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Stabilization of a Wave-Heat Cascade System

Hugo Lhachemi, Christophe Prieur, and Emmanuel Trélat ^{*†‡}

Abstract

We consider the output-feedback stabilization of a one-dimensional cascade coupling a reaction-diffusion equation and a wave equation through an internal term, with Neumann boundary control acting at the wave endpoint. Two measurements are available: the wave velocity at the controlled boundary and a temperature-type observation of the reaction-diffusion component, either distributed or pointwise. Under explicit, necessary and sufficient conditions on the coupling and observation profiles, we show that the generator of the open-loop system is a Riesz-spectral operator. Exploiting this structure, we design a finite-dimensional dynamic output-feedback law, based on a finite number of parabolic modes, which achieves arbitrary exponential decay in both the natural energy space and a stronger parabolic norm. The construction relies on a spectral reduction and a Lyapunov argument in Riesz bases. We also extend the design to pointwise temperature or heat-flux measurements.

Keywords: PDE cascade, reaction-diffusion equation, wave equation, output-feedback.

The present work is the result of the split of [21] into the feedback stabilization study reported in this paper and the study of the controllability properties of the cascade in the companion work [22].

1 Introduction

1.1 Studied problem

We fix $L > 0$, $c \in \mathbb{R}$ and $\beta \in L^\infty(0, L)$. We consider the cascade system

$$\partial_t y(t, x) = \partial_{xx} y(t, x) + cy(t, x) + \beta(x)z(t, x), \quad (1a)$$

$$\partial_{tt} z(t, x) = \partial_{xx} z(t, x), \quad (1b)$$

for $t > 0$ and $x \in (0, L)$, with boundary conditions

$$y(t, 0) = y(t, L) = 0, \quad (1c)$$

$$z(t, 0) = 0, \quad \partial_x z(t, L) = u(t). \quad (1d)$$

The state $y(t, \cdot)$ solves a one-dimensional reaction-diffusion equation, forced by the wave state $z(t, \cdot)$ that solves a one-dimensional wave equation. The control input u acts on the right Neumann trace

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of z . Such wave-reaction-diffusion cascades appear for instance in simplified models of microwave heating, see e.g. [4, 13, 30, 33]. We assume that two outputs are available: the first one is the wave velocity at the right endpoint,

$$z_o(t) = \partial_t z(t, L), \quad (2)$$

and the second one is a heat measurement of the form

$$y_o(t) = \int_0^L c_o(x) y(t, x) dx \quad (3)$$

where $c_o \in L^2(0, L)$ models a weighted temperature observation.

1.2 State-of-the-art and contribution

To the best of our knowledge, controllability and stabilization results for coupled PDEs of different types (such as heat-wave cascades) remain rather scarce. This contrasts with the rich theory available for coupled parabolic systems; see for instance [1, 3, 20] and the references therein. The works [31, 32] on a heat-wave system coupled at the boundary have strongly inspired the present paper. Hyperbolic-elliptic couplings have been investigated in [6, 24]. Backstepping-based stabilization has been developed in [5, 10] for hyperbolic-parabolic systems where the hyperbolic component is a transport equation.

To the best of the authors' knowledge, the present paper is the first to address the design of an explicit output-feedback controller for a one-dimensional wave-heat cascade. Our main contribution is the construction of a finite-dimensional dynamic output-feedback law ensuring exponential stabilization of the full cascade (1), with an arbitrary prescribed decay rate, under spectral controllability and observability conditions that are explicit, necessary, and sufficient. The key structural ingredient is that the generator of the closed-loop system is a Riesz-spectral operator. We establish the following result that, at this stage, we state in a rather informal way.

Theorem 1. *Given any $\delta > 0$, under explicit controllability and observability conditions, there exists an output-feedback control, using the measurements (2) and (3), explicitly built from a finite number of modes (depending on δ), such that the closed-loop controlled system (1) is exponentially stable in $\mathcal{H}^0 = L^2(0, L) \times H_{(0)}^1(0, L) \times L^2(0, L)$ and also in $\mathcal{H}^1 = H_0^1(0, L) \times H_{(0)}^1(0, L) \times L^2(0, L)$, with the decay rate δ .*

The precise statement is given in Theorem 2 in Section 4. Based on spectral considerations, we design a Lyapunov function and build the feedback control using only a finite number of modes. We also extend the result of Theorem 1 to the case of a pointwise temperature measurement $y_o(t) = y(t, \xi_p)$, or a pointwise heat flux measurement $y_o(t) = \partial_x y(t, \xi_p)$, at some location $\xi_p \in [0, L]$.

1.3 Paper organization and technical outline

The paper is organized as follow. In Section 2, we design a preliminary boundary velocity feedback acting on the wave equation and analyze the spectrum of the resulting generator. Section 3 is devoted to the spectral reduction of the problem and to the derivation of finite-dimensional models capturing the unstable parabolic modes. The output-feedback design and the stability analysis of the closed-loop system are carried out in Section 4. In Section 5 we extend the results from distributed temperature measurements to pointwise temperature or heat-flux measurements. Section 6 presents a numerical example illustrating the performance of the proposed controller. Section 7 gathers some concluding remarks and perspectives. In the Appendix, we provide further comments on the exact controllability properties of the wave-heat cascade.

For the reader's convenience, we provide below a brief technical outline of the main steps leading to our stabilization result (Theorem 2 in Section 4):

- Using the boundary wave measurement (2), we first design a preliminary velocity feedback (4) to stabilize the hyperbolic part and define the corresponding generator \mathcal{A} .
- We compute the eigenvalues and eigenvectors of \mathcal{A} (Lemma 1).
- We prove that the family of eigenvectors of \mathcal{A} forms a Riesz basis Φ of the energy space \mathcal{H}^0 , so that \mathcal{A} is a Riesz-spectral operator generating a C_0 -semigroup (Lemma 2).
- We compute the dual Riesz basis Ψ by analysing the adjoint operator \mathcal{A}^* (Lemma 3).
- For the parabolic component, we construct a Riesz basis Φ^1 of a stronger Hilbert space \mathcal{H}^1 (Lemma 4).
- We perform a spectral reduction of the closed-loop system and identify finite-dimensional models capturing the relevant parabolic modes (Subsections 3.1 and 3.2).
- We derive necessary and sufficient controllability and observability conditions for these models (Lemmas 5-7).
- We introduce a finite-dimensional output-feedback controller based on a Luenberger-type observer and a state-feedback on the truncated model (Section 4).
- Finally, we prove exponential stability of the full PDE cascade in both \mathcal{H}^0 and \mathcal{H}^1 norms by means of a Lyapunov functional built in the Riesz bases (Theorem 2).

2 Preliminary velocity feedback and spectral properties

2.1 Preliminary velocity feedback

The control strategy consists of first shifting the spectrum of the wave equation (1b), by applying a velocity feedback on the right Neumann trace (1d), namely, by setting

$$u(t) = -\alpha z_o(t) + v(t) = -\alpha \partial_t z(t, L) + v(t) \quad (4)$$

where v is another control to be designed, and $\alpha > 1$ is chosen such that (see Remark 1)

$$c \neq \frac{1}{2L} \log\left(\frac{\alpha-1}{\alpha+1}\right) + \frac{n^2 \pi^2}{L^2} \quad \forall n \in \mathbb{N}^*. \quad (5)$$

When $v = 0$, the output-feedback u given by (4) is known to stabilize the wave equation (1b). The additional control v is going to be used and designed from the output (3) in order to stabilize the full cascade system (1). Hence, with (4), the system is now

$$\partial_t y(t, x) = \partial_{xx} y(t, x) + cy(t, x) + \beta(x)z(t, x), \quad (6a)$$

$$\partial_{tt} z(t, x) = \partial_{xx} z(t, x), \quad (6b)$$

$$y(t, 0) = y(t, L) = 0, \quad (6c)$$

$$z(t, 0) = 0, \quad \partial_x z(t, L) + \alpha \partial_t z(t, L) = v(t), \quad (6d)$$

and the new control is v .

To design an effective feedback control v based on a finite number of modes, we follow a spectral reduction approach developed in [7, 8, 19, 25] in other contexts. In turn, we will design a Lyapunov function.

We define the Hilbert space of complex-valued functions

$$\mathcal{H}^0 = L^2(0, L) \times H_{(0)}^1(0, L) \times L^2(0, L) \quad (7a)$$

where $H_{(0)}^1(0, L) = \{g \in H^1(0, L) \mid g(0) = 0\}$, with the inner product

$$\langle (f_1, g_1, h_1), (f_2, g_2, h_2) \rangle_{\mathcal{H}^0} = \int_0^L (f_1 \bar{f}_2 + g_1' \bar{g}_2' + h_1 \bar{h}_2) \quad (7b)$$

and corresponding norm denoted $\|\cdot\|_{\mathcal{H}^0}$. When the context is clear, we simply denote by $\langle \cdot, \cdot \rangle$ the inner product of \mathcal{H}_0 . Setting $X = (y, z, \partial_t z)^\top$, the control system (6) is written in the abstract form $\dot{X}(t) = \mathcal{A}X(t) + \mathcal{B}v(t)$, where the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H}^0 \rightarrow \mathcal{H}^0$ is defined by

$$\mathcal{A} = \begin{pmatrix} \partial_{xx} + c \text{id} & \beta \text{id} & 0 \\ 0 & 0 & \text{id} \\ 0 & \partial_{xx} & 0 \end{pmatrix} \quad (8a)$$

(id denotes the identity operator on $L^2(0, L)$) and domain

$$D(\mathcal{A}) = \{(f, g, h) \in H^2(0, L) \times H^2(0, L) \times H^1(0, L) \mid f(0) = f(L) = g(0) = h(0) = 0, g'(L) + ah(L) = 0\}, \quad (8b)$$

and the control operator \mathcal{B} is defined by transposition (see Appendix .1).

2.2 Spectral properties of \mathcal{A}

Lemma 1. *The eigenvalues of \mathcal{A} are*

$$\begin{aligned} \lambda_{1,n} &= c - \frac{n^2 \pi^2}{L^2}, \quad n \in \mathbb{N}^*, \\ \lambda_{2,m} &= \frac{1}{2L} \log\left(\frac{\alpha-1}{\alpha+1}\right) + i \frac{m\pi}{L}, \quad m \in \mathbb{Z}, \end{aligned}$$

with associated eigenvectors $\phi_{1,n} = (\phi_{1,n}^1, \phi_{1,n}^2, \phi_{1,n}^3)$ and $\phi_{2,m} = (\phi_{2,m}^1, \phi_{2,m}^2, \phi_{2,m}^3)$ respectively given by

$$\begin{aligned} \phi_{1,n}^1(x) &= \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right), \quad \phi_{1,n}^2(x) = 0, \quad \phi_{1,n}^3(x) = 0, \\ \phi_{2,m}^1(x) &= \frac{1}{A_m r_m} \int_x^L \beta(s) \sinh(\lambda_{2,m}s) \sinh(r_m(x-s)) ds \\ &\quad + \frac{\sinh(r_m(L-x))}{A_m r_m \sinh(r_m L)} \int_0^L \beta(s) \sinh(\lambda_{2,m}s) \sinh(r_m s) ds \end{aligned}$$

where r_m is a square root of $\lambda_{2,m} - c$ with $\text{Re}(r_m) \geq 0$,

$$\begin{aligned} \phi_{2,m}^2(x) &= \frac{1}{A_m} \sinh(\lambda_{2,m}x), \quad \phi_{2,m}^3(x) = \frac{\lambda_{2,m}}{A_m} \sinh(\lambda_{2,m}x), \\ A_m &= \frac{\sqrt{(\mu^2 L^2 + m^2 \pi^2) \sinh(2\mu L)}}{L\sqrt{2\mu}}, \quad \mu = -\frac{1}{2L} \log\left(\frac{\alpha-1}{\alpha+1}\right) > 0. \end{aligned}$$

Remark 1. The constraint (5) is introduced to avoid that the real eigenvalue $\lambda_{2,0}$ associated with the wave equation coincide with an eigenvalue $\lambda_{1,n}$ of the reaction-diffusion equation. When equality holds for some $n \in \mathbb{N}^*$, the eigenvalue $\lambda_{2,0} = \lambda_{1,n}$ is of geometric multiplicity one but of algebraic multiplicity two, leading to a Jordan block of dimension 2. Since all other eigenvalues are simple, our approach works. It also applies to the case $\alpha \in (0, 1)$ with the change $\lambda_{2,m} = \frac{1}{2L} \log\left(\frac{1-\alpha}{1+\alpha}\right) + i \frac{(2m+1)\pi}{2L}$ for $m \in \mathbb{Z}$. \circ

Proof. Let $\lambda \in \mathbb{C}$ and $(f, g, h) \in D(\mathcal{A})$ be such that $\mathcal{A}(f, g, h) = \lambda(f, g, h)$, i.e., $f, g \in H^2(0, L)$ and $h \in H^1(0, L)$ such that

$$f'' + cf + \beta g = \lambda f, \quad (9a)$$

$$h = \lambda g, \quad (9b)$$

$$g'' = \lambda h = \lambda^2 g, \quad (9c)$$

$$f(0) = f(L) = g(0) = h(0) = 0, \quad (9d)$$

$$g'(L) + \alpha h(L) = g'(L) + \alpha \lambda g(L) = 0. \quad (9e)$$

Assume first that $\lambda = 0$. By (9c) we have $g'' = 0$ with $g(0) = g'(L) = 0$, implying along with (9b) that $g = h = 0$. Hence, from (9a), $f'' + cf = 0$ and $f(0) = f(L) = 0$, from which we deduce that $\frac{n^2\pi^2}{L^2} = c$ and $f(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right)$ for some $n \in \mathbb{N}^*$. In particular, $\lambda = c - \frac{n^2\pi^2}{L^2} = 0$.

Assume now that $\lambda \neq 0$. Using (9c) and the fact that $g(0) = 0$, we have $g(x) = \delta(e^{\lambda x} - e^{-\lambda x})$ for some $\delta \in \mathbb{R}$. Furthermore, by (9e), $0 = g'(L) + \alpha \lambda g(L) = \lambda \delta(e^{\lambda L} + e^{-\lambda L} + \alpha(e^{\lambda L} - e^{-\lambda L}))$. Since $\lambda \neq 0$, there are two cases.

If $\delta = 0$ then $g = h = 0$ and, by (9a), $f'' = (\lambda - c)f$ and $f(0) = f(L) = 0$ hence $\lambda = c - \frac{n^2\pi^2}{L^2}$ and $f(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right)$ for some $n \in \mathbb{N}^*$ provided $\lambda \neq 0$.

If $\delta \neq 0$ then $e^{\lambda L} + e^{-\lambda L} + \alpha(e^{\lambda L} - e^{-\lambda L}) = 0$, which is equivalent to $e^{2\lambda L} = \frac{\alpha-1}{\alpha+1}$. Since $\alpha > 1$, this gives $\lambda = \frac{1}{2L} \log\left(\frac{\alpha-1}{\alpha+1}\right) + i\frac{m\pi}{L}$ for some $m \in \mathbb{Z}$. Moreover $g(x) = \sinh(\lambda x)$ and $h(x) = \lambda \sinh(\lambda x)$, where we take (without loss of generality) $\delta = 1/2$. Finally, $f'' + (c - \lambda)f = -\beta(x)g$ with $f(0) = f(L) = 0$. Recalling that $\alpha > 1$ is selected so that $c \neq \frac{1}{2L} \log\left(\frac{\alpha-1}{\alpha+1}\right)$, we have $\lambda - c \neq 0$. Hence, denoting by r one of its two distinct square roots, i.e., $r^2 = \lambda - c$ with $r \neq 0$, we obtain $f(x) = (\delta_1 - \frac{1}{2r} \int_L^x \beta(s)g(s)e^{-rs} ds)e^{rx} + (\delta_2 + \frac{1}{2r} \int_L^x \beta(s)g(s)e^{rs} ds)e^{-rx}$ for some constants $\delta_1, \delta_2 \in \mathbb{C}$ that must be selected such that $f(0) = 0$ and $f(L) = 0$. The latter equation yields $\delta_2 = -\delta_1 e^{2rL}$, implying that $f(x) = 2\delta_1 e^{rL} \sinh(r(x-L)) - \frac{1}{r} \int_L^x \beta(s)g(s) \sinh(r(x-s)) ds$. Then, $f(0) = 0$ gives $-2\delta_1 e^{rL} \sinh(rL) - \frac{1}{r} \int_0^L \beta(s)g(s) \sinh(rs) ds = 0$. We note that $\sinh(rL) = 0$ if and only if $e^{2rL} = 1$, i.e., if and only if $2rL = 2ik\pi$ for some $k \in \mathbb{Z}$. Then, we must have $\lambda = c + r^2 = c - \frac{k^2\pi^2}{L^2}$. Since $\lambda = \frac{1}{2L} \log\left(\frac{\alpha-1}{\alpha+1}\right) + i\frac{m\pi}{L}$, this is possible only if $m = 0$, which contradicts the assumption that $\alpha > 1$ has been selected so that $c \neq \frac{1}{2L} \log\left(\frac{\alpha-1}{\alpha+1}\right) + \frac{k^2\pi^2}{L^2}$ for any $k \in \mathbb{N}$. Hence, $\sinh(rL) \neq 0$, which gives $\delta_1 = -\frac{1}{2e^{rL}r \sinh(rL)} \int_0^L \beta(s)g(s) \sinh(rs) ds$. \square

Remark 2. The sequence of eigenvalues $(\lambda_{1,n})_{n \in \mathbb{N}^*}$ (resp., $(\lambda_{1,m})_{m \in \mathbb{Z}}$) is associated with the reaction-diffusion part (1a) (resp., the wave part (1b)) of the system (1) and is referred to as the *parabolic spectrum* (resp., the *hyperbolic spectrum*). \circ

Lemma 2. *The family of generalized eigenvectors $\Phi = \{\phi_{1,n} \mid n \in \mathbb{N}^*\} \cup \{\phi_{2,m} \mid m \in \mathbb{Z}\}$ is a Riesz basis of \mathcal{H}^0 . Hence, \mathcal{A} is a Riesz spectral operator that generates a C_0 -semigroup.*

Proof. Noting that $\{\phi_{1,n}^1 \mid n \in \mathbb{N}^*\}$ is a Hilbert basis of $L^2(0, L)$ and that $\{(\phi_{2,m}^2, \phi_{2,m}^3) \mid m \in \mathbb{Z}\}$ is a Riesz basis of $H_{(0)}^1(0, L) \times L^2(0, L)$ we infer, first, that Φ is ω -linearly independent, and second, defining $\tilde{\phi}_{1,n} = \phi_{1,n}$ and $\tilde{\phi}_{2,m} = (0, \phi_{2,m}^2, \phi_{2,m}^3)$, that $\tilde{\Phi} = \{\tilde{\phi}_{1,n} \mid n \in \mathbb{N}^*\} \cup \{\tilde{\phi}_{2,m} \mid m \in \mathbb{Z}\}$ is a Riesz basis of \mathcal{H}^0 . By Bari's theorem [11], it then follows that Φ is a Riesz basis provided that

$$\sum_{n \in \mathbb{N}^*} \|\phi_{1,n} - \tilde{\phi}_{1,n}\|_{\mathcal{H}^0}^2 + \sum_{m \in \mathbb{Z}} \|\phi_{2,m} - \tilde{\phi}_{2,m}\|_{\mathcal{H}^0}^2 = \sum_{m \in \mathbb{Z}} \|\phi_{2,m}^1\|_{L^2}^2 < \infty. \quad (10)$$

To prove (10), we first note that $\phi_{2,m}^1 = f_{m,1} + f_{m,2}$ with

$$f_{m,1}(x) = \frac{\sinh(r_m(L-x))}{A_m r_m \sinh(r_m L)} \int_0^x \beta(s) \sinh(\lambda_{2,m} s) \sinh(r_m s) ds,$$

$$f_{m,2}(x) = \frac{1}{A_m r_m \sinh(r_m L)} \times \int_x^L \beta(s) \sinh(\lambda_{2,m} s) (\sinh(r_m(L-x)) \sinh(r_m s) - \sinh(r_m L) \sinh(r_m(s-x))) ds.$$

We study the two terms separately. Since $r_m^2 = \lambda_{2,m} - c = \frac{1}{2L} \log\left(\frac{\alpha-1}{\alpha+1}\right) - c + i \frac{m\pi}{L}$ with $\operatorname{Re}(r_m) \geq 0$, we infer that $|r_m| \sim \sqrt{\frac{|m|\pi}{L}}$ and $\operatorname{Re} r_m \sim \sqrt{\frac{|m|\pi}{2L}}$ as $|m| \rightarrow +\infty$. Recall that $|\sinh(\operatorname{Re} z)| \leq |\cosh(z)|$, $|\sinh(z)| \leq |\cosh(\operatorname{Re} z)|$ for any $z \in \mathbb{C}$, and $\cosh(x) \leq e^x$ and $\frac{1}{2}(e^x - 1) \leq \sinh(x) \leq e^x/2$ for any $x \geq 0$. For $|m|$ large, we obtain

$$\begin{aligned} |f_{m,1}(x)| &\leq \frac{|\sinh(r_m(L-x))|}{A_m |r_m| |\sinh(r_m L)|} \int_0^x |\beta(s)| |\sinh(\lambda_{2,m} s)| |\sinh(r_m s)| ds \\ &\leq \frac{\|\beta\|_{L^\infty} \cosh(\operatorname{Re} r_m(L-x))}{A_m |r_m| \sinh(\operatorname{Re} r_m L)} \int_0^x \cosh(\operatorname{Re} \lambda_{2,m} s) \cosh(\operatorname{Re} r_m s) ds \\ &\leq \frac{2\|\beta\|_{L^\infty} \cosh(\mu L) e^{\operatorname{Re} r_m(L-x)}}{A_m |r_m| (e^{\operatorname{Re} r_m L} - 1)} \int_0^x \cosh(\operatorname{Re} r_m s) ds \\ &\leq \frac{2\|\beta\|_{L^\infty} \cosh(\mu L) e^{\operatorname{Re} r_m(L-x)}}{A_m |r_m| (e^{\operatorname{Re} r_m L} - 1)} \times \frac{\sinh(\operatorname{Re} r_m x)}{\operatorname{Re} r_m} \\ &\leq \frac{\|\beta\|_{L^\infty} \cosh(\mu L) e^{\operatorname{Re} r_m(L-x)}}{A_m |r_m| (e^{\operatorname{Re} r_m L} - 1)} \frac{e^{\operatorname{Re} r_m x}}{\operatorname{Re} r_m} \\ &\leq \|\beta\|_{L^\infty} \cosh(\mu L) \frac{e^{\operatorname{Re} r_m L}}{e^{\operatorname{Re} r_m L} - 1} \frac{1}{A_m |r_m| \operatorname{Re} r_m} \end{aligned}$$

hence $\|f_{m,1}\|_{L^\infty} = O(1/m^2)$ as $|m| \rightarrow +\infty$. For the second term, noting that $\sinh(r_m(L-x)) \sinh(r_m s) - \sinh(r_m L) \sinh(r_m(s-x)) = \sinh(r_m x) \sinh(r_m(L-s))$, we have

$$\begin{aligned} |f_{m,2}(x)| &\leq \frac{\int_x^L |\beta(s)| |\sinh(\lambda_{2,m} s)| |\sinh(r_m x)| |\sinh(r_m(L-s))| ds}{A_m |r_m| |\sinh(r_m L)|} \\ &\leq \frac{\|\beta\|_{L^\infty} \int_x^L \cosh(\operatorname{Re} \lambda_{2,m} s) \cosh(\operatorname{Re} r_m x) \cosh(\operatorname{Re} r_m(L-s)) ds}{A_m |r_m| \sinh(\operatorname{Re} r_m L)} \\ &\leq \frac{2\|\beta\|_{L^\infty} \cosh(\mu L) e^{\operatorname{Re} r_m x}}{A_m |r_m| (e^{\operatorname{Re} r_m L} - 1)} \int_x^L e^{\operatorname{Re} r_m(L-s)} ds \\ &\leq \frac{2\|\beta\|_{L^\infty} \cosh(\mu L) e^{\operatorname{Re} r_m x}}{A_m |r_m| (e^{\operatorname{Re} r_m L} - 1)} \frac{e^{\operatorname{Re} r_m(L-x)}}{\operatorname{Re} r_m} \\ &\leq 2\|\beta\|_{L^\infty} \cosh(\mu L) \frac{e^{\operatorname{Re} r_m L}}{e^{\operatorname{Re} r_m L} - 1} \frac{1}{A_m |r_m| \operatorname{Re} r_m} \end{aligned}$$

hence $\|f_{m,2}\|_{L^\infty} = O(1/m^2)$ as $|m| \rightarrow +\infty$. \square

Since the controllability properties of the system are captured by the properties of the dual Riesz basis Ψ of Φ , we next compute the eigenvectors of the adjoint operator \mathcal{A}^* .

Lemma 3. *Identifying the Hilbert space \mathcal{H}^0 with its dual, the adjoint operator \mathcal{A}^* is given by*

$$\mathcal{A}^* = \begin{pmatrix} \partial_{xx} + c \operatorname{id} & 0 & 0 \\ P_\beta & 0 & -\operatorname{id} \\ 0 & -\partial_{xx} & 0 \end{pmatrix} \quad (11a)$$

with $P_\beta f = \int_0^{(\cdot)} \int_\tau^L \beta(s) f(s) ds d\tau$, and domain

$$D(\mathcal{A}^*) = \{(f, g, h) \in H^2(0, L) \times H^2(0, L) \times H^1(0, L) \mid f(0) = f(L) = g(0) = h(0) = 0, g'(L) - \alpha h(L) = 0\}. \quad (11b)$$

Its eigenfunctions are given by the dual Riesz basis $\Psi = \{\psi_{1,n} \mid n \in \mathbb{N}^*\} \cup \{\psi_{2,m} \mid m \in \mathbb{Z}\}$ of Φ , associated with the eigenvalues

$$\begin{aligned}\mu_{1,n} &= \lambda_{1,n} = c - \frac{n^2\pi^2}{L^2}, & n \in \mathbb{N}^*, \\ \mu_{2,m} &= \bar{\lambda}_{2,m} = \frac{1}{2L} \log\left(\frac{\alpha-1}{\alpha+1}\right) - i\frac{m\pi}{L}, & m \in \mathbb{Z},\end{aligned}$$

where, setting $\psi_{1,n} = (\psi_{1,n}^1, \psi_{1,n}^2, \psi_{1,n}^3)$ and $\psi_{2,m} = (\psi_{2,m}^1, \psi_{2,m}^2, \psi_{2,m}^3)$,

$$\begin{aligned}\psi_{1,n}^1(x) &= \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right), \\ \psi_{1,n}^2(x) &= -\frac{\gamma_n}{\lambda_{1,n}^2(\cosh(\lambda_{1,n}L) + \alpha \sinh(\lambda_{1,n}L))} \sqrt{\frac{2}{L}} (\cosh(\lambda_{1,n}(x-L)) - \alpha \sinh(\lambda_{1,n}(x-L))) \\ &\quad - \frac{1}{\lambda_{1,n}^2} \sqrt{\frac{2}{L}} \int_x^L \beta(s) \sin\left(\frac{n\pi}{L}s\right) \sinh(\lambda_{1,n}(x-s)) ds \\ &\quad + \frac{1}{\lambda_{1,n}} \sqrt{\frac{2}{L}} \int_0^x \int_\tau^L \beta(s) \sin\left(\frac{n\pi}{L}s\right) ds d\tau, \\ \psi_{1,n}^3(x) &= \frac{\gamma_n}{\lambda_{1,n}(\cosh(\lambda_{1,n}L) + \alpha \sinh(\lambda_{1,n}L))} \sqrt{\frac{2}{L}} (\cosh(\lambda_{1,n}(x-L)) - \alpha \sinh(\lambda_{1,n}(x-L))) \\ &\quad + \frac{1}{\lambda_{1,n}} \sqrt{\frac{2}{L}} \int_x^L \beta(s) \sin\left(\frac{n\pi}{L}s\right) \sinh(\lambda_{1,n}(x-s)) ds,\end{aligned}$$

with

$$\gamma_n = \int_0^L \beta(s) \sin\left(\frac{n\pi}{L}s\right) \sinh(\lambda_{1,n}s) ds \quad (12a)$$

whenever $\frac{n^2\pi^2}{L^2} \neq c$, i.e., when $\lambda_{1,n} \neq 0$, and

$$\begin{aligned}\psi_{1,n}^1(x) &= \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right), & \psi_{1,n}^2(x) &= \alpha\gamma_n \sqrt{\frac{2}{L}}x, \\ \psi_{1,n}^3(x) &= \sqrt{\frac{2}{L}} \int_0^x \int_\tau^L \beta(s) \sin\left(\frac{n\pi}{L}s\right) ds d\tau,\end{aligned}$$

with

$$\gamma_n = \int_0^L \int_\tau^L \beta(s) \sin\left(\frac{n\pi}{L}s\right) ds d\tau \quad (12b)$$

whenever $\frac{n^2\pi^2}{L^2} = c$, i.e., when $\lambda_{1,n} = 0$, and

$$\begin{aligned}\psi_{2,m}^1(x) &= 0, & \psi_{2,m}^2(x) &= \frac{A_m}{L(\lambda_{2,m})^2} \sinh(\bar{\lambda}_{2,m}x), \\ \psi_{2,m}^3(x) &= -\frac{A_m}{L\lambda_{2,m}} \sinh(\bar{\lambda}_{2,m}x).\end{aligned}$$

The eigenvectors satisfy $\mathcal{A}^*\psi_{1,n} = \lambda_{1,n}\psi_{1,n}$ and $\mathcal{A}^*\psi_{2,m} = \bar{\lambda}_{2,m}\psi_{2,m}$ and have been normalized so that $\langle\phi_{1,n}, \psi_{1,n}\rangle = 1$ and $\langle\phi_{2,m}, \psi_{2,m}\rangle = 1$ for all $n \in \mathbb{N}^*$ and $m \in \mathbb{Z}$.

Proof. Using (11), it can be checked that $\langle\mathcal{A}(f_1, g_1, h_1), (f_2, g_2, h_2)\rangle = \langle(f_1, g_1, h_1), \mathcal{A}^*(f_2, g_2, h_2)\rangle$ for all $(f_1, g_1, h_1) \in D(\mathcal{A})$ and all $(f_2, g_2, h_2) \in D(\mathcal{A}^*)$. Let us now solve $\mathcal{A}^*(f, g, h) = \lambda(f, g, h)$ for some $\lambda \in \mathbb{C}$ and $(f, g, h) \in D(\mathcal{A}^*)$. This gives $f'' + cf = \lambda f$, $-g'' = \lambda h$ and $-\int_0^x \int_L^\tau \beta(s)f(s) ds d\tau - h = \lambda g$. Proceeding similarly to the proof of Lemma 1, the case $f = 0$ gives $\lambda = \lambda_{2,-m}$ for some $m \in \mathbb{Z}$ with $(f, g, h) = \psi_{2,m}$ while the case $f \neq 0$ gives $\lambda = \lambda_{1,n}$ for some $n \in \mathbb{N}^*$. Studying separately the cases $\lambda_{1,n} = 0$ and $\lambda_{1,n} \neq 0$, we infer the claimed conclusion. Note that in the case $\lambda_{1,n} \neq 0$ the introduced eigenfunctions are well-defined because it follows from (5) that $\cosh(\lambda_{1,n}L) + \alpha \sinh(\lambda_{1,n}L) \neq 0$. \square

Lemma 2 shows that \mathcal{A} is a Riesz operator. This will allow us in the next sections to perform stabilization in \mathcal{H}^0 norm. But actually, the stabilization of the parabolic part of the system can also be achieved in H^1 norm (compared to the L^2 norm in the case of the space \mathcal{H}^0) thanks to the following result.

Lemma 4. $\Phi^1 = \{\frac{L}{n\pi}\phi_{1,n} \mid n \in \mathbb{N}^*\} \cup \{\phi_{2,m} \mid m \in \mathbb{Z}\}$ is a Riesz basis of the Hilbert space

$$\mathcal{H}^1 = H_0^1(0, L) \times H_{(0)}^1(0, L) \times L^2(0, L) \quad (13a)$$

endowed with the inner product

$$\langle (f_1, g_1, h_1), (f_2, g_2, h_2) \rangle = \int_0^L (f_1' \overline{f_2'} + g_1' \overline{g_2'} + h_1 \overline{h_2}). \quad (13b)$$

Proof. Denoting by $\Delta : H_0^1(0, L) \cap H^2(0, L) \rightarrow L^2(0, L)$ the usual Dirichlet operator, the operator $J : \mathcal{H}^1 \rightarrow \mathcal{H}^0$ defined by $J(f, g, h) = (\sqrt{-\Delta}f, g, h)$ is easily seen to be a surjective isometry. Indeed, by integration by parts,

$$\begin{aligned} \langle J(f_1, g_1, h_1), J(f_2, g_2, h_2) \rangle &= \langle (-\Delta f_1, g_1, h_1), (f_2, g_2, h_2) \rangle \\ &= \int_0^L f_1' f_2' + g_1' g_2' + h_1 h_2 \, dx = \langle (f_1, g_1, h_1), (f_2, g_2, h_2) \rangle. \end{aligned}$$

Hence $\{J^{-1}\phi_{1,n} \mid n \in \mathbb{N}^*\} \cup \{J^{-1}\phi_{2,m} \mid m \in \mathbb{Z}\}$ is a Riesz basis of \mathcal{H}^1 , and we have $J^{-1}\phi_{1,n} = \frac{L}{n\pi}\phi_{1,n}$ and $J^{-1}\phi_{2,m} = ((\sqrt{-\Delta})^{-1}\phi_{2,m}^1, \phi_{2,m}^2, \phi_{2,m}^3)$. To use the Bari theorem, we have to show that $\sum_{m \in \mathbb{Z}} \|\phi_{2,m}^1 - (\sqrt{-\Delta})^{-1}\phi_{2,m}^1\|_{H_0^1}^2 < +\infty$. Since $\|(\sqrt{-\Delta})^{-1}\phi_{2,m}^1\|_{H_0^1} = \|\phi_{2,m}^1\|_{L^2}$, it is sufficient to show that $\sum_{m \in \mathbb{Z}} \|\phi_{2,m}^1\|_{H_0^1}^2 < +\infty$. Following the proof of Lemma 2, we have $\phi_{2,m}^1 = f_{m,1} + f_{m,2}$ with $\|f_{m,1}'\|_{L^\infty} = O(1/|m|^{3/2})$ and $\|f_{m,2}'\|_{L^\infty} = O(1/|m|^{3/2})$. \square

3 Spectral reduction

3.1 Homogeneous representation and spectral reduction

In order to work with a homogeneous representation of (6), we lift the control from the boundary into the domain (see e.g. [9, Sec. 3.3]) by making the change of variables:

$$w^1(t, x) = y(t, x), \quad w^2(t, x) = z(t, x), \quad (14a)$$

$$w^3(t, x) = \partial_t z(t, x) - \frac{x}{\alpha L} v(t). \quad (14b)$$

Assuming that v is of class \mathcal{C}^1 (this assumption will be fulfilled by the upcoming adopted feedback control strategy), the control system becomes

$$\partial_t w^1 = \partial_{xx} w^1 + c w^1 + \beta w^2, \quad (15a)$$

$$\partial_t w^2 = w^3 + \frac{x}{\alpha L} v, \quad (15b)$$

$$\partial_t w^3 = \partial_{xx} w^2 - \frac{x}{\alpha L} \dot{v}, \quad (15c)$$

$$w^1(t, 0) = w^1(t, L) = 0, \quad (15d)$$

$$w^2(t, 0) = 0, \quad \partial_x w^2(t, L) + \alpha w^3(t, L) = 0, \quad (15e)$$

$$w^1(0, x) = y_0(x), \quad w^2(0, x) = z_0(x), \quad (15f)$$

$$w^3(0, x) = z_1(x) - \frac{x}{\alpha L} v(0). \quad (15g)$$

Setting $W(t) = (w^1(t, \cdot), w^2(t, \cdot), w^3(t, \cdot))$, $a(x) = (0, \frac{x}{\alpha L}, 0) \in \mathcal{H}^0$ and $b(x) = (0, 0, -\frac{x}{\alpha L}) \in \mathcal{H}^0$, one has

$$\dot{W}(t) = \mathcal{A}W(t) + av(t) + b\dot{v}(t), \quad (16a)$$

$$W(0, x) = (y_0(x), z_0(x), z_1(x) - \frac{x}{\alpha L}v(0)). \quad (16b)$$

We now use the Riesz basis Φ and Ψ defined in Section 2.2, expanding

$$W(t, \cdot) = \sum_{n \in \mathbb{N}^*} w_{1,n}(t)\phi_{1,n} + \sum_{m \in \mathbb{Z}} w_{2,m}(t)\phi_{2,m} \quad (17)$$

with $w_{1,n} = \langle W(t, \cdot), \psi_{1,n} \rangle$, $w_{2,m} = \langle W(t, \cdot), \psi_{2,m} \rangle$, $a_{1,n} = \langle a, \psi_{1,n} \rangle$, $a_{2,m} = \langle a, \psi_{2,m} \rangle$, $b_{1,n} = \langle b, \psi_{1,n} \rangle$ and $b_{2,m} = \langle b, \psi_{2,m} \rangle$ for all $n \in \mathbb{N}^*$ and $m \in \mathbb{Z}$. Defining the new control $v_d = \dot{v}$ (auxiliary input for control design), we thus have

$$\dot{w}_{1,n} = \lambda_{1,n}w_{1,n} + a_{1,n}v + b_{1,n}v_d, \quad n \in \mathbb{N}^*, \quad (18a)$$

$$\dot{w}_{2,m} = \lambda_{2,m}w_{2,m} + a_{2,m}v + b_{2,m}v_d, \quad m \in \mathbb{Z}, \quad (18b)$$

$$\dot{v} = v_d. \quad (19)$$

3.2 Finite-dimensional model and controllability properties

3.2.1 Finite-dimensional model

For a given $\delta > 0$, let us choose an integer $N_0 \in \mathbb{N}^*$, large enough so that

$$\lambda_{1,n} \leq \lambda_{1,N_0+1} = c - \frac{(N_0+1)^2\pi^2}{L^2} < -\delta < 0 \quad \forall n \geq N_0 + 1.$$

We next consider the finite-dimensional system consisting of the N_0 first modes of the plant (18a) associated with the eigenvalues $\lambda_{1,n}$ of the reaction-diffusion equation. Setting

$$\begin{aligned} W_0 &= (w_{1,1} \quad w_{1,2} \quad \dots \quad w_{1,N_0})^\top \in \mathbb{R}^{N_0}, \\ A_0 &= \text{diag}(\lambda_{1,1}, \lambda_{1,2}, \dots, \lambda_{1,N_0}) \in \mathbb{R}^{N_0 \times N_0}, \\ B_{a,0} &= (a_{1,1} \quad a_{1,2} \quad \dots \quad a_{1,N_0})^\top \in \mathbb{R}^{N_0}, \\ B_{b,0} &= (b_{1,1} \quad b_{1,2} \quad \dots \quad b_{1,N_0})^\top \in \mathbb{R}^{N_0}, \end{aligned}$$

we have

$$\dot{W}_0(t) = A_0W_0(t) + B_{a,0}v(t) + B_{b,0}v_d(t). \quad (20)$$

Note in particular that these dynamics do not capture any mode $\lambda_{2,m}$ of the hyperbolic part of the plant as described by (18b). Augmenting the state vector and the matrices by setting

$$W_1 = \begin{pmatrix} v \\ W_0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ B_{a,0} & A_0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 \\ B_{b,0} \end{pmatrix},$$

we deduce that

$$\dot{W}_1(t) = A_1W_1(t) + B_1v_d(t). \quad (21)$$

3.2.2 Controllability

Recalling that v_d is viewed as an auxiliary control input, the following result characterizes the controllability property of (21).

Lemma 5. *The pair (A_1, B_1) satisfies the Kalman condition if and only if $\gamma_n \neq 0$ for any $n \in \{1, \dots, N_0\}$, where γ_n is defined by (12a) if $\frac{n^2\pi^2}{L^2} \neq c$ and by (12b) if $\frac{n^2\pi^2}{L^2} = c$.*

Proof. By the Hautus test, (A_1, B_1) does not satisfy the Kalman condition if and only if there exist $\lambda \in \mathbb{C}$ and $(x_1, x_2) \in \mathbb{C} \times \mathbb{C}^{N_0} \setminus \{(0, 0)\}$ such that $\overline{x_2}^\top B_{a,0} = \lambda \overline{x_1}^\top$, $\overline{x_2}^\top A_0 = \lambda \overline{x_2}^\top$, and $\overline{x_1}^\top + \overline{x_2}^\top B_{b,0} = 0$. This is possible if and only if there exists $x_2 \neq 0$ such that $A_0^* x_2 = \overline{\lambda} x_2$ and $\overline{x_2}^\top (B_{a,0} + \lambda B_{b,0}) = 0$, i.e., if and only if $\lambda \in \Lambda = \{\lambda_{1,1}, \lambda_{1,2}, \dots, \lambda_{1,N_0}\}$ while (because the eigenvalues of \mathcal{A}^* are simple) there exists $z \in D(\mathcal{A}^*) \setminus \{0\}$ such that $\mathcal{A}^* z = \overline{\lambda} z$ and $\langle a + \lambda b, z \rangle = 0$. Setting $z = (z^1, z^2, z^3)$, the equation $\mathcal{A}^* z = \overline{\lambda} z$ gives

$$(z^1)'' + cz^1 = \overline{\lambda} z^1 \quad (22a)$$

$$-z^3 + \int_0^{(\cdot)} \int_\tau^L \beta(s) z^1(s) ds d\tau = \overline{\lambda} z^2 \quad (22b)$$

$$-(z^2)'' = \overline{\lambda} z^3 \quad (22c)$$

$$z^1(0) = z^1(L) = z^2(0) = z^3(0) = 0 \quad (22d)$$

$$(z^2)'(L) - \alpha z^3(L) = 0 \quad (22e)$$

while

$$\begin{aligned} 0 &= \langle a + \lambda b, z \rangle = \int_0^L \left(\left(\frac{x}{\alpha L} \right)' \overline{(z^2)'(x)} - \frac{\lambda x}{\alpha L} \overline{z^3(x)} \right) dx \\ &= \left[\frac{x}{\alpha L} \overline{(z^2)'(x)} \right]_{x=0}^{x=L} - \int_0^L \frac{x}{\alpha L} \overline{((z^2)''(x) + \overline{\lambda} z^3(x))} dx \\ &= \frac{1}{\alpha} \overline{(z^2)'(L)} \end{aligned}$$

where we have used (22c). Combining with (22e), we get

$$(z^2)'(L) = z^3(L) = 0. \quad (23)$$

Let us discard the case $z^1 = 0$ by noting that, in this case, (22b) gives $-z^3 = \overline{\lambda} z^2$. Combining this result with (22c) and (23) we infer that $(z^3)'' - (\overline{\lambda})^2 z^3 = 0$ with $z^3(L) = (z^3)'(L) = 0$. By Cauchy uniqueness, we deduce that $z^3 = 0$. Finally, (22c) gives $(z^2)'' = 0$ with $z^2(0) = (z^2)'(L) = 0$, hence $z^2 = 0$. This is a contradiction with the initial assumption that $z \neq 0$.

Hence, we must have $z^1 \neq 0$. Based on (22a) and (22d) we have $(z^1)'' = (\overline{\lambda} - c)z^1$ with $z^1(0) = z^1(L) = 0$. Then there exist $A \neq 0$ and $n \in \mathbb{N}^*$ such that $z^1(x) = A \sin\left(\frac{n\pi}{L}x\right)$ and $\lambda = c - \frac{n^2\pi^2}{L^2}$. Since $\lambda \in \Lambda$, we have $\lambda = \lambda_{1,n}$ for some $n \in \{1, \dots, N_0\}$. We infer from (22b), (22c), and (23) that

$$(z^3)''(x) - \lambda_{1,n}^2 z^3(x) = -A\beta(x) \sin\left(\frac{n\pi}{L}x\right), \quad (24a)$$

$$z^3(0) = z^3(L) = (z^3)'(L) = 0, \quad (24b)$$

where we recall that $\lambda_{1,n} = c - \frac{n^2\pi^2}{L^2}$.

Assume first that $\lambda_{1,n} \neq 0$. In this case, integrating (24a), there exists $\delta_1, \delta_2 \in \mathbb{R}$ such that

$$z^3(x) = \left(\delta_1 + \frac{A}{2\lambda_{1,n}} \int_0^x e^{\lambda_{1,n}s} \beta(s) \sin\left(\frac{n\pi}{L}s\right) ds \right) e^{-\lambda_{1,n}x}$$

$$+ \left(\delta_2 - \frac{A}{2\lambda_{1,n}} \int_0^x e^{-\lambda_{1,n}s} \beta(s) \sin\left(\frac{n\pi}{L}s\right) ds \right) e^{\lambda_{1,n}x}.$$

The condition $z^3(0) = 0$ gives $\delta_1 + \delta_2 = 0$, hence

$$z^3(x) = -2\delta_1 \sinh(\lambda_{1,n}x) + \frac{A}{\lambda_{1,n}} \int_0^x \beta(s) \sinh(\lambda_{1,n}(s-x)) \sin\left(\frac{n\pi}{L}s\right) ds.$$

The conditions $z^3(L) = 0$ and $(z^3)'(L) = 0$ give the system

$$\begin{pmatrix} \sinh(\lambda_{1,n}L) & \chi_1 \\ \cosh(\lambda_{1,n}L) & \chi_2 \end{pmatrix} \begin{pmatrix} -2\delta_1\lambda_{1,n} \\ A \end{pmatrix} = 0$$

where $\chi_1 = \int_0^L \beta(s) \sinh(\lambda_{1,n}(s-L)) \sin\left(\frac{n\pi}{L}s\right) ds$ and $\chi_2 = -\int_0^L \beta(s) \cosh(\lambda_{1,n}(s-L)) \sin\left(\frac{n\pi}{L}s\right) ds$. Since $A \neq 0$ and $\lambda_{1,n} \neq 0$, the determinant of the above 2×2 matrix must be zero, i.e.,

$$0 = \int_0^L \beta(s) \sinh(\lambda_{1,n}s) \sin\left(\frac{n\pi}{L}s\right) ds = \gamma_n.$$

Under this condition, one can compute $A \neq 0$ and δ_1 , and thus obtain z^3 . Finally, z^2 is obtained by integrating twice (22c) and using the conditions $z^2(0) = (z^2)'(L) = 0$ borrowed from (22d) and (23). The obtained $z = (z^1, z^2, z^3) \neq 0$ satisfies (22) and (23).

Assume now that $\lambda_{1,n} = 0$. In this case, (24a) reduces to $(z^3)'' = -A\beta(x) \sin\left(\frac{n\pi}{L}x\right)$. Owing to (24b), we infer from the first and third conditions that $z^3(x) = A \int_0^x \int_\tau^L \beta(s) \sin\left(\frac{n\pi}{L}s\right) ds d\tau$ while the second condition gives

$$0 = \int_0^L \int_\tau^L \beta(s) \sin\left(\frac{n\pi}{L}s\right) ds d\tau = \gamma_n$$

because $A \neq 0$. As previously, we then obtain z^2 by integrating twice (22c) with the conditions $z^2(0) = (z^2)'(L) = 0$ borrowed from (22d) and (23). Hence, the computed $z = (z^1, z^2, z^3) \neq 0$ satisfies (22) and (23). \square

Remark 3. The result of Lemma 5 can be interpreted as follows. Since $X(t) = (y(t, \cdot), z(t, \cdot), \partial_t z(t, \cdot))^\top$, we have $W = X + bv$. Hence

$$\dot{x}_{1,n} = \lambda_{1,n}x_{1,n} + \beta_{1,n}v, \quad n \in \mathbb{N}^*, \quad (25a)$$

$$\dot{x}_{2,m} = \lambda_{2,m}x_{2,m} + \beta_{2,m}v, \quad m \in \mathbb{Z}, \quad (25b)$$

where $x_{1,n}(t) = \langle X(t, \cdot), \psi_{1,n} \rangle$, $x_{2,m}(t) = \langle X(t, \cdot), \psi_{2,m} \rangle$, $\beta_{1,n} = a_{1,n} + \lambda_{1,n}b_{1,n}$ and $\beta_{2,m} = \frac{a_{2,m} + \lambda_{2,m}b_{2,m}}{\alpha}$. The computations done in the proof of Lemma 5 show that $\beta_{i,l} = \frac{1}{\alpha}(\psi_{i,l}^2)'(L) = \psi_{i,l}^3(L)$. This means that the control has no impact on the mode $\lambda_{i,l}$ if and only if $\psi_{i,l}^3(L) = 0$. \circ

3.3 Discussion on the coefficients γ_n

We discuss the behavior of the coefficients γ_n for particular examples of functions β . Let us assume, in the present section, that

$$\beta(x) = \beta_0 \mathbf{1}_{[a,b]}(x) \quad \forall x \in (0, L) \quad (26)$$

for some $\beta_0 \in \mathbb{R} \setminus \{0\}$ and $0 \leq a < b \leq L$. Recalling that $\lambda_{1,n} = c - \frac{n^2\pi^2}{L^2}$, we infer from (12a) that

$$\gamma_n = \frac{\beta_0}{\lambda_{1,n}^2 + \frac{n^2\pi^2}{L^2}} \left(-\frac{n\pi}{L} \sinh(\lambda_{1,n}b) \cos\left(\frac{n\pi b}{L}\right) + \frac{n\pi}{L} \sinh(\lambda_{1,n}a) \cos\left(\frac{n\pi a}{L}\right) \right)$$

$$+ \lambda_{1,n} \cosh(\lambda_{1,n}b) \sin\left(\frac{n\pi b}{L}\right) - \lambda_{1,n} \cosh(\lambda_{1,n}a) \sin\left(\frac{n\pi a}{L}\right) \quad (27a)$$

if $\frac{n^2\pi^2}{L^2} \neq c$, and

$$\gamma_n = -\frac{\beta_0 L}{n\pi} \left(b \cos\left(\frac{n\pi b}{L}\right) - a \cos\left(\frac{n\pi a}{L}\right) \right) + \frac{\beta_0 L^2}{n^2\pi^2} \left(\sin\left(\frac{n\pi b}{L}\right) - \sin\left(\frac{n\pi a}{L}\right) \right) \quad (27b)$$

if $\frac{n^2\pi^2}{L^2} = c$. There are some critical values of a, b for which there exists $n \in \mathbb{N}^*$ such that $\gamma_n = 0$ and thus controllability is lost for the mode n (see Lemma 5). Let us elaborate further.

In the particular case $a = 0$ and $b = L$, i.e., $\beta \equiv \beta_0$ on $(0, L)$, we get from (27) that

$$\gamma_n = \frac{(-1)^n \beta_0 \frac{n\pi}{L}}{\left(\frac{n^2\pi^2}{L^2} - c\right)^2 + \frac{n^2\pi^2}{L^2}} \sinh\left(\left(\frac{n^2\pi^2}{L^2} - c\right)L\right) \neq 0$$

if $\frac{n^2\pi^2}{L^2} \neq c$, and $\gamma_n = \frac{(-1)^{n+1} \beta_0 L^2}{n\pi} \neq 0$ if $\frac{n^2\pi^2}{L^2} = c$. Hence, the result of Lemma 5 ensures that the finite-dimensional system (21) is always controllable, whatever the numbers N_0 of modes of the parabolic spectrum captured by the dynamics.

Consider now the case $L = 1$, $c = 50$, $\beta_0 = 1$, and $a = 0$. Figure 1 shows γ_2 as a function of $b \in [0, L]$; γ_2 vanishes at $b \approx 0.586$ and thus, controllability is lost for this value of b .

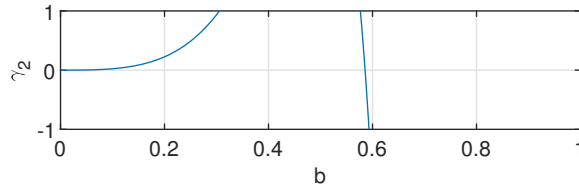


Figure 1: γ_2 as a function of $b \in [0, 1]$, with $L = 1$, $c = 50$, $\beta_0 = 1$, and $a = 0$.

The next lemma shows that loss of controllability rarely happens.

Lemma 6. Define $S = \{(a, b) \mid 0 \leq a < b \leq L\}$ and recall γ_n is defined by (12). The subset \hat{S} of S such that $\gamma_n \neq 0$ for any $n \in \mathbb{N}^*$ is dense and of full Lebesgue measure in S .

Proof. Given any $n \in \mathbb{N}^*$, the set $Z_n = \{(a, b) \in S \mid \gamma_n = 0\}$ is closed, of zero measure and of empty interior because γ_n is a nontrivial analytic function of (a, b) . The union Z of all Z_n , $n \in \mathbb{N}^*$, is of zero measure, and of empty interior by the Baire theorem. The set \hat{S} is then defined as the complement of Z in S . \square

Remark 4. Lemmas 5 and 6 imply that, when $(a, b) \in \hat{S}$, the finite-dimensional system (21) is always controllable, whatever the numbers N_0 of modes of the parabolic spectrum captured by the dynamics. However, although \hat{S} is dense and of full Lebesgue measure in S , it may fail to be open, and thus robustness of controllability with respect to (a, b) may fail. \circ

3.4 Measurement

In view of the control design, based on (17), we note that the system output $y_o(t)$ defined by (3) is expanded as

$$y_o(t) = \int_0^L c_o(x) y(t, x) dx = \int_0^L c_o(x) w^1(t, x) dx$$

$$\begin{aligned}
&= \sum_{n \in \mathbb{N}^*} c_{1,n} w_{1,n}(t) + \sum_{m \in \mathbb{Z}} c_{2,m} w_{2,m}(t) \\
&= C_0 W_0(t) + \sum_{n \geq N_0+1} c_{1,n} w_{1,n}(t) + \sum_{m \in \mathbb{Z}} c_{2,m} w_{2,m}(t)
\end{aligned}$$

with

$$c_{1,n} = \int_0^L c_o(x) \phi_{1,n}^1(x) dx, \quad n \in \mathbb{N}^*, \quad (28a)$$

$$c_{2,m} = \int_0^L c_o(x) \phi_{2,m}^1(x) dx, \quad m \in \mathbb{Z}. \quad (28b)$$

and $C_0 = (c_{1,1} \ c_{1,2} \ \dots \ c_{1,N_0}) \in \mathbb{R}^{1 \times N_0}$. Since the matrix A_0 is diagonal with simple eigenvalues, we have the following result which will be instrumental for the proposed output-feedback control strategy.

Lemma 7. *The pair (A_0, C_0) satisfies the Kalman condition if and only if*

$$c_{1,n} = \sqrt{\frac{2}{L}} \int_0^L c_o(x) \sin\left(\frac{n\pi}{L}x\right) dx \neq 0, \quad \forall n \in \{1, \dots, N_0\}.$$

4 Feedback stabilization

Given two integers $N \geq N_0 + 1$ and $M \in \mathbb{N}$ to be chosen later, we define the following control strategy:

$$\begin{aligned}
\dot{\hat{w}}_{1,n} &= \lambda_{1,n} \hat{w}_{1,n} + a_{1,n} v + b_{1,n} v_d \\
&\quad - l_{1,n} \left(\sum_{k=1}^N c_{1,k} \hat{w}_{1,k} + \sum_{|l| \leq M} c_{2,l} \hat{w}_{2,l} - y_o \right), \quad 1 \leq n \leq N_0, \quad (29a)
\end{aligned}$$

$$\dot{\hat{w}}_{1,n} = \lambda_{1,n} \hat{w}_{1,n} + a_{1,n} v + b_{1,n} v_d, \quad N_0 + 1 \leq n \leq N, \quad (29b)$$

$$\dot{\hat{w}}_{2,m} = \lambda_{2,m} \hat{w}_{2,m} + a_{2,m} v + b_{2,m} v_d, \quad |m| \leq M, \quad (29c)$$

$$v_d = k_v v + \sum_{n=1}^{N_0} k_{1,n} \hat{w}_{1,n}, \quad (29d)$$

where k_v and $k_{1,n}$, for $1 \leq n \leq N_0$, are the feedback gains and $l_{1,n}$, for $1 \leq n \leq N_0$, are the observer gains. This finite-dimensional control strategy leveraging a Luenberger-type observer on a finite number of modes is inspired by the seminal work [26] and its more recent developments [12, 15, 17, 18]. We set

$$\begin{aligned}
K &= (k_v \ k_{1,1} \ k_{1,2} \ \dots \ k_{1,N_0}) \in \mathbb{R}^{1 \times (N_0+1)}, \\
L &= (l_{1,1} \ l_{1,2} \ \dots \ l_{1,N_0})^\top \in \mathbb{R}^{N_0}.
\end{aligned}$$

We now state the main result of this paper.

Theorem 2. *Let $\delta > 0$ be arbitrary. Let $N_0 \in \mathbb{N}^*$ and $\alpha > 1$ be such that $\lambda_{1,N_0+1} < -\delta$ and $\rho = \frac{1}{2L} \log\left(\frac{\alpha-1}{\alpha+1}\right) = \operatorname{Re} \lambda_{2,m} < -\delta$. Assume that:*

- $\gamma_n \neq 0$ for any $n \in \{1, \dots, N_0\}$, where γ_n is defined by (12a) if $\frac{n^2 \pi^2}{L^2} \neq c$ and by (12b) if $\frac{n^2 \pi^2}{L^2} = c$;

- $c_{1,n} \neq 0$ for any $n \in \{1, \dots, N_0\}$, where $c_{1,n}$ is defined by (28).

Let $K \in \mathbb{R}^{1 \times (N_0+1)}$ and $L \in \mathbb{R}^{N_0}$ be such that $A_1 + B_1 K$ and $A_0 - LC_0$ are Hurwitz with eigenvalues of real part less than $-\delta < 0$.

Then, for all integers $N \geq N_0 + 1$ and M sufficiently large¹, there exists $C > 0$ such that any solution of the system (1) in closed-loop with the output-feedback control (3), (4), (19), (29) satisfies

$$\begin{aligned} & \left\| (y(t, \cdot), z(t, \cdot), \partial_t z(t, \cdot)) \right\|_H + |v(t)| + \sum_{n=1}^N |\hat{w}_{1,n}(t)| + \sum_{|m| \leq M} |\hat{w}_{2,m}(t)| \\ & \leq C e^{-\delta t} \left(\left\| (y(0, \cdot), z(0, \cdot), \partial_t z(0, \cdot)) \right\|_H + |v(0)| + \sum_{n=1}^N |\hat{w}_{1,n}(0)| + \sum_{|m| \leq M} |\hat{w}_{2,m}(0)| \right) \end{aligned} \quad (30)$$

for every $t \geq 0$, where H is either the Hilbert space \mathcal{H}^0 defined by (7) or \mathcal{H}^1 defined by (13).

Remark 5. As it follows from the proof, for any initial condition $(y(0, \cdot), z(0, \cdot), \partial_t z(0, \cdot)) \in H$, there is a unique solution living in H . \circ

Proof. Let $\kappa \in [0, 2)$ be such that $(c_{1,n}/n^\kappa)_{n \in \mathbb{N}^*} \in \ell^2(\mathbb{N})$. In this proof we set $\kappa = 0$; different values will be chosen later in the proof of Theorem 3. We define the observation discrepancies $e_{1,n} = w_{1,n} - \hat{w}_{1,n}$ and $e_{2,m} = w_{2,m} - \hat{w}_{2,m}$, the scaled quantity $\tilde{e}_{1,n} = n^\kappa e_{1,n}$, as well as the vectors

$$\begin{aligned} \hat{W}_0 &= (\hat{w}_{1,1} \quad \hat{w}_{1,2} \quad \dots \quad \hat{w}_{1,N_0})^\top, \\ E_1 &= (e_{1,1} \quad e_{1,2} \quad \dots \quad e_{1,N_0})^\top, \\ \hat{W}_2 &= (\hat{w}_{1,N_0+1} \quad \dots \quad \hat{w}_{1,N} \quad \hat{w}_{2,0} \quad \dots \quad \hat{w}_{2,-M} \quad \hat{w}_{2,M})^\top, \\ E_2 &= (\tilde{e}_{1,N_0+1} \quad \dots \quad \tilde{e}_{1,N} \quad e_{2,0} \quad \dots \quad e_{2,-M} \quad e_{2,M})^\top. \end{aligned}$$

We infer from (18) and (29) that

$$\dot{\hat{W}}_0 = A_0 \hat{W}_0 + B_{a,0} v + B_{b,0} v_d + LC_0 E_1 + LC_1 E_2 + L\zeta_1 + L\zeta_2, \quad (31a)$$

$$\dot{E}_1 = (A_0 - LC_0) E_1 - LC_1 E_2 - L\zeta_1 - L\zeta_2, \quad (31b)$$

$$\dot{\hat{W}}_2 = A_2 \hat{W}_2 + B_{a,1} v + B_{b,1} v_d, \quad (31c)$$

$$\dot{E}_2 = A_2 E_2, \quad (31d)$$

where

$$\begin{aligned} A_2 &= \text{diag}(\lambda_{1,N_0+1}, \dots, \lambda_{1,N}, \lambda_{2,0}, \dots, \lambda_{2,-M}, \lambda_{2,M}), \\ B_{a,1} &= (a_{1,N_0+1} \quad \dots \quad a_{1,N} \quad a_{2,0} \quad \dots \quad a_{2,-M} \quad a_{2,M})^\top, \\ B_{b,1} &= (b_{1,N_0+1} \quad \dots \quad b_{1,N} \quad b_{2,0} \quad \dots \quad b_{2,-M} \quad b_{2,M})^\top, \\ C_1 &= \left(\frac{c_{1,N_0+1}}{(N_0+1)^\kappa} \quad \dots \quad \frac{c_{1,N}}{N^\kappa} \quad c_{2,0} \quad \dots \quad c_{2,-M} \quad c_{2,M} \right), \\ \zeta_1 &= \sum_{k \geq N+1} c_{1,k} w_{1,k}, \quad \zeta_2 = \sum_{|l| \geq M+1} c_{2,l} w_{2,l}, \end{aligned}$$

¹They must be chosen large enough so that the inequalities (37) are satisfied.

with $A_2 \in \mathbb{C}^{N-N_0+2M+1}$, $B_{a,1}, B_{b,1} \in \mathbb{C}^{N-N_0+2M+1}$ and $C_1 \in \mathbb{C}^{N-N_0+2M+1}$. We note in particular that $\|C_1\| = O(1)$ as $N, M \rightarrow \infty$. Defining the augmented vectors

$$\hat{W}_1 = \begin{pmatrix} v \\ \hat{W}_0 \end{pmatrix}, \quad \tilde{L} = \begin{pmatrix} 0 \\ L \end{pmatrix},$$

we obtain

$$\begin{aligned} v_d &= K\hat{W}_1, \\ \dot{\hat{W}}_1 &= (A_1 + B_1K)\hat{W}_1 + \tilde{L}C_0E_1 + \tilde{L}C_1E_2 + \tilde{L}\zeta_1 + \tilde{L}\zeta_2. \end{aligned}$$

Finally, setting $Y = \text{col}(\hat{W}_1, E_1, \hat{W}_2, E_2)$, we have

$$\dot{Y} = FY + \mathcal{L}\zeta_1 + \mathcal{L}\zeta_2$$

where

$$\begin{aligned} F &= \begin{pmatrix} A_1 + B_1K & \tilde{L}C_0 & 0 & \tilde{L}C_1 \\ 0 & A_0 - LC_0 & 0 & -LC_1 \\ B_{a,1}(1 \ 0 \ \cdots \ 0) + B_{b,1}K & 0 & A_2 & 0 \\ 0 & 0 & 0 & A_2 \end{pmatrix} \\ \mathcal{L} &= (\tilde{L}^\top \quad -L^\top \quad 0 \quad 0)^\top. \end{aligned}$$

Let us now define a suitable Lyapunov functional. Let P be a Hermitian positive definite matrix (to be chosen later).

For the parabolic part of the system evaluated in L^2 norm (i.e., for the state of the system evaluated in \mathcal{H}^0 norm), we define

$$V(Y, w) = \bar{Y}^\top PY + \sum_{n \geq N+1} |w_{1,n}|^2 + \sum_{|m| \geq M+1} |w_{2,m}|^2. \quad (33)$$

Since Φ is a Riesz basis of \mathcal{H}^0 (see Lemma 2), of dual basis Ψ (see Lemma 3), it follows that \sqrt{V} is a norm, equivalent to the norm of $\mathbb{C}^{2N+1} \times \mathcal{H}^0$.

For the parabolic part of the system evaluated in H^1 norm (i.e., for the state of the system evaluated in \mathcal{H}^1 norm), we define

$$V(Y, w) = \bar{Y}^\top PY + \sum_{n \geq N+1} n^2 |w_{1,n}|^2 + \sum_{|m| \geq M+1} |w_{2,m}|^2. \quad (34)$$

Since Φ^1 is a Riesz basis of \mathcal{H}^1 (see Lemma 4), \sqrt{V} is a norm, equivalent to the norm of $\mathbb{C}^{2N+1} \times \mathcal{H}^1$. Indeed, using (17), we have

$$\begin{aligned} Y(t, \cdot) &= (w^1(t, \cdot), w^2(t, \cdot), w^3(t, \cdot)) \\ &= \sum_{n \in \mathbb{N}^*} w_{1,n}(t) \phi_{1,n} + \sum_{m \in \mathbb{Z}} w_{2,m}(t) \phi_{2,m} \\ &= \sum_{n \in \mathbb{N}^*} \frac{n\pi}{L} w_{1,n}(t) \frac{L}{n\pi} \phi_{1,n} + \sum_{m \in \mathbb{Z}} w_{2,m}(t) \phi_{2,m}. \end{aligned}$$

The above two series converge *a priori* in \mathcal{H}^0 norm. However, for a classical solution $W(t, \cdot) \in D(\mathcal{A})$, using Lemma 4, the series converge in \mathcal{H}^1 norm. Hence, thanks to Lemma 4, $\|W(t, \cdot)\|_{\mathcal{H}^1}^2$ is equivalent to $\sum_{n \in \mathbb{N}^*} n^2 |w_{1,n}(t)|^2 + \sum_{m \in \mathbb{Z}} |w_{2,m}(t)|^2$. This justifies the definition of (34).

Since the proofs in \mathcal{H}^0 norm and in \mathcal{H}^1 norm are now similar, we focus on the second case. Setting $\tilde{Y} = \text{col}(Y, \zeta_1, \zeta_2)$, the computation of the time derivative of V along the system trajectories gives

$$\begin{aligned}
\dot{V} &= \tilde{Y}^\top \begin{pmatrix} \bar{F}^\top P + PF & P\mathcal{L} & P\mathcal{L} \\ \mathcal{L}^\top P & 0 & 0 \\ \mathcal{L}^\top P & 0 & 0 \end{pmatrix} \tilde{Y} \\
&+ 2 \sum_{n \geq N+1} n^2 \text{Re}((\lambda_{1,n} w_{1,n} + a_{1,n} v + b_{1,n} v_d) \overline{w_{1,n}}) \\
&+ 2 \sum_{|m| \geq M+1} \text{Re}((\lambda_{2,m} w_{2,m} + a_{2,m} v + b_{2,m} v_d) \overline{w_{2,m}}) \\
&\leq \tilde{Y}^\top \begin{pmatrix} \bar{F}^\top P + PF & P\mathcal{L} & P\mathcal{L} \\ \mathcal{L}^\top P & 0 & 0 \\ \mathcal{L}^\top P & 0 & 0 \end{pmatrix} \tilde{Y} \\
&+ 2 \sum_{n \geq N+1} n^2 \lambda_{1,n} |w_{1,n}|^2 + 2 \sum_{|m| \geq M+1} \underbrace{\text{Re} \lambda_{2,m}}_{=\rho} |w_{2,m}|^2 \\
&+ \frac{2}{\epsilon} \left(\sum_{n \geq N+1} n^4 |w_{1,n}|^2 + \sum_{|m| \geq M+1} |w_{2,m}|^2 \right) \\
&+ \epsilon \left(\sum_{n \geq N+1} |a_{1,n}|^2 + \sum_{|m| \geq M+1} |a_{2,m}|^2 \right) |v|^2 \\
&+ \epsilon \left(\sum_{n \geq N+1} |b_{1,n}|^2 + \sum_{|m| \geq M+1} |b_{2,m}|^2 \right) |v_d|^2
\end{aligned}$$

with $\epsilon > 0$ arbitrary and $\rho = \frac{1}{2L} \log\left(\frac{\alpha-1}{\alpha+1}\right) = \text{Re} \lambda_{2,m}$. The latter inequality has been obtained by using four times Young's inequality $ab \leq \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2}$. Defining now $E = (1 \ 0 \ \dots \ 0)$ and $\tilde{K} = (K \ 0 \ 0 \ 0)$, we infer that $v = EY$ and $v_d = \tilde{K}Y$. Furthermore, using the Cauchy-Schwarz inequality,

$$\begin{aligned}
\zeta_1^2 &= \left(\sum_{k \geq N+1} c_{1,k} w_{1,k} \right)^2 \leq \underbrace{\sum_{k \geq N+1} |c_{1,k}|^2}_{=S_{c,1,N}} \times \sum_{k \geq N+1} |w_{1,k}|^2, \\
\zeta_2^2 &= \left(\sum_{|l| \geq M+1} c_{2,l} w_{2,l} \right)^2 \leq \underbrace{\sum_{|l| \geq M+1} |c_{2,l}|^2}_{=S_{c,2,M}} \times \sum_{|l| \geq M+1} |w_{2,l}|^2.
\end{aligned}$$

Defining $S_{a,N,M} = \sum_{n \geq N+1} |a_{1,n}|^2 + \sum_{|m| \geq M+1} |a_{2,m}|^2$ and $S_{b,N,M} = \sum_{n \geq N+1} |b_{1,n}|^2 + \sum_{|m| \geq M+1} |b_{2,m}|^2$, the combination of all above estimates gives

$$\dot{V} + 2\delta V \leq \tilde{Y}^\top \Theta \tilde{Y} + \sum_{n \geq N+1} n^2 \Gamma_{1,n} |w_{1,n}|^2 + \Gamma_{2,M} \sum_{|m| \geq M+1} |w_{2,m}|^2 \quad (35)$$

with

$$\Theta = \begin{pmatrix} \Theta_{1,1} & P\mathcal{L} & P\mathcal{L} \\ \mathcal{L}^\top P & -\eta_1 & 0 \\ \mathcal{L}^\top P & 0 & -\eta_2 \end{pmatrix}, \quad (36a)$$

$$\Gamma_{1,n} = 2\left(\lambda_{1,n} + \frac{n^2}{\epsilon} + \delta\right) + \frac{\eta_1 S_{c,1,N}}{n^2}, \quad (36b)$$

$$\Gamma_{2,M} = 2\left(\rho + \frac{1}{\epsilon} + \delta\right) + \eta_2 S_{c,2,M}, \quad (36c)$$

where $\Theta_{1,1} = \bar{F}^\top P + PF + 2\delta P + \epsilon S_{a,N,M} \bar{E}^\top E + \epsilon S_{b,N,M} \tilde{K}^\top \tilde{K}$ and $\eta_1, \eta_2 > 0$ arbitrary. Choosing $\epsilon > L^2/\pi^2$, we have $\Gamma_{1,n} = 2\left(-\left(\frac{\pi^2}{L^2} - \frac{1}{\epsilon}\right)n^2 + c + \delta\right) + \frac{\eta_1 S_{c,1,N}}{n^2} \leq \Gamma_{1,N+1}$ for any $n \geq N+1$. Hence, $\dot{V} + 2\delta V \leq 0$ (yielding the claimed stability estimate (30)) provided there exist integers $N \geq N_0 + 1$ and $M \in \mathbb{N}$, real numbers $\epsilon > L^2/\pi^2$ and $\eta_1, \eta_2 > 0$, and a Hermitian matrix $P \succ 0$ such that

$$\Theta \preceq 0, \quad \Gamma_{1,N+1} \leq 0, \quad \Gamma_{2,M} \leq 0. \quad (37)$$

To conclude the proof, it remains to prove that the constraints (37) are always feasible for N, M chosen large enough. Recalling that the matrix F is defined by (32), it is easy to see that F is Hurwitz with eigenvalues of real part less than $-\delta < 0$. Moreover, since $\|C_1\| = O(1)$ as $N, M \rightarrow \infty$, the application of [17, Appendix] shows that the solution $P \succ 0$ of the Lyapunov equation $\bar{F}^\top P + PF + 2\delta P = -I$ is such that $\|P\| = O(1)$ as $N, M \rightarrow +\infty$. Furthermore, $\|E\|$, $\|\mathcal{L}\|$, and $\|\tilde{K}\|$ are constants, not depending on N, M while $S_{a,N,M}, S_{b,N,M}, S_{c,1,N}, S_{c,2,M} \rightarrow 0$ as $N, M \rightarrow +\infty$. Recalling that, by assumption, $\rho < -\delta$, we take $\epsilon > L^2/\pi^2$ large enough so that $\rho + \frac{1}{\epsilon} + \delta < 0$. We finally take

$$\eta_1 = \begin{cases} \frac{1}{\sqrt{S_{c,1,N}}} & \text{if } S_{c,1,N} \neq 0, \\ N & \text{otherwise,} \end{cases}$$

$$\eta_2 = \begin{cases} \frac{1}{\sqrt{S_{c,2,M}}} & \text{if } S_{c,2,M} \neq 0, \\ M & \text{otherwise,} \end{cases}$$

which, in particular, implies that $\eta_1, \eta_2 \rightarrow +\infty$ while $\eta_1 S_{c,1,N}, \eta_2 S_{c,2,M} \rightarrow 0$ as $N, M \rightarrow +\infty$. With these choices, it can be seen that $\Gamma_{1,N+1} \rightarrow -\infty$ as $N \rightarrow +\infty$ and $\Gamma_{2,M} \rightarrow 2\left(\rho + \frac{1}{\epsilon} + \delta\right) < 0$ as $M \rightarrow +\infty$. Hence $\Gamma_{1,N+1} \leq 0$ and $\Gamma_{2,M} \leq 0$ for all N, M large enough. Finally, the Schur complement theorem applied to

$$\Theta = \begin{pmatrix} -I + \epsilon S_{a,N,M} \bar{E}^\top E + \epsilon S_{b,N,M} \tilde{K}^\top \tilde{K} & P\mathcal{L} & P\mathcal{L} \\ \mathcal{L}^\top P & -\eta_1 & 0 \\ \mathcal{L}^\top P & 0 & -\eta_2 \end{pmatrix}.$$

shows that $\Theta \preceq 0$ if and only if

$$-I + \epsilon S_{a,N,M} \bar{E}^\top E + \epsilon S_{b,N,M} \tilde{K}^\top \tilde{K} + (P\mathcal{L} \quad P\mathcal{L}) \begin{pmatrix} \frac{1}{\eta_1} & 0 \\ 0 & \frac{1}{\eta_2} \end{pmatrix} \begin{pmatrix} \mathcal{L}^\top P \\ \mathcal{L}^\top P \end{pmatrix} \preceq 0.$$

We note that

$$-I + \epsilon S_{a,N,M} \bar{E}^\top E + \epsilon S_{b,N,M} \tilde{K}^\top \tilde{K} \preceq -(1 - \epsilon(S_{a,N,M} \|E\|^2 + S_{b,N,M} \|\tilde{K}\|^2))I \preceq -\frac{1}{2}I$$

for N, M taken large enough, because $\epsilon(S_{a,N,M} \|E\|^2 + S_{b,N,M} \|\tilde{K}\|^2) \rightarrow 0$ as $N, M \rightarrow +\infty$. In this case,

$$\begin{aligned} -I + \epsilon S_{a,N,M} \bar{E}^\top E + \epsilon S_{b,N,M} \tilde{K}^\top \tilde{K} + (P\mathcal{L} \quad P\mathcal{L}) \begin{pmatrix} \frac{1}{\eta_1} & 0 \\ 0 & \frac{1}{\eta_2} \end{pmatrix} \begin{pmatrix} \mathcal{L}^\top P \\ \mathcal{L}^\top P \end{pmatrix} \\ \preceq -\frac{1}{2}I + \left(\frac{1}{\eta_1} + \frac{1}{\eta_2}\right) P\mathcal{L}\mathcal{L}^\top P. \end{aligned}$$

Recalling that $\eta_1, \eta_2 \rightarrow +\infty$ and $\|P\| = O(1)$ as $N, M \rightarrow +\infty$ while $\|\mathcal{L}\|$ is a constant, not depending on N, M , the latter matrix is $\preceq 0$ for N, M chosen large enough. This completes the proof. \square

5 Extension to a pointwise measurement for the heat equation

In this section, instead of the distributed measurement (3) of the reaction-diffusion PDE, we consider the case of pointwise measurement described either by

$$y_o(t) = y(t, \xi_p) \quad (38)$$

or by

$$y_o(t) = \partial_x y(t, \xi_p) \quad (39)$$

for a fixed $\xi_p \in [0, L]$. We infer from (14) and (17) that either

$$\begin{aligned} y_o(t) &= w(t, \xi_p) \\ &= \sum_{n \in \mathbb{N}^*} w_{1,n}(t) \underbrace{\phi_{1,n}^1(\xi_p)}_{=c_{1,n}} + \sum_{m \in \mathbb{Z}} w_{2,m}(t) \underbrace{\phi_{2,m}^1(\xi_p)}_{=c_{2,m}} \end{aligned}$$

or

$$\begin{aligned} y_o(t) &= \partial_x w(t, \xi_p) \\ &= \sum_{n \in \mathbb{N}^*} w_{1,n}(t) \underbrace{(\phi_{1,n}^1)'(\xi_p)}_{=c_{1,n}} + \sum_{m \in \mathbb{Z}} w_{2,m}(t) \underbrace{(\phi_{2,m}^1)'(\xi_p)}_{=c_{2,m}}. \end{aligned}$$

Theorem 3. *Considering the pointwise measurement described either by (38) or (39) for some $\xi_p \in [0, L]$, the same statement as Theorem 2 holds true for the Hilbert space $H = \mathcal{H}^1$ defined by (13).*

Proof. The assumption that $c_{1,n} \neq 0$ for any $n \in \{1, \dots, N_0\}$ implies the controllability of the pair (A_0, C_0) . Now, the proof follows the one of Theorem 2 by using the Lyapunov functional (34). The only differences lie in the estimates of $\zeta_1 = \sum_{n \geq N+1} c_{1,n} w_{1,n}$ and of $\zeta_2 = \sum_{|m| \geq M+1} c_{2,m}^1(\xi_p) w_{2,m}$, as well as in the selection of $\kappa \in [0, 2)$ that appears in the definition of the matrix C_1 and which must be selected to ensure that $\|C_1\| = O(1)$ as $N \rightarrow +\infty$ for applying the Lemma in the appendix of [17].

In the case of the pointwise measurement (38), by Lemma 1, we have $\phi_{1,n}^1(\xi_p) = O(1)$, hence

$$\zeta_1^2 \leq \underbrace{\sum_{n \geq N+1} \frac{|\phi_{1,n}^1(\xi_p)|^2}{n^2}}_{=S_{c,1,N} < +\infty} \times \sum_{n \geq N+1} n^2 |w_{1,n}|^2.$$

Moreover, we showed in the proof of Lemma 2 that $\|\phi_{2,m}^1\|_{L^\infty} = O(1/m^2)$, yielding

$$\zeta_2^2 \leq \underbrace{\sum_{|m| \geq M+1} |\phi_{2,m}^1(\xi_p)|^2}_{=S_{c,2,M} < +\infty} \times \sum_{|m| \geq M+1} |w_{2,m}|^2.$$

Then, proceeding as in the proof of Theorem 2, we obtain (35) with $\Gamma_{1,n} = 2(\lambda_{1,n} + \frac{n^2}{\epsilon} + \delta) + \eta_1 S_{c,1,N}$, while Θ and $\Gamma_{2,M}$ are defined by (36a) and (36c), respectively. The above discussion also shows that selecting $\kappa = 1$ ensures that $\|C_1\| = O(1)$ as $N, M \rightarrow +\infty$. The proof is then similar to the one of Theorem 2.

In the case of the pointwise measurement (39), by Lemma 1, we have $(\phi_{1,n}^1)'(\xi_p) = O(n)$, hence

$$\zeta_1^2 \leq \underbrace{\sum_{n \geq N+1} \frac{|(\phi_{1,n}^1)'(\xi_p)|^2}{n^{7/2}}}_{=S_{c,1,N} < \infty} \times \sum_{n \geq N+1} n^{7/2} |w_{1,n}|^2.$$

Moreover, following the proof in Lemma 2 that $\|(\phi_{2,m}^1)'\|_{L^\infty} = O(1/|m|^{3/2})$, we have

$$\zeta_2^2 \leq \underbrace{\sum_{|m| \geq M+1} |\phi_{2,m}^1(\xi_p)|^2}_{=S_{c,2,M} < \infty} \times \sum_{|m| \geq M+1} |w_{2,m}|^2.$$

Then, proceeding as in the proof of Theorem 2, we obtain (35) with $\Gamma_{1,n} = 2\left(\lambda_{1,n} + \frac{n^2}{\epsilon} + \delta\right) + n^{3/2}\eta_1 S_{c,1,N}$, while Θ and $\Gamma_{2,M}$ are defined by (36a) and (36c), respectively. Selecting $\kappa = 7/4$, the above discussion also implies that $\|C_1\| = O(1)$ as $N, M \rightarrow +\infty$. The proof is then similar to the one of Theorem 2. \square

6 Numerical illustration

We consider the wave-heat cascade system (1) with $L = 1$, $c = 10$, and $\beta(x) = 1 + x^2$. The considered output is the pointwise measurement (38) localized at $\xi_p = \sqrt{3}/2$. In this setting the first element of the parabolic spectrum is $\lambda_{1,1} = 10 - \pi^2 > 0$, indicating that the heat subsystem is open-loop unstable. In order to achieve the targeted closed-loop decay rate $\delta = 1$, we set for the preliminary feedback (4) the constant $\alpha = 1.1$, ensuring that $\text{Re } \lambda_{2,m} < -\delta$. We next build the finite dimensional model (20) of dimension $N_0 = 1$ that captures the eigenvalue $\lambda_{1,1}$. It is checked that $\gamma_1 \neq 0$, which enables based on Lemma 5 the computation of a feedback matrix K_1 so that $A_1 - B_1 K$ has its eigenvalues located at $\{-2, -3\}$. Moreover, it is checked that $c_{1,1} \neq 0$, which enables us to compute the observer gain L so that $A_0 - LC_0$ has its eigenvalue located at -4 . Writing the constraints (37) from Theorem 3 into linear matrix inequalities of the decision variables ϵ, η_1, η_2 , it is found that the closed-loop system is exponentially stable with $\delta = 1$ for $N = 2$ and $M = 8$.

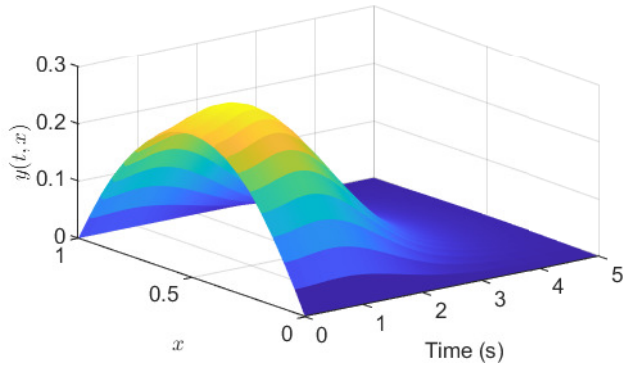
For numerical simulation, we consider the initial condition $y(0, x) = x(L - x)$ for the heat equation while $z(0, x) = \partial_t z(0, x) = 0$ for the wave equation. The simulation is performed by considering the 8 first modes of the parabolic spectrum and the $2 \times 20 + 1 = 41$ first modes of the hyperbolic spectrum. The obtained results are depicted in Fig. 2. In accordance with the theoretical predictions of Theorem 3, we observe the decay of the system trajectory to zero.

7 Conclusion

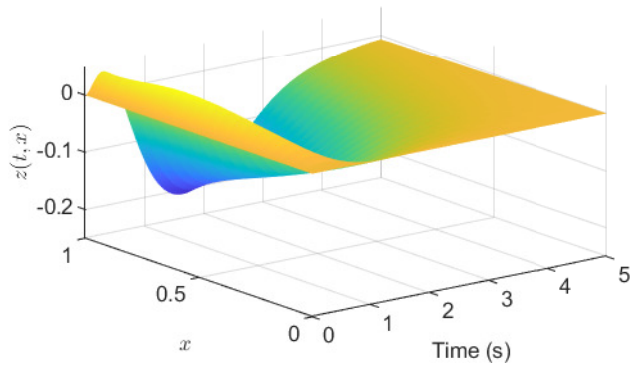
We have studied the stabilization problem for the wave-reaction-diffusion cascade system (1), where the solution of the wave equation appears as a source term in the heat equation. Exploiting the Riesz-spectral structure of the system, we derived explicit, necessary, and sufficient controllability and observability conditions under which we designed an explicit finite-dimensional output-feedback law ensuring arbitrary exponential decay in suitable Hilbert norms.

A natural perspective is to consider the ‘‘symmetric’’ cascade model where the solution of the heat equation appears as a source term in the wave equation:

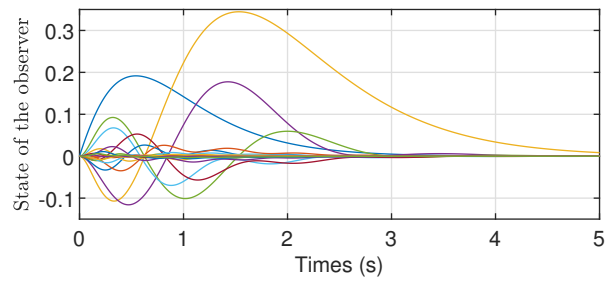
$$\partial_{tt} z(t, x) = \partial_{xx} z(t, x) + \beta(x)y(t, x), \quad (40a)$$



(a) State of the PDE $y(t, \cdot)$



(b) State of the PDE $z(t, \cdot)$



(c) State of the observer

Figure 2: Trajectory of the wave-heat cascade (1) in closed loop

$$\partial_t y(t, x) = \partial_{xx} y(t, x) + cy(t, x), \quad (40b)$$

$$z(t, 0) = \partial_x z(t, L) = 0, \quad (40c)$$

$$y(t, 0) = 0, \quad y(t, L) = u(t). \quad (40d)$$

While the study of the exact controllability of the system seems to be achievable using the same approach, the actual design of an explicit feedback control strategy seems much more challenging because the preliminary shift of the spectrum of the wave equation achieved by (4) cannot be applied anymore. This is left as an open question for future research.

Throughout the article, we have assumed c constant in (1a). Actually, it is not difficult to generalize all results of this paper to the case $c \in L^\infty(0, L)$, but this complicates the spectral analysis of the operators. For large values of n and m , the eigenvalues and eigenfunctions keep asymptotically the expression given in Lemmas 1 and 3, but for low frequencies their expression may differ, thus having an impact on the coefficients γ_n defined by (12). Then, more technical assumptions appear to characterize controllability.

Another question is to study multi-dimensional versions of (1). We believe this is much more challenging because the formalism of Riesz bases cannot be used.

.1 Defining \mathcal{B} by transposition

At the beginning of Section 2, following the theory of well-posed linear systems developed in [29] (see also [27]), we have written the control system (6) in the abstract form $\dot{X}(t) = \mathcal{A}X(t) + \mathcal{B}v(t)$ with $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H}^0 \rightarrow \mathcal{H}^0$ defined by (8), and the Hilbert space \mathcal{H}^0 is defined by (7). In this appendix, we show how to define the control operator \mathcal{B} by transposition. Recall that the adjoint operator $\mathcal{A}^* : D(\mathcal{A}^*) \rightarrow \mathcal{H}^0$ is defined by (11). Following [27, 29], identifying \mathcal{H}^0 with its dual, we search $\mathcal{B} \in L(\mathbb{R}, D(\mathcal{A}^*)')$, or equivalently, $\mathcal{B}^* \in L(D(\mathcal{A}^*), \mathbb{R})$, where $D(\mathcal{A}^*)'$ is the dual of $D(\mathcal{A}^*)$ with respect to the pivot space \mathcal{H}^0 . Note that $D(\mathcal{A}^*) \subset \mathcal{H}^0 \subset D(\mathcal{A}^*)'$ with continuous and dense embeddings. Since the equality $\dot{X} = \mathcal{A}X + \mathcal{B}v$ is written in the space $D(\mathcal{A}^*)'$, using the duality bracket $\langle \cdot, \cdot \rangle_{D(\mathcal{A}^*)', D(\mathcal{A}^*)}$, we have

$$\langle \dot{X}(t), \psi \rangle_{D(\mathcal{A}^*)', D(\mathcal{A}^*)} = \langle X(t), \mathcal{A}^* \psi \rangle_{\mathcal{H}^0} + v(t) \mathcal{B}^* \psi$$

for any $\psi \in D(\mathcal{A}^*)$. Taking a sufficiently regular solution $X(t, \cdot) = (y(t, \cdot), z(t, \cdot), \partial_t z(t, \cdot))^\top$ of (1) (so that $\dot{X}(t) \in \mathcal{H}^0$), and denoting $\psi = (\psi_1, \psi_2, \psi_3)^\top$, a straightforward computation using (6) and integrations by parts show that

$$\langle \dot{X}(t), \psi \rangle_{\mathcal{H}^0} = \langle X(t), \mathcal{A}^* \psi \rangle_{\mathcal{H}^0} + v(t) \psi_3(L).$$

Since $\langle \dot{X}(t), \psi \rangle_{D(\mathcal{A}^*)', D(\mathcal{A}^*)} = \langle \dot{X}(t), \psi \rangle_{\mathcal{H}^0}$, a density argument finally leads to

$$\mathcal{B}^* \psi = \psi_3(L) \quad \forall \psi = (\psi_1, \psi_2, \psi_3)^\top \in D(\mathcal{A}^*).$$

This formula defines the control operator $\mathcal{B} \in L(\mathbb{R}, D(\mathcal{A}^*)')$ by transposition. Following [27, Section 5.1.4] or [29, Proposition 10.9.1], one could define \mathcal{B} with an abstract formula, but this is not useful in this paper.

.2 Controllability properties of the wave-heat cascade (1)

The main objective of this paper is the explicit design of an output-feedback law that stabilizes the wave-heat cascade system (1) with a prescribed exponential decay rate. Theorem 2 shows that the system (1) is completely stabilizable in the energy space \mathcal{H}^0 defined by (7), i.e., can be stabilized at any decay rate by means of a linear feedback, provided that $\gamma_n \neq 0$ for all $n \in \mathbb{N}^*$. This raises

the natural question of whether corresponding exact controllability and exact null-controllability properties hold (see [27, 29]).

We give below, for completeness, a controllability result for the cascade system (1). Its proof, which is given in the companion work [22], relies on an observability inequality for the adjoint system, obtained by applying an Ingham-Müntz type inequality [2, 14, 16] to Riesz-spectral expansions of the solutions.

Theorem 4. *Assume that $T > 2L$ and that*

$$\gamma_n \neq 0 \quad \forall n \in \mathbb{N}^*. \quad (41)$$

(Recall that γ_n is defined by (12a) if $\frac{n^2\pi^2}{L^2} \neq c$ and by (12b) if $\frac{n^2\pi^2}{L^2} = c$.) Define the Hilbert space

$$V = \left\{ \sum_{n \in \mathbb{N}^*} a_n \phi_{1,n} + \sum_{m \in \mathbb{Z}} b_m \phi_{2,m} \mid a_n \in \mathbb{R}, b_m \in \mathbb{C}, \sum_{n \in \mathbb{N}^*} |a_n|^2 \frac{n^4}{\gamma_n^2} e^{\nu n^2} + \sum_{m \in \mathbb{Z}} |b_m|^2 < +\infty \right\}$$

with $\nu = 2\frac{\pi^2}{L^2}(1 + \frac{T}{L})$, endowed with the norm

$$\|X\|_V^2 = \sum_{n \in \mathbb{N}^*} |\langle X, \psi_{1,n} \rangle|^2 \frac{n^4}{\gamma_n^2} e^{\nu n^2} + \sum_{m \in \mathbb{Z}} |\langle X, \psi_{2,m} \rangle|^2.$$

The Hilbert space V_0 is defined as V but with $\nu = \frac{2\pi^2}{L^2}$. Then:

1. The control system (1) is exactly controllable in time T in the space V , i.e., given any $X_0, X_1 \in V$, there exists $u \in L^2(0, T)$ such that the solution $X = (y, z, \partial_t z)^\top$ of (1) starting at $X(0) = X_0$ satisfies $X(T) = X_1$.
2. The control system (1) is exactly null controllable in time T in the space V_0 , i.e., given any $X_0 \in V_0$, there exists $u \in L^2(0, T)$ such that the solution $X = (y, z, \partial_t z)^\top$ of (1) starting at $X(0) = X_0$ satisfies $X(T) = 0$.
3. The control system (1) is approximately controllable in time T in the Hilbert space \mathcal{H}^0 (or in any other Hilbert space H such that $H \subset \mathcal{H}^0$ or $\mathcal{H}^0 \subset H$ with continuous and dense embeddings), i.e., given any $X_0, X_1 \in \mathcal{H}^0$ and any $\varepsilon > 0$, there exists $u \in L^2(0, T)$ such that the solution $X = (y, z, \partial_t z)^\top$ of (1) starting at $X(0) = X_0$ satisfies $\|X(T) - X_1\|_{\mathcal{H}^0} \leq \varepsilon$.

Moreover, if one of the two following holds:

- $0 < T < 2L$;
- $T > 0$ with $\gamma_n = 0$ for some $n \in \mathbb{N}^*$,

then the control system (1) is neither exactly controllable, nor exactly null controllable, nor approximately controllable in time T in any Hilbert space \mathcal{H} such that $\mathcal{H} \subset \mathcal{H}^0$ or $\mathcal{H}^0 \subset \mathcal{H}$ with continuous and dense embeddings.

Remark 6. Thanks to Lemma 2, the spaces V and V_0 coincide with \mathcal{H}^0 on the hyperbolic components, whereas they are strict subspaces on the parabolic component. This reflects the stronger regularity requirement imposed by the reaction-diffusion part of the cascade.

It is interesting to note that the spaces V and V_0 are not conventional, due to the fact that, depending on the coupling function $\beta \in L^\infty(0, L)$, the coefficients γ_n may be wildly oscillating. ◦

It is known that exact null controllability implies complete stabilizability (see [23,28]) for initial data in the same space. Hence, Theorem 4 implies that, under the assumption of nontriviality of the coefficients γ_n , there exists a linear feedback control u such that the closed-loop system (1) is exponentially stable for initial data in V_0 , with a decay rate that can be chosen arbitrarily large. However, this result remains purely theoretical and does not provide an explicit way to construct a feedback law. Moreover, stabilization is obtained only for initial data in V_0 , which is a rather small subspace of \mathcal{H}^0 . These facts further motivate the explicit output-feedback construction developed in the present article.

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