x] & A_{x\lambda} & ::= A_\lambda A_v \mid A_{x\lambda}[A_{x\lambda}/x] \\ A_\lambda & ::= x \mid A_\lambda A_v \mid A_\lambda[A_{x\lambda}/x] \end{aligned}$$

In \mathbf{dCbV} , as well as in the \mathbf{CbV} case (and its variant with permutation rules), there is no equivalent to (weak) head normal forms. We define our approximants as the normal forms potentially containing \perp . This is similar to the ones used for Böhm trees in [30] (for a variant of \mathbf{CbV} with permutation rules).

We then define approximation trees in the usual way, setting $\mathbf{AT}_\circ(M) = \cup\{A_\circ \mid M \rightarrow_\circ^* N, A_\circ \sqsubseteq N\}$, for $\circ \in \{n, v\}$, (and where $M \sqsubseteq N$ means again that M is obtained by replacing

¹¹ In fact, as for \mathbf{dBang} , having $\mathbf{AT}(\lambda x M) = \lambda x \mathbf{AT}(M)$ holding globally is convenient for technical reasons.

subterms of N by \perp). We leave it to the reader to verify that the proof of Lemma 36 can be adapted to dCbV and dCbN , incorporating the distant setting into these standard results.

In the following, we extend translations $()^n$ and $()^v$ to approximants by setting $\perp^n = \perp^v = \perp$, and study the commutation between approximation and the embedding (Theorem 59), that will allow us to transport our Commutation Theorem (Theorem 46) into dCbV and dCbN . Notice that M is a dCbN (resp. dCbV) approximant if and only if M^n (resp. M^v) is a dBang approximant.

Now we can prove the weaker form of reverse simulation mentioned in the preamble of this section. Notice that this result corresponds to the embedding notion of Dufour and Mazza [21].

► **Lemma 58** (Embedding).

1. Let $M \in \text{dCbN}$. If $M^n \rightarrow_! N$, then there is some $P \in \text{dCbN}$ such that $M \rightarrow_n^* P$ and $N \rightarrow_!^* P^n$.
2. Let $M \in \text{dCbV}$. If $M^v \rightarrow_! N$, then there is some $P \in \text{dCbV}$ such that $M \rightarrow_v^* P$ and $N \rightarrow_!^* P^v$.

► **Theorem 59.**

1. Let $M \in \text{dCbN}$. $(\text{AT}_n(M))^n = \text{AT}_!(M^n)$.
2. Let $M \in \text{dCbV}$. Then $(\text{AT}_v(M))^v = \text{AT}_!(M^v)$.

Thanks to the compatibility of approximation trees and Taylor expansion with the translations of dCbN and dCbV into dBang we can apply our commutation result for dBang to both calculi. Although these results have been well established (to our knowledge, only for non-distant CbN and CbV), this application illustrates that the subsuming power of dBang has been brought in the approximation theory, which was our purpose.

► **Theorem 60.**

1. Let $M \in \text{dCbV}$. $\mathcal{T}^v(\text{AT}_v(M)) = \mathbf{nf}(\mathcal{T}^v(M))$.
2. Let $M \in \text{dCbN}$. $\mathcal{T}^n(\text{AT}_n(M)) = \mathbf{nf}(\mathcal{T}^n(M))$.

4 Meaningfulness and Taylor expansion

In CbN and CbV , solvability and scrutability are characterized by non-empty Taylor normal forms [27, 19]. We aim to establish an analogous link in dBang , for which a characterization of meaningfulness has been provided [31], but without an approximation theory related to this notion. The first part of this result is achieved by Theorem 61. The second part presents significant challenges, which this section aims to address.

► **Theorem 61.** *Let $M \in \text{dBang}$. If M is meaningful, then $\text{TNF}(M) \neq \emptyset$.*

This result is encouraging for our study of Taylor expansion in dBang framework, as it applies to dCbN and dCbV through our previous results (Corollary 55): Theorem 61 applies to both settings.

The converse fails in general: some non-empty Taylor normal forms correspond to meaningless terms. As mentioned in [31], some elementary terms, such as xx , are meaningless, whereas xy is not. Their intersection type system can distinguish between these terms, but it is unlikely that such a distinction can be captured at the syntactic level using Taylor expansion. Another approach would be to restrict ourselves to a (clash-free) fragment of

dBang excluding patterns that do not make sense from a **dCbV** nor a **dCbN** discipline (such as xx), but again we can exhibit terms such as $(x!x)(x!x)$ (Recall that $(xM)^v = !x(M^v)$ and $(Mx)^n = M^n!x.$) that are meaningless but cannot reasonably be assigned an empty Taylor normal form.

We prove in the remaining of this section that the result holds independently for the two sublanguages of **dBang** consisting of terms translated from $()^v$ and $()^n$. Our proof employs techniques adapted from **CbN** [27] and **CbV** [19], providing an initial characterization of the relationship between meaningfulness and Taylor expression in a distant setting. Although it is frustrating that we cannot prove the equivalence once for **dBang** and to apply it directly to its fragments; this limitation also raises an open question which we find to be of interest: is there a significant, bigger fragment of **dBang** for which the equivalence can be proven generically? Would this fragment cover terms not coming from a **dCbV** or **dCbN** translations? This is, for now, an open question.

We consider two strict subsets of **dBang**: **dBang_V** and **dBang_N** corresponding to terms obtained by translating from **dCbN** and **dCbV**, respectively. These fragments also have the advantage of excluding *clashes* - problematic **dBang** terms such as **der**(λxM) - which are often omitted from the analysis [16, 24] (see Remark 13).

► **Definition 62.**

$$\begin{aligned} \mathbf{dBang}_N: M_n &:= x \mid \lambda x M_n \mid M_n ! M_n \mid M_n [!M_n/x] \\ \mathbf{dBang}_V: M_v &:= !x \mid !(\lambda x M_v) \mid L_v \langle \lambda x M_v \rangle M_v \mid L_v \langle x \rangle M_v \mid \mathbf{der}(M_v) M_v \mid M_v [M_v/x] \\ L_v &:= \square \mid L_v [M_v/x] \end{aligned}$$

A simple inspection of the definitions yields the following property:

► **Lemma 63.** *For any $M \in \mathbf{dCbV}$, $M^v \in \mathbf{dBang}_V$, and for any $M \in \mathbf{dCbN}$, $M^n \in \mathbf{dBang}_N$.*

Note that the converse holds for **dBang_N**, but not for **dBang_V**. For example **der**($!x$) $M \in \mathbf{dBang}_V$, but no term of **dCbV** translates to this term due of the side condition of $(-)^v$ which ensures the preservation of normal forms. However, we cannot exclude these patterns from **dBang_V** as they can be obtained from some reduction as shown in Section 3.

We aim to ensure that our fragments are closed under reduction. Otherwise, a term in **dBang_V**, for example, could reduce in a term in **dBang** for which meaningfulness cannot be guaranteed (such as xx or clashes like **der**(λxM)). The following lemma can be proven by a standard induction.

► **Lemma 64.** ***dBang_V** and **dBang_N** are closed under $\rightarrow!$.*

4.1 Meaningfulness and Taylor Expansion for **dCbN**

The case of **dCbN** is relatively easy to handle, as we can adapt to the distant case the following properties; which correspond to well-known features of λ calculus: resource normal form correspond to head normal forms and terms with head normal forms are meaningful.

► **Lemma 65.** *The normal forms of **dBang_N** are of shape $\lambda x_1 \dots x_k(x)!N_1 \dots !N_l$, where $k, l \in \mathbb{N}$.*

Naturally, full normal form requires the N_i to be in normal form too, but as we shall see, this is not relevant for studying meaningfulness, as these terms will be erased by an appropriate testing context. Previous observations can be brought at a resource level.

► **Lemma 66.** *Let $m \triangleleft_n M$ with $M \in \mathbf{dBang}_N$. If $\mathbf{nf}(m) \neq \emptyset$, it is of shape $\lambda x_1 \dots x_k(x)\bar{n}_1 \dots \bar{n}_l$.*

We are now able to state the theorem establishing the classical link between Taylor expansion and meaningfulness in the case of \mathbf{dBang}_N .

► **Theorem 67.** *Let $M \in \mathbf{dBang}_N$. If $TNF(M) \neq \emptyset$, then M is meaningful.*

► **Corollary 68.** *For any M in \mathbf{dCbN} , $TNF(M) \neq \emptyset$ if and only if M is meaningful.*

4.2 Meaningfulness and Taylor expansion for \mathbf{dCbV}

The \mathbf{dCbV} case requires a finer analysis of normal forms. We first characterize normal forms of \mathbf{dBang}_V and their counterparts in resource calculus. The following definition and lemma are derived by a standard induction over the syntax of \mathbf{dBang}_V .

► **Definition 69.** *The normal forms of \mathbf{dBang}_V are described by the following syntax:*

$$\begin{aligned} B &:= B_! \mid !\lambda x B \mid !x \mid L\langle B \rangle & L &:= \square \mid L[B_!/x] \\ B_! &:= L\langle x \rangle B \mid \mathbf{der}(B_!)B \mid L\langle B_! \rangle \end{aligned}$$

► **Lemma 70.** *Let $m \triangleleft_! M \in \mathbf{dBang}_V$. If $\mathbf{nf}(m) \neq \emptyset$ then it is of the following form:*

$$\begin{aligned} b &:= b_! \mid [\lambda x b, \dots, \lambda x b] \mid [x, \dots, x] \mid l\langle b \rangle & l &:= \square \mid l[b_!/x] \\ b_! &:= l\langle x \rangle b \mid \mathbf{der}(b_!)b \mid l\langle b_! \rangle \end{aligned}$$

We consider a family of terms which are suitable for providing an appropriate testing context for any term of \mathbf{dBang}_V with non-empty Taylor normal form, in which it eventually reduces to a value. We define $\circ_0 = \lambda x_0 x_0$ and $\circ_{k+1} = \lambda x_{k+1} !\circ_k$.

We establish the testing context with the following lemma. The proof (see Appendix C) follows Carraro and Guerrieri's technique of mutual induction [19, Lemmas 26 and 27].

► **Lemma 71.** *Let $\{x_1, \dots, x_n\}$ be a set of variables and $M \in B$ (Definition 69) with $\mathbf{fv}(M) \subseteq \{x_1, \dots, x_n\}$. There exists $c \in \mathbb{N}$ such that for any $k_1, \dots, k_n \geq c$ we have $M\{\circ_{k_1}/x_1 \cdots \circ_{k_n}/x_n\} \rightarrow_!^* !P$ for some P .*

We can now state the central theorem of this section.

► **Theorem 72.** *Let $M \in \mathbf{dBang}_V$. If $TNF(M) \neq \emptyset$, then M is meaningful.*

► **Corollary 73.** *Let $M \in \mathbf{dCbV}$ then M is meaningful if and only if $TNF(M) \neq \emptyset$.*

5 Conclusion and Discussions

We developed an approximation theory for the distant \mathbf{dBang} -calculus, defining approximation trees and Taylor expansion and relating them to meaningfulness and to each other. These results are part of a wider effort to generalize the theory of the \mathbf{CbN} and \mathbf{CbV} λ -calculi. And indeed, we retrieve similar results in distant \mathbf{CbN} and distant \mathbf{CbV} , via translations in \mathbf{dBang} .

Future work includes extending meaningfulness to proof structures, pursuing Dufour and Mazza's work [21] using non-inductive variants of classical approximation notions (in particular, their Böhm trees do not have an actual tree structure since approximations cannot be described through an tree-like inductive syntax). Also, we mentioned in Section 4 an open question about the possibility to characterize a significant fragment of \mathbf{dBang} for which meaningful terms coincide with terms having a non-empty Taylor normal form; this line of work should be explored to develop the general understanding of \mathbf{dBang} .

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A

 Proofs of Section 2

► **Lemma 74.** *If $M \mapsto_! N$, then for any $m \triangleleft_! M$, either $m \rightarrow_\delta \emptyset$, or there is some $n \triangleleft_! N$ such that $m \rightarrow_\delta n$.*

Proof of Lemma 74. By induction on the definition $\mapsto_!$:

- If $m \triangleleft_! \mathbf{der}(L\langle !N \rangle)$, then there exist $k \in \mathbb{N}, n_1, \dots, n_k \triangleleft_! N$, and $l \triangleleft_! L$ such that $m = \mathbf{der}(l\langle [n_1, \dots, n_k] \rangle)$, and then $m \mapsto_! \emptyset$ if $k \neq 1$, otherwise $m \mapsto_! l\langle n_1 \rangle \triangleleft_! L\langle N \rangle$.
- If $m \triangleleft_! (L\langle \lambda x N \rangle P)$, then $m = l\langle \lambda x n \rangle p$ for some $l \triangleleft_! L, n \triangleleft_! N$ and $p \triangleleft_! P$. Then, $m \rightarrow_\delta l\langle n[p/x] \rangle \triangleleft_! L\langle N[P/x] \rangle$.
- If $m \triangleleft_! N[L\langle !P \rangle/x]$ then there are $k \in \mathbb{N}, p_1, \dots, p_k \triangleleft_! P, n \triangleleft_! N, l \triangleleft_! N$ such that $m = n[l\langle [p_1, \dots, p_k]/x \rangle]$. Then $m \rightarrow_\delta \emptyset$ if $d_x(m) \neq k$, and otherwise for any $\sigma \in P_k$, $m \rightarrow_\delta l\langle n\{p_{\sigma(1)}/x_1, \dots, p_{\sigma(k)}/x_k\} \rangle$. ◀

This simulation property can be extended to surface contexts:

► **Lemma 75.** *If $M \rightarrow_{!s} N$, then for any $m \triangleleft_! M$, either $m \rightarrow_{\delta s} \emptyset$ or there is some $n \triangleleft_! N$ such that $m \rightarrow_{\delta s} n$.*

Proof of Lemma 75. Let $M = S\langle M' \rangle$ and $N = S\langle N' \rangle$ with $M \mapsto_! N$; and then $m = s\langle m' \rangle$ for $s \triangleleft_! S$. By induction on surface contexts:

- $S = \square$. Then $M \mapsto_! N$, we apply Lemma 74.
- $S = \lambda x S'$. Then $M = \lambda x S'\langle M' \rangle$ and $N = \lambda x S'\langle N' \rangle$. By induction hypothesis, either $m' \rightarrow_{\delta s} \emptyset$, and then $\lambda x s'\langle m' \rangle \rightarrow_{\delta s} \emptyset$, either we have some $n' \triangleleft_! N'$ such that $m' \rightarrow_{\delta s} n'$, and then $\lambda x s'\langle m' \rangle \rightarrow_\delta \lambda x s'\langle n' \rangle$ by definition of resource surface reduction.

All remainder cases are similar, since for any resource surface context s and term $n \triangleleft_! N$, there exist S such that $s\langle n \rangle \triangleleft_! S\langle N \rangle$ which enables the induction hypothesis. Again, this fails for full contexts. ◀

We can obtain a symmetric property with the same arguments. We could qualify, following Dufour and Mazza [21], the previous lemma as *push forward*, and the next lemmas as *pull back*.

► **Lemma 76.** *If $M \rightarrow_{!s} N$, then for any $n \triangleleft_! N$, there is some $m \triangleleft_! M$ such that $m \rightarrow_{\delta s} n$.*

Proof of Lemma 76. Let $M = S\langle M' \rangle, N = S\langle N' \rangle$ with $M' \mapsto_! N'$ then we can reason by induction on S .

- $S = \square$. Then $M \mapsto_! N$ and by a straightforward case analysis on $\mapsto_!$.
- $S = \lambda x S'$. Then $M = \lambda x S'\langle M' \rangle, N = \lambda x S'\langle N' \rangle$ and $m = \lambda x s'\langle m' \rangle$ with $s' \triangleleft_! S'$. By induction hypothesis there exists $m' \triangleleft_! M'$ such that $m' \rightarrow_\delta n'$ then let $m = \lambda x s'\langle m' \rangle$ we have that $m \rightarrow_{\delta s} n$ by closure.

All the remainder cases are similar. ◀

Proof of Lemma 19. We proceed by induction on the exponential depth where the reduction occurs. By lemma 75, and since $\rightarrow_{\delta s} \subseteq \Rightarrow_\delta$, the only case we need to show is the full context closure, where $M = F\langle M' \rangle = !F'\langle M' \rangle$ and $N = F\langle N' \rangle$.

Then, $m = [p_1, \dots, p_k]$ for some $k \in \mathbb{N}$, where $p_i \triangleleft_! F'\langle M' \rangle$.

By induction hypothesis, either some p_i reduces to \emptyset , and then also $m \Rightarrow_\delta \emptyset$, either for any $i \leq k$, there is some $p'_i \triangleleft_! F'\langle N' \rangle$ such that $p_i \Rightarrow_\delta p'_i$. Then, $m \Rightarrow_\delta [p'_1, \dots, p'_k] \triangleleft_! !F'\langle N' \rangle = N$. ◀

Proof of Lemma 23. By Theorem 21, we immediately have $\text{TNF}(M) \subseteq \text{TNF}(N)$. Then, let $n \in \text{TNF}(N)$. By iteration of Lemma 20, there is some $m \in \mathcal{T}(M)$ such that $m \rightrightarrows_{\delta}^* n$. Then n must be in $\text{TNF}(M)$. ◀

Proof of Lemma 24. Consider $m_0 \triangleleft_! M$ such that $m_0 \rightarrow_{\delta}^* m$. We reason by induction over the exponential depth under which the reduction occurs. If this depth is 0, then $m_0 \rightarrow_{\delta_s}^* m$ and we apply (iteratively) Lemma 15 to conclude.

Otherwise, we use the factorization property : by Proposition 22, we have some n such that $m_0 \rightarrow_{\delta_s}^* n = s\langle \bar{n}_1, \dots, \bar{n}_k \rangle \rightrightarrows_{\delta}^* s\langle \bar{n}'_1, \dots, \bar{n}'_k \rangle = m$. By Lemma 15, we have some $N = S\langle !N_1, \dots, !N_k \rangle$ such that $M \rightarrow_{!s}^* N$ and $n \triangleleft_! N$ (thus $n_i \triangleleft_! N_i$). Then, for any $i \leq k$, $\bar{n}_i = [n_{i,1}, \dots, n_{i,l_i}]$ and $\bar{n}'_i = [n'_{i,1}, \dots, n'_{i,l_i}]$ (in normal form, since these are subterms of m) with $n_{i,j} \triangleleft_! N_i$ and $n_{i,j} \rightarrow_{\delta}^* n'_{i,j}$. This reduction occurs under an exponential (*i.e.* inside a multiset), we then can apply our induction hypothesis to assert that there is some N'_i such that $N_i \rightarrow_! N'_i$ and $n'_{i,j} \triangleleft_! N'_i$. We can conclude, by setting $M' = S\langle !N'_1, \dots, !N'_k \rangle$. ◀

A.1 Proof of Section 2.3

Proof of Lemma 30. We show the lemma holds for one-step reductions, assuming $M \rightarrow_! N$. The closure is easily obtained by induction on the number of steps. By definition, $M = A\{P_1/\perp_1, \dots, P_k/\perp_k\}$, and by Lemma 28, there must be some $j \in \{1, \dots, k\}$ such that $N = A(\{P_i/\perp_i\}_{i \neq j} \{P'_j/\perp_j\})$ with $P_j \rightarrow_! P'_j$. Consequently, we have $A \sqsubseteq N$. ◀

Proof of Lemma 33. From Definition 31 we deduce $\mathcal{A}(N) \subseteq \mathcal{A}(M)$, and from Lemma 30 the other inclusion. ◀

Proof of Lemma 36. The downwards closure is by definition of $\mathcal{A}(M)$. For directedness, let us assume $A_1, A_2 \in \mathcal{A}(M)$, we show that there exists $A_3 \in \mathcal{A}(M)$ such that $A_1 \sqsubseteq A_3$ and $A_2 \sqsubseteq A_3$, by induction on A_1 .

- If $A_1 = x$ then $M \rightarrow_!^* x$ and $A_2 = x$ or \perp (Remark 34), then we set $A_3 = x$.
- If $A_1 = \perp$, by definition of \sqsubseteq we have $A_1 \sqsubseteq A_2$, we set $A_3 = A_2$.
- If $A_1 = \lambda x A'_1$, by Lemma 28 we have $M \rightarrow_!^* \lambda x N$ and $A'_1 \in \mathcal{A}(N)$. Then, by Lemma 33 we have $A_2 \in \mathcal{A}(\lambda x N)$ so either $A_2 = \perp$ and we set $A_3 = A_2$ or $\lambda x A'_2$ so $A'_2 \in \mathcal{A}(N)$. Then, by induction hypothesis, there is some A'_3 such that $A'_1 \sqsubseteq A'_3$ and $A'_2 \sqsubseteq A'_3$. We then set $A_3 = \lambda x A'_3$.
- If $A_1 = !A'_1$ we reason as in the previous case.
- If $A_1 = \mathbf{der}(A'_1)$. Then $M \rightarrow_!^* \mathbf{der}(N)$ and $A'_1 \in \mathcal{A}(N)$. By Lemma 33 we have $A_2 \in \mathcal{A}(\mathbf{der}(N))$ so either $A_2 = \perp$ or $A_2 = \mathbf{der}(A'_2)$ such that $A'_2 \in \mathbf{der}(A_1)$, and therefore we can use our induction hypothesis to obtain an upper bound A'_3 of A'_1 and A'_2 and set $A_3 = \mathbf{der}(A'_3)$, which is indeed an upper bound of $\{A_1, A_2\}$ by contextual closure of the approximation order.
- If $A_1 = A'_1[A''_1/x]$, then $M \rightarrow_!^* N'[N''/x] = N$ with $A'_1 \in \mathcal{A}(N')$ and $A''_1 \in \mathcal{A}(N'')$. A'_1 is not of shape $L(!-)$, then neither is N'' . Again, $A_2 \in \mathcal{A}(N_1)$, then $A_2 = A'_2[A''_2/x]$ with $A'_2 \in \mathcal{A}(N')$ and $A''_2 \in \mathcal{A}(N'')$. By induction hypothesis, we can find A'_3 an upper bound of $\{A'_1, A'_2\}$ and A''_3 an upper bound of $\{A''_1, A''_2\}$. We conclude by setting $A_3 = A'_3[A''_3/x]$.
- If $A_1 = A'_1 A''_1$, we reason similarly, but using the fact that A'_1 cannot be of shape $L(\lambda x -)$. ◀

A.2 Proof of Section 2.4

Proof of Lemma 43. By induction on A . If $A = \perp$ then $\mathcal{T}(A) = \emptyset \subseteq \mathcal{T}(M)$. If $A = x$ then $M = x$ and $\mathcal{T}(A) = \mathcal{T}(M) = \{x\}$. If $A = \lambda x A'$, then $M = \lambda x M'$ with $A' \sqsubseteq M'$. By induction hypothesis, $\mathcal{T}(A') \subseteq \mathcal{T}(M')$. Then, $\mathcal{T}(A) = \{\lambda x a' \mid a' \in \mathcal{T}(A')\} \subseteq \mathcal{T}(M) = \{\lambda x m' \mid m' \in \mathcal{T}(M')\}$. The other cases are treated similarly by routine induction. \blacktriangleleft

Proof of Lemma 44. We have some M' such that $M \rightarrow_!^* M'$ and $A \sqsubseteq M'$. We have that $\mathbf{nf}(\mathcal{T}(M)) = \mathbf{nf}(\mathcal{T}(M'))$ by Lemma 23. By Lemma 43 we have $\mathcal{T}(A) \subseteq \mathcal{T}(M')$. We conclude by observing that terms in $\mathcal{T}(A)$ are in normal form, and that normal terms in $\mathcal{T}(M')$ must also belong to $\mathbf{nf}(\mathcal{T}(M'))$ as they are not affected by any reduction. \blacktriangleleft

Proof of Lemma 45. By induction on m .

- If $m = x$, then $M = x$ and we set $A = x$.
- If $m = \lambda x n$, then $M = \lambda x N$ with $n \triangleleft_! N$. Since n must be in normal form, we can apply the induction hypothesis to obtain some $A' \sqsubseteq N$ such that $n \triangleleft_! A'$. We then set $A = \lambda x A'$.
- If $m = m_1 m_2$, then $M = M_1 M_2$ with $m_i \triangleleft_! M_i$. Again, by induction hypothesis, we have $A_1 \sqsubseteq M_1$ and $A_2 \sqsubseteq M_2$ with $m_i \triangleleft_! A_i$. It remains to show that $A_1 A_2$ belongs to the set of approximants described in Definition 26. Notice that m_1 cannot be of shape $l\langle \lambda x^- \rangle$, then since $m_1 \triangleleft_! A_1$, A_1 cannot be a bottom or an abstraction. A simple examination of the syntax of approximants is enough to conclude that $A_1 A_2$ indeed belongs to it.
- If $m = \mathbf{der}(n)$, then we again obtain $n \triangleleft_! N$ and some $A' \sqsubseteq N$ with $n \triangleleft_! A'$. Since n cannot be of shape $l\langle [-] \rangle$. (since m is normal), we can again check Definition 26 to conclude that $A = \mathbf{der}(A)'$ is an approximant, and that $m \triangleleft_! A$.
- If $m = n[p/x]$, we reason as for the application case, but using the case that p cannot be of shape $l\langle [-] \rangle$.
- If $m = [n_1, \dots, n_k]$, then $M = !N$ with $n_i \triangleleft_! N$ for all $i \in \{1, \dots, k\}$. Then by induction hypothesis, there is $A_i \sqsubseteq N$ with $n_i \triangleleft_! A_i$. We then take $A = !A_i$, which works for any i . \blacktriangleleft

Proof of Theorem 46. We proceed by double inclusion.

- Take $m \in \mathcal{T}(\mathbf{AT}_1 M)$, then there exists some $A_0 \in \mathcal{A}(M)$ such that $m \in \mathcal{T}(A_0)$, by Definition 41. There is M_0 such that $M \rightarrow_!^* M_0$ and $A_0 \sqsubseteq M_0$. We can therefore apply Lemma 44 to conclude that $m \in \mathbf{TNF}(M_0)$, which is equal to $\mathbf{TNF}(M)$ by Lemma 23.
- Assume $m \in \mathbf{TNF}(M)$. By Lemma 24 there exists M_0 such that $M \rightarrow_!^* M_0$ and $m \in \mathcal{T}(M_0)$. By Lemma 45, there is $A \sqsubseteq M_0$ such that $m \in \mathcal{T}(A)$. By definition, $A \in \mathcal{A}(M)$, so we conclude that $m \in \mathcal{T}(\mathbf{AT}_1 M)$. \blacktriangleleft

B Proofs of Section 3

Proof of Lemma 54. For the \mathbf{dCbN} case: Considering a resource term $m \in \delta\mathbf{Bang}$, we can show that $m \triangleleft_n M$ (see Definition 50) if and only if $m \triangleleft_! M^n$ (see Figure 1), by an immediate induction on M .

- x is the only approximation of $x = x^n$.
- $m \triangleleft_n \lambda x N \in \mathbf{dCbN}$ if and only if $m = \lambda x n$ with $n \triangleleft_n N$, if and only if (induction hypothesis) $n \triangleleft_! N^n$, and then if and only if $m \triangleleft_! \lambda x N^n = (\lambda x N)^n$.
- $m \triangleleft_n NP \in \mathbf{dCbN}$ if and only if $m = n[p_1, \dots, p_k]$ with $n \triangleleft_n N, p_i \triangleleft_! P$ for any $i \leq k$ if and only if (induction hypothesis) $n \triangleleft_! N^n$ and $p_i \triangleleft_! P^n$, and then if and only if $m \triangleleft_! (NP)^n$.

- Case $M = N[P/x]$ is similar to the previous one. ◀

For the \mathbf{dCbV} case: By induction on M :

- For any $k \in \mathbb{N}$, $[x]_k \triangleleft_v x$ and $[x]_k \triangleleft_! x^v = !x$.
- For any $k \in \mathbb{N}$, $[\lambda x m_1, \dots, \lambda x m_k] \triangleleft_v \lambda x M$ iff $m_i \triangleleft_v M$ for all i (induction hypothesis) $m_i \triangleleft_! M^v$ iff $[\lambda x m_1, \dots, \lambda x m_k] \triangleleft_! (\lambda x M)^v = !(\lambda x M^v)$.
- $m \triangleleft_v NP$.
 - Either N is an application. Then $m \triangleleft_v NP$ iff $m = \mathbf{der}(n)p$ with $n \triangleleft_v N$ and $p \triangleleft_v P$ iff (induction hypothesis) $n \triangleleft_! N^v$ and $p \triangleleft_! P^v$ iff $\mathbf{der}(n)p \triangleleft_! (NP)^v$.
 - Either $N = !N'$, and then $m \triangleleft_v NP$ iff $m = n'p$ with $n' \triangleleft_v N'$ and $p \triangleleft_v P$ iff $n' \triangleleft_! N'^v$, and $p \triangleleft_! P^v$ iff $n'p \triangleleft_! (NP)^v$.

► **Lemma 77** (Substitution).

1. Let $M, N \in \mathbf{dCbN}$. $M^n\{N^n/x\} = M\{N/x\}^n$.
2. Let $M, N \in \mathbf{dCbV}$. If $N^v = !P$, then $M\{N/x\}^v = M^v\{P/x\}$

Proof. Standard induction on M . This substitution lemma only holds for value in the case of \mathbf{dCbV} , because substituting a term to (the traduction of) a variable necessarily puts it under an exponential. This is not an issue because in \mathbf{dCbV} , these substitutions occur only if it is the case. ◀

Proof of Theorem 58. For $\circ \in \{v, n\}$, the statements of the lemma can be depicted as follows, where the dashed lines and the term P are the ones to be established:

$$\begin{array}{ccc}
 M^\circ & \longrightarrow & N \dashrightarrow^* P^\circ \\
 \uparrow \scriptstyle{()^\circ} & & \uparrow \scriptstyle{()^\circ} \\
 M & \dashrightarrow & P
 \end{array}$$

First, notice that the translations $()^v$ and $()^n$ generate only redexes of shape $L\langle \lambda x M \rangle N$ and $M[L\langle !N \rangle/x]$ ¹².

(1) By induction on the reduction $M^n \rightarrow_! N$.

- $M^n = L^n\langle \lambda x N_1^n \rangle !N_2^n$ and $N = L^n\langle N_1^n [!N_2^n/x] \rangle$. We have $N = (L\langle N_1 [N_2/x] \rangle)^n$ by definition of $()^n$. Since $M = L\langle \lambda x N_1 \rangle N_2$, we have $M \rightarrow_! L\langle N_1 [N_2/x] \rangle$, and we are done, setting $P = L\langle N_1 [N_2/x] \rangle$.
- $M^n = N_1^n [L^n\langle !N_2^n \rangle/x]$ and $N = L^n\langle N_1^n \{N_2^n/x\} \rangle$. By Lemma 77, $N = (L\langle N_1 \{N_2/x\} \rangle)^n$, and again we are done, setting $P = L\langle N_1 \{N_2/x\} \rangle$, since $M = N_1 [L\langle N_2 \rangle/x] \rightarrow_! L\langle N_1 \{N_2/x\} \rangle$.
- The reduction is contextual:
 - $M^n = N_1^n !N_2^n$ and $N = N_1^n !N_2^n$ with $N_1 \rightarrow_n N_1'$. By induction hypothesis, there is some P_1 such that $N_1 \rightarrow_n^* P_1$ and $N_1' \rightarrow_!^* P_1^n$. Then we set $P = P_1 N_2$, and we have indeed $M = N_1 N_1 \rightarrow_n^* P$ and $N = N_1^n !N_2^n \rightarrow^* P_1^n !N_2^n = (P_1 N_2)^n = P^n$.
 - $M^n = N_1^n !N_2^n$ and $N = N_1^n !N_2^n$ with $N_2 \rightarrow_! N_2'$. By induction hypothesis, we have some P_2 such that $N_2 \rightarrow_n^* P_2$ and $N_2' \rightarrow_!^* P_2^n$. We then set $P = N_1 P_2$.
 - $M^n = N_1^n [!N_2^n/x]$, $N = N_1^n [!N_2^n/x]$ with $N_2 \rightarrow_! N_2'$. We reason as in the previous case.

¹²Redexes like $\mathbf{der}(L\langle !N \rangle)$ can however appear during reductions, from translations $()^v$, but not in the translation itself.

- $M^n = \lambda x N_0^n$, $N = \lambda x N'_0$ with $N_0^n \rightarrow! N'_0$. By induction hypothesis, there is P_0 such that $N_0 \rightarrow_n^* P_0$ and $N'_0 \rightarrow!^* P_0^n$. We then set $P = \lambda x P_0$.

(2) By induction on the reduction $M^v \rightarrow! N$. The first two cases are similar to before except the position of the exponential.

- $M^v = L^v \langle \lambda x N_1^v \rangle N_2^v$ and $N = L^v \langle N_1^v [N_2^v/x] \rangle$. We have $N = (L \langle N_1 [N_2/x] \rangle)^v$ by definition of $()^v$, and $M = L \langle \lambda x N_1 \rangle N_2$. We set $P = L \langle N_1 [N_2/x] \rangle$, satisfying $M \rightarrow_v P$ and $N \rightarrow!^0 P^v$.
- $M^v = N_1^v [L^v \langle !N_2^v \rangle /x]$ and $N = L^v \langle N_1^v \{N_2^v/x\} \rangle$, then $M = N_1 [L \langle N'_2 \rangle /x]$ with N'_2 being a variable or an application. By Lemma 77 $N_1 \{N'_2/x\}^v = N_1^v \{N_2^v/x\}$, so $N = (L \langle N_1 \{N_2/x\} \rangle)^v$. We then set $P = L \langle N_1 \{N_2/x\} \rangle$.
- The reduction is contextual. We only detail the case where the reduction occurs in the left member of an application and under a dereliction; the other cases follow from an application of the induction hypothesis as before. The second of these two cases is important, as it is the responsible for the only configuration where P must be distinct from N .
 - $M^v = \mathbf{der}(N_1^v) N_2^v$ (in that case $M = N_1 N_2$ with N_1 not being of shape $L \langle V \rangle$ for any value V) and $N = \mathbf{der}(N'_1) N_2^v$. By induction hypothesis, there is some P_1 such that $N_1 \rightarrow_v^* P_1$ and $N'_1 \rightarrow!^* P_1^v$. We then have two possibilities:
 - * $P_1^v \neq L \langle !- \rangle$ (P_1 is not a value). Then, $(P_1 N_2)^v = \mathbf{der}(P_1^v) N_2^v$. We then set $P = P_1 N_2$, and we have $M = N_1 N_2 \rightarrow_v^* P_1 N_2$ and $N \rightarrow!^* \mathbf{der}(P_1^v) N_2^v = P^v$.
 - * $P_1^v = L^v \langle !Q^v \rangle$. Then, $(P_1 N_2)^v = L^v \langle Q^v \rangle N_2^v$. We have $N \rightarrow!^* \mathbf{der}(L^v \langle !Q^v \rangle) N_2^v \rightarrow! L^v \langle Q^v \rangle N_2^v$ by a single reduction step¹³. We then set $P = L \langle Q \rangle N_2$. It verifies $N \rightarrow!^* P^v$ as we just saw. We also have $M \rightarrow_v^* P$, because $M = N_1 N_2$ and $N_1 \rightarrow_v^* P_1$. Since $P_1^v = L^v \langle !Q^v \rangle$, it follows that Q is a value (either a variable or an abstraction) and that $P_1 = L \langle Q \rangle$, by definition of $()^v$. ◀

Proof of Theorem 59. For this proof we will benefit from the properties of Böhm trees stated in Proposition 38.

(1) We proceed by coinduction on $\mathbf{AT}_n(M)$.

- If $\mathbf{AT}_n(M) = \perp$, then $\mathcal{A}(M) = \{\perp\}$. We need to show that $\mathcal{A}(M^n) = \{\perp\}$. Consider $A \in \mathcal{A}(M^n)$, we have $M^n \rightarrow!^* N$ with $A \sqsubseteq N$. By Lemma 58, we have some P such that $M \rightarrow!^* P$ and $N \rightarrow!^* P^n$. By Lemma 30, $A \sqsubseteq P^n$. Now, observe that P must be some dCbN redex, otherwise $\mathcal{A}(M)$ would contain other approximations than \perp . Then, by simulation (Theorem 49), P^n also is a redex, and since the syntax of approximants contains no redex, necessarily $A = \perp$. We conclude that $\mathbf{AT}_1(M^n) = \perp$.
- If $\mathbf{AT}_n(M) = x$, then $\mathbf{AT}_n(M)^n = x = \mathbf{AT}_1(x) = \mathbf{AT}_1(x^n)$.
- If $\mathbf{AT}_n(M) = \lambda x \mathbf{AT}_n(N)$, then $(\mathbf{AT}_n(M))^n = (\lambda x \mathbf{AT}_n(N))^n = \lambda x (\mathbf{AT}_n(N))^n$ (by definition of $()^n$). By coinduction hypothesis, $(\mathbf{AT}_n(N))^n = \mathbf{AT}_1(N^n)$. Then, $(\mathbf{AT}_n(M))^n = \lambda x \mathbf{AT}_1(N^n) = \mathbf{AT}_1(\lambda x N^n) = \mathbf{AT}_1((\lambda x N)^n) = \mathbf{AT}_1(M^n)$.
- If $\mathbf{AT}_n(M)$ is an application, then it is equal to some $(x) \mathbf{AT}_n(N)$. Then, we have $(\mathbf{AT}_n(M))^n = (x)!(\mathbf{AT}_n(N))^n = (x)!\mathbf{AT}_1(N^n)$, by coinduction hypothesis, which is equal to $\mathbf{AT}_1((x)!N^n) = \mathbf{AT}_1((x)N^n) = \mathbf{AT}_1(M^n)$.
- $\mathbf{AT}_n(M)$ cannot contain any explicit substitution, as they always correspond to redexes in dCbN.

¹³These steps are called *administrative* in Arrial, Guerrieri and Kessner's work [9].

(2) We proceed by coinduction on $\text{AT}_v(M)$.

- If $\text{AT}_v(M) = \perp$, we reason as above, using this time the second item of Lemma 58.
- If $\text{AT}_v(M) = x$, then $(\text{AT}_v(M))^v = x^v = !x = \text{AT}_!(!x) = \text{AT}_!(x^v)$.
- If $\text{AT}_v(M) = \lambda x \text{AT}_v(N)$, then $(\text{AT}_v(M))^v = !(\lambda x (\text{AT}_v(N))^v)$. By coinduction hypothesis, it is equal to $!(\lambda x \text{AT}_!(N^v)) = \text{AT}_!(!(\lambda x N^v)) = \text{AT}_!(\lambda x N^v) = \text{AT}_!(M^v)$.
- If $\text{AT}_v(M)$ is an application, then it must be equal to some $x([\text{AT}_v(N_i)/y_i]_{1 \leq i \leq k} \text{AT}_v(N_0^v))$ (because in this case M reduces to some $L\langle x \rangle N_0$).
Then $(\text{AT}_v(M))^v = x([\text{AT}_v(N_i)^v/y_i]_{1 \leq i \leq k} (\text{AT}_v(N_0^v))^v)$. By coinduction hypothesis, $(\text{AT}_v(N_j))^v = \text{AT}_!(N_j^v)$ for $j \in \{0, \dots, k\}$. Then $(\text{AT}_v(M))^v = x([\text{AT}_!(N_i^v)/y_i]_{1 \leq i \leq k} \text{AT}_!(N_0^v))$. Again, for $i \in \{1, \dots, k\}$, $\text{AT}_!(N_i^v)$ cannot be an exponential, since those explicit substitutions must not be reducible. Then, $(\text{AT}_v(M))^v = \text{AT}_!(x([\text{AT}_!(N_i^v)/x]_{i \in \{1, \dots, k\}} N_0^v)) = \text{AT}_!(M^v)$.
- If $\text{AT}_v(M) = \text{AT}_v(N_1)[\text{AT}_v(N_2)/x]$, then again $\text{AT}_v(N_2)$ cannot be a value, hence $(\text{AT}_v(M))^v = \text{AT}_!(N_1^v)[\text{AT}_!(N_2^v)/x]$ (by coinduction hypothesis) $= \text{AT}_!(N_1^v[N_2^v/x]) = \text{AT}_!(N_1[N_2/x]^v)$. ◀

Proof of Theorem 60. Let $\circ \in \{n, v\}$. We have the following equalities:

$$\begin{aligned}
 \mathbf{nf}(\mathcal{T}^\circ(M)) &= \mathbf{nf}(\mathcal{T}(M^\circ)) && \text{By Lemma 54} \\
 &= \mathcal{T}(BT(M^\circ)) && \text{By Theorem 46} \\
 &= \mathcal{T}((BT_\circ(M))^\circ) && \text{By Theorem 59} \\
 &= \mathcal{T}_\circ(BT_\circ(M)) && \text{By Lemma 54}
 \end{aligned}$$

C Proofs of Section 4

Proof of Theorem 61. We show the contrapositive of the statement: assume that $\text{TNF}(M) = \emptyset$. By definition, for every $m \in \mathcal{T}(M)$, we have $m \rightarrow_\delta^* \emptyset$. Consider now any resource testing context t . We can easily establish that $t\langle m \rangle \rightarrow_\delta^* \emptyset$; since $\emptyset m = \emptyset$, $(\lambda x \emptyset)m = \emptyset$, and then by induction.

For M to be meaningful, there must exist a testing context T such that $T\langle M \rangle \rightarrow_{!s}^k !P$ for some P and $k \in \mathbb{N}$. We have $\llbracket \cdot \rrbracket \triangleleft_! !P$. By iteratively applying Lemma 76, there is some term $s \triangleleft_! T\langle M \rangle$ such that $s \rightarrow_{\delta_s}^k \llbracket \cdot \rrbracket$. By Lemma 9, and because testing contexts are included in surface contexts, we have some $t \triangleleft_! T, m \triangleleft_! M$ such that $s = t\langle m \rangle$.

This leads to a contradiction: $t\langle m \rangle \rightarrow_{\delta_s} \llbracket \cdot \rrbracket$, yet we have shown that $t\langle m \rangle \rightarrow_{\delta_s} \emptyset$ for any t . ◀

Proof of Lemma 65. The following observations suffice:

- terms in \mathbf{dBang}_N containing explicit substitutions are reducible.
- If the leftmost subterm of an application is not a variable, the entire term is reducible. ◀

Proof of Theorem 67. Consider some $p \in \text{TNF}(M)$, assumed non-empty. Then there is some $m \triangleleft_! M$ such that $m \rightarrow_\delta^* p$.

By Lemma 66, $p = \lambda x_1 \dots x_k(x) \bar{n}_1 \dots \bar{n}_l$ for some $k, l \in \mathbb{N}$. Proposition 22 allows us to focus on surface reduction: there are some $m' = \lambda x_1 \dots x_k(x) \bar{n}'_1, \dots, \bar{n}'_l$ such that $m \rightarrow_{\delta_s}^* m' \rightrightarrows_\delta^* p$ (the second part of the reduction acting inside the bags).

Then, by iteratively applying Lemma 15, we obtain $M' \in \mathbf{dBang}$ such that $m' \triangleleft_! M'$ and $M \rightarrow_{!s}^* M'$. By the definition of the approximation relation $\triangleleft_!$, we have $M' = \lambda x_1 \dots x_k(x) !N'_1 \dots !N'_l$ where $\bar{n}'_i \triangleleft_! !N'_i$.

We now define the appropriate testing context $T = ((\lambda x \square)!(\lambda y_1 \dots y_l!z_0))!z_1 \dots!z_k$ where the z_i are chosen distinct from the x_j and y_j .

We observe that $T\langle M' \rangle \rightarrow_{!s} (\lambda x_1 \dots x_k(\lambda y_1 \dots y_l!z_0))!z_1 \dots!z_k!N'_1 \dots!N'_k \rightarrow_{!s}^2 !z_0$. We conclude as follows: since $T\langle M' \rangle \rightarrow_{!s}^2 !z_0$, $M \rightarrow_{!s}^* M'$ and T is a surface context, we have $T\langle M \rangle \rightarrow_{!s}^* !z_0$ by the contextual closure of $\rightarrow_{!s}$. Therefore, M is meaningful. \blacktriangleleft

Proof of Corollary 68. Recall that, by Corollary 55 and Theorem 49, $\text{TNF}(M) \neq \emptyset$ if and only if $\text{TNF}(M^n) \neq \emptyset$, and M is meaningful if and only if M^n is meaningful.

(\rightarrow) Assume M is meaningful, then M^n is also meaningful, and by Theorem 61, $\text{TNF}(M^n) \neq \emptyset$. It follows that $\text{TNF}(M) \neq \emptyset$.

(\leftarrow) If $\text{TNF}(M) \neq \emptyset$, then $\text{TNF}(M^n) \neq \emptyset$. By Lemma 63, $M^n \in \mathbf{dBang}_N$, and Theorem 67 implies that M^n must be meaningful. Therefore, M is also meaningful. \blacktriangleleft

In what follows, the variables in \circ_i are always taken fresh, so as they do not interfere with variables in the terms where the \circ_i are substituted. In particular, we use the fact that for any $i > 1$ and any M , $\circ_k!M \rightarrow_{!}^2 !\circ_{k-1}$.

Proof of Lemma 71. By induction on the syntax B :

- If M is of the form $!x$ or $!\lambda x B$, then we are done since $M\sigma \rightarrow_{!}^0 !P$ for some P and any substitution σ .
- If $M \in B_{!}$, we apply Lemma 78, which guarantees the existence of a substitution σ such that $M\sigma \rightarrow_{!}^* !\circ_j$ for some j .
- If $M = N[P_1/x_1] \dots [P_k/x_k]$, then $N \in B$ and $P_i \in B_{!}$ for all i . Let $\{y_1, \dots, y_l\} = \mathbf{fv}(N) \cup \bigcup_{i \leq k} \mathbf{fv}(P_k)$. By induction hypothesis, we have c such that for any $k_1, \dots, k_l \geq c$, $N\{\circ_{k_1}/y_1, \dots, \circ_{k_l}/y_l\} \rightarrow_{!} !P$ for some P . By Lemma 78, for $i \leq k$, there exist c_i and n_i such that for any $k_{i,1}, \dots, k_{i,l} \geq c_i$, and for some $j_i \geq n_i$, $P_i\{\circ_{k_{i,1}}/y_1, \dots, \circ_{k_{i,l}}/y_l\} \rightarrow_{!}^* !\circ_{j_i}$. Consider then $m_i = \max\{k_i, k_{i,1}, \dots, k_{i,k}\}$ for any $i \leq l$. We then have some P' and some r_i with $N[P_1/x_1] \dots [P_k/x_k]\{\circ_{m_1}/y_1, \dots, \circ_{m_l}/y_l\} \rightarrow_{!}^* !P'[\circ_{r_1}/y_1] \dots [\circ_{r_l}/y_l] \rightarrow_{!}^* !P'\{\circ_{r_1}/y_1\} \dots \{\circ_{r_l}/y_l\}$, which is again a value as required. \blacktriangleleft

► **Lemma 78.** Let $\{x_1, \dots, x_n\}$ be a set of variables and $M \in B_{!}$ (Definition 69) with $\mathbf{fv}(M) \subseteq \{x_1, \dots, x_n\}$. There exist $k, c \in \mathbb{N}$ such that for any $k_1, \dots, k_n \geq c$, there is some $j \geq k$ with $M\{\circ_{k_1}/x_1 \dots \circ_{k_n}/x_n\} \rightarrow_{!}^* !\circ_j$.

Proof. By induction on $B_{!}$:

- $M = L\langle x \rangle N = (x[N_1/y_1] \dots [N_m/y_m])N$ with $N \in B$, $N_i \in B_{!}$ for all $x \leq m$, and $\{x_1, \dots, x_n\} = \mathbf{fv}(M)$. By Lemma 71, there exists k such that $N\{\circ_{k_1}/x_1, \dots, \circ_{k_n}/x_n\} \rightarrow_{!}^* !P$ for any $k_i \geq k$ and some P .

By induction hypothesis, we have for each $i \leq m$, some l_i and c_i such that for any $l_{i,1}, \dots, l_{i,n} \geq l_i$ $N_i\{\circ_{l_{i,1}}/y_1, \dots, \circ_{l_{i,n}}/y_n\} \rightarrow_{!}^* !\circ_{j_i}$ for some $j_i \geq c_i$.

Let n_x be the index of x in $\{x_1, \dots, x_n\}$ (of course, $x \in \mathbf{fv}(M)$).

We then set $r_i = \max\{k_i, l_{i,1}, \dots, l_{i,n}\}$ for each $i \neq n_x$; and we consider r_{n_x} an arbitrary integer greater or equal to $\max\{k_{n_x}, l_{n_x,1}, \dots, l_{n_x,n}\}$.

We find that $M\{\circ_{r_1}/x_1, \dots, \circ_{r_m}/x_m\} \rightarrow_{!}^* \circ_{n_x}[\circ_{r'_1}/x_1] \dots [\circ_{r'_m}/x_m]!P'$, with $r'_i \geq r_i$ for all $i \leq m$. The reduction then yields $\circ_{n_x}!P'$, which reduces immediately to $!\circ_{r_{n_x}-1}$. This concludes the case, as the reduction holds for any $r_{n_x} \geq \max\{k_{n_x}, l_{n_x,1}, \dots, l_{n_x,n}\}$.

- $M = \mathbf{der}(N)N'$ with $N \in B_{!}$, $N' \in B$, and $\{x_1, \dots, x_m\} = \mathbf{fv}(M)$. By induction hypothesis, we have some n, n', c such that for any $n_1, \dots, n_m \geq n$ and $N\{\circ_{n_1}/x_1, \dots, \circ_{n_m}/x_m\} \rightarrow_{!}^* !\circ_j$ for all $j \geq c$, and for any $n'_1, \dots, n'_m \geq n'$, $N'\{\circ_{n'_1}/x_1, \dots, \circ_{n'_m}/x_m\} \rightarrow_{!}^* !P$ for some P .

Then, consider $k_i = \max\{n_i, n'_i\}$ for any $i \leq m$, we have that $M\{\circ_{k_1}/x_1, \dots, \circ_{k_m}/x_m\} \rightarrow_1^* \mathbf{der}(!\circ_j)!P' \rightarrow_1 \circ_j!P' \rightarrow_1^2 !\circ_{j-1}$ (notice that we need here to take $j > 1$, which is allowed by our hypothesis).

- $M = N[P_1/x_1] \dots [P_k/x_k]$. This case is similar to the third case of Lemma 71: the explicit substitutions are removed after an application of the induction hypothesis. ◀

Proof of Theorem 72. Consider some $p \in \text{TNF}(M)$, assumed non-empty. There is some $m \triangleleft_1 M$ such that $m \rightarrow_\delta^* p$. By Lemma 70, p belongs to the syntax b . Proposition 22 ensures that there is some $p' \in b$ such that $m \rightarrow_{\delta_s} p'$ (as in Theorem 67, we focus on internal reduction).

By iteratively applying Lemma 76, we obtain P' such that $M \rightarrow_1^* P'$ and $p' \triangleleft_1 P'$. By the definition of \triangleleft_1 , we also have that $P' \in B$.

Let $\{x_1, \dots, x_k\} = \mathbf{fv}(P')$. Lemma 71 implies that there are some terms N_1, \dots, N_k such that $P'\{N_1/x_1, \dots, N_k/x_k\} \rightarrow_1^* !Q$ for some term Q .

We define the testing context $C = (\lambda x_1 \dots \lambda x_k \square)!N_1 \dots !N_k$, which satisfies $C\langle P' \rangle \rightarrow_1^* !Q$. We can conclude that M is meaningful by the contextuality of reduction, since $C\langle M \rangle \rightarrow_1^* C\langle P' \rangle \rightarrow_1^* !Q$. ◀

Proof of Corollary 73. Recall that, by Corollary 55 and Theorem 49, $\text{TNF}(M) \neq \emptyset$ if and only if $\text{TNF}(M^v) \neq \emptyset$, and M is meaningful if and only if M^v is meaningful.

(\rightarrow) Assume M is meaningful, then M^v is meaningful, and by Theorem 61, $\text{TNF}(M^v) \neq \emptyset$. It follows that $\text{TNF}(M) \neq \emptyset$.

(\leftarrow) If $\text{TNF}(M) \neq \emptyset$, then $\text{TNF}(M^v) \neq \emptyset$. By Lemma 63, $M^v \in \mathbf{dBang}_V$, and Theorem 72 implies that M^v is meaningful. Therefore, M is also meaningful. ◀