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### ► To cite this version:

Beatrice Zambotti, Vincent Andrieu, Daniele Astolfi, Laurent Bako, Madiha Nadri, et al.. Exact Outlier Cancellation in Discrete-Time Observers via Stubborn Redesign. 2026. <hal-05509568>

**HAL Id: hal-05509568**

**<https://hal.science/hal-05509568v1>**

Preprint submitted on 13 Feb 2026

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# Exact Outlier Cancellation in Discrete-Time Observers via Stubborn Redesign

Beatrice Zambotti, Vincent Andrieu, Daniele Astolfi, Laurent Bako, Madiha Nadri, Luca Zaccarian

**Abstract**—This paper addresses the problem of robust state estimation for nonlinear discrete-time systems subject to sporadic high-magnitude measurement outliers. Given a nominal convergent observer, we propose a “stubborn” redesign by saturating the output injection term with a dynamic threshold. In contrast to previous approaches that rely on scalar filters, we introduce a higher order augmentation based on a finite-memory buffer. Our redesign ensures a deadbeat recovery property, where the influence of an outlier on the saturation threshold vanishes exactly after a finite number of steps determined by the system’s observability index. This allows for perfect rejection, at the steady state, of sporadic disturbances enjoying a suitable dwell-time property. In the absence of disturbances, we recover the global convergence properties of the nominal observer. The proposed redesign is illustrated through simulation on a single-link flexible-joint manipulator.

## I. INTRODUCTION

State estimation in nonlinear discrete-time systems can be highly sensitive to measurement signals affected by rare but high-magnitude disturbances. Such events, commonly referred to as outliers, arise in a wide range of practical scenarios, including sensor malfunctions, communication dropouts, electromagnetic interference, or malicious attacks [1]–[3]. Despite their sporadic nature, outliers can have a critical impact in closed-loop operation: a single outlier can instantaneously corrupt the state estimate and propagate through the feedback loop, degrading the performance.

From a signal processing perspective, handling outliers is largely a post-processing task. When data are analyzed offline, batch optimization or smoothing techniques can effectively identify and remove anomalous samples by exploiting non-causal information. Traditionally, outlier rejection is framed within the context of Fault Detection and Identification (FDI) [4]–[6]. In this framework, the objective is to isolate anomalies by comparing measurements with an analytical model to generate residual signals. However, as argued in [7], relying on banks of estimators to test all possible failure

hypotheses becomes intractable when the number of potential corruption patterns is large. Modern approaches reframe this task as a sparse recovery problem. In this context, the objective is to recover the nominal measurement sequence from corrupted data by exploiting the intrinsic sparsity of the outlier vector. Strategies rooted in convex optimization [7]–[11] isolate anomalies over a time window without explicitly considering failure modes. However, these approaches are generally limited to linear systems and operate on data batches, rather than single measurements. To extend these robust capabilities to the online control of nonlinear systems, strategies such as Moving Horizon Estimation (MHE) formulate the estimation problem over a finite sliding window, enabling the explicit handling of nonlinearities and constraints [12], [13]. Recent extensions incorporate robust Bayesian inference and divergence measures to weight data according to their reliability [14]. However, this increased modeling flexibility comes at a high computational cost: solving a numerical optimization problem recursively introduces latency that is often incompatible with fast feedback loops, which instead demand low-complexity, fully recursive algorithms.

Within the class of recursive estimators, however, the existing literature on robust observers is quite misaligned with the problem of rejecting sporadic measurement outliers. Most approaches are designed to attenuate persistent disturbances or stochastic noise and are therefore ill-suited to handle rare, high-magnitude corruptions. The Extended Kalman Filter (EKF) [15]–[18], for instance, represents the gold standard for handling Gaussian noise. However, as it relies on accurate stochastic models of process and measurement noise, the presence of high-amplitude outliers significantly degrades the filter’s performance. Similarly,  $\mathcal{H}_\infty$  approaches [19], [20] focus on worst-case attenuation, and are therefore inherently conservative for sporadic events. On the deterministic side, most approaches focus exclusively on nominal performance. For instance, Deadbeat estimators [21], [22], Kazantzis-Kravaris-Luenberger (KKL) [23], High-Gain observers [24], Luenberger-like designs [25], and standard LMI-based designs [26], [27] target finite-time, arbitrarily fast, or global asymptotic convergence. Furthermore, specific observer redesigns have been proposed to address domain limitations, for instance, by enforcing state constraints [28]. These methods lack explicit mechanisms for rejecting anomalous measurements and, as a result, high-magnitude outliers are directly injected into the observer dynamics, leading to large estimation errors.

In this paper, we propose a systematic method for redesign-

Work funded by the ANR under grant OLYMPIA ANR-23-CE48-0006.

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ing an existing observer to enhance its robustness against sporadic outliers. This approach builds upon the “stubborn” redesign paradigm, introduced in [29]–[31], which limits the influence of inconsistent measurements by saturating the innovation term. This mechanism allows the observer to adaptively adjust the saturation level in response to detected anomalies, improving robustness while preserving the stability properties. Existing designs employ dynamic saturation thresholds governed by scalar update laws, ensuring asymptotic recovery once disturbances vanish [29]–[33]. While the continuous-time framework has been extensively studied for both linear [30] and nonlinear systems [31], [33], [34], the discrete-time counterpart has received less attention. Early works in the linear discrete-time setting rely on LMI-based designs [30], but often result in complex design procedures without feasibility guarantees. More recently, we revisited the linear discrete-time setting showing that standard detectability is sufficient to ensure global exponential stability in a noise-free scenario [32].

Nevertheless, these stubborn designs suffer from an intrinsic drawback: the scalar filter imposes an exponential recovery, which keeps the saturation level unnecessarily large even after an outlier has disappeared, delaying the recovery of the nominal performance. In this paper, we overcome this limitation by replacing the scalar filter with a finite-memory vector buffer of past output error energies. By employing such augmentation, we achieve two main advantages. First, the observer exhibits a deadbeat recovery property. Due to the nilpotent structure of the memory dynamics, the influence of a sporadic outlier on the threshold is completely eliminated after a fixed number of steps defined by the system’s observability index. This allows us to rigorously prove exact rejection of outliers at the steady state, provided the dwell time between disturbances is sufficiently large. When the dwell-time condition is not satisfied, we establish Input-to-State Stability (ISS) with respect to the measurement disturbances, ensuring bounded estimation errors despite repeated corruptions. Second, the stability analysis leverages the finite memory window to establish a strict dominance of the threshold over the estimation error. This allows us to prove finite-time desaturation and Global Exponential Stability (GES) using a composite Lyapunov function, providing constructive tuning guidelines for the saturation parameters.

This paper is organized as follows. Section II introduces the problem formulation and states the main result in Theorem 1. Section III, with Theorem 2, formalizes the exact outlier cancellation property. The proof of Theorem 1 is then developed in two parts: the unperturbed case, including the proof of GES, is addressed in Section IV, while the perturbed case, with the proof of ISS, is presented in Section V. Finally, in Section VI, we provide a numerical example that illustrates the effectiveness of the proposed stubborn observer redesign.

**Notation.** Given a vector  $v$ ,  $|v|$  denotes its Euclidean norm and  $|v|_1$  denotes its 1-norm. For a matrix  $M$ ,  $\|M\|$  denotes the induced Euclidean norm (spectral norm), while  $\|M\|_\infty$  denotes its induced infinity norm (maximum absolute row sum). The notation  $P > 0$  indicates that  $P$  is a symmetric positive definite matrix. Given a vector field  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , we denote recursively  $f^{i+1}(x) := f(f^i(x))$  with  $f^0(x) := x$ .

## II. PROBLEM STATEMENT

### A. System description and nominal observer

Consider a nonlinear discrete-time system of the form

$$\begin{aligned} x_{k+1} &= f(x_k) \\ y_k &= h(x_k) + d_k \end{aligned} \quad (1)$$

where  $k \in \mathbb{Z}_{\geq 0}$  is the discrete-time index,  $x_k \in \mathbb{R}^n$  is the state,  $y_k \in \mathbb{R}^m$  is the output, and  $d_k \in \mathbb{R}^{n_d}$  are external disturbances. For system (1), we require the following structural assumption, which establishes Lipschitz regularity and a uniform observability property.

*Assumption 1:* Functions  $f, h$  are globally Lipschitz (with constants  $\ell_f, \ell_h$ ). Furthermore, there exist an integer  $\tau > 0$  (called observability index) and a function  $\gamma \in \mathcal{K}_\infty$  (called observability modulus) such that

$$\sum_{i=0}^{\tau-1} |h(f^i(x_a)) - h(f^i(x_b))| \geq \gamma(|x_a - x_b|), \quad \forall (x_a, x_b) \in \mathbb{R}^{2n}, \quad (2)$$

where  $f^{i+1}(x) := f(f^i(x))$  for all  $i \in \mathbb{N}$  and  $f^0(x) = x$ .

*Remark 1:* Condition (2) guarantees that any two distinct initial states generate output trajectories that are distinguishable over a finite horizon of length  $\tau$ . As a result, the system is observable in an incremental sense: differences in the state are reflected in accumulated differences of the resulting outputs.

This condition can be interpreted as a nonlinear generalization of finite-horizon observability in linear systems. In the linear case, the observability Gramian provides a quadratic lower bound on the output trajectories generated by a nonzero initial state mismatch, with its minimum eigenvalue quantifying the degree of observability [35]. Here, the observability modulus  $\gamma$  plays an analogous role by providing a nonlinear lower bound on the output trajectories mismatch, without relying on linearity or a quadratic structure.  $\circ$

The stubborn augmentation that we propose is based on the existence of a generic nominal observer of the form

$$\begin{aligned} \hat{x}_{k+1} &= f(\hat{x}_k) + g(\hat{x}_k, y_k - \hat{y}_k) =: \hat{f}(\hat{x}_k, \tilde{y}_k + d_k) \\ \hat{y}_k &= h(\hat{x}_k), \quad \tilde{y}_k = h(x_k) - h(\hat{x}_k), \end{aligned} \quad (3)$$

enjoying the following regularity property.

*Assumption 2:* The function  $g : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  is globally Lipschitz with constants  $\ell_{g_x} > 0, \ell_{g_y} > 0$ , i.e.

$$|g(x, y) - g(x', y')| \leq \ell_{g_x} |x - x'| + \ell_{g_y} |y - y'|, \quad (4)$$

for all  $x, x' \in \mathbb{R}^n$ , and for all  $y, y' \in \mathbb{R}^m$ . Moreover,  $g(x, 0) = 0$  for all  $x \in \mathbb{R}^n$ .

*Remark 2:* The assumption  $g(x, 0) = 0$  guarantees that the nonlinearity does not introduce any bias when the second argument is zero, ensuring that the origin is a valid equilibrium point for the error dynamics and preserving the forward invariance of the zero-error set. Combined with (4), this assumption guarantees that

$$|g(x, y)| \leq \ell_{g_y} |y|, \quad \forall x \in \mathbb{R}^n, \quad \forall y \in \mathbb{R}^m. \quad (5)$$

We assume that observer (3) has been designed to ensure exponential stability of the origin for the estimation error

dynamics in the absence of disturbances (namely  $d \equiv 0$ ). We assume that exponential stability is certified by a Lyapunov function enjoying a specific regularity condition with respect to the measurement disturbances  $d$ . This property quantifies the sensitivity of the estimation error to perturbations and is essential for establishing Input-to-State Stability (ISS) of the error dynamics.

*Assumption 3:* There exists a function  $U : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  and positive constants  $\underline{u}, \bar{u}, c_u$  such that:

- For all  $x, \hat{x} \in \mathbb{R}^n$ :

$$\underline{u}|x - \hat{x}|^2 \leq U(x, x - \hat{x}) \leq \bar{u}|x - \hat{x}|^2. \quad (6a)$$

- For all  $x, \hat{x} \in \mathbb{R}^n$ :

$$\begin{aligned} U(f(x), f(x) - \hat{f}(\hat{x}, h(x) - h(\hat{x}))) - U(x, x - \hat{x}) \\ \leq -c_u U(x, x - \hat{x}). \end{aligned} \quad (6b)$$

- For all  $\varepsilon > 0$ , there exists a function  $\gamma_\varepsilon \in \mathcal{K}_\infty$  such that for all  $x, e, \delta \in \mathbb{R}^n$ :

$$|U(x, e + \delta) - U(x, e)| \leq \varepsilon|e|^2 + \gamma_\varepsilon(|\delta|). \quad (6c)$$

*Remark 3:* Assumption 3 defines the requirements for the nominal Lyapunov function  $U$  to ensure compatibility with the stubborn augmentation. While properties (6a) and (6b) are standard for certifying uniform Global Exponential Stability (GES) of the set where  $x = \hat{x}$  for the ideal unperturbed dynamics, condition (6c) characterizes the sensitivity of  $U$  with respect to a generic additive mismatch  $\delta$ . This term may represent measurement noise, or the structural discrepancy induced by the saturation in the stubborn augmentation. Specifically, (6c) ensures that the variation in the Lyapunov increment can be partially dominated by the nominal decay rate through an appropriate choice of  $\varepsilon$ .  $\circ$

## B. Stubborn redesign and main result

Building upon the nominal observer (3), we introduce a redesign paradigm to enhance its robustness against sporadic disturbances. Our redesign incorporates a dynamic saturation mechanism in the output injection term, making the observer stubborn in the sense of [30]–[32]. Unlike previous approaches relying on scalar infinite-memory filters [32], we propose here a finite-memory architecture. The saturation threshold is adjusted based on a vector state  $\varsigma_k$  acting as a finite memory buffer of the past output error magnitudes.

To this end, consider the standard decentralized symmetric scalar saturation function with limit  $\Delta \in \mathbb{R}_{\geq 0}$ , namely

$$\sigma_\Delta(\tilde{y}_i) := \max\{-\Delta, \min\{\Delta, \tilde{y}_i\}\}, \quad \Delta \in \mathbb{R}_{\geq 0}, \quad (7)$$

acting on each component  $\tilde{y}_i$  of  $\tilde{y}$ . Then, introduce the following notation for the vector saturation function  $\text{sat}_\theta$ , parametrized by the scalar parameter  $\theta \in \mathbb{R}_{> 0}$ ,

$$\begin{aligned} \text{sat}_\theta(\tilde{y}; \varsigma) &:= [\sigma_{\Delta(\varsigma)}(\tilde{y}_1) \cdots \sigma_{\Delta(\varsigma)}(\tilde{y}_m)]^\top \\ \Delta(\varsigma) &:= \theta\gamma^{-1}(\theta|\varsigma|_1) \end{aligned} \quad (8)$$

where  $\Delta \in \mathbb{R}_{\geq 0}$  denotes the adaptive saturation threshold and  $\gamma \in \mathcal{K}_\infty$  is the function introduced in Assumption 1. For compactness of notation, from now on, we omit the

dependence of  $\Delta$  on  $\varsigma$ . The saturation operator defined above acts component-wise on the output error  $\tilde{y}$  and allows us to define the finite-memory stubborn redesign of the observer (3) as:

$$\begin{aligned} \hat{x}_{k+1} &= f(\hat{x}_k) + g(\hat{x}_k, \text{sat}_\theta(\tilde{y}_k + d_k; \varsigma_k)), \\ \varsigma_{k+1} &= S\varsigma_k + G|\tilde{y}_k + d_k|, \quad \varsigma_k \in \mathbb{R}_{\geq 0}^\tau \\ \tilde{y}_k &= h(x_k) - h(\hat{x}_k) \end{aligned} \quad (9)$$

where the pair  $(S, G)$  defines the memory buffer of dimension  $\tau$  ( $\tau$  is the observability index from Assumption 1):

$$S = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \quad (10)$$

Note that this is a shift matrix, therefore the state  $\varsigma_k$  stores the history of the output error norms over the window  $\{k - \tau, \dots, k - 1\}$ , specifically  $\varsigma_k = [|\tilde{y}_{k-1} + d_{k-1}|, \dots, |\tilde{y}_{k-\tau} + d_{k-\tau}|]^\top$ . The stubborn observer (9) explicitly accounts for the system nonlinearity through the inverse function  $\gamma^{-1} \in \mathcal{K}_\infty$  and the parameter  $\theta \in \mathbb{R}_{\geq 0}$ , which shapes the saturation threshold.

We present the main stability properties of the proposed stubborn observer. The following theorem first establishes Global Exponential Stability (GES) of the origin for the estimation error dynamics in the unperturbed scenario (namely  $d = 0$ ). Furthermore, it demonstrates that under specific conditions, the observer achieves Input-to-State Stability (ISS) with respect to the measurement disturbances  $d$ .

*Theorem 1:* Consider the coupled system (1)–(9). Suppose that Assumptions 1, 2 and 3 hold. Then, there exists a parameter  $\theta^* > 0$  such that for all  $\theta > \theta^*$ :

- 1) In the absence of disturbances ( $d \equiv 0$ ), the set  $\mathcal{E} := \{(x, \hat{x}, \varsigma) \in \mathbb{R}^{2n+\tau} : x = \hat{x}, \varsigma = 0\}$  is Globally Exponentially Stable (GES).
- 2) In the case where  $\gamma(s) = c_\gamma s$ , for some scalar  $c_\gamma > 0$ , the error system is Input-to-State Stable (ISS) with respect to the disturbance  $d$ .

*Remark 4:* (Practical Tuning and Performance Trade-offs) The explicit evaluation of the lower bound  $\theta^*$  in Theorem 1 relies on the knowledge of the observability modulus  $\gamma$  and on the system's Lipschitz constants (Assumption 1), which are often uncertain or difficult to estimate in practice. Moreover, since  $\theta^*$  is derived via a worst-case Lyapunov analysis, the resulting theoretical bound is generally conservative.

In practice, when a linear observability modulus is assumed (i.e., Item 2 of Theorem 1 with  $\gamma(s) = c_\gamma s$ ), the explicit knowledge of  $\gamma$  is not required. In this scenario, we may implement the threshold as  $\Delta_k = \Theta|\varsigma_k|_1$ , where, the scalar gain  $\Theta > 0$  combines the effect of  $\theta$  and  $c_\gamma$  into a single design parameter to be tuned, similar to the gain proposed in [31, Equation 10]. The tuning of  $\Theta$  reflects an intrinsic trade-off of the proposed approach: large values of  $\Theta$  recover the fast convergence rate of the nominal observer but reduce the filtering capabilities; conversely, smaller values of  $\Theta$  enhance robustness against outliers and impulsive noise, at the cost of a slower convergence. Finally, we note that, as  $\Theta \rightarrow 0$ , the

threshold  $\Delta_k \rightarrow 0$ , which causes the observer to degenerate towards an open-loop predictor.  $\circ$

### III. EXACT OUTLIER CANCELLATION

Beyond the stability properties established in Theorem 1, the proposed finite-memory architecture offers a decisive performance advantage: the ability to perfectly reject sporadic outliers and recover the exact steady state in finite time. This “deadbeat” behavior is structurally guaranteed by the nilpotent dynamics of the memory buffer, regardless of the system’s nonlinearity.

#### A. Sporadic Disturbance Model

We consider the scenario where the measurement disturbance  $d_k$  in (1) represents sporadic high-magnitude outliers.

*Definition 1 (Sporadic Disturbance with Dwell Time):*

A disturbance sequence  $d = \{d_k\}_{k \in \mathbb{N}}$  is called *sporadic with dwell time*  $T_{\text{dwell}}$  if there exists a set of time instants  $\mathcal{K} = \{k_1, k_2, \dots\}$  such that:

- 1)  $d_k = 0$  for all  $k \notin \mathcal{K}$ ;
- 2)  $k_{i+1} - k_i > T_{\text{dwell}}$  for all  $i \geq 1$ .

This definition implies that the outliers occur at times  $k \in \mathcal{K}$  and are separated by at least  $T_{\text{dwell}}$  steps of clean measurements.

#### B. Exact Rejection Property

The following theorem establishes that if the system is at the steady state, a sporadic outlier has zero effect on the state estimation, provided that the memory size is small enough to flush the disturbance before the next occurrence.

*Theorem 2 (Perfect Rejection and Recovery):* Consider the nonlinear system (1) and the stubborn observer (9). Assume Assumptions 1, 2 and 3 hold and that the observer is at the steady state at time  $k = 0$  (i.e.,  $\hat{x}_0 = x_0$  and  $\varsigma_0 = 0$ ). Let  $\tau \in \mathbb{R}_{>0}$  be the system’s observability index. If the disturbance  $d_k$  is sporadic with a dwell time  $T_{\text{dwell}} > \tau$ , then exact rejection is achieved, namely

$$\hat{x}_k = x_k, \quad \forall k \geq 0. \quad (11)$$

*Remark 5:* Theorem 2 highlights a key advantage of the finite-memory architecture compared to asymptotically decaying filters. While scalar filters would asymptotically decay, theoretically maintaining a non-zero threshold indefinitely, the proposed design ensures that the influence of an outlier is completely eliminated from the observer dynamics after exactly  $\tau$  steps.  $\circ$

*Proof.* We proceed by induction. Assume that the system is at the steady state at time  $k_i \in \mathcal{K}$  (an instant where an outlier occurs), i.e.,  $\hat{x}_{k_i} = x_{k_i}$  and  $\varsigma_{k_i} = 0$ . By splitting the analysis into four time windows, we show next that the estimation error remains zero until the next outlier  $k_{i+1} \in \mathcal{K}$  and that  $\varsigma_{k_{i+1}} = 0$ .

*a) Time window 1: rejection at impact (time  $k_i$ )* Since the memory is empty ( $\varsigma_{k_i} = 0$ ), the threshold is  $\Delta_{k_i} = 0$ . Consequently, the saturation blocks the disturbance completely:  $\text{sat}_\theta(\tilde{y}_{k_i} + d_{k_i}; 0) = 0$ . The observer update becomes:

$$\hat{x}_{k_i+1} = f(\hat{x}_{k_i}) + g(\hat{x}_{k_i}, 0). \quad (12)$$

Using Assumption 2 ( $g(\cdot, 0) = 0$ ) and the fact that  $\hat{x}_{k_i} = x_{k_i}$ , we have:

$$\hat{x}_{k_i+1} = f(x_{k_i}) = x_{k_i+1}. \quad (13)$$

Thus, the estimation error remains exactly zero.

*b) Time window 2: memory update (times  $k_i + 1$  to  $k_i + \tau$ ).* The outlier magnitude  $|d_{k_i}|$  enters the memory vector via the update  $\varsigma_{k_i+1} = G|d_{k_i}|$ . This results in a positive threshold  $\Delta > 0$  for the subsequent steps. However, due to the dwell time condition ( $T_{\text{dwell}} > \tau$ ), we have  $d_{k_i+j} = 0$  for all  $j \in \{1, \dots, \tau\}$ . Since the state estimate is exact ( $\hat{x} = x$ ), the output prediction error is null:  $\tilde{y}_{k_i+j} = h(x_{k_i+j}) - h(\hat{x}_{k_i+j}) = 0$ . Consequently, the input to the correction term is zero, and  $g(\hat{x}_{k_i+j}, \text{sat}_\theta(0; \sigma_{k_i+j})) = 0$ , regardless of the threshold magnitude. It follows that the estimation error remains exactly at zero.

*c) Time window 3: deadbeat recovery (time  $k_i + \tau + 1$ ).* The memory state  $\varsigma$  evolves according to a linear nilpotent dynamic driven by the history of the output errors and disturbances. At step  $k = k_i + \tau + 1$ , the state is equal to

$$\varsigma_k = S^\tau \varsigma_{k_i+1} + \sum_{m=0}^{\tau-1} S^{\tau-1-m} G \underbrace{|\tilde{y}_{k_i+m+1} + d_{k_i+m+1}|}_0.$$

Since  $S^\tau = 0$  and the driving term is zero during the interval, we obtain  $\varsigma_k = 0$ .

*d) Time window 4: steady state (times  $k_i + \tau + 1$  to  $k_{i+1}$ ).* At step  $k_i + \tau + 1$ , the system has returned to the exact steady state ( $\hat{x} = x, \varsigma = 0$ ). Since the next outlier cannot occur before  $k_i + T_{\text{dwell}} + 1 > k_i + \tau + 1$ , the system is fully reset and ready to reject the next disturbance perfectly.  $\blacksquare$

## IV. ANALYSIS OF THE UNPERTURBED CASE (PROOF OF THEOREM 1, ITEM 1)

In this section, we analyze the behavior of the proposed nonlinear stubborn observer (9) in the disturbance-free scenario ( $d \equiv 0$ ). Our primary objective is to show that the introduction of the adaptive saturation mechanism preserves the Global Exponential Stability (GES) of the origin for the estimation error dynamics established in Assumption 3.

We consider system (1) and observer (9) under disturbance-free conditions, i.e., with  $d = 0$  for all  $k$ . Defining the error  $e = x - \hat{x}$ , we may write the error dynamics as:

$$\begin{aligned} e_{k+1} &= f(x_k) - \hat{f}(\hat{x}_k, \text{sat}_\theta(\tilde{y}_k; \varsigma_k)) \\ &=: \tilde{f}(x_k, \hat{x}_k, \text{sat}_\theta(\tilde{y}_k; \varsigma_k)) \\ \varsigma_{k+1} &= S\varsigma_k + G|h(x_k) - h(\hat{x}_k)|, \quad \varsigma_k \in \mathbb{R}_{\geq 0}^\tau. \end{aligned} \quad (14)$$

The main result for this scenario corresponds to Item 1 of Theorem 1. To address the finite-memory architecture, we extend the analysis of the scalar case [32] by constructing a composite Lyapunov function that incorporates a quadratic term for the vector memory state  $\varsigma$ . Specifically, we exploit the nilpotent dynamics of  $\varsigma$  to establish a dominance condition of the threshold over the estimation error. This allows us to prove that the saturation becomes inactive in finite time, thereby recovering the nominal exponential convergence properties.

## A. Structural Properties

To facilitate the proof of the main result, we introduce three technical results that establish fundamental structural properties of the dynamics (14), whose proofs are given in Section IV-C to avoid breaking the flow of the exposition. Importantly, these properties depend primarily on the system  $(f, h)$  and the nominal correction term of the observer  $(g)$ , and are independent of the stubborn tuning parameter  $\theta$ . Lemma 1 provides a recursive bound on the error dynamics, while Lemma 2 leverages the finite-memory structure of  $\varsigma$  to show that the threshold variable dominates the estimation error after  $\tau$  steps.

*Lemma 1:* Consider the unperturbed error dynamics (14) and suppose that Assumptions 1 and 2 hold. Then, there exists a constant  $\ell_e \geq 0$ , independent of parameter  $\theta$ , such that, for any initial condition  $(x_0, \hat{x}_0, \varsigma_0) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_{\geq 0}^\tau$ , the corresponding solution satisfies:

$$|e_{k+1}| \leq \ell_e |e_k|, \quad \forall k \geq 0. \quad (15)$$

*Lemma 2 (Observability and History Dominance):* Let Assumptions 1 and 2 hold. Then, there exist an integer  $\tau > 0$  and a constant  $b_\tau > 0$  such that, for any trajectory of the unperturbed error dynamics (1), (14):

- 1) The accumulated output error satisfies

$$\sum_{i=0}^{\tau-1} |h(x_{k+i}) - h(\hat{x}_{k+i})| \geq b_\tau \gamma (|x_k - \hat{x}_k|), \quad \forall k \geq 0. \quad (16)$$

- 2) For all  $k \geq \tau$ , the memory state satisfies

$$|\varsigma_k|_1 \geq b_\tau \gamma (\ell_e^{-\tau} |e_k|). \quad (17)$$

## B. Lyapunov analysis and main proof

In this section, we establish global exponential stability of the set  $\mathcal{E}$ , defined in Item 1 of Theorem 1 for the coupled system (1)-(9), using a composite Lyapunov function. This function includes the estimation error energy  $U(x, e)$  and a quadratic form of the memory state  $\varsigma$ , weighted by a scalar  $\rho$  to manage the coupling terms. The following Proposition shows the decay properties of such Lyapunov function, which will next be used to prove Item 1 of Theorem 1.

*Proposition 1:* Suppose that Assumptions 1, 2 and 3 hold. Then, there exist a symmetric positive definite matrix  $P \in \mathbb{R}^{\tau \times \tau}$  and scalars  $\theta^* > 0, \rho > 0$  such that for all  $\theta > \theta^*$ , the function:

$$V(x, e, \varsigma) := U(x, e) + \rho \varsigma^\top P \varsigma \quad (18)$$

satisfies the following properties:

- 1) There exist  $\underline{v}, \bar{v} > 0$  such that:

$$\underline{v} |(e, \varsigma)|^2 \leq V(x, e, \varsigma) \leq \bar{v} |(e, \varsigma)|^2. \quad (19)$$

- 2) There exists  $c_v > 0$  such that for any trajectory of the unperturbed error dynamics (1), (14), the inequality

$$\begin{aligned} V(x_{k+1}, e_{k+1}, \varsigma_{k+1}) - V(x_k, e_k, \varsigma_k) \\ \leq -c_v V(x_k, e_k, \varsigma_k) \end{aligned} \quad (20)$$

holds for all  $k \geq \tau$ .

*Proof of Proposition 1. Step 1: Finite-time Desaturation.* Leveraging the History Dominance Lemma (Lemma 2), we show that the saturation becomes inactive after an initial transient of length  $\tau$ . Consider the saturation threshold defined in (8) as  $\Delta_k := \theta \gamma^{-1} (\theta |\varsigma_k|_1)$ . By definition,  $\text{sat}_\theta(\tilde{y}_k; \varsigma_k) = \tilde{y}_k$  provided that  $\Delta_k \geq |\tilde{y}_k|_\infty$ . From the Lipschitz continuity of  $h$  given in Assumption 1, a sufficient condition for desaturation is  $\Delta_k \geq \ell_h |e_k| \geq |\tilde{y}_k| \geq |\tilde{y}_k|_\infty$ .

For any  $k \geq \tau$ , equation (17) in Lemma 2 yields, for all  $\theta > 0$ ,

$$\theta |\varsigma_k|_1 \geq \theta b_\tau \gamma (\ell_e^{-\tau} |e_k|). \quad (21)$$

Assume  $\theta$  is chosen sufficiently large so that  $\theta b_\tau \geq 1$ . Then, it holds that  $\theta |\varsigma_k|_1 \geq \gamma (\ell_e^{-\tau} |e_k|)$ . Since  $\gamma^{-1} \in \mathcal{K}_\infty$ , we can apply it to both sides, which gives

$$\gamma^{-1} (\theta |\varsigma_k|_1) \geq \ell_e^{-\tau} |e_k|. \quad (22)$$

Finally, multiplying by  $\theta$ , we obtain the lower bound for the threshold

$$\Delta_k \geq \theta \ell_e^{-\tau} |e_k|. \quad (23)$$

Consequently, by defining  $\theta^* := \max\{b_\tau^{-1}, \ell_h \ell_e^\tau\}$ , we ensure that, for all  $\theta \geq \theta^*$ , condition  $\Delta_k \geq \ell_h |e_k| \geq |\tilde{y}_k|_\infty$  is satisfied for all  $k \geq \tau$ , and thus:

$$\text{sat}_\theta(\tilde{y}_k; \varsigma_k) = \tilde{y}_k, \quad \forall k \geq \tau. \quad (24)$$

*Step 2: Lyapunov Decrease.* Since the shift matrix  $S$  has all eigenvalues at zero, it is Schur stable. Therefore, for any positive definite matrix  $Q > 0$ , there exists a unique positive definite matrix  $P > 0$  satisfying the discrete Lyapunov equation

$$S^\top P S - P = -I. \quad (25)$$

For  $k \geq \tau$ , the saturation is inactive by virtue of (24). Consequently, the error system evolves nominally, and the memory evolves as  $\varsigma_{k+1} = S \varsigma_k + G |\tilde{y}_k|$ . Evaluating the increment  $\Delta V = V_{k+1} - V_k$  under (24), and using Assumption 3 for the  $U$ -component, yields, for all  $k \geq \tau$ ,

$$\Delta V \leq -c_u U(x_k, e_k) + \rho (\varsigma_{k+1}^\top P \varsigma_{k+1} - \varsigma_k^\top P \varsigma_k). \quad (26)$$

Substituting the dynamics of  $\varsigma$  and expanding the quadratic term using (25), we obtain

$$\begin{aligned} \varsigma_{k+1}^\top P \varsigma_{k+1} - \varsigma_k^\top P \varsigma_k &= -\varsigma_k^\top S_k \\ &+ 2\varsigma_k^\top S^\top P G |\tilde{y}_k| + G^\top P G |\tilde{y}_k|^2. \end{aligned} \quad (27)$$

Using Young's inequality on the cross term, for any  $\varepsilon > 0$ , yields

$$2\varsigma_k^\top S^\top P G |\tilde{y}_k| \leq \varepsilon |\varsigma_k|^2 + \frac{1}{\varepsilon} |S^\top P G|^2 |\tilde{y}_k|^2. \quad (28)$$

By choosing  $\varepsilon$  such that  $0 < \varepsilon < 1$ , the term  $-\varsigma_k^\top S_k$  dominates  $\varepsilon |\varsigma_k|^2$ . Specifically, since  $P > 0$ , there exists  $c_\sigma > 0$  such that:

$$-\varsigma_k^\top S_k + \varepsilon |\varsigma_k|^2 \leq -c_\sigma \varsigma_k^\top P \varsigma_k. \quad (29)$$

Combining these bounds into the expression of  $\Delta V$ , we obtain

$$\Delta V \leq -c_u U(x_k, e_k) - \rho c_\sigma \varsigma_k^\top P \varsigma_k + \rho K_P |\tilde{y}_k|^2, \quad \forall k \geq \tau, \quad (30)$$

where  $K_P := G^\top P G + \frac{1}{\varepsilon} |S^\top P G|^2$ . From the Lipschitz property of  $h$  in Assumption 2 and the lower bound of  $U$

in (6a), we have  $|\tilde{y}_k|^2 \leq \ell_h^2 |e_k|^2 \leq \frac{\ell_h^2}{\underline{u}} U(x_k, e_k)$ . Letting  $K_u := \frac{\ell_h^2}{\underline{u}}$ , we obtain

$$\Delta V \leq -(c_u - \rho K_P K_u) U(x_k, e_k) - \rho c_\sigma s_k^\top P s_k, \quad \forall k \geq \tau. \quad (31)$$

Finally, by choosing  $\rho < \frac{c_u}{K_P K_u}$ , and defining  $c_v := \min\{c_u - \rho K_P K_u, c_\sigma\}$ , we conclude:

$$\Delta V \leq -c_v V(x_k, e_k, s_k), \quad \forall k \geq \tau, \quad (32)$$

which proves the exponential decay for  $k \geq \tau$ . ■

*Proof of Theorem 1, Item 1.* Let  $\chi_k := (x_k, e_k, s_k)$  be a trajectory of system (1), (14). From Proposition 1, the Lyapunov function decreases exponentially after  $k = \tau$  steps. Specifically:

$$V(\chi_k) \leq (1 - c_v)^{k-\tau} V(\chi_\tau), \quad \forall k \geq \tau. \quad (33)$$

Given the definition of the function  $V$  in (18) and exploiting the quadratic upper bound for  $U$  from (6a), we have, for any  $k \leq \tau$ ,

$$V(\chi_k) \leq \bar{u} |e_k|^2 + \rho \lambda_{\max}(P) |s_k|^2. \quad (34)$$

Since by Lemma 1,  $|e_k| \leq \ell_e^k |e_0|$  for all  $k$ , and given that, by construction, the memory vector  $s_k$  stores the past output errors (which implies that  $|s_k|^2 = \sum_{j=1}^{\tau-k} |\tilde{y}_{k-j}|^2$ ), we can write, due to the Lipschitz property of  $h$ ,

$$|s_k|^2 \leq \sum_{j=1}^{\tau-k} \ell_h^2 (\ell_e^{k-j} |e_0|)^2 \leq \ell_e^{2k} \ell_\varsigma |e_0|^2, \quad (35)$$

with  $\ell_\varsigma := \ell_h^2 \left( \frac{1 - \ell_e^{-2\tau}}{\ell_e^2 - 1} \right)$ . Substituting bound (35) in (34) yields

$$V(\chi_k) \leq |e_0|^2 (\bar{u} \ell_e^{2k} + \rho \lambda_{\max}(P) \ell_e^{2k} \ell_\varsigma), \quad \forall k \leq \tau. \quad (36)$$

Finally, exploiting (6a) we can define the constant  $M_0 := \max_{k \leq \tau} \frac{1}{\underline{u}} (\bar{u} \ell_e^{2k} + \rho \lambda_{\max}(P) \ell_e^{2k} \ell_\varsigma)$  which satisfies

$$V(\chi_k) \leq M_0 V(\chi_0), \quad \forall k \leq \tau. \quad (37)$$

Substituting the transient bound evaluated at  $k = \tau$  into the asymptotic decay equation (33), we obtain for all  $k \geq \tau$ :

$$V(\chi_k) \leq M_0 (1 - c_v)^{-\tau} (1 - c_v)^k V(\chi_0). \quad (38)$$

Combining this with the finite-time bound (37) for  $k < \tau$  and the quadratic bounds of  $V$  in (19), we conclude that there exist constants  $K > 0$  and  $\lambda \in (0, 1)$  such that  $|(e_k, s_k)| \leq K \lambda^k |(e_0, s_0)|$  for all  $k \geq 0$ . Thus, the set  $\mathcal{E}$  is Globally Exponentially Stable for (1), (14) ■

### C. Technical Proofs

The proof of Lemma 1 follows from the Lipschitz properties specified in Assumptions 1 and 2.

*Proof of Lemma 1.* From the error dynamics (14), using the triangle inequality along with (5), we have:

$$|e_{k+1}| \stackrel{(4)}{\leq} \ell_f |x_k - \hat{x}_k| + \ell_{g_y} |\text{sat}_\theta(\tilde{y}_k; s_k)|. \quad (39)$$

By definition, the saturation operator satisfies  $|\sigma_\Delta(s)| \leq |s|$  for any threshold  $\Delta \geq 0$ . Therefore,  $|\text{sat}_\theta(\tilde{y}_k; s_k)| \leq |\tilde{y}_k| = |h(x_k) - h(\hat{x}_k)|$ . Substituting this into (39) yields:

$$\begin{aligned} |e_{k+1}| &\leq \ell_f |e_k| + \ell_{g_y} |h(x_k) - h(\hat{x}_k)| \\ &\leq (\ell_f + \ell_{g_y} \ell_h) |e_k|, \end{aligned} \quad (40)$$

which proves the lemma with  $\ell_e := \ell_f + \ell_{g_y} \ell_h$ . ■

*Proof of Lemma 2.* We start by showing by induction that, for each  $j \in \{1, \dots, \tau\}$ , there exist a scalar  $\kappa_j > 0$  and a correction term  $\phi_j$  such that for every trajectory  $(\hat{x}_k, s_k)_{k \in \mathbb{N}}$  of the observer (1), (9) and every  $k \geq 0$ , it holds that

$$\hat{x}_{k+j} = f^j(\hat{x}_k) + \phi_j, \quad (41a)$$

$$|\phi_j| \leq \kappa_j \sum_{i=0}^{j-1} |h(x_{k+i}) - h(\hat{x}_{k+i})|, \quad (41b)$$

with  $f^j(\cdot)$  defined as in Assumption 2. The base case  $j = 1$  holds trivially, with  $\phi_1 := g(\hat{x}_k, \text{sat}_\theta(\tilde{y}_k; s_k))$ . From the Lipschitz continuity of  $g$  and  $g(\hat{x}, 0) = 0$  (Assumption 2), and the globally Lipschitz saturation bound (Remark 2), it follows that  $|\phi_1| \leq \ell_{g_y} |h(x_k) - h(\hat{x}_k)|$ , which satisfies (41b) with  $\kappa_1 := \ell_{g_y}$ .

For the inductive step, assume (41) holds for some  $j \in \{1, \dots, \tau - 1\}$ . We have:

$$\begin{aligned} \hat{x}_{k+j+1} &= f(\hat{x}_{k+j}) + g(\hat{x}_{k+j}, \text{sat}_\theta(\tilde{y}_{k+j}; s_{k+j})) \\ &\stackrel{(41a)}{=} g(\hat{x}_{k+j}, \text{sat}_\theta(h(f^j(x_k)) - h(f^j(\hat{x}_k)) + \phi_j)) \\ &\quad + f(f^j(\hat{x}_k) + \phi_j; s_{k+j}). \end{aligned} \quad (42)$$

Using the Lipschitz continuity of  $f$  and  $h$ , we define the perturbation terms:

$$\begin{aligned} f(f^j(\hat{x}_k) + \phi_j) &= f^{j+1}(\hat{x}_k) + \delta_f, \quad \text{with } |\delta_f| \leq \ell_f |\phi_j|, \\ h(f^j(\hat{x}_k) + \phi_j) &= h(f^j(\hat{x}_k)) + \delta_h, \quad \text{with } |\delta_h| \leq \ell_h |\phi_j|. \end{aligned} \quad (43)$$

Similarly, the second argument of the correction term involving  $g$ , is perturbed by  $\delta_h$ . By the Lipschitz continuity of  $g$ , and the saturation function (which has a Lipschitz constant of 1), we can define a perturbation  $\delta_g$  such that

$$\begin{aligned} g(\hat{x}_{k+j}, \text{sat}_\theta(h(f^j(x_k)) - h(f^j(\hat{x}_k)) + \phi_j); s_{k+j}) &= \\ g(\hat{x}_{k+j}, \text{sat}_\theta(h(f^j(x_k)) - h(f^j(\hat{x}_k)); s_{k+j}) + \delta_g, \end{aligned} \quad (44)$$

with  $|\delta_g| \leq \ell_{g_y} \ell_h |\phi_j|$ . Collecting the terms in (43) and (44), we can define

$$\begin{aligned} \phi_{j+1} &:= \delta_f + g(\hat{x}_{k+j}, \text{sat}_\theta(h(f^j(x_k)) - h(f^j(\hat{x}_k)); s_{k+j})) \\ &\quad + \delta_g \end{aligned} \quad (45)$$

Then, the observer dynamics (42) can be written as

$$\hat{x}_{k+j+1} = f^{j+1}(\hat{x}_k) + \phi_{j+1}$$

which proves (41a) for the step  $j + 1$ . Due to the stated bound on  $\delta_f$  and  $\delta_g$  and (5), we may bound  $\phi_{j+1}$  in (45) as follows

$$|\phi_{j+1}| \leq (\ell_f + \ell_{g_y} \ell_h) |\phi_j| + \ell_{g_y} |h(f^j(x_k)) - h(f^j(\hat{x}_k))|$$

Then, (41b) holds at step  $j + 1$ , with  $\kappa_{j+1} := \max\{(\ell_f + \ell_{g_y} \ell_h) \kappa_j, \ell_{g_y}\}$ .

Using (41), we show next, by induction, that for each  $j \in \{1, \dots, \tau\}$  there exists a scalar  $0 < b_j \leq 1$ , such that

$$\sum_{i=0}^{j-1} |h(x_{k+i}) - h(\hat{x}_{k+i})| \geq b_j \sum_{i=0}^{j-1} |h(f^i(x_k)) - h(f^i(\hat{x}_k))|. \quad (46)$$

For the base case, where  $j = 1$ , Equation (46) simplifies to  $|h(x_k) - h(\hat{x}_k)| = |h(x_k) - h(\hat{x}_k)|$  (because  $f^0(x_k) = x_k$ ), so any  $b_1 \in (0, 1]$  satisfies (46). For the induction step, assume that (46) holds for some  $j$  with  $b_j > 0$ . At step  $j + 1$ , we will explicitly construct  $b_{j+1} > 0$  such that (46) holds with  $j$  replaced by  $j + 1$ . First note that, for any scalar  $c_j \geq 1$  it holds that

$$\begin{aligned} c_j \sum_{i=0}^j |h(x_{k+i}) - h(\hat{x}_{k+i})| \\ \geq |h(x_{k+j}) - h(\hat{x}_{k+j})| + c_j \sum_{i=0}^{j-1} |h(x_{k+i}) - h(\hat{x}_{k+i})|. \end{aligned} \quad (47)$$

Given that  $h(x_{k+j}) = h(f^j(x_k) + \phi_j)$ , and by adding and subtracting the term  $h(f^j(\hat{x}_k))$ , we can establish the following bound

$$\begin{aligned} & |h(x_{k+j}) - h(\hat{x}_{k+j})| \\ & \geq |h(f^j(x_k)) - h(f^j(\hat{x}_k))| - |h(f^j(\hat{x}_k)) - h(\hat{x}_{k+j})| \\ & \stackrel{(41a)}{\geq} |h(f^j(x_k)) - h(f^j(\hat{x}_k))| - \ell_h |\phi_j| \\ & \stackrel{(41b)}{\geq} |h(f^j(x_k)) - h(f^j(\hat{x}_k))| \\ & \quad - \ell_h \kappa_j \sum_{i=0}^{j-1} |h(x_{k+i}) - h(\hat{x}_{k+i})|. \end{aligned} \quad (48)$$

Substituting (48) and the inductive hypothesis (46) (assumed valid at the step  $j$ ) in (47), we obtain

$$\begin{aligned} c_j \sum_{i=0}^j |h(x_{k+i}) - h(\hat{x}_{k+i})| & \stackrel{(48)}{\geq} |h(f^j(x_k)) - h(f^j(\hat{x}_k))| \\ & + (c_j - \ell_h \kappa_j) \sum_{i=0}^{j-1} |h(x_{k+i}) - h(\hat{x}_{k+i})| \\ & \stackrel{(46)}{\geq} |h(f^j(x_k)) - h(f^j(\hat{x}_k))| \\ & + b_j (c_j - \ell_h \kappa_j) \sum_{i=0}^{j-1} |h(f^i(x_k)) - h(f^i(\hat{x}_k))|. \end{aligned} \quad (49)$$

We now select  $c_j$  such that the second term on the right-hand side of (49) is non-negative. Setting  $c_j = \ell_h \kappa_j + 1$ , we have  $c_j \geq 1$  and  $b_j(c_j - \ell_h \kappa_j) = b_j$ . Thus, (49) becomes

$$\begin{aligned} c_j \sum_{i=0}^j |h(x_{k+i}) - h(\hat{x}_{k+i})| & \geq |h(f^j(x_k)) - h(f^j(\hat{x}_k))| \\ & + b_j \sum_{i=0}^{j-1} |h(f^i(x_k)) - h(f^i(\hat{x}_k))|. \end{aligned}$$

Dividing by  $c_j$  and recalling that  $b_j \leq 1$  implies  $\frac{1}{c_j} \geq \frac{b_j}{c_j}$ , we

get:

$$\begin{aligned} \sum_{i=0}^j |h(x_{k+i}) - h(\hat{x}_{k+i})| & \geq \frac{b_j}{c_j} (|h(f^j(x_k)) - h(f^j(\hat{x}_k))| \\ & + \sum_{i=0}^{j-1} |h(f^i(x_k)) - h(f^i(\hat{x}_k))|). \end{aligned} \quad (50)$$

By defining  $b_{j+1} := \frac{b_j}{c_j}$ , we conclude that (46) holds for the step  $j + 1$ . Finally, we can use the observability inequality (2) along with (46), evaluated with  $j = \tau$ , to conclude that

$$\begin{aligned} \sum_{i=0}^{\tau-1} |h(x_{k+i}) - h(\hat{x}_{k+i})| & \geq b_\tau \sum_{i=0}^{\tau-1} |h(f^i(x_k)) - h(f^i(\hat{x}_k))| \\ & \geq b_\tau \gamma (|x_k - \hat{x}_k|), \end{aligned}$$

which proves (16). Consider now a time step  $k \geq \tau$ . By construction, the state  $\varsigma_k$  stores the history of the previous  $\tau$  output injection errors. Specifically, we have:

$$|\varsigma_k|_1 = \sum_{j=1}^{\tau} |\tilde{y}_{k-j}| = \sum_{j=1}^{\tau} |h(x_{k-j}) - h(\hat{x}_{k-j})|. \quad (51)$$

Therefore, for any  $k \geq \tau$ , we may apply equation (16) to get

$$|\varsigma_k|_1 \geq \sum_{i=0}^{\tau-1} |h(x_{k-\tau+i}) - h(\hat{x}_{k-\tau+i})| \geq b_\tau \gamma (|e_{k-\tau}|).$$

Next, by Lemma 1, we have  $|e_k| \leq \ell_e^\tau |e_{k-\tau}|$ , which implies that  $|e_{k-\tau}| \geq \ell_e^{-\tau} |e_k|$ ,  $\forall k \geq \tau$ . Since  $\gamma$  is a class  $\mathcal{K}_\infty$  function (thus non-decreasing), we obtain (17), as to be proven.  $\blacksquare$

## V. ANALYSIS OF THE PERTURBED CASE (PROOF OF THEOREM 1, ITEM 2)

In this section, we provide the proof for Item 2 of Theorem 1, establishing Input-to-State Stability (ISS) of the error dynamics. We analyze the robustness of the observer with respect to measurement disturbances  $d$  and we focus on the case where the observability modulus  $\gamma$  in Assumption 1 is linear, i.e.,  $\gamma(s) = c_\gamma s$ . This linearity assumption is crucial to ensure that the saturation threshold map is globally Lipschitz, which allows obtaining global uniform bounds.

### A. Notations and Nominal Reference

Consider system (1) and observer (9) and let  $e = x - \hat{x}$  be the estimation error. Then, we may write the perturbed error dynamics as

$$\begin{aligned} e_{k+1} & = f(x_k) - \hat{f}(\hat{x}_k, \text{sat}_\theta(\tilde{y}_k + d_k; \varsigma_k)) \\ \varsigma_{k+1} & = S\varsigma_k + G|\tilde{y}_k + d_k|, \end{aligned} \quad (52)$$

where  $\tilde{y}_k = h(x_k) - h(\hat{x}_k)$ .

Our proof relies on comparing the perturbed trajectory against a nominal ‘‘ghost’’ trajectory that would have evolved if the disturbances had ceased  $\tau$  steps ago. For compactness, we define the extended state  $\chi_k := (x_k, e_k, \varsigma_k)$ . For each  $j \geq k - \tau$ , we denote by  $\phi(j; \chi_{k-\tau}, \mathbf{d})$  the solution of (52) at

step  $j$ , which equals  $\chi_{k-\tau}$  at time  $k-\tau$ , and is driven by the sequence  $\mathbf{d} := \{d_k\}_{k \in \mathbb{N}}$ .

Consider a current time step  $k \geq \tau$ . We focus the attention on two trajectories initialized at the same past state  $\chi_{k-\tau}$ :

$$\chi_j := (x_j, e_j, \varsigma_j) := \phi(j; \chi_{k-\tau}, \mathbf{d}) \quad (\text{Perturbed}) \quad (53a)$$

$$\chi_j^\circ := (x_j^\circ, e_j^\circ, \varsigma_j^\circ) := \phi(j; \chi_{k-\tau}, \mathbf{0}) \quad (\text{Nominal}) \quad (53b)$$

for  $j \in \{k-\tau, \dots, k+1\}$ . Since (53a) and (53b) start from the same past state, by definition,  $\chi_{k-\tau} = \chi_{k-\tau}^\circ$ .

## B. Finite-Time Deviation Analysis

With the following lemma, we quantify how the Lyapunov function  $V(\chi)$  along the perturbed trajectory can deviate from the nominal one  $V(\chi^\circ)$  over the interval  $\{k-\tau, \dots, k\}$ . With a slight abuse of notation, we denote the disturbance magnitude over such interval as  $\|\mathbf{d}_{k-\tau}^k\|_\infty := \max_{k-\tau \leq i \leq k} |d_i|$ .

*Lemma 3 (Finite-Time Deviation):* Suppose that Assumptions 1, 2 and 3 hold with  $\gamma(s) = c_\gamma s$ . Let  $\tau \geq 1$ . For any scalar  $\nu > 0$ , there exists a function  $\beta_\nu \in \mathcal{K}_\infty$  such that, for each  $k \geq \tau$ , for any initial condition  $\chi_{k-\tau}$  and for any disturbance sequence  $\mathbf{d}$ , the following bound holds for all  $j \in \{k, k+1\}$ :

$$|V(\chi_j) - V(\chi_j^\circ)| \leq \nu V(\chi_{k-\tau}) + \beta_\nu(\|\mathbf{d}_{k-\tau}^k\|_\infty). \quad (54)$$

*Proof of Lemma 3.* Fix  $k \geq \tau$  and, based on the definition in (53), let

$$\delta e_i := |e_{k-\tau+i} - e_{k-\tau+i}^\circ|, \quad \delta \varsigma_i := |\varsigma_{k-\tau+i} - \varsigma_{k-\tau+i}^\circ| \quad (55)$$

for  $i \in \{0, \dots, \tau\}$  denote the deviations between the perturbed and nominal trajectories. Since both trajectories are initialized at the same state  $\chi_{k-\tau} = \chi_{k-\tau}^\circ$ , we have  $\delta e_0 = 0$  and  $\delta \varsigma_0 = 0$ .

Due to the assumption of linear observability,  $\gamma(s) = c_\gamma s$ , the saturation threshold in (8) is a linear function of the memory state norm

$$\Delta_k = \frac{\theta^2}{c_\gamma} |\varsigma_k|_1. \quad (56)$$

First, recall that for any vector  $\varsigma \in \mathbb{R}^\tau$ , the 1-norm is related to the Euclidean norm by  $|\varsigma|_1 \leq \sqrt{\tau} |\varsigma|$ . Consequently, using the reverse triangle inequality, we obtain the bound

$$||\varsigma_a|_1 - |\varsigma_b|_1| \leq |\varsigma_a - \varsigma_b|_1 \leq \sqrt{\tau} |\varsigma_a - \varsigma_b|. \quad (57)$$

Then, by leveraging the non-expansive property of the saturation function (defined as in (7)), we get  $|\sigma_{\Delta_a}(u_a) - \sigma_{\Delta_b}(u_b)| \leq |u_a - u_b| + |\Delta_a - \Delta_b|$ , which implies

$$\begin{aligned} |\text{sat}_\theta(y_a; \varsigma_a) - \text{sat}_\theta(y_b; \varsigma_b)| &\leq |\text{sat}_\theta(y_a; \varsigma_a) - \text{sat}_\theta(y_b; \varsigma_b)|_1 \\ &\leq \sum_{j=1}^m |\sigma_{\Delta_a}(y_{a,j}) - \sigma_{\Delta_b}(y_{b,j})| \\ &\leq \sum_{j=1}^m (|y_{a,j} - y_{b,j}| + |\Delta_a - \Delta_b|) \\ &\leq \sqrt{m} |y_a - y_b| + \frac{m\theta^2}{c_\gamma} \left| |\varsigma_a|_1 - |\varsigma_b|_1 \right|. \end{aligned} \quad (58)$$

Using (58) and the Lipschitz continuity of  $f, g$  (from Assumption 2) and (57), the error deviations in (55) evolve, for all  $i = 0, \dots, \tau$ , as

$$\begin{aligned} \delta e_{i+1} &\leq (\ell_f + \ell_{g_x}) \delta e_i + \ell_{g_y} \left( \frac{m\theta^2}{c_\gamma} \left| |\varsigma_i|_1 - |\varsigma_i^\circ|_1 \right| \right. \\ &\quad \left. + \sqrt{m} |\tilde{y}_{k-\tau+i} + d_{k-\tau+i} - \tilde{y}_{k-\tau+i}^\circ| \right) \\ &\leq (\ell_f + \ell_{g_x} + \sqrt{m} \ell_{g_y} \ell_h) \delta e_i + \frac{m\theta^2 \ell_{g_y} \sqrt{\tau}}{c_\gamma} \delta \varsigma_i \\ &\quad + \sqrt{m} \ell_{g_y} |d_{k-\tau+i}|, \end{aligned} \quad (59)$$

Similarly, the memory dynamics deviation in (55) satisfies, for all  $i = 0, \dots, \tau$ ,

$$\begin{aligned} \delta \varsigma_{i+1} &\leq \|S\| \delta \varsigma_i + |G| \left( |\tilde{y}_{k-\tau+i} + d_{k-\tau+i} - \tilde{y}_{k-\tau+i}^\circ| \right) \\ &\leq \|S\| \delta \varsigma_i + |G| \ell_h \delta e_i + |G| |d_{k-\tau+i}|. \end{aligned} \quad (60)$$

By defining the vector  $\delta_i := [\delta e_i, \delta \varsigma_i]^\top \in \mathbb{R}_{\geq 0}^2$ , and taking the Euclidean norm of the component-wise inequalities (59), (60), we obtain

$$|\delta_{i+1}| \leq \|M\| |\delta_i| + |N| |d_{k-\tau+i}|, \quad \forall i = 0, \dots, \tau \quad (61)$$

where  $M := \begin{bmatrix} \ell_f + \ell_{g_x} + \sqrt{m} \ell_{g_y} \ell_h & \frac{m\theta^2 \ell_{g_y} \sqrt{\tau}}{c_\gamma} \\ |G| \ell_h & \|S\| \end{bmatrix}$  and  $N := \begin{bmatrix} \sqrt{m} \ell_{g_y} \\ |G| \end{bmatrix}$ . Iterating the scalar inequality (61) and denoting  $\bar{M} = \max\{1, \|M\|\}$ , we get from  $|\delta_0| = 0$ ,  $|\delta_1| \leq \|M\| |N| |d_{k-\tau}| \leq \bar{M} |N| |d_{k-\tau}|$ ,  $|\delta_2| \leq 2\bar{M}^2 |N| \max\{|d_{k-\tau}|, |d_{k-\tau+1}|\}$  and so on, which generalizes to

$$|\delta_i| \leq \underbrace{(\tau+1) \bar{M}^{\tau+1} |N|}_{:= M_\delta} \|\mathbf{d}_{k-\tau}^k\|_\infty, \quad \forall i = 0, \dots, \tau+1.$$

It follows that both state deviations in (55) are globally bounded by the disturbance as

$$\delta e_i \leq M_\delta \|\mathbf{d}_{k-\tau}^k\|_\infty, \quad \delta \varsigma_i \leq M_\delta \|\mathbf{d}_{k-\tau}^k\|_\infty, \quad \forall i = 0, \dots, \tau+1 \quad (62)$$

Given the definition of the function  $V$  in Equation (18), for a compact notation, define  $V_j := V(\chi_j)$  and  $V_j^\circ := V(\chi_j^\circ)$ ,  $U_j := U(x_j, e_j)$  and  $U_j^\circ := U(x_j, e_j^\circ)$ . From (18), we get

$$|V_j^\circ - V_j| \leq |U_j^\circ - U_j| + \rho |(\varsigma_j^\circ)^\top P \varsigma_j^\circ - \varsigma_j^\top P \varsigma_j|. \quad (63)$$

From the finite-time bound (37) obtained in Lemma 1, there exists a scalar  $c_0 > 0$  such that

$$|e_j^\circ|^2 \leq c_0 V(\chi_{k-\tau}), \quad \forall j \in \{k-\tau, \dots, k+1\} \quad (64)$$

$$|\varsigma_j^\circ|^2 \leq c_0 V(\chi_{k-\tau}) \quad \forall j \in \{k-\tau, \dots, k+1\} \quad (65)$$

Given the prescribed scalar  $\nu \geq 0$  in (54), select now two scalars  $\eta_e > 0$  and  $\eta_\varsigma > 0$  sufficiently small to satisfy

$$\eta_e c_0 \leq \frac{\nu}{2}, \quad \rho \|P\| \eta_\varsigma c_0 \leq \frac{\nu}{2}. \quad (66)$$

Exploiting Assumption 3 with (55) and the previous inequalities, we bound the  $U$ -component of inequality (63), for all

$j = k - \tau, \dots, k + 1$ , as

$$\begin{aligned} |U_j^\circ - U_j| &\stackrel{(6c)}{\leq} \eta_\varepsilon |e_j^\circ|^2 + \gamma_\varepsilon (\delta e_{j-k+\tau}) \\ &\stackrel{(62),(64)}{\leq} \eta_\varepsilon c_0 V(\chi_{k-\tau}) + \gamma_\varepsilon (M_\delta |\mathbf{d}_{k-\tau}^k|_\infty) \\ &\stackrel{(66)}{\leq} \frac{\nu}{2} V(\chi_{k-\tau}) + \gamma_\varepsilon (M_\delta |\mathbf{d}_{k-\tau}^k|_\infty). \end{aligned} \quad (67)$$

For the quadratic memory term in (63), let us denote  $|\Delta V_{\zeta,j}| := |\zeta_j^\top P \zeta_j - (\zeta_j^\circ)^\top P \zeta_j^\circ|$ . Moreover, defining the memory state deviation  $\tilde{\zeta}_j := \zeta_j - \zeta_j^\circ$ , for all  $j = k - \tau, \dots, k + 1$ , so that  $|\tilde{\zeta}_j| = \delta \zeta_{j-k+\tau}$  for all such  $j$ , we get

$$|\Delta V_{\zeta,j}| = |\tilde{\zeta}_j^\top P (\zeta_j + 2\zeta_j^\circ)| \leq \|P\| |\tilde{\zeta}_j|^2 + 2\|P\| |\tilde{\zeta}_j| |\zeta_j^\circ|.$$

Considering the parameter  $\eta_\zeta$  selected to satisfy (66), we may apply a Young inequality to obtain  $2\|P\| |\tilde{\zeta}_j| |\zeta_j^\circ| \leq \frac{1}{\eta_\zeta} \|P\| |\tilde{\zeta}_j|^2 + \eta_\zeta \|P\| |\zeta_j^\circ|^2$ . Substituting  $|\tilde{\zeta}_j| = \delta \zeta_{j-k+\tau} \leq M_\delta |\mathbf{d}_{k-\tau}^k|_\infty$ , which follows from (55) and (62), yields

$$\begin{aligned} |\Delta V_{\zeta,j}| &\leq \|P\| \eta_\zeta |\zeta_j^\circ|^2 + \|P\| \left(1 + \frac{1}{\eta_\zeta}\right) \delta \zeta_{j-k+\tau}^2 \\ &\stackrel{(65),(66)}{\leq} \frac{\nu}{2\rho} V(\chi_{k-\tau}) + \|P\| \left(1 + \frac{1}{\eta_\zeta}\right) M_\delta^2 |\mathbf{d}_{k-\tau}^k|_\infty^2. \end{aligned} \quad (68)$$

Combining (63) with (67) and (68), we prove (54) with  $\beta_\nu(s) := \gamma_\varepsilon (M_\delta s) + \rho \|P\| (1 + \eta_\zeta^{-1}) M_\delta^2 s^2$ . ■

### C. Recursive Lyapunov Evolution

*Proposition 2:* Suppose that Assumptions 1–3 hold with  $\gamma(s) = c_\nu s$ . Let  $c_v \in (0, 1)$  be the nominal decay rate from Proposition 1. Then, for any scalar  $\varepsilon > 0$ , there exists a function  $\Gamma_\varepsilon \in \mathcal{K}_\infty$  such that, for all  $k \geq \tau$ ,

$$V_{k+1} \leq (1 - c_v) V_k + \varepsilon V_{k-\tau} + \Gamma_\varepsilon (|\mathbf{d}_{k-\tau}^k|_\infty). \quad (69)$$

*Proof of Proposition 2.* Consider the trajectories initialized at  $k - \tau$ . We express the value of  $V$  at step  $k + 1$  as  $V_{k+1} = V(\chi_{k+1}^\circ) + \tilde{v}_{k+1}$ , where  $\tilde{v}_j := V(\chi_j) - V(\chi_j^\circ)$  represents the deviation from the nominal ghost trajectory. From Proposition 1, the nominal trajectory decays exponentially, specifically we have  $V(\chi_{k+1}^\circ) \leq (1 - c_v) V(\chi_k^\circ)$ . Substituting the identity  $V(\chi_k^\circ) = V_k - \tilde{v}_k$ , we obtain

$$\begin{aligned} V_{k+1} &= V(\chi_{k+1}^\circ) + \tilde{v}_{k+1} \\ &\leq (1 - c_v) (V_k - \tilde{v}_k) + \tilde{v}_{k+1} \\ &\leq (1 - c_v) V_k + (1 - c_v) |\tilde{v}_k| + |\tilde{v}_{k+1}|. \end{aligned} \quad (70)$$

where the last inequality holds since  $(1 - c_v) > 0$ .

Define the scalar  $\nu := \frac{\varepsilon}{2 - c_v}$ . Lemma 3 ensures the existence of  $\beta_\nu \in \mathcal{K}_\infty$  such that  $|\tilde{v}_j| \leq \nu V_{k-\tau} + \beta_\nu (|\mathbf{d}_{k-\tau}^k|_\infty)$  for both  $j = k$  and  $j = k + 1$ . Substituting such bounds, evaluated for  $|\tilde{v}_k|$  and  $|\tilde{v}_{k+1}|$ , into (70), we get

$$V_{k+1} \leq (1 - c_v) V_k + [(1 - c_v) + 1] (\nu V_{k-\tau} + \beta_\nu (|\mathbf{d}_{k-\tau}^k|_\infty)) \quad (71)$$

Recalling that we chose  $\nu$  such that  $(2 - c_v)\nu = \varepsilon$ , the final recurrence becomes:

$$V_{k+1} \leq (1 - c_v) V_k + \varepsilon V(\chi_{k-\tau}) + \Gamma_\varepsilon (|\mathbf{d}_{k-\tau}^k|_\infty), \quad (72)$$

where  $\Gamma_\varepsilon(\cdot) := (2 - c_v)\beta_\nu(\cdot)$  is a class  $\mathcal{K}_\infty$  function. ■

### D. Proof of Theorem 1, Item 2

*Proof of Theorem 1, Item 2.* We analyze the stability of the recurrence established in Proposition 2 by defining an augmented state vector  $Z_k := [V_k \ V_{k-1} \ \dots \ V_{k-\tau}]^\top \in \mathbb{R}_{\geq 0}^{\tau+1}$ . Equation (69) implies the following component-wise inequalities, for each  $k \geq \tau$ ,

$$0 \preceq Z_{k+1} \preceq A_\varepsilon Z_k + B \Gamma_\varepsilon (|\mathbf{d}_{k-\tau}^k|_\infty), \quad (73)$$

where  $B = [1, 0, \dots, 0]^\top$  and  $A_\varepsilon \in \mathbb{R}^{(\tau+1) \times (\tau+1)}$  is a companion-like matrix

$$A_\varepsilon = \begin{bmatrix} 1 - c_v & 0 & \dots & 0 & \varepsilon \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix}.$$

Since matrix  $A_\varepsilon$  in (73) is Metzler, because it only has nonnegative entries, then from [36, Proposition 2]  $A_\varepsilon$  is Schur stable if and only if there exists a  $p \in \mathbb{R}^{\tau+1}$  having positive entries ( $p \succ 0$ ), such that  $A_\varepsilon^\top p \prec p$  (component-wise).

We now look for such a vector  $p := [p_1 \ p_2 \ \dots \ p_{\tau+1}]^\top$ , with  $p_i > 0$  for all  $i = 1, \dots, \tau + 1$ . By expanding the condition  $A_\varepsilon^\top p \prec p$ , we obtain the following set of linear inequalities:

$$p_2 < c_v p_1, \quad (74a)$$

$$p_i < p_{i-1}, \quad \forall i = 3, \dots, \tau + 1, \quad (74b)$$

$$\varepsilon p_1 < p_{\tau+1}. \quad (74c)$$

Combining the inequalities (74), we obtain the chain condition  $\varepsilon p_1 < p_{\tau+1} < \dots < p_2 < c_v p_1$ . This implies that, for any  $\varepsilon$  satisfying  $\varepsilon < c_v$ , there exists a vector  $p$  satisfying  $A_\varepsilon^\top p \prec p$ . Since  $c_v \in (0, 1)$  is a system constant and  $\varepsilon$  can be chosen arbitrarily small (as shown in Proposition 2), this condition is always satisfiable.

For any  $p \succ 0$  satisfying (74), define now the Lyapunov function  $W_k := p^\top Z_k$ . We now analyze the one-step increment of  $W_k$  along the trajectories of (73). Since  $p \succ 0$ , pre-multiplying the component-wise inequality (73) by  $p^\top$  preserves the order of the inequality. Therefore, the one-step difference satisfies:

$$\begin{aligned} \Delta W_k &:= W_{k+1} - W_k = p^\top Z_{k+1} - p^\top Z_k \\ &\stackrel{(73)}{\leq} p^\top A_\varepsilon Z_k - p^\top Z_k + p^\top B \Gamma_\varepsilon (|\mathbf{d}_{k-\tau}^k|_\infty), \quad \forall k \geq \tau. \end{aligned} \quad (75)$$

Furthermore, since the strict inequality  $A_\varepsilon^\top p \prec p$  holds, there exists a scalar  $c_w \in (0, 1)$  such that  $A_\varepsilon^\top p \preceq (1 - c_w)p$  or, equivalently,  $p^\top A_\varepsilon \preceq (1 - c_w)p^\top$ . Substituting this, along with  $p^\top B = p_1$ , in (75), yields

$$\Delta W_k \leq -c_w p^\top Z_k + p_1 \Gamma_\varepsilon (|\mathbf{d}_{k-\tau}^k|_\infty), \quad \forall k \geq \tau.$$

By defining  $\bar{\gamma}(s) := p_1 \Gamma_\varepsilon(s)$ , we obtain:

$$\Delta W_k \leq -c_w W_k + \bar{\gamma} (|\mathbf{d}_{k-\tau}^k|_\infty), \quad \forall k \geq \tau. \quad (76)$$

Inequality (76), combined with the equivalence of norms  $\min_i (p_i) |Z_k|_1 \leq p^\top Z_k \leq \max_i (p_i) |Z_k|_1$  (where  $|Z_k|_1 = \sum Z_{i,k}$ , since  $Z_k \geq 0$ ), implies, by [37, Lemma 3.5], that the auxiliary system  $Z_k$  (and thus  $V_k$ ) is ISS with respect to

$|d_{k-\tau}^k|$ . Specifically, there exist  $\tilde{\beta} \in \mathcal{KL}$  and  $\tilde{\gamma} \in \mathcal{K}_\infty$  such that

$$|Z_k| \leq \tilde{\beta}(|Z_\tau|, k - \tau) + \tilde{\gamma} \left( \sup_{0 \leq j \leq k} |d_{j-\tau}^j|_\infty \right), \quad \forall k \geq \tau. \quad (77)$$

Using the upper bound (19), we can bound the initial condition  $|Z_\tau|$  of the augmented state as

$$\begin{aligned} |Z_\tau| &\leq \sum_{k=0}^{\tau} V_k \stackrel{(19)}{\leq} \bar{v} \sum_{k=0}^{\tau} (|e_k|^2 + |\varsigma_k|^2) \\ &\leq \bar{v}(\tau + 1) \max_{0 \leq k \leq \tau} (|e_k|^2 + |\varsigma_k|^2). \end{aligned} \quad (78)$$

Moreover, due to the Lipschitz continuity of the system dynamics (Assumption 2), the error growth  $e$  is bounded in the interval  $[0, \tau]$ . Specifically, iterating the dynamics (52) yields

$$|e_k| \stackrel{(15)}{\leq} \ell_e^k |e_0| + \sum_{i=0}^{k-1} \ell_e^{k-1-i} |d_0^i|_\infty \leq \tilde{\ell}_e |e_0| + \ell_d |d_0^\tau|_\infty, \quad \forall k \leq \tau, \quad (79)$$

with  $\ell_d := \ell_{g,y} \sum_{j=0}^{\tau-1} \ell_e^j$  and  $\tilde{\ell}_e := \max\{1, \ell_e^\tau\}$ . Similarly, given that  $\|S\| = 1$  and  $|G| = 1$ , the state  $\varsigma$  can be bounded in the interval  $[0, \tau]$  as

$$\begin{aligned} |\varsigma_k| &\leq |\varsigma_0| + \sum_{i=0}^{k-1} (\ell_h |e_i| + |d_i|) \\ &\stackrel{(79)}{\leq} |\varsigma_0| + \tilde{\ell}_\varsigma |e_0| + \ell_{\varsigma,d} |d_0^\tau|_\infty, \quad \forall k \leq \tau, \end{aligned} \quad (80)$$

with  $\tilde{\ell}_\varsigma := \tau \ell_h \tilde{\ell}_e$  and  $\ell_{\varsigma,d} := \tau(\ell_h \ell_d + 1)$ . Substituting (79) and (80) into (78), and applying a Young inequality, yields

$$|Z_\tau| \leq c_e |e_0|^2 + c_\varsigma |\varsigma_0|^2 + c_d |d_0^\tau|_\infty^2, \quad (81)$$

with  $c_e := \bar{v}(\tau+1)(2\tilde{\ell}_e^2 + 3\tilde{\ell}_\varsigma^2)$ ,  $c_\varsigma := 3\bar{v}(\tau+1)$  and  $c_d := \bar{v}(\tau+1)(2\ell_d^2 + 3\ell_{\varsigma,d}^2)$ . Next, substitute inequality (81) into the ISS bound for  $Z_k$  given in (77). Using the weak triangle inequality for class  $\mathcal{K}$  functions of [38, Equation (12)], we can separate the contributions of the initial state and the disturbance as

$$\begin{aligned} |Z_k| &\leq \tilde{\beta}(c_e |e_0|^2 + c_\varsigma |\varsigma_0|^2 + c_d |d_0^\tau|_\infty^2, k - \tau) + \tilde{\gamma}(|d_0^k|_\infty) \\ &\leq \tilde{\beta}(2(c_e |e_0|^2 + c_\varsigma |\varsigma_0|^2), k - \tau) + \tilde{\beta}(2c_d |d_0^\tau|_\infty^2, k - \tau) \\ &\quad + \tilde{\gamma}(|d_0^k|_\infty), \quad \forall k \geq \tau. \end{aligned} \quad (82)$$

Since  $\tilde{\beta}(s, k) \in \mathcal{KL}$  is decreasing with respect to time  $k$ , it holds that  $\tilde{\beta}(\cdot, k - \tau) \leq \tilde{\beta}(\cdot, 0)$  for all  $k \geq \tau$ . Moreover, since  $V_k$  is a non-negative component of the vector  $Z_k$ , it satisfies  $V_k \leq |Z_k|_1 \leq |Z_k|$  and, using the lower bound from Assumption 3, we get  $\underline{|e_k|} \leq V_k \leq |Z_k|$ . It follows that the estimation error satisfies the ISS bound  $|e_k| \leq \beta(|e_0, \varsigma_0|, k - \tau) + \gamma(|d_0^k|_\infty)$  with the functions  $\beta$  and  $\gamma$  defined as

$$\begin{aligned} \beta(s, k) &:= \frac{1}{\sqrt{u}} \sqrt{\tilde{\beta}(c_s s^2, k)} \in \mathcal{KL} \\ \gamma(s) &:= \frac{1}{\sqrt{u}} \left( \sqrt{\tilde{\beta}(2c_d s^2, 0)} + \sqrt{\tilde{\gamma}(s)} \right) \in \mathcal{K}, \end{aligned} \quad (83)$$

with  $c_s := \max\{2c_e, 2c_\varsigma\}$ .  $\blacksquare$

## VI. APPLICATION ON A SINGLE-LINK FLEXIBLE-JOINT MANIPULATOR

To illustrate the performance of the stubborn redesign compared to a nominal observer, we consider a Single-Link Flexible-Joint Manipulator (FJM) dynamics subject to nonlinear friction phenomena, taking inspiration from [39], [40].

### A. System Modeling

The continuous-time dynamics of the FJM is described as in [39] by

$$\begin{aligned} \dot{x} &= A_{\text{ct}} x + B_{\text{ct}} u + \phi(x), \\ y &= Cx, \end{aligned} \quad (84)$$

where the state vector  $x = [\theta_m, \dot{\theta}_m, \theta_l, \dot{\theta}_l]^\top \in \mathbb{R}^4$  contains the motor and link positions and velocities. The control input  $u$  is the motor torque, and the output  $y \in \mathbb{R}^2$  consists of the motor states. The system matrices are described as

$$\left[ \begin{array}{c|c} A_{\text{ct}} & B_{\text{ct}} \\ \hline C & \mathbf{0} \end{array} \right] = \left[ \begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ -\frac{K}{J_m} & -\frac{B_m}{J_m} & \frac{K}{J_m} & 0 & \frac{1}{J_m} \\ 0 & 0 & 0 & 1 & 0 \\ \frac{K}{J_l} & 0 & -\frac{K}{J_l} & -\frac{B_l}{J_l} & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array} \right] \quad (85)$$

where  $J_m, J_l$  denote respectively the motor and link inertia,  $K$  is the joint stiffness, and  $B_m, B_l$  are the viscous friction coefficients. The term  $\phi$  accounts for gravity effect and nonlinear friction (Stribeck model) [40]. While including nonlinear friction allows us to assess the proposed approach in the presence of nonlinearities, its inherent stabilizing effect could mask the observer's performance. Therefore, we introduce the friction term exclusively on the motor dynamics, leaving the link dynamics undamped. This yields

$$\phi(x) = \begin{bmatrix} 0 \\ -\frac{1}{J_m} \tau_{f_m}(x_2) \\ 0 \\ -\frac{mgl}{J_l} \sin(x_3) \end{bmatrix}. \quad (86)$$

where the friction torque is modeled as

$$\tau_{f_m}(x_2) = \left( F_C + (F_S - F_C) e^{-|x_2/v_s|^p} \right) \tanh\left(\frac{x_2}{\varepsilon_f}\right), \quad (87)$$

where  $F_C > 0$  is the Coulomb friction torque,  $F_S > F_C$  is the static friction torque,  $v_s$  is the Stribeck velocity,  $p$  is a positive scalar shaping the Stribeck curve, and  $\varepsilon_f$  is a regularization parameter to smoothen the Coulomb-induced sign function at zero velocity.

Applying a forward Euler discretization with period  $T_s$ , we obtain the following discrete-time model

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + T_s \phi(x_k) \\ y_k &= Cx_k, \end{aligned} \quad (88)$$

with  $A := I + T_s A_{\text{ct}}$  and  $B := T_s B_{\text{ct}}$ .

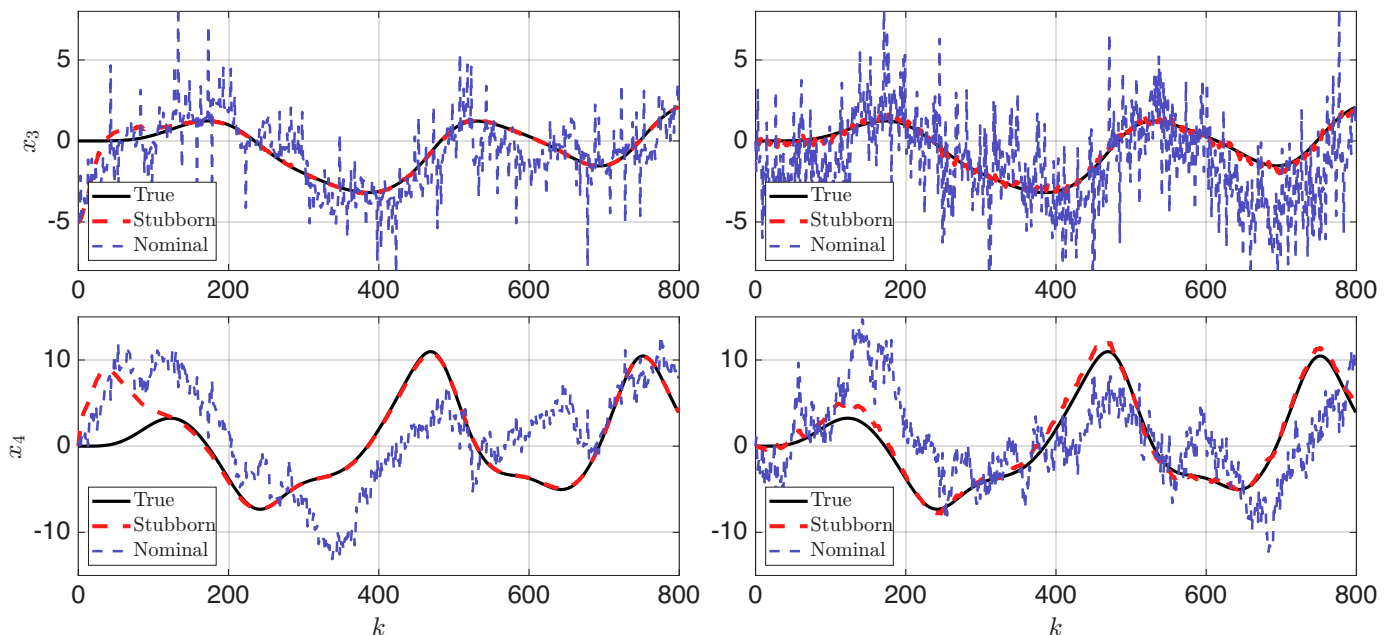


Fig. 1. Comparison between the evolution of the state estimate for the nominal observer (88) (blue dashed line) and the proposed stubborn redesign (90) (red dashed line), with  $\Theta = 0.02$ . Left:  $T_{\text{dwell}} = 5$  and non-zero initial conditions. Right:  $T_{\text{dwell}} = 2$  and zero initial conditions.

## B. Observer Design

We start by designing a nominal baseline observer by computing the gain  $L$  by solving the LMIs from [26, Theorem 3]. This yields:

$$\bar{x}_{k+1} = A\bar{x}_k + Bu_k + T_s\phi(\bar{x}_k) + L(y_k - \bar{y}_k). \quad (89)$$

Before implementing the stubborn redesign, we validate the structural assumptions required by our framework. First, the discretized linear pair  $(A, C)$  satisfies the observability rank condition. We select the memory horizon  $\tau = 4$  (equal to the state dimension) to conservatively upper-bound the observability index and to ensure that the collected output history provides sufficient information for the state reconstruction. Furthermore, the nonlinear term  $\phi$  is modeled using sine and hyperbolic tangent functions, which are inherently bounded with bounded derivatives and guarantees global Lipschitz continuity of the system dynamics. With these conditions met, the finite-memory stubborn observer can be implemented as

$$\begin{aligned} \hat{x}_{k+1} &= A\hat{x}_k + Bu_k + T_s\phi(\hat{x}_k) + L \text{sat}_{\Delta_k}(\tilde{y}_k) \\ \varsigma_{k+1} &= S\varsigma_k + G|\tilde{y}_k|, \quad \varsigma_k \in \mathbb{R}^4 \\ \Delta_k &= \Theta|\varsigma_k|_1, \end{aligned} \quad (90)$$

where  $\tilde{y}_k = y_k - \hat{y}_k$ . The scalar parameter  $\Theta$  replaces the nonlinear gain due to the observations in Remark 4. The scalar  $\Theta$  is not chosen using the explicit theoretical lower bound obtained with Theorem 1, which is conservative and difficult to compute, but is instead tuned following the qualitative trade-off described in Remark 4. In this example, we empirically select  $\Theta = 0.02$ , a value sufficiently small to effectively reject sporadic outliers, yet large enough to maintain a convergence rate comparable to the one of the nominal observer.

## C. Tuning and Simulation

The system physical parameters are chosen as in [39], while the friction coefficients are taken from [40]. We simulate the system driven by a sinusoidal control torque, with output measurements corrupted by sporadic disturbances  $d$ , modeled as described by Definition 1. Specifically, we consider  $d$  as a periodic outlier sequence with period  $T_{\text{dwell}}$ , where the initial occurrence is randomly chosen between 1 and  $T_{\text{dwell}}$  using a uniform distribution. The outlier amplitudes are drawn from a zero-mean normal distribution with a variance of  $0.5f^2$ , where  $f = 20$  is the dispersion factor.

a) *Exact outlier cancellation:* To numerically validate the property of Theorem 2, we first consider a scenario with non-zero initial estimation errors and we set the outlier period to  $T_{\text{dwell}} = 5$  steps. Since the memory horizon is  $\tau = 4$ , the condition  $T_{\text{dwell}} > \tau$  is satisfied, allowing the observer to fully flush the memory between consecutive outliers. As shown in the left two graphs of Figure 1, the stubborn observer (red dashed line) converges rapidly to the true state trajectories (black solid line) despite the non-zero initial condition. After the transient phase, the estimate remains tightly coupled to the true state, effectively ignoring the impulsive disturbances. In contrast, the nominal observer (blue dashed line) performs visibly worse: its unsaturated injection term feeds through the outliers, resulting in erratic fluctuations and large estimation errors. In the second scenario, we start with a zero initial estimation error (i.e.,  $x_0 = \bar{x}_0 = \hat{x}_0$ ,  $\varsigma_0 = \mathbf{0}$ ) and we increase the outlier frequency by setting  $T_{\text{dwell}} = 2$ . In this case,  $T_{\text{dwell}} < \tau$ , which implies that a new outlier impacts the system before the previous one has been cleared from the memory shift buffer, violating the dwell-time condition of Definition 1, required for perfect deadbeat recovery Theorem 2.

b) *Transient response and tuning trade-offs:* Figure 2 shows the evolution of the estimation error norm  $|e_k|$  in a noise-

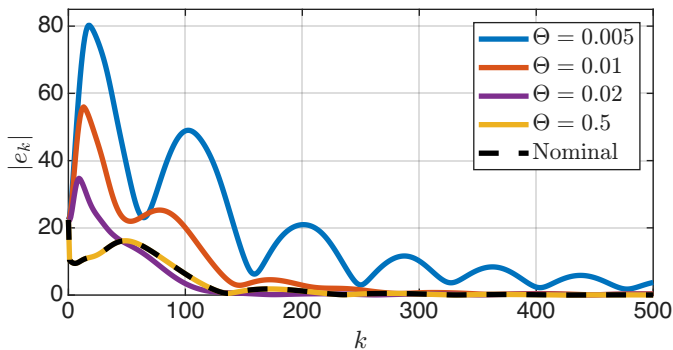


Fig. 2. Comparison of the transient response of the estimation error norm  $|e_k|$  in a noise-free scenario between the nominal observer (89) (black dashed line) and the stubborn redesign (90) (solid line), for different values of  $\Theta$ .

free scenario. For conservative settings (e.g.,  $\Theta = 0.01$ , blue line), the strict saturation threshold limits the magnitude of the correction term, resulting in a slower convergence, as compared to the nominal observer. However, as  $\Theta$  increases, the saturation becomes inactive during the transient, and the stubborn observer recovers the fast convergence rate of the nominal design (black dashed line). This result highlights a structural trade-off in the stubborn redesign: achieving stronger outlier rejection requires a tighter saturation threshold, which inevitably limits the correction term and reduces the convergence speed during large transients. Note that, according to the theoretical analysis, the stubborn observer is expected to coincide with the nominal observer after exactly  $\tau$  steps. However, as observed in Figure 2, the simulated convergence time is longer. This delay is a direct consequence of the tuning of the parameter  $\Theta$ , which was not selected to satisfy the theoretical bound, but following the discussion in Remark 4.

*c) Monte Carlo robustness analysis:* To assess the robustness of the proposed observer, we conducted a Monte Carlo analysis over  $10^4$  simulation runs. To ensure a fair comparison, both the nominal and stubborn observers were initialized with identical estimation errors  $e_0$  and subjected to the same noise realization within each run. Specifically, the initial error was sampled from a uniform distribution  $e_0 \sim \mathcal{U}(-\delta, \delta)$ , where  $\delta \in \mathbb{R}^4$  represents the maximum admissible error vector based on the structural limits of the FMJ. Measurement noise was generated to preserve the sporadic-outlier characteristics defined in the previous section. Figure 3 depicts the resulting distributions of the Root Mean Squared Error (RMSE) for the two distinct dwell-times.

The left plot ( $T_{\text{dwell}} = 5$ ) confirms that when the theoretical dwell-time condition is satisfied, and for a small value of  $\Theta$ , the stubborn observer effectively rejects outliers, converging to a near-zero RMSE despite a nonzero initial error. In contrast, the nominal observer exhibits large error excursions due to its sensitivity to corrupted measurements. The right plot of Figure 3 illustrates the case with  $T_{\text{dwell}} = 2$ , where the theoretical recovery condition is violated. Although exact deadbeat recovery is no longer guaranteed, for a small value of  $\Theta$ , the stubborn observer maintains a consistently low estimation error. This highlights the practical effectiveness of

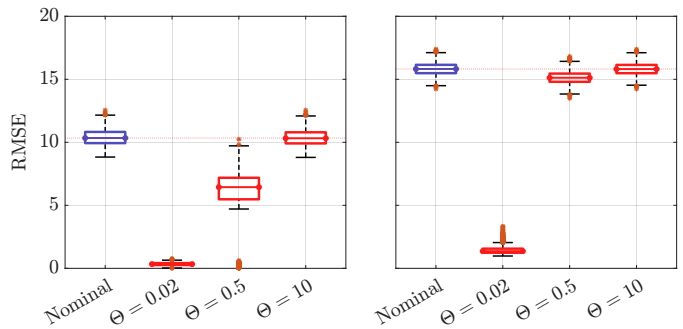


Fig. 3. Boxplots of the RMSE distribution for the nominal (blue) and stubborn (red) observers, for different values of the tuning parameter  $\Theta$ , computed over  $10^4$  simulation runs. Left:  $T_{\text{dwell}} = 5$  and non-zero initial conditions. Right:  $T_{\text{dwell}} = 2$  and non-zero initial conditions.

the saturation mechanism in preventing the oscillation in the nominal design, providing robustness even beyond the strict theoretical bounds. Finally, the results of Figure 3 further highlight the impact of the tuning parameter  $\Theta$ . As  $\Theta$  increases, the saturation constraint is progressively relaxed, leading to an increase in the RMSE in the presence of outliers. For sufficiently large values of  $\Theta$ , the saturation becomes inactive, and the performance of the stubborn observer asymptotically recovers that of the nominal nominal observer (indicated by the horizontal dotted line). This behavior confirms that  $\Theta$  governs a fundamental trade-off: it must be chosen small enough to suppress the effect of outliers, while remaining large enough to avoid degrading or even destabilizing the nominal estimation error dynamics.

## VII. CONCLUSION

This paper introduced a robust stubborn redesign for non-linear discrete-time observers based on a finite-memory architecture and dynamic saturation. By augmenting a nominal observer with a finite-memory buffer and a dynamic saturation level, the proposed scheme fundamentally departs from classical infinite-memory filtering strategies and achieves intrinsic robustness against sporadic measurement outliers. The key contribution lies in the finite-memory architecture, which guarantees a deadbeat recovery property: the influence of any isolated disturbance is exactly flushed after a finite number of steps, determined by the observability index, and independently of its magnitude. A rigorous analysis establishes Global Exponential Stability in nominal conditions and Input-to-State Stability with respect to bounded measurement noise. Simulation results on a flexible-joint manipulator confirmed the theoretical predictions, highlighting the superiority of the proposed approach over standard linear filtering techniques in handling large, impulsive disturbances.

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