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Geometric eigenvalue estimates of Kuttler-Sigillito type on differential forms

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Abstract. We introduce a new biharmonic Steklov problem on differential forms with Dirichlet-type boundary conditions and show that it is elliptic. We prove the existence of a discrete spectrum for this problem and give variational characterizations for eigenvalues associated to it. We establish eigenvalue estimates known as Kuttler-Sigillito inequalities, that connect the eigenvalues of different problems on differential forms with curvature quantities on the manifold.

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Keywords: Riemannian manifolds with boundary, biharmonic Steklov operator, discrete spectrum, eigenvalue estimates

1 Introduction

On planar domains, J. Kuttler and V. Sigillito [12, 13] established in 1968 a series of inequalities relating the eigenvalues of the membrane problems (both Dirichlet and Neumann Laplacian eigenvalues), the biharmonic Steklov problems, and the classical Steklov problem. More recently, Hassannezhad and Siffert [10] extended some of these results to compact Riemannian manifolds with C^2 boundary. In [1], we have lately extended to differential forms Kuttler-Sigillito inequalities that do not involve curvature.

Let (M^n, g) be a compact Riemannian manifold with a smooth boundary and let ν be the inward unit normal along the boundary ∂M . Recall that the scalar biharmonic Steklov problem with Dirichlet boundary conditions (BSD) [7, 10, 13, 16] is defined as:

$$\begin{cases} \Delta^2 u = 0 & \text{on } M \\ u = 0 & \text{on } \partial M \\ \Delta u - q\partial_\nu u = 0 & \text{on } \partial M, \end{cases}$$

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and the scalar biharmonic Steklov problem with Neumann boundary conditions (BSN) [10, 13] is:

$$\begin{cases} \Delta^2 u = 0 & \text{on } M \\ \partial_\nu u = 0 & \text{on } \partial M \\ \partial_\nu(\Delta u) - \ell u = 0 & \text{on } \partial M, \end{cases}$$

where u is a smooth function on M , and Δ is the positive scalar Laplacian operator. In both problems, the real numbers q and ℓ are the eigenvalues of the operator.

We denote by λ_k , μ_k , σ_k , q_k and ℓ_k the k^{th} eigenvalues associated with scalar eigenfunctions for the Dirichlet, Neumann, Steklov, BSD and BSN problems, respectively

We recall some of Kuttler-Sigillito inequalities.

Theorem 1.1. [10, Theorem 1.1] *Let (M^n, g) be a compact Riemannian manifold of dimension $n \geq 2$ with a C^2 boundary. For all integer $k \geq 1$, we have:*

$$\mu_k \sigma_1 \leq \ell_k \quad \text{and} \quad \mu_1 \sigma_k \leq \ell_k.$$

The theorem was proven in [13] for $k = 1$. On the other hand, we have the following:

Theorem 1.2. [13, Table 1] *On a bounded domain M of the plane with piecewise C^1 boundary ∂M , the following inequalities hold:*

- (i) $q_1 \sigma_1^2 \leq \ell_1$,
- (ii) $\mu_1^{-1} \leq \lambda_1^{-1} + (q_1 \ell_1)^{-\frac{1}{2}}$,
- (iii) $\mu_1^{-1} \leq \lambda_1^{-1} + (q_1 \sigma_1)^{-1}$.

We recall the definition of a star-shaped manifold:

- A domain $U \subset \mathbb{R}^n$ is said to be star-shaped with respect to a point x_0 if for every $x \in U$, the line segment connecting x_0 to x is entirely contained in U .
- A manifold M with C^2 boundary is said to be *star-shaped* with respect to $x_0 \in M$ if there exists a star-shaped domain $\Omega \subset \mathbb{R}^n \simeq T_{x_0}M$ with respect to 0, such that the exponential map \exp_{x_0} is defined on Ω and $\exp_{x_0} : \Omega \rightarrow M$ is a diffeomorphism.

Let us fix $x_0 \in M$ and assume that M is star-shaped with respect to x_0 , we define the following notation: $d_{x_0} : M \rightarrow [0, \infty)$; $y \mapsto d_{x_0}(y) := d(x_0, y)$ that is the Riemannian distance induced by the metric g , $\rho_{x_0}(y) := \frac{1}{2}d_{x_0}(y)^2$, $r_{\max} := \max_{y \in M} d_{x_0}(y) = \max_{y \in \partial M} d_{x_0}(y)$. We also denote by $h : \partial M \rightarrow \mathbb{R}$ the map given by

$$h := \langle -\nabla \rho_{x_0}, \nu \rangle, \quad h_{\max} := \max_{y \in \partial M} \langle -\nabla \rho_{x_0}, \nu \rangle, \quad h_{\min} := \min_{y \in \partial M} \langle -\nabla \rho_{x_0}, \nu \rangle. \quad (1)$$

In particular, this implies $\langle \nabla \rho_{x_0}(x), \nu(x) \rangle \geq 0$ for all $x \in \partial M$.

Theorem 1.3. [10, Theorem 1.3] *Let (M^n, g) be a compact Riemannian manifold whose Ricci curvature Ric_g satisfies $\text{Ric}_g \geq (n-1)\kappa$, for $\kappa \in \mathbb{R}$. Assume that M is star shaped with respect to $x_0 \in M$, then we have*

$$\sigma_1 \geq \frac{h_{\min} \mu_1}{2r_{\max} \mu_1^{\frac{1}{2}} + C_0},$$

where $C_0 := C_0(n, \kappa, r_{\max})$ is a positive constant depending only on n , κ and r_{\max} .

When M is a domain in \mathbb{R}^2 the constant becomes $C_0 = 2$ and this was previously shown in inequality (VIII) of [13]. The following theorem generalizes the inequalities (V) and (VI) of [13].

Theorem 1.4. [10, Theorem 1.4] *Let (M^n, g) be a compact Riemannian manifold whose sectional curvature K_g satisfies $\kappa_1 \leq K_g \leq \kappa_2$. Moreover, assume that there exists $x_0 \in M$ such that M is star shaped with respect to x_0 and the cut locus of x_0 in M is the empty set. Then there exist positive constants $C_i := C_i(n, \kappa_1, \kappa_2, r_{\max})$, $i = 1, 2$ such that*

1. $\lambda_k \geq \frac{C_1 q_m}{h_{\max}}$
2. $\lambda_k \leq \frac{4r_{\max}^2 q_k^2 - 2C_2 h_{\min} q_k}{h_{\min}}$.

Here, m is the multiplicity of λ_k .

In the same spirit of [1] where we have extended Theorem 1.1 and Theorem 1.2 to differential forms, we establish in this paper estimates that generalize the curvature-dependent inequalities of Theorem 1.3 and Theorem 1.4.

We denote by d the exterior differential operator and by δ its L^2 -adjoint. We also let $\iota : \partial M \rightarrow M$ the inclusion map, and \lrcorner the interior product (see Section 2.1). Now, we recall the biharmonic Steklov problem with Dirichlet-type conditions (BSD1) which has been extended to differential forms by F. El Chami, N. Ginoux, G. Habib, and O. Makhoul in [5]. For a compact Riemannian manifold (M^n, g) with smooth boundary ∂M , the boundary problem

$$(BSD1) \begin{cases} \Delta^2 \omega = 0 & \text{on } M \\ \omega = 0 & \text{on } \partial M \\ \nu \lrcorner \Delta \omega + q \iota^* \delta \omega = 0 & \text{on } \partial M \\ \iota^* \Delta \omega - q \nu \lrcorner d \omega = 0 & \text{on } \partial M, \end{cases} \quad (2)$$

on p -forms, $p \in \{0, \dots, n\}$ has a discrete spectrum consisting of an unbounded monotonously nondecreasing sequence of positive eigenvalues of finite multiplicities $(q_{j,p})_{j \geq 1}$.

Theorem 1.5. [5, Theorem 2.6] *The first eigenvalue $q_{1,p}$ of (2) can be characterized as follows:*

$$q_{1,p} = \inf \left\{ \frac{\|\Delta \omega\|_{L^2(M)}^2}{\|\nu \lrcorner d \omega\|_{L^2(\partial M)}^2 + \|\iota^* \delta \omega\|_{L^2(\partial M)}^2} \mid \omega \in \Omega^p(M), \omega|_{\partial M} = 0 \text{ and } \nabla_\nu \omega \neq 0 \right\} \quad (3)$$

$$= \inf \left\{ \frac{\|\omega\|_{L^2(\partial M)}^2}{\|\omega\|_{L^2(M)}^2} \mid \omega \in \Omega^p(M) \setminus \{0\}, \Delta \omega = 0 \text{ on } M \right\}. \quad (4)$$

Both infima are indeed minima: (3) is attained by an eigenform of (2), associated to $q_{1,p}$ and (4) is attained by $\Delta \omega$, where ω is an eigenform of (2), associated to $q_{1,p}$.

In Section 4 of this paper, we introduce a new BSD problem as follows

$$(BSD2) \begin{cases} \Delta^2 \omega = 0 & \text{on } M \\ \omega = 0 & \text{on } \partial M \\ \iota^* \delta \omega = 0 & \text{on } \partial M \\ \iota^* \Delta \omega - q \nu \lrcorner d \omega = 0 & \text{on } \partial M, \end{cases}$$

that we call BSD2.

Of particular note, the (BSN) problem for p -differential forms is generalized in [1]. We have studied three different BSN problems, each of which has a discrete spectrum consisting of an

unbounded, non-decreasing sequence of positive real eigenvalues with finite multiplicities. These problems (BSN1),(BSN2) and (BSN3) respectively are as follows:

$$\left\{ \begin{array}{ll} \Delta^2 \omega = 0 & \text{on } M \\ \nu \lrcorner \omega = 0 & \text{on } \partial M \\ \nu \lrcorner d\omega = 0 & \text{on } \partial M \\ \nu \lrcorner \Delta \omega + \ell \iota^* \delta \omega = 0 & \text{on } \partial M \\ \nu \lrcorner \Delta d\omega + \ell \iota^* \omega = 0 & \text{on } \partial M \end{array} \right. ; \quad \left\{ \begin{array}{ll} \Delta^2 \omega = 0 & \text{on } M \\ \nu \lrcorner \omega = 0 & \text{on } \partial M \\ \nu \lrcorner d\omega = 0 & \text{on } \partial M \\ \iota^* \delta \omega = 0 & \text{on } \partial M \\ \nu \lrcorner \Delta d\omega + \ell \iota^* \omega = 0 & \text{on } \partial M \end{array} \right. \text{ and } \left\{ \begin{array}{ll} \Delta^2 \omega = 0 & \text{on } M \\ \nu \lrcorner \omega = 0 & \text{on } \partial M \\ \nu \lrcorner d\omega = 0 & \text{on } \partial M \\ \nu \lrcorner \Delta \omega = 0 & \text{on } \partial M \\ \nu \lrcorner \Delta d\omega + \ell \iota^* \omega = 0 & \text{on } \partial M. \end{array} \right.$$

Notation 1.6. [14, Paragraph 2] We recall the Bochner formula: On differential forms, we have

$$\Delta = \nabla^* \nabla + W^{[p]}$$

where $W^{[p]}$ is the curvature term, which is a self-adjoint endomorphism acting on p -forms.

We also recall that the shape operator S acting on $T(\partial M)$ is defined by $S(X) = -\nabla_X \nu$ for all $x \in \partial M$, $X \in T_x \partial M$.

It is extended to all $\alpha \in \Omega^p(\partial M)$ by

$$(S^{[p]} \alpha)(X_1, \dots, X_p) = \sum_{i=1}^p \alpha(X_1, \dots, S(X_i), \dots, X_p).$$

We denote by $\lambda_{k,p}$, $\mu_{k,p}$, $\sigma_{k,p}$, $q_{k,p}$ and $\mathbf{q}_{k,p}$ the k^{th} eigenvalues for differential forms for arbitrary degree p of the following problems, respectively: Dirichlet, Neumann, Steklov, BSD1 and BSD2.

The following generalizations of [Theorem 1.3](#) and [Theorem 1.4](#) are the main results of this note. Recall that h_{\min} and h_{\max} have been defined in (1).

Theorem 1.7. Let (M^n, g) be a compact Riemannian manifold with a smooth boundary ∂M . Assume that M is star-shaped with respect to x_0 and that its Ricci curvature satisfies $\text{Ric}_g \geq (n-1)\kappa$ where $\kappa \in \mathbb{R}$, $W^{[p]} \geq 0$, and $S^{[p]} \geq 0$. Then for $p \geq 1$, we have

$$\sigma_{1,p} > \frac{\frac{1}{2} h_{\min} \mu_{1,p}}{r_{\max} \mu_{1,p}^{\frac{1}{2}} + \frac{1}{2} \max(1 + d_{x_0} \cdot H_{\kappa} \circ d_{x_0})}. \quad (5)$$

Theorem 1.8. Let (M^n, g) be a compact Riemannian manifold with a smooth boundary ∂M . Assume that M is star-shaped with respect to x_0 and that $\kappa_1 \leq K_g \leq \kappa_2$ where K_g is the sectional curvature, then for all $k \in \mathbb{N}$, $p \geq 1$, $x_0 \in M$ there exists a constant $C_2 = C_2(p, n, \kappa_1, \kappa_2)$ such that:

$$\lambda_{k,p} \leq \frac{4\mathbf{q}_{k,p}^2 r_{\max}^2 + 2\mathbf{q}_{k,p} h_{\min} C_2}{h_{\min}^2}. \quad (6)$$

Extending the proof of the first inequality of [Theorem 1.4](#) yields a negative constant C_1 for $p \geq 1$. However, the expected result would be as follows:

Conjecture 1.9. Let (M^n, g) be a compact Riemannian manifold, whose sectional curvature K_g satisfies $\kappa_1 \leq K_g \leq \kappa_2$. Assume that M is star-shaped with respect to $x_0 \in M$ then, for all $k \in \mathbb{N}$, $p \geq 1$, there exists a constant $C_1 = C_1(p, n, \kappa_1, \kappa_2) > 0$ such that:

$$\lambda_{k,p} \geq \frac{C_1 q_{m_k,p}}{r_{\max} + h_{\max}}. \quad (7)$$

Here m_k is the multiplicity of $\lambda_{k,p}$.

This inequality could be of interest when m is sufficiently large compared with k . To show inequality (5), we need the variational characterizations for the eigenvalues of the Steklov and Neumann problems on differential forms, see [Section 2.2](#). For inequality (6), we use the variational characterizations of the Dirichlet problem (see [Section 2.2](#)) and the new biharmonic Steklov problem with Dirichlet boundary conditions (BSD2). It is not clear whether an analogue of (6) holds for (BSD1).

Remark: We are currently studying the spectrum of the Neumann, BSN and BSD problems on differential forms on the unit ball, in order to check whether the inequalities established both in this paper and in [1] are optimal.

Outline of the Article: In Section 2 of this paper, we recall some definitions and ingredients that we use, such as integration by parts formulas and comparison theorems. In section Section 3 we establish the Rellich identity for differential forms. In Section 4 we introduce the new biharmonic Steklov problem with Dirichlet boundary conditions on differential forms. Finally, in section Section 5, we show Theorem 1.7 and Theorem 1.8.

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2 Preliminaries

In this section, we provide some results and tools that are used in our work. The results of Section 2.1 and Section 2.2 are already in [1].

2.1 Review on differential forms

For differential forms, we denote by $\langle \cdot, \cdot \rangle$ the natural inner product induced by the Riemannian metric and by $d\mu_g$ the Riemannian volume form. We define $(\omega, \omega') := \int_M \langle \omega, \omega' \rangle d\mu_g$. The codifferential operator δ , is the formal adjoint of the exterior derivative d , i.e., for all $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^{p+1}(M)$ with support in $M \setminus \partial M$, we have $(d\alpha, \beta) = (\alpha, \delta\beta)$. Throughout this paper, we identify the tangent space TM with its cotangent space T^*M via the musical isomorphism. That is, for a vector field X , we denote by $X^\flat = g(X, \cdot)$ the corresponding 1-form. Given a local orthonormal basis $(e_i)_{i=1}^n$, we have the local expressions of $d = \sum_i e_i^\flat \wedge \nabla_{e_i}$ and $\delta = -\sum_i e_i \lrcorner \nabla_{e_i}$, see for instance [8, Proposition 2.61]. The Hodge-de-Rham Laplacian is the operator Δ acting on $\Omega^p(M)$, defined by

$$\Delta = d\delta + \delta d.$$

We recall the following integration by parts formulas that are essential in our work:

Proposition 2.1. *For $\omega \in \Omega^p(M)$ and $\omega' \in \Omega^{p+1}(M)$, we have*

$$\int_M \langle d\omega, \omega' \rangle d\mu_g = \int_M \langle \omega, \delta\omega' \rangle d\mu_g - \int_{\partial M} \langle \iota^* \omega, \nu \lrcorner \omega' \rangle d\mu_g. \quad (8)$$

For all $\omega, \omega' \in \Omega^p(M)$ we have

$$\begin{aligned} \int_M \langle \Delta\omega, \omega' \rangle d\mu_g &= \int_M \langle d\omega, d\omega' \rangle d\mu_g + \int_M \langle \delta\omega, \delta\omega' \rangle d\mu_g \\ &\quad + \int_{\partial M} \langle \nu \lrcorner d\omega, \iota^* \omega' \rangle d\mu_g - \int_{\partial M} \langle \iota^* \delta\omega, \nu \lrcorner \omega' \rangle d\mu_g. \end{aligned} \quad (9)$$

Identity (8) is shown in [18, p.182]. It implies (9), as shown in [4, Equation 30] and [18, p.421]. In particular, for any $\omega, \omega' \in \Omega^p(M)$, we deduce [18, p.182]

$$\begin{aligned} &\int_M \left(\langle \Delta\omega, \omega' \rangle - \langle \omega, \Delta\omega' \rangle \right) d\mu_g \\ &= \int_{\partial M} \left(\langle \nu \lrcorner d\omega, \iota^* \omega' \rangle - \langle \iota^* \omega, \nu \lrcorner d\omega' \rangle + \langle \nu \lrcorner \omega, \iota^* \delta\omega' \rangle - \langle \iota^* \delta\omega, \nu \lrcorner \omega' \rangle \right) d\mu_g. \end{aligned} \quad (10)$$

Replacing ω with $\Delta\omega$ in (10), we obtain:

$$\begin{aligned} \int_M \langle \Delta^2\omega, \omega' \rangle d\mu_g &= \int_M \langle \Delta\omega, \Delta\omega' \rangle d\mu_g \\ &\quad + \int_{\partial M} \left(\langle \nu \lrcorner d\Delta\omega, \iota^* \omega' \rangle - \langle \iota^* \Delta\omega, \nu \lrcorner d\omega' \rangle + \langle \nu \lrcorner \Delta\omega, \iota^* \delta\omega' \rangle - \langle \iota^* \delta\Delta\omega, \nu \lrcorner \omega' \rangle \right) d\mu_g. \end{aligned} \quad (11)$$

In addition, for $\omega = \omega'$ in (9), we deduce the following formula:

$$\begin{aligned} \int_M \langle \Delta \omega, \omega \rangle d\mu_g &= \int_M (|d\omega|^2 + |\delta\omega|^2) d\mu_g + \int_{\partial M} \langle \nu \lrcorner d\omega, \iota^* \omega \rangle d\mu_g \\ &\quad - \int_{\partial M} \langle \iota^* \delta\omega, \nu \lrcorner \omega \rangle d\mu_g. \end{aligned} \quad (12)$$

Note that, in [18], the identities are given using ω on the boundary rather than $\iota^* \omega$, as we do in this paper. This difference is because the normal component is always zero in our three BSN problems.

2.2 Boundary problems and variational characterizations

In this section, we recall the classical boundary problems and present the variational characterizations of their eigenvalues.

Definition 2.2. *Let (M^n, g) be a compact connected Riemannian manifold with boundary ∂M . The Dirichlet problem acting on differential forms is defined as follows:*

$$\begin{cases} \Delta \omega = \lambda \omega & \text{on } M \\ \omega = 0 & \text{on } \partial M, \end{cases} \quad (13)$$

where $\lambda \in \mathbb{R}$. The Neumann problem for differential forms (with absolute boundary conditions) is defined by

$$\begin{cases} \Delta \omega = \mu \omega & \text{on } M \\ \nu \lrcorner \omega = 0 & \text{on } \partial M \\ \nu \lrcorner d\omega = 0 & \text{on } \partial M, \end{cases} \quad (14)$$

where $\mu \in \mathbb{R}$.

The spectrum of the Laplacian for the Dirichlet and Neumann problems on forms is discrete, consisting respectively of eigenvalues (see [9])

$$0 < \lambda_{1,p} \leq \lambda_{2,p} \leq \dots \quad \text{and} \quad 0 \leq \mu_{1,p} \leq \mu_{2,p} \leq \dots$$

Given any p -form ω on the boundary, it is shown in [15] that there exists a unique p -form $\hat{\omega}$ on M solving the following problem:

$$\begin{cases} \Delta \hat{\omega} = 0 & \text{on } M \\ \iota^* \hat{\omega} = \omega & \text{on } \partial M \\ \nu \lrcorner \hat{\omega} = 0 & \text{on } \partial M. \end{cases} \quad (15)$$

The differential form $\hat{\omega}$ is called the tangential harmonic extension of ω .

The kernel of the Dirichlet problem is trivial, while the kernels of the Neumann and Steklov problems are given by the absolute de Rham cohomology

$$H_A^p(M) = \{\omega \in \Omega^p(M) \mid d\omega = 0 \text{ on } M, \delta\omega = 0 \text{ on } M, \nu \lrcorner \omega = 0 \text{ on } \partial M\}. \quad (16)$$

Definition 2.3. *The Steklov operator $T^{[p]} : \Lambda^p(\partial M) \rightarrow \Lambda^p(\partial M)$ is defined by $T^{[p]}(\omega) = -\nu \lrcorner d\hat{\omega}$.*

It is proved in [15, Section 2], that the operator $T^{[p]}$ is an elliptic, self-adjoint and pseudodifferential operator with a discrete spectrum consisting of eigenvalues

$$0 \leq \sigma_{1,p} \leq \sigma_{2,p} \leq \dots$$

We provide some of the variational characterizations of the eigenvalues of the previous problems on differential forms, derived from the min-max principle (see [2, Chapter 1]), that we will use later on. Several of these expressions can also be found in [4, 9, 11, 15]. The proofs primarily follow the same lines as those used for the scalar case and we omit them here.

Let us set $H_N^1(M) := \{\omega \in H^1(M) \mid \nu \lrcorner \omega = 0\}$ and recall that the absolute de Rham cohomology of order p , $H_A^p(M)$ is given by (16).

Proposition 2.4. *The k^{th} eigenvalue $\lambda_{k,p}$ of the problem (13) is given by*

$$\lambda_{k,p} = \inf_{\substack{V \subset H_0^1(M) \\ \dim(V)=k}} \sup_{0 \neq \omega \in V} \frac{\|d\omega\|_{L^2(M)}^2 + \|\delta\omega\|_{L^2(M)}^2}{\|\omega\|_{L^2(M)}^2} \quad (17)$$

$$= \inf_{\substack{V \subset H^2(M) \cap H_0^1(M) \\ \dim(V)=k}} \sup_{0 \neq \nabla \omega \in V} \frac{\|\Delta \omega\|_{L^2(M)}^2}{\|d\omega\|_{L^2(M)}^2 + \|\delta\omega\|_{L^2(M)}^2}. \quad (18)$$

In addition, the first non-zero eigenvalue $\mu_{1,p}$ of the problem (14) is given by

$$\mu_{1,p} = \inf \left\{ \frac{\|d\omega\|_{L^2(M)}^2 + \|\delta\omega\|_{L^2(M)}^2}{\|\omega\|_{L^2(M)}^2} \mid \omega \in H_N^1(M) \text{ and } \omega \perp_{L^2(M)} H_A^p(M) \right\}. \quad (19)$$

Its k^{th} non-zero eigenvalue $\mu_{k,p}$ is given by

$$\mu_{k,p} = \inf_{\substack{V \subset H_N^1(M) \\ \dim(V)=k+\dim H_A^p(M)}} \sup_{0 \neq \omega \in V} \frac{\|d\omega\|_{L^2(M)}^2 + \|\delta\omega\|_{L^2(M)}^2}{\|\omega\|_{L^2(M)}^2}. \quad (20)$$

Moreover, the first non-zero eigenvalue of the Steklov problem on differential forms is given by

$$\sigma_{1,p} = \inf \left\{ \frac{\|d\omega\|_{L^2(M)}^2 + \|\delta\omega\|_{L^2(M)}^2}{\|\omega\|_{L^2(\partial M)}^2} \mid \omega \in H_N^1(M) \text{ and } \omega \perp_{L^2(\partial M)} H_A^p(M) \right\}. \quad (21)$$

2.3 Comparison theorems

Below are the comparison theorems for the Laplacian and the Hessian that will be used to prove the inequalities stated in [Theorem 1.7](#) and [Theorem 1.8](#).

Definition 2.5. [[10](#), Section 2.2] For all $\kappa \in \mathbb{R}$, we denote by $H_\kappa : [0, \infty) \rightarrow \mathbb{R}$ the function that satisfies the following Riccati differential equation

$$H'_\kappa + H_\kappa^2 + \kappa = 0, \text{ with } \lim_{r \rightarrow 0} \frac{rH_\kappa(r)}{n-1} = 1.$$

Actually, we have

$$H_\kappa(r) = \begin{cases} (n-1)\sqrt{\kappa} \cot(\sqrt{\kappa}r) & \text{if } \kappa > 0, \\ \frac{n-1}{r} & \text{if } \kappa = 0, \\ (n-1)\sqrt{|\kappa|} \coth(\sqrt{|\kappa|}r) & \text{if } \kappa < 0. \end{cases}$$

We denote by $A \wedge B := \min\{A, B\}$ and $A \vee B := \max\{A, B\}$, for $A, B \in \mathbb{R}$. The following theorems are stated in [[3](#), Theorem 1.59 and 1.82] and [[10](#), Theorem 2.1 and 2.2].

Theorem 2.6. Let (M^n, g) be a complete Riemannian manifold. Let $\gamma : [0, L] \rightarrow M$ be a minimal geodesic from a point x_0 on M , such that the intersection of its image with the cut-locus of x_0 is empty. Suppose also that $\kappa_1 \leq K_g(X, \gamma'(t)) \leq \kappa_2$ for all $t \in [0, L]$ and $X \in T_{\gamma(t)}M$ orthogonal to $\gamma'(t)$. Then:

1. The function d_{x_0} satisfies the following inequalities:

$$\begin{aligned} \nabla^2 d_{x_0}(X, X) &\leq \frac{H_{\kappa_1}(t)}{n-1} g(X, X), & \forall t \in [0, L], X \in \langle \gamma'(t) \rangle^\perp \subset T_{\gamma(t)}M. \\ \nabla^2 d_{x_0}(X, X) &\geq \frac{H_{\kappa_2}(t)}{n-1} g(X, X), & \forall t \in \left[0, L \wedge \frac{\pi}{2\sqrt{\kappa_2 \vee 0}}\right], X \in \langle \gamma'(t) \rangle^\perp \subset T_{\gamma(t)}M. \end{aligned}$$

2. The function ρ_{x_0} satisfies the following estimates:

$$\begin{aligned} \nabla^2 \rho_{x_0}(X, X) &\leq \frac{tH_{\kappa_1}(t)}{n-1} g(X, X), & \forall t \in [0, L], X \in \langle \gamma'(t) \rangle^\perp \subset T_{\gamma(t)}M. \\ \nabla^2 \rho_{x_0}(X, X) &\geq \frac{tH_{\kappa_2}(t)}{n-1} g(X, X), & \forall t \in [0, L \wedge \frac{\pi}{2\sqrt{\kappa_2 \vee 0}}], X \in \langle \gamma'(t) \rangle^\perp \subset T_{\gamma(t)}M. \end{aligned}$$

Theorem 2.7. *Let (M^n, g) be a complete Riemannian manifold. The distance function and the squared distance function satisfy the following inequalities:*

1. *If $\text{Ric}_g \geq (n-1)\kappa$, $\kappa \in \mathbb{R}$, then, for all $x_0 \in M$, we have*

$$-\Delta d_{x_0}(y) \leq H_\kappa(d_{x_0}(y)) \quad \text{and} \quad -\Delta \rho_{x_0}(y) \leq 1 + d_{x_0}(y) \cdot H_\kappa(d_{x_0}(y)).$$

2. *Under the same assumptions as in Theorem 2.6, we have:*

a) *For all $t \in [0, L]$,*

$$-\Delta d_{x_0}(\gamma(t)) \leq H_{\kappa_1}(t) \quad \text{and} \quad -\Delta \rho_{x_0}(\gamma(t)) \leq 1 + tH_{\kappa_1}(t).$$

b) *For all $t \in \left[0, L \wedge \frac{\pi}{2\sqrt{\kappa_2 \vee 0}}\right]$, we have*

$$-\Delta d_{x_0}(\gamma(t)) \geq H_{\kappa_2}(t) \quad \text{and} \quad -\Delta \rho_{x_0}(\gamma(t)) \geq 1 + tH_{\kappa_2}(t).$$

3 Rellich identity for differential forms

In this section we give the generalization of the Rellich identity for differential forms, which is a particular expression of integration by parts. It was proven in [10, Section 3] in the case of functions on a compact Riemannian manifold (M, g) with C^2 boundary.

Let (M^n, g) be a compact manifold with smooth boundary $\partial M = \Sigma$. Given a smooth vector field F on M , we define a self-adjoint endomorphism $T_F^{[p]}$ of $\Omega^p(M)$ as follows (see [17]): if ω is a p -form and X_1, \dots, X_p are tangent vector fields, then

$$(T_F^{[p]} \omega)(X_1, \dots, X_p) = \sum_{k=1}^p \omega(X_1, \dots, \nabla_{X_k} F, \dots, X_p).$$

Lemma 3.1. [17, Lemma 2.1] *Let ω be a p -form and F a vector field on M . We have:*

$$\mathcal{L}_F \omega = \nabla_F \omega + T_F^{[p]} \omega,$$

where \mathcal{L}_F is the Lie derivative in the direction of F .

We now state the Rellich (or Pohozaev) identity on differential forms. This formula expresses the Laplacian of a differential form in terms of expressions of order at most 1.

Theorem 3.2. [6] *Let (M^n, g) be a compact Riemannian manifold with smooth boundary ∂M and let $F : M \rightarrow TM$ be a Lipschitz vector field on M . Then for every $\omega \in \Omega^p(M)$ we have*

$$\begin{aligned} & \int_M \langle \Delta \omega, F \lrcorner d\omega \rangle d\mu_g + \int_M \langle \delta \omega, F \lrcorner \Delta \omega \rangle d\mu_g \\ &= -\frac{1}{2} \int_{\partial M} (|d\omega|^2 + |\delta \omega|^2) \langle F, \nu \rangle d\mu_g + \int_{\partial M} \langle F \wedge i^*(\delta \omega), \nu \lrcorner d\omega \rangle d\mu_g + \int_{\partial M} \langle i^*(F \lrcorner d\omega), \nu \lrcorner d\omega \rangle d\mu_g \\ & \quad + \int_{\partial M} \langle i^*(F \lrcorner \delta \omega), \nu \lrcorner \delta \omega \rangle d\mu_g - \frac{1}{2} \int_M (|d\omega|^2 + |\delta \omega|^2) \text{div } F d\mu_g + \int_M \langle \delta \omega, dF \lrcorner d\omega \rangle d\mu_g \\ & \quad + \int_M \langle T_F^{[p+1]} d\omega, d\omega \rangle d\mu_g + \int_M \langle T_F^{[p-1]} \delta \omega, \delta \omega \rangle d\mu_g. \end{aligned} \tag{22}$$

Proof. The proof is essentially based on the integration by parts formulas stated in Proposition 2.1. We

compute

$$\begin{aligned}
\int_M \langle \Delta \omega, F \lrcorner d\omega \rangle d\mu_g &= \int_M \langle \delta d\omega, F \lrcorner d\omega \rangle d\mu_g + \int_M \langle d\delta\omega, F \lrcorner d\omega \rangle d\mu_g \\
&= \int_M \langle d\omega, d(F \lrcorner d\omega) \rangle d\mu_g + \int_{\partial M} \langle \iota^*(F \lrcorner d\omega), \nu \lrcorner d\omega \rangle d\mu_g \\
&\quad + \int_M \langle \delta\omega, \delta(F \lrcorner d\omega) \rangle d\mu_g - \int_{\partial M} \langle \iota^*(\delta\omega), \nu \lrcorner F \lrcorner d\omega \rangle d\mu_g \\
&= \int_M \langle d\omega, \mathcal{L}_F(d\omega) \rangle d\mu_g + \int_{\partial M} \langle \iota^*(F \lrcorner d\omega), \nu \lrcorner d\omega \rangle d\mu_g \\
&\quad + \int_M \langle \delta\omega, \delta(F \lrcorner d\omega) \rangle d\mu_g - \int_{\partial M} \langle \iota^*(\delta\omega), \nu \lrcorner F \lrcorner d\omega \rangle d\mu_g \\
&= \int_M \langle d\omega, \nabla_F d\omega \rangle d\mu_g + \int_M \langle d\omega, T_F^{[p+1]} d\omega \rangle d\mu_g + \int_{\partial M} \langle \iota^*(F \lrcorner d\omega), \nu \lrcorner d\omega \rangle d\mu_g \\
&\quad + \int_M \langle \delta\omega, \delta(F \lrcorner d\omega) \rangle d\mu_g - \int_{\partial M} \langle \iota^*(\delta\omega), \nu \lrcorner F \lrcorner d\omega \rangle d\mu_g. \tag{23}
\end{aligned}$$

We evaluate now the term $\int_M \langle d\omega, \nabla_F d\omega \rangle d\mu_g$:

$$\begin{aligned}
\int_M \langle d\omega, \nabla_F d\omega \rangle d\mu_g &= \frac{1}{2} \int_M F(|d\omega|^2) d\mu_g \\
&= -\frac{1}{2} \int_M |d\omega|^2 \operatorname{div} F d\mu_g - \frac{1}{2} \int_{\partial M} |d\omega|^2 \langle F, \nu \rangle d\mu_g.
\end{aligned}$$

We recall now that for any vector field X and $(e_i)_{1 \leq i \leq n}$ an orthonormal frame we have

$$\begin{aligned}
\delta(X \lrcorner \omega) &= -e_i \lrcorner \nabla_{e_i}(X \lrcorner \omega) \\
&= -e_i \lrcorner (\nabla_{e_i} X \lrcorner \omega + X \lrcorner \nabla_{e_i} \omega) \\
&= -e_i \lrcorner \nabla_{e_i} X \lrcorner \omega + X \lrcorner \underbrace{e_i \lrcorner \nabla_{e_i} \omega}_{-\delta\omega}.
\end{aligned}$$

In our case we need $\delta(F \lrcorner d\omega)$, which will be equal to $-e_i \lrcorner \nabla_{e_i} F \lrcorner d\omega - F \lrcorner \delta d\omega$. Now to compute the term $\int_M \langle \delta\omega, \delta(F \lrcorner d\omega) \rangle d\mu_g$ we use that

$$\delta(F \lrcorner d\omega) = dF \lrcorner d\omega - F \lrcorner \delta d\omega = dF \lrcorner d\omega - F \lrcorner \Delta \omega + F \lrcorner \delta d\omega,$$

where $dF = \sum_{i=1}^n e_i \wedge \nabla_{e_i} F$. By replacing this last equalities in (23), we obtain

$$\begin{aligned}
&\int_M \langle \Delta \omega, F \lrcorner d\omega \rangle d\mu_g + \int_M \langle \delta\omega, F \lrcorner \Delta \omega \rangle d\mu_g \\
&= -\frac{1}{2} \int_{\partial M} |d\omega|^2 \langle F, \nu \rangle d\mu_g - \frac{1}{2} \int_M |d\omega|^2 \operatorname{div} F d\mu_g + \int_M \langle d\omega, T_F^{[p+1]} d\omega \rangle d\mu_g + \int_{\partial M} \langle \iota^*(F \lrcorner d\omega), \nu \lrcorner d\omega \rangle d\mu_g \\
&\quad + \int_M \langle \delta\omega, dF \lrcorner d\omega \rangle d\mu_g + \int_M \langle \delta\omega, F \lrcorner \delta d\omega \rangle d\mu_g - \int_{\partial M} \langle \iota^*(\delta\omega), \nu \lrcorner F \lrcorner d\omega \rangle d\mu_g. \tag{24}
\end{aligned}$$

It remains to calculate the term $\int_M \langle \delta\omega, F \lrcorner \delta d\omega \rangle d\mu_g$. We have

$$\begin{aligned}
\int_M \langle \delta\omega, F \lrcorner \delta d\omega \rangle d\mu_g &= \int_M \langle \delta\omega, \mathcal{L}_F(\delta\omega) \rangle d\mu_g - \int_M \langle \delta\omega, d(F \lrcorner \delta\omega) \rangle d\mu_g \\
&= \int_M \langle \delta\omega, \nabla_F \delta\omega \rangle d\mu_g + \int_M \langle \delta\omega, T_F^{[p-1]} \delta\omega \rangle d\mu_g + \int_{\partial M} \langle \iota^*(F \lrcorner \delta\omega), \nu \lrcorner \delta\omega \rangle d\mu_g \\
&= \int_M \frac{1}{2} F(|\delta\omega|^2) d\mu_g + \int_M \langle \delta\omega, T_F^{[p-1]} \delta\omega \rangle d\mu_g + \int_{\partial M} \langle \iota^*(F \lrcorner \delta\omega), \nu \lrcorner \delta\omega \rangle d\mu_g \\
&= \frac{1}{2} \int_M |\delta\omega|^2 \delta F d\mu_g - \frac{1}{2} \int_{\partial M} |\delta\omega|^2 \langle F, \nu \rangle d\mu_g + \int_M \langle \delta\omega, T_F^{[p-1]} \delta\omega \rangle d\mu_g + \int_{\partial M} \langle \iota^*(F \lrcorner \delta\omega), \nu \lrcorner \delta\omega \rangle d\mu_g \\
&= -\frac{1}{2} \int_M |\delta\omega|^2 \operatorname{div} F d\mu_g - \frac{1}{2} \int_{\partial M} |\delta\omega|^2 \langle F, \nu \rangle d\mu_g + \int_M \langle \delta\omega, T_F^{[p-1]} \delta\omega \rangle d\mu_g + \int_{\partial M} \langle \iota^*(F \lrcorner \delta\omega), \nu \lrcorner \delta\omega \rangle d\mu_g.
\end{aligned}$$

This last identity combined to (24) leads to the result. \square

4 A new biharmonic Steklov problem with Dirichlet boundary conditions on differential forms

In this section, we follow the same methods that are used for the study of the Steklov biharmonic problem with Dirichlet-type boundary conditions as in [5]. These methods have also been used for other problems in [1].

In order to establish Inequality (6), we need a variational characterization of the BSD different from (3) and (4). This characterization is needed to be of the following form:

$$\mathfrak{q}_{1,p} = \inf \left\{ \frac{\|\Delta \omega\|_{L^2(M)}^2}{\|\nu \lrcorner d\omega\|_{L^2(\partial M)}^2} \mid \omega \in \Omega^p(M), \omega|_{\partial M} = 0, \iota^* \delta \omega = 0 \text{ on } \partial M \text{ and } \nabla_\nu \omega \neq 0 \right\}.$$

To find a spectral problem with such a variational characterization, we compute the critical points of the following functional:

$$Q(\omega) := \frac{\|\Delta \omega\|_{L^2(M)}^2}{\|\nu \lrcorner d\omega\|_{L^2(\partial M)}^2},$$

that will be the eigenforms of the new problem that we denote by BSD2. Let the space

$$H_{BSD2}^2(M) = \{\omega \in H^2(M) \mid \iota^* \delta \omega = 0 \text{ on } \partial M \text{ and } \omega|_{\partial M} = 0\}.$$

The form ω is a critical point if and only if for all $\omega' \in H_{BSD2}^2(M)$ we have

$$d_\omega Q(\omega') = \frac{2(\Delta \omega, \Delta \omega')_{L^2(M)} \|\nu \lrcorner d\omega\|_{L^2(\partial M)}^2 - 2(\nu \lrcorner d\omega, \nu \lrcorner d\omega')_{L^2(\partial M)} \|\Delta \omega\|_{L^2(M)}^2}{\|\nu \lrcorner d\omega\|_{L^2(\partial M)}^4} = 0.$$

This implies that

$$(\Delta \omega, \Delta \omega')_{L^2(M)} \|\nu \lrcorner d\omega\|_{L^2(\partial M)}^2 = (\nu \lrcorner d\omega, \nu \lrcorner d\omega')_{L^2(\partial M)} \|\Delta \omega\|_{L^2(M)}^2.$$

Hence, performing an integration by parts using (11) we get

$$(\Delta^2 \omega, \omega')_{L^2(M)} - (\iota^* \Delta \omega, \nu \lrcorner d\omega')_{L^2(\partial M)} = \frac{\|\Delta \omega\|_{L^2(M)}^2}{\|\nu \lrcorner d\omega\|_{L^2(\partial M)}^2} (\nu \lrcorner d\omega, \nu \lrcorner d\omega')_{L^2(\partial M)},$$

and thus,

$$(\Delta^2 \omega, \omega')_{L^2(M)} = \left(\frac{\|\Delta \omega\|_{L^2(M)}^2}{\|\nu \lrcorner d\omega\|_{L^2(\partial M)}^2} \nu \lrcorner d\omega - \iota^* \Delta \omega, \nu \lrcorner d\omega' \right)_{L^2(\partial M)}. \quad (25)$$

Since $\omega'|_{\partial M} = 0$ and $\iota^* \delta \omega' = 0$, this implies that $\nu \lrcorner \nabla_\nu \omega' = 0$ and $\nu \lrcorner d\omega' = \iota^*(\nabla_\nu \omega')$. Following the argument in [1, p.37], we fix a positive ϵ and define the set $D_\epsilon := \{x \in M \mid d(x, \partial M) \geq \epsilon\}$. We consider a test form ω' vanishing on the set of points whose distance from the boundary is less than ϵ . We have from (25) that $(\Delta^2 \omega, \omega')_{L^2(M)} = 0$ for all ω' , hence $\Delta^2 \omega = 0$ on D_ϵ . Letting ϵ tend to zero, we obtain that $\Delta^2 \omega = 0$ on M , thus the differential form $\nu \lrcorner d\omega'$ can take any value, so $\frac{\|\Delta \omega\|_{L^2(M)}^2}{\|\nu \lrcorner d\omega\|_{L^2(\partial M)}^2} \nu \lrcorner d\omega = \iota^* \Delta \omega$. We then conclude the following problem:

$$(BSD2) \begin{cases} \Delta^2 \omega = 0 & \text{on } M \\ \omega = 0 & \text{on } \partial M \\ \iota^* \delta \omega = 0 & \text{on } \partial M \\ \iota^* \Delta \omega - \mathfrak{q} \nu \lrcorner d\omega = 0 & \text{on } \partial M. \end{cases} \quad (26)$$

We follow the same steps as in [1] and evaluate the principal symbols of $B_1 \omega = \omega|_{\partial M}$, $B_2 \omega = \iota^* \delta \omega$, and $B_3 \omega = \iota^* \Delta \omega$, for $v \in T^*M \setminus \{0\}$ and $y(t) = e^{-|v|t}(at + b)\omega_0$ with $a, b \in \mathbb{R}$, $\omega_0 \in \Lambda^p T_x^* M|_{\partial M}$. We obtain:

$$\sigma_{B_1}(-iv + \partial_t \nu^b)(y)(0) = b\omega_0,$$

$$\sigma_{B_2}(-iv + \partial_t \nu^b)(y)(0) = -a \nu \lrcorner \omega_0 + |v| b \nu \lrcorner \omega_0 + ib \nu \lrcorner \iota^* \omega_0,$$

$$\sigma_{B_3}(-iv + \partial_t \nu^b)(y)(0) = 2a|v| \iota^* \omega_0.$$

Thus, we set the map

$$\begin{aligned}\Phi : \mathcal{M}_{\Delta^2, v}^+ &\longrightarrow \bigoplus_{j=1}^4 E_j \\ y &\longmapsto \Phi(y)\end{aligned}$$

given by

$$\Phi(y) = (b\omega_0, -a\nu \lrcorner \omega_0 + |v|b\nu \lrcorner \omega_0 + ibv\nu \iota^* \omega_0, 2a|v|\iota^* \omega_0).$$

It is easy to check that Φ is injective and thus is an isomorphism as the dimensions of the spaces is $2\binom{n}{p}$. Thus, the problem (BSD2) is elliptic in the sense of Shapiro-Lopatinskij.

Now to show that the problem (BSD2) has a discrete spectrum of eigenvalues, we follow the same steps as in [1, Section 3.1] and [5, Section 2.2]. We define the space

$$Z_{BSD2} = \{\omega \in \Omega^p(M) \mid \Delta^2 \omega = 0, \omega|_{\partial M} = 0, \text{ and } \iota^* \delta \omega = 0 \text{ on } \partial M\}.$$

For ω an eigenform and $\omega' \in Z_{BSD2}$, we easily obtain through integration by parts (11) that

$$\int_M \langle \Delta \omega, \Delta \omega' \rangle d\mu_g = \mathbf{q} \int_{\partial M} \langle \nu \lrcorner d\omega, \nu \lrcorner d\omega' \rangle d\mu_g.$$

Referring to the proof of [1, Section 3], we get the following theorem:

Theorem 4.1. *Let (M^n, g) be a compact Riemannian manifold with a smooth boundary ∂M . The following problem:*

$$\begin{cases} \Delta^2 \omega = 0 & \text{on } M \\ \omega = 0 & \text{on } \partial M \\ \iota^* \delta \omega = 0 & \text{on } \partial M \\ \iota^* \Delta \omega - \mathbf{q} \nu \lrcorner d\omega = 0 & \text{on } \partial M, \end{cases} \quad (27)$$

for differential p -forms, has a discrete spectrum consisting of a non decreasing sequence of positive eigenvalues of finite multiplicities, denoted by $(\mathbf{q}_{i, \mathbf{p}})_{i \geq 1}$.

Thus, we have the following variational characterizations, whose proofs are identical to those of the variational characterizations (3) treated in [1, Theorem 1.17] and [5, Theorem 2.6] :

$$\mathbf{q}_{1, p} = \inf \left\{ \frac{\|\Delta \omega\|_{L^2(M)}^2}{\|\nu \lrcorner d\omega\|_{L^2(\partial M)}^2} \mid \omega \in \Omega^p(M), \omega|_{\partial M} = 0, \iota^* \delta \omega = 0 \text{ on } \partial M \text{ and } \nabla_\nu \omega \neq 0 \right\}. \quad (28)$$

For all $k \in \mathbb{N}$, the variational characterization of the k^{th} eigenvalue of the BSD2 on differential forms is given by:

$$\mathbf{q}_{k, p} = \inf_{\substack{\mathcal{V}_k \subset H_{BSD2}^2(M) \\ \dim(\mathcal{V}_k / H_{BSD2}^2(M) \cap \mathcal{V}_k) = k}} \sup_{\omega \in \mathcal{V}_k \setminus H_{BSD2}^2(M)} \frac{\|\Delta \omega\|_{L^2(M)}^2}{\|\nu \lrcorner d\omega\|_{L^2(\partial M)}^2}. \quad (29)$$

Lemma 4.2. *Using the notations defined previously, we have the following inequality:*

$$q_{k, p} \leq \mathbf{q}_{k, p}.$$

Proof. First of all the quotient of $\mathbf{q}_{k, p}$ is bigger than the one of $q_{k, p}$ as we omit from the denominator the term $\|\iota^* \delta \omega\|_{L^2(\partial M)}^2$. In addition, the space where $\mathbf{q}_{k, p}$ is defined is smaller than the one of $q_{k, p}$, and we know that the infimum over the larger set is the smaller, which explains the order given for the variational characterizations. \square

5 Eigenvalue estimates

In this section, we establish Kuttler-Sigillito inequalities with geometric conditions on the curvature of the manifold. We start by the following remark that will be used later on.

Lemma 5.1. *Let $\omega \in \Omega^p(M)$ and $X \in \chi(M)$, then the following inequality holds:*

$$|\nabla_X \omega| \leq |X| \cdot |\nabla \omega|.$$

Proof. Let $\{e_j\}_{1 \leq j \leq n}$ be an orthonormal frame of TM . We compute

$$\begin{aligned} |\nabla_X \omega|^2 &= \sum_{j,k=1}^n \langle X, e_j \rangle \cdot \langle X, e_k \rangle \cdot \langle \nabla_{e_j} \omega, \nabla_{e_k} \omega \rangle \\ &\leq \sum_{j,k=1}^n |\langle X, e_j \rangle| \cdot |\langle X, e_k \rangle| \cdot |\nabla_{e_j} \omega| \cdot |\nabla_{e_k} \omega| \\ &= \left(\sum_{j=1}^n |\langle X, e_j \rangle| \cdot |\nabla_{e_j} \omega| \right)^2 \\ &\leq \left(\sum_{j=1}^n \langle X, e_j \rangle^2 \right) \cdot \left(\sum_{j=1}^n |\nabla_{e_j} \omega|^2 \right) \\ &= |X|^2 \cdot |\nabla \omega|^2. \end{aligned}$$

□

For the definitions of $W^{[p]}$ and $S^{[p]}$ see [Notation 1.6](#). Recall that $\sigma_{1,p}$ is the first non-zero eigenvalue of the Steklov operator, see [Definition 2.3](#). The following theorem gives us Inequality (5).

Theorem 5.2. *Let (M^n, g) be a compact Riemannian manifold with smooth boundary ∂M . Assume that M is star-shaped with respect to x_0 and that $\text{Ric}_g \geq (n-1)\kappa$ where $\kappa \in \mathbb{R}$, $W^{[p]} \geq 0$ and $S^{[p]} \geq 0$. Then, for $p \geq 1$, we have the following inequality:*

$$\sigma_{1,p} > \frac{\frac{1}{2} h_{\min} \mu_{1,p}}{r_{\max} \mu_{1,p}^{\frac{1}{2}} + \frac{1}{2} \max_M (1 + d_{x_0} \cdot H_\kappa \circ d_{x_0})}. \quad (30)$$

Remark 5.3. *We have particular cases, where the hypothesis of [Theorem 5.2](#) can be reformulated as follows:*

- i. If $p = 1$, the hypothesis of the theorem expresses into the following way: $W^{[p]} = \text{Ric}_g \geq 0$, $S^{[p]} = S \geq 0$ and $\kappa > 0$.*
- ii. If $p = 1$, $W^{[p]} = \text{Ric}_g \geq 0$, $S^{[p]} \geq 0$ and $\kappa \leq 0$, the best lower bound to be used is with H_0 , so the inequality becomes*

$$\sigma_{1,p} > \frac{\frac{1}{2} h_{\min} \mu_{1,p}}{r_{\max} \mu_{1,p}^{\frac{1}{2}} + \frac{1}{2} \max_M (1 + d_{x_0} \cdot H_0 \circ d_{x_0})}.$$

Proof of [Theorem 5.2](#). Let $u \in \Omega^p(M)$ be a $\sigma_{1,p}$ -eigenform such that u is $L^2(\partial M)$ -orthogonal to $H_A^p(M)$ and let $\omega := \text{proj}_{\perp_{L^2(M)}}(u)$ in $H_A^p(M)^\perp$. We use the following identity which follows from integration by parts:

$$\int_M \frac{1}{2} |\omega|^2 \Delta \rho_{x_0} d\mu_g - \int_M \langle \nabla \rho_{x_0}, \nabla (\frac{1}{2} |\omega|^2) \rangle d\mu_g = \frac{1}{2} \int_{\partial M} |\omega|^2 \langle \nu, \nabla \rho_{x_0} \rangle d\mu_g,$$

and since $X(\frac{1}{2} |\omega|^2) = \langle \nabla_X \omega, \omega \rangle$, replacing X by $\nabla \rho_{x_0}$ we get

$$\int_M \frac{1}{2} |\omega|^2 \Delta \rho_{x_0} d\mu_g - \int_M \langle \nabla_{\nabla \rho_{x_0}} \omega, \omega \rangle d\mu_g = \frac{1}{2} \int_{\partial M} |\omega|^2 \cdot \partial_\nu \rho_{x_0} d\mu_g.$$

Using [Theorem 2.7](#), we get

$$\begin{aligned} -\frac{1}{2} \int_{\partial M} |\omega|^2 \cdot \partial_\nu \rho_{x_0} d\mu_g &\leq \int_M \langle \nabla_{\nabla \rho_{x_0}} \omega, \omega \rangle d\mu_g + \frac{1}{2} \int_M |\omega|^2 (1 + d_{x_0} \cdot H_\kappa \circ d_{x_0}) d\mu_g \\ &\leq \int_M \langle \nabla_{\nabla \rho_{x_0}} \omega, \omega \rangle d\mu_g + \frac{1}{2} \max_M (1 + d_{x_0} \cdot H_\kappa \circ d_{x_0}) \int_M |\omega|^2 d\mu_g. \end{aligned} \quad (31)$$

Now, according to Lemma 5.1, we obtain $|\langle \nabla_{\nabla \rho_{x_0}} \omega, \omega \rangle| \leq |\nabla \rho_{x_0}| \cdot |\nabla \omega| \cdot |\omega|$. Notice that $|\nabla \rho_{x_0}| = d_{x_0}$. We then compute using Hölder inequality:

$$\begin{aligned}
\left(\int_M \langle \nabla_{\nabla \rho_{x_0}} \omega, \omega \rangle d\mu_g \right)^2 &\leq \left(\int_M d_{x_0} |\nabla \omega| \cdot |\omega| d\mu_g \right)^2 \\
&\leq \int_M |\omega|^2 d\mu_g \int_M d_{x_0}^2 |\nabla \omega|^2 d\mu_g \\
&\leq \left(\max_M d_{x_0}^2 \right) \left(\int_M |\omega|^2 d\mu_g \right) \left(\int_M |\nabla \omega|^2 d\mu_g \right) \\
&= r_{\max}^2 \left(\int_M |\omega|^2 d\mu_g \right) \left(\int_M |\nabla \omega|^2 d\mu_g \right). \tag{32}
\end{aligned}$$

According to Reilly's formula [14, Theorem 3], and for $\nu \lrcorner \omega = 0$, we have:

$$\left(\int_M |d\omega|^2 + |\delta\omega|^2 \right) d\mu_g = \int_M |\nabla \omega|^2 d\mu_g + \int_M \langle W^{[p]} \omega, \omega \rangle d\mu_g + \int_{\partial M} \langle S^{[p]}(\iota^* \omega), \iota^* \omega \rangle d\mu_g.$$

Hence, for $S^{[p]} \geq 0$ and $W^{[p]} \geq 0$ we find

$$\int_M |\nabla \omega|^2 d\mu_g \leq \int_M (|d\omega|^2 + |\delta\omega|^2) d\mu_g. \tag{33}$$

Plugging Equations (32) and (33) into (31), we get

$$\begin{aligned}
\frac{1}{2} \int_{\partial M} h_{\min} |\omega|^2 d\mu_g &\leq -\frac{1}{2} \int_{\partial M} |\omega|^2 \partial_{\nu} \rho_{x_0} d\mu_g \leq r_{\max} \left(\int_M |\omega|^2 d\mu_g \right)^{\frac{1}{2}} \left(\int_M (|d\omega|^2 + |\delta\omega|^2) d\mu_g \right)^{\frac{1}{2}} \\
&\quad + \frac{1}{2} \max_M (1 + d_{x_0} \cdot H_{\kappa} \circ d_{x_0}) \int_M |\omega|^2 d\mu_g \\
&\stackrel{(19)}{\leq} r_{\max} \mu_{1,p}^{-\frac{1}{2}} \int_M (|d\omega|^2 + |\delta\omega|^2) d\mu_g \\
&\quad + \frac{1}{2} \max_M (1 + d_{x_0} \cdot H_{\kappa} \circ d_{x_0}) \mu_{1,p}^{-1} \int_M (|d\omega|^2 + |\delta\omega|^2) d\mu_g \\
&\leq \left(r_{\max} \mu_{1,p}^{-\frac{1}{2}} + \frac{1}{2} \max_M (1 + d_{x_0} \cdot H_{\kappa} \circ d_{x_0}) \mu_{1,p}^{-1} \right) \int_M (|d\omega|^2 + |\delta\omega|^2) d\mu_g.
\end{aligned}$$

Recall that $u = \omega + u_0$ with $u_0 \in H_A^p(M)$ and $\omega \perp_{L^2(M)} u_0$, then

$$\| \omega \|_{L^2(\partial M)}^2 = \| u - u_0 \|_{L^2(\partial M)}^2 = \| u \|_{L^2(\partial M)}^2 + \| u_0 \|_{L^2(\partial M)}^2 - \underbrace{2 \langle u, u_0 \rangle_{L^2(\partial M)}}_0 \geq \| u \|_{L^2(\partial M)}^2. \tag{34}$$

Hence we find

$$\begin{aligned}
\frac{1}{2} h_{\min} \int_{\partial M} |u|^2 d\mu_g &\leq \frac{1}{2} h_{\min} \int_{\partial M} |\omega|^2 d\mu_g \\
&\leq \left(r_{\max} \mu_{1,p}^{-\frac{1}{2}} + \frac{1}{2} \max_M (1 + d_{x_0} \cdot H_{\kappa} \circ d_{x_0}) \mu_{1,p}^{-1} \right) \int_M (|d\omega|^2 + |\delta\omega|^2) d\mu_g \\
&= \left(r_{\max} \mu_{1,p}^{-\frac{1}{2}} + \frac{1}{2} \max_M (1 + d_{x_0} \cdot H_{\kappa} \circ d_{x_0}) \mu_{1,p}^{-1} \right) \int_M (|du|^2 + |\delta u|^2) d\mu_g \\
&\stackrel{(21)}{=} \sigma_{1,p} \left(r_{\max} \mu_{1,p}^{-\frac{1}{2}} + \frac{1}{2} \max_M (1 + d_{x_0} \cdot H_{\kappa} \circ d_{x_0}) \mu_{1,p}^{-1} \right) \int_{\partial M} |u|^2 d\mu_g.
\end{aligned}$$

Thus, we deduce the following inequality:

$$\sigma_{1,p} \geq \frac{\frac{1}{2} h_{\min} \mu_{1,p}}{r_{\max} \mu_{1,p}^{\frac{1}{2}} + \frac{1}{2} \max_M (1 + d_{x_0} \cdot H_{\kappa} \circ d_{x_0})}.$$

Next, we show that this inequality is strict. Suppose that

$$\sigma_{1,p} = \frac{\frac{1}{2}h_{\min}\mu_{1,p}}{r_{\max}\mu_{1,p}^{\frac{1}{2}} + \frac{1}{2}\max_M(1 + d_{x_0} \cdot H_{\kappa} \circ d_{x_0})}.$$

Then we have equalities in all the above inequalities, which leads to $\omega = u$, resulting in a contradiction, because u would simultaneously be a Steklov and a Neumann eigenform and therefore must be zero. Hence, the inequality must be strict. \square

Corollary 5.4. *If we let M to be a bounded domain in \mathbb{R}^n with convex smooth boundary ∂M . Then, for $p \geq 1$, we have the following inequality:*

$$\sigma_{1,p} > \frac{\frac{1}{2}h_{\min}\mu_{1,p}}{r_{\max}\mu_{1,p}^{\frac{1}{2}} + \frac{n}{2}}.$$

This holds since $H_{\kappa}(r) = \frac{n-1}{r}$ on \mathbb{R}^n . In particular, on the ball of radius r , the inequality becomes

$$\sigma_{1,p} > \frac{r\mu_{1,p}}{2r\mu_{1,p}^{\frac{1}{2}} + n}.$$

Let us show Inequality (6) of [Theorem 1.8](#).

Theorem 5.5. *Let (M^n, g) be a compact Riemannian manifold. Assume that M is star-shaped with respect to $x_0 \in M$ and that its sectional curvature K_g satisfies $\kappa_1 \leq K_g \leq \kappa_2$. Then, for all $k \in \mathbb{N}$, $p \geq 1$, there exists a constant $C_2 = C_2(p, n, \kappa_1, \kappa_2)$ such that:*

$$\lambda_{k,p} \leq \frac{4\mathbf{q}_{k,p}^2 r_{\max}^2 + 2\mathbf{q}_{k,p} h_{\min} C_2}{h_{\min}^2}. \quad (35)$$

Let $(e_{j_k})_{j_k}$ be a local eigenvector basis of $\nabla^2 \rho_{x_0}$, such that $\nabla^2 \rho_{x_0} e_{j_k} = \gamma_{j_k} e_{j_k}$. For $p \geq 1$, in order to bound the term $\langle T^{[p]}, \cdot \rangle$ appearing in the Rellich identity (22), we define

$$\beta_{\min,p}(y) := \min_{j_1 < \dots < j_p} \sum_{k=1}^p \gamma_{j_k}(y) \quad \text{and} \quad \beta_{\max,p}(y) := \max_{j_1 < \dots < j_p} \sum_{k=1}^p \gamma_{j_k}(y).$$

Lemma 5.6. *Let (M^n, g) be a compact Riemannian manifold. Assume that M is star-shaped with respect to $x_0 \in M$ and that its sectional curvature K_g satisfies $\kappa_1 \leq K_g \leq \kappa_2$. Then we get*

$$\frac{p}{n-1} \min_M (d_{x_0} \cdot H_{\kappa_2}(d_{x_0})) \leq \beta_{\max,p} \leq \frac{p}{n-1} \max_M (d_{x_0} \cdot H_{\kappa_1}(d_{x_0})). \quad (36)$$

Proof. We compute the following for $F = \nabla \rho_{x_0}$:

$$\begin{aligned} (T_F^{[p]} \omega)(e_{j_1}, \dots, e_{j_p}) &= \sum_{k=1}^p \omega(e_{j_1}, \dots, \nabla_{e_{j_k}} F, \dots, e_{j_p}) \\ &= \sum_{k=1}^p \omega(e_{j_1}, \dots, \nabla^2 \rho_{x_0}(e_{j_k}), \dots, e_{j_p}) \\ &= \sum_{k=1}^p \gamma_{j_k} \omega(e_{j_1}, \dots, e_{j_p}). \end{aligned}$$

Thus, we can write the following bounds using [Theorem 2.6](#):

$$\underbrace{\left(\min_{j_1 < \dots < j_p} \sum_{k=1}^p \gamma_{j_k}(y) \right)}_{\beta_{\min,p}(y)} \|\omega\|_{L^2(M)}^2 \leq \langle T_F^{[p]} \omega, \omega \rangle(y) \leq \underbrace{\left(\max_{j_1 < \dots < j_p} \sum_{k=1}^p \gamma_{j_k}(y) \right)}_{\beta_{\max,p}(y)} \|\omega\|_{L^2(M)}^2. \quad (37)$$

\square

Proof of Theorem 5.5. Let $\omega_1, \dots, \omega_k \in H^2(M)$ be $\mathbf{q}_{1,p}, \dots, \mathbf{q}_{k,p}$ -eigenforms and $E_k = \text{Span}(\omega_1, \dots, \omega_k)$. Thus for any $\omega \in E_k$

$$\begin{aligned}
\lambda_{k,p} &\stackrel{(18)}{\leq} \sup_{\omega \in E_k} \frac{\|\Delta \omega\|_{L^2(M)}^2}{\|d\omega\|_{L^2(M)}^2 + \|\delta \omega\|_{L^2(M)}^2} \\
&\stackrel{(29)}{\leq} \mathbf{q}_{k,p} \sup_{\omega \in E_k} \frac{\|\nu \lrcorner d\omega\|_{L^2(\partial M)}^2}{\|d\omega\|_{L^2(M)}^2 + \|\delta \omega\|_{L^2(M)}^2} \\
&= \mathbf{q}_{k,p} \sup_{\omega \in E_k} \frac{\|\nabla_\nu \omega\|_{L^2(\partial M)}^2}{\|d\omega\|_{L^2(M)}^2 + \|\delta \omega\|_{L^2(M)}^2}. \tag{38}
\end{aligned}$$

We take $F = \nabla \rho_{x_0}$, the Rellich identity (22) gives us

$$\begin{aligned}
&\int_M \langle \Delta \omega, F \lrcorner d\omega \rangle d\mu_g + \int_M \langle \delta \omega, F \lrcorner \Delta \omega \rangle d\mu_g \\
&= -\frac{1}{2} \int_{\partial M} (|d\omega|^2 + |\delta \omega|^2) \langle F, \nu \rangle d\mu_g + \int_{\partial M} \langle F \wedge i^*(\delta \omega), \nu \lrcorner d\omega \rangle d\mu_g + \int_{\partial M} \langle i^*(F \lrcorner d\omega), \nu \lrcorner d\omega \rangle d\mu_g + \int_{\partial M} \langle i^*(F \lrcorner \delta \omega), \nu \lrcorner \delta \omega \rangle d\mu_g \\
&\quad - \frac{1}{2} \int_M (|d\omega|^2 + |\delta \omega|^2) \text{div} F d\mu_g + \int_M \langle \delta \omega, dF \lrcorner d\omega \rangle d\mu_g + \int_M \langle T_F^{[p+1]} d\omega, d\omega \rangle d\mu_g + \int_M \langle T_F^{[p-1]} \delta \omega, \delta \omega \rangle d\mu_g. \tag{39}
\end{aligned}$$

We first control the boundary terms:

$$\begin{aligned}
&\underbrace{-\frac{1}{2} \int_{\partial M} (|d\omega|^2 + |\delta \omega|^2) \langle F, \nu \rangle d\mu_g}_{A} + \underbrace{\int_{\partial M} \langle F \wedge i^*(\delta \omega), \nu \lrcorner d\omega \rangle d\mu_g}_{B} \\
&\quad + \underbrace{\int_{\partial M} \langle i^*(F \lrcorner d\omega), \nu \lrcorner d\omega \rangle d\mu_g}_{C} + \underbrace{\int_{\partial M} \langle i^*(F \lrcorner \delta \omega), \nu \lrcorner \delta \omega \rangle d\mu_g}_{D}.
\end{aligned}$$

A: Since $\omega \in E_k$, we have $\nu \lrcorner d\omega = i^* \nabla_\nu \omega$ and $i^* \delta \omega = -\nu \lrcorner \nabla_\nu \omega = 0$. Thus, $\|\nu \lrcorner d\omega\|_{L^2(\partial M)}^2 = \|\nabla_\nu \omega\|_{L^2(\partial M)}^2$. We also have $|d\omega|^2 = |\nu \lrcorner d\omega|^2 = |i^*(\nabla_\nu \omega)|^2$. Now by [15], we have $\nu \lrcorner \delta \omega = -\delta(\nu \lrcorner \omega)$, hence,

$$|\delta \omega|^2 = |\nu \lrcorner \delta \omega|^2 + |i^* \delta \omega|^2 = |-\delta(\underbrace{\nu \lrcorner \omega}_0)|^2 + |\underbrace{i^* \delta \omega}_0|^2 = |\nu \lrcorner \nabla_\nu \omega|^2 = 0.$$

Thus, we finally get that A is equal to

$$-\frac{1}{2} \int_{\partial M} |\nabla_\nu \omega|^2 \langle F, \nu \rangle d\mu_g. \tag{40}$$

B: Since $i^* \delta \omega = 0$, then the term B vanishes.

C: We compute

$$i^*(F \lrcorner d\omega) = \langle F, \nu \rangle i^*(\nu \lrcorner d\omega) + i^* F \lrcorner i^* d\omega = \langle F, \nu \rangle (\nu \lrcorner d\omega) = \langle F, \nu \rangle i^*(\nabla_\nu \omega).$$

This gives,

$$\begin{aligned}
\int_{\partial M} \langle i^*(F \lrcorner d\omega), \nu \lrcorner d\omega \rangle d\mu_g &= \int_{\partial M} \langle F, \nu \rangle |i^*(\nabla_\nu \omega)|^2 d\mu_g \\
&= \int_{\partial M} \langle F, \nu \rangle (|\nabla_\nu \omega|^2 - \underbrace{|\nu \lrcorner \nabla_\nu \omega|^2}_0) d\mu_g \\
&= \int_{\partial M} \langle F, \nu \rangle |\nabla_\nu \omega|^2 d\mu_g.
\end{aligned}$$

D: Since $\nu \lrcorner \delta \omega = -\delta(\nu \lrcorner \omega) = 0$, the integral (D) vanishes.

Combining the results and the fact that $dF = 0$, (39) reduces to

$$\begin{aligned}
& - \underbrace{\int_M \langle \Delta\omega, F \lrcorner d\omega \rangle d\mu_g - \int_M \langle \delta\omega, F \lrcorner \Delta\omega \rangle d\mu_g}_E + \underbrace{\frac{1}{2} \int_M (|d\omega|^2 + |\delta\omega|^2) \Delta\rho_{x_0} d\mu_g}_F + \underbrace{\int_M \langle T_F^{[p+1]} d\omega, d\omega \rangle d\mu_g}_G \\
& \quad + \underbrace{\int_M \langle T_F^{[p-1]} \delta\omega, \delta\omega \rangle d\mu_g}_G = -\frac{1}{2} \int_{\partial M} |\nabla_\nu \omega|^2 \langle F, \nu \rangle d\mu_g.
\end{aligned}$$

Using the inequalities

$$|F \lrcorner \alpha| \leq |F| \cdot |\alpha| \text{ and } |F \wedge \alpha| \leq |F| \cdot |\alpha|, \quad (41)$$

that are valid for any form α , we estimate the remaining integrals separately using Hölder inequality and taking $F = \nabla\rho_{x_0}$. We begin with

E:

$$\begin{aligned}
& - \int_M \langle \Delta\omega, F \lrcorner d\omega \rangle d\mu_g - \int_M \langle \delta\omega, F \lrcorner \Delta\omega \rangle d\mu_g \\
& = - \int_M \langle \Delta\omega, F \lrcorner d\omega + F \wedge \delta\omega \rangle d\mu_g \\
& \leq \int_M |\Delta\omega| \cdot |F \lrcorner d\omega + F \wedge \delta\omega| d\mu_g \\
& \leq \left(\int_M |\Delta\omega|^2 d\mu_g \right)^{\frac{1}{2}} \left(\int_M (|F \lrcorner d\omega + F \wedge \delta\omega|^2) d\mu_g \right)^{\frac{1}{2}} \\
& \leq \left(\int_M |\Delta\omega|^2 d\mu_g \right)^{\frac{1}{2}} \left(\int_M (|F \lrcorner d\omega|^2 + |F \wedge \delta\omega|^2 + 2 \underbrace{\langle F \lrcorner d\omega, F \wedge \delta\omega \rangle}_0) d\mu_g \right)^{\frac{1}{2}} \\
& \stackrel{(41)}{\leq} \left(\int_M |\Delta\omega|^2 d\mu_g \right)^{\frac{1}{2}} \left(\int_M (|F|^2 |d\omega|^2 + |F|^2 |\delta\omega|^2) d\mu_g \right)^{\frac{1}{2}} \\
& \leq r_{\max} \left(\int_M |\Delta\omega|^2 d\mu_g \right)^{\frac{1}{2}} \left(\int_M (|d\omega|^2 + |\delta\omega|^2) d\mu_g \right)^{\frac{1}{2}} \\
& \stackrel{(29)}{\leq} \mathbf{q}_{k,p}^{\frac{1}{2}} r_{\max} \left(\int_{\partial M} |\nabla_\nu \omega|^2 d\mu_g \right)^{\frac{1}{2}} \left(\int_M (|d\omega|^2 + |\delta\omega|^2) d\mu_g \right)^{\frac{1}{2}}.
\end{aligned}$$

F: Using again [Theorem 2.7](#), we find

$$\frac{1}{2} \int_M (|d\omega|^2 + |\delta\omega|^2) \Delta\rho_{x_0} d\mu_g \leq -\frac{1}{2} \int_M (|d\omega|^2 + |\delta\omega|^2) (1 + d_{x_0} \cdot H_{\kappa_2}(d_{x_0})) d\mu_g.$$

G: Using (37), we have

$$\int_M \langle T_F^{[p+1]} d\omega, d\omega \rangle d\mu_g \leq \beta_{\max, p+1} \int_M |d\omega|^2 d\mu_g$$

and

$$\int_M \langle T_F^{[p-1]} \delta\omega, \delta\omega \rangle d\mu_g \leq \beta_{\max, p-1} \int_M |\delta\omega|^2 d\mu_g$$

thus

$$\begin{aligned}
\int_M \langle T_F^{[p+1]} d\omega, d\omega \rangle d\mu_g + \int_M \langle T_F^{[p-1]} \delta\omega, \delta\omega \rangle d\mu_g & \leq \max_M(\beta_{\max, p+1}; \beta_{\max, p-1}) \\
& \quad \times \int_M (|d\omega|^2 + |\delta\omega|^2) d\mu_g.
\end{aligned}$$

Therefore, by regrouping the terms we finally get,

$$\begin{aligned}
-\frac{1}{2} \int_{\partial M} |\nabla_\nu \omega|^2 \langle \nabla \rho_{x_0}, \nu \rangle d\mu_g &\leq \mathbf{q}_{k,p}^{\frac{1}{2}} r_{\max} \left(\int_{\partial M} |\nabla_\nu \omega|^2 d\mu_g \right)^{\frac{1}{2}} \left(\int_M (|d\omega|^2 + |\delta\omega|^2) d\mu_g \right)^{\frac{1}{2}} \\
&\quad + \left(\max_M (\beta_{\max, p+1}; \beta_{\max, p-1}) - \frac{1}{2} (1 + \min_M (d_{x_0} \cdot H_{\kappa_2}(d_{x_0}))) \right) \\
&\quad \times \int_M (|d\omega|^2 + |\delta\omega|^2) d\mu_g \\
&\stackrel{(36)}{\leq} \mathbf{q}_{k,p}^{\frac{1}{2}} r_{\max} \left(\int_{\partial M} |\nabla_\nu \omega|^2 d\mu_g \right)^{\frac{1}{2}} \left(\int_M (|d\omega|^2 + |\delta\omega|^2) d\mu_g \right)^{\frac{1}{2}} \\
&\quad + \left(\frac{p+1}{n-1} \max_M (d_{x_0} \cdot H_{\kappa_1}(d_{x_0})) - \frac{1}{2} (1 + \min_M (d_{x_0} \cdot H_{\kappa_2}(d_{x_0}))) \right) \\
&\quad \times \int_M (|d\omega|^2 + |\delta\omega|^2) d\mu_g.
\end{aligned}$$

This implies that

$$\begin{aligned}
h_{\min} \int_{\partial M} |\nabla_\nu \omega|^2 d\mu_g &\leq 2\mathbf{q}_{k,p}^{\frac{1}{2}} r_{\max} \left(\int_{\partial M} |\nabla_\nu \omega|^2 d\mu_g \right)^{\frac{1}{2}} \left(\int_M (|d\omega|^2 + |\delta\omega|^2) d\mu_g \right)^{\frac{1}{2}} \\
&\quad + \underbrace{\left(\frac{2(p+1)}{n-1} \max_M (d_{x_0} \cdot H_{\kappa_1}(d_{x_0})) - (1 + \min_M (d_{x_0} \cdot H_{\kappa_2}(d_{x_0}))) \right)}_{C_2} \int_M (|d\omega|^2 + |\delta\omega|^2) d\mu_g.
\end{aligned}$$

Now let $\mathcal{A}^2 = \frac{\int_{\partial M} |\nabla_\nu \omega|^2 d\mu_g}{\int_M (|d\omega|^2 + |\delta\omega|^2) d\mu_g}$. Dividing by $\int_M (|d\omega|^2 + |\delta\omega|^2) d\mu_g$, we obtain the second-degree polynomial in \mathcal{A} :

$$h_{\min} \mathcal{A}^2 - 2\mathbf{q}_{k,p}^{\frac{1}{2}} r_{\max} \mathcal{A} - C_2 \leq 0.$$

By calculating the discriminant and then the bigger root of this polynomial, we obtain that

$$\begin{aligned}
\mathcal{A}^2 &\leq \frac{\left(\mathbf{q}_{k,p}^{\frac{1}{2}} r_{\max} + \sqrt{\mathbf{q}_{k,p} r_{\max}^2 + h_{\min} C_2} \right)^2}{h_{\min}^2} \\
&\leq \frac{2\mathbf{q}_{k,p} r_{\max}^2 + 2\mathbf{q}_{k,p} r_{\max}^2 + 2h_{\min} C_2}{h_{\min}^2} = \frac{4\mathbf{q}_{k,p} r_{\max}^2 + 2h_{\min} C_2}{h_{\min}^2}.
\end{aligned}$$

Replacing the result in (38), we finally get

$$\lambda_{k,p} \leq \frac{4\mathbf{q}_{k,p}^2 r_{\max}^2 + 2\mathbf{q}_{k,p} h_{\min} C_2}{h_{\min}^2}. \quad (42)$$

This completes the proof. \square

Remark 5.7. As $\mathbf{q}_{k,p}$ goes to infinity when k goes to infinity, the upper bound in (42) is positive regardless the sign of C_2 . We know that the limit of $r_{\max} H_\kappa(r_{\max})$ when r_{\max} goes to zero is $n-1$ hence for $p > \frac{n}{2} - 1$, the limit of C_2 when r_{\max} goes to zero becomes $2(p+1) - n$ which is positive. The function $r H_\kappa(r)$ is constant if $\kappa = 0$, increasing on $[0, \infty)$ if $\kappa < 0$, and decreasing on $[0, \infty)$ if $\kappa > 0$. Here are some values of C_2 depending on the variation of the sign of κ_1 and κ_2 .

- $\kappa_1 = \kappa_2 = 0$ et $p \geq 1$: $C_2 = 2(p+1) - n$. This includes the case when M is a domain in \mathbb{R}^n .
- $\kappa_1 \leq \kappa_2 \leq 0$: $C_2 = \frac{2(p+1)}{n-1} r_{\max} H_{\kappa_1}(r_{\max}) - n$.
- $0 \leq \kappa_1 \leq \kappa_2$: $C_2 = 2(p+1) - (1 + r_{\max} H_{\kappa_2}(r_{\max}))$.
- $\kappa_1 \leq 0 \leq \kappa_2$: $C_2 = \frac{2(p+1)}{n-1} r_{\max} H_{\kappa_1}(r_{\max}) - (1 + r_{\max} H_{\kappa_2}(r_{\max}))$.

Remark 5.8. We tried to prove (7) using the same idea as in the proof of (6), applying the Rellich identity, but at the end we obtain a negative constant C_1 of the form

$$C_1 = \left(1 + \frac{2(p-1)}{n-1}\right) \min_{r \in [0, r_{\max})} d_{x_0} H_{\kappa_2}(d_{x_0}) - \left(1 + \frac{2p}{n-1}\right) \max_{r \in [0, r_{\max})} d_{x_0} H_{\kappa_1}(d_{x_0}).$$

Using this technique, it seems that the first inequality of [Theorem 1.4](#) could not be extended to differential forms of degree $p \geq 1$.

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