



HAL
open science

The critical case for the concentration of eigenfunctions on singular Riemannian manifolds

Charlotte Dietze

► **To cite this version:**

Charlotte Dietze. The critical case for the concentration of eigenfunctions on singular Riemannian manifolds. 2026. <hal-05471643>

HAL Id: hal-05471643

<https://hal.science/hal-05471643v1>

Preprint submitted on 22 Jan 2026

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Copyright - All rights reserved

THE CRITICAL CASE FOR THE CONCENTRATION OF EIGENFUNCTIONS ON SINGULAR RIEMANNIAN MANIFOLDS

CHARLOTTE DIETZE

ABSTRACT. We consider a compact Riemannian manifold with boundary with a certain class of critical singular Riemannian metrics that are singular at the boundary. The corresponding Laplace-Beltrami operator can be seen as a Grushin-type operator plus a potential. We show in the critical case that the average density of eigenfunctions for the Laplace-Beltrami operator with eigenvalues below $\lambda > 0$ is distributed over all length scales between $\lambda^{-1/2}$ and 1 near the boundary. We give a precise description of this distribution as $\lambda \rightarrow \infty$.

1. INTRODUCTION

Let X be an $(n+1)$ -dimensional compact smooth Riemannian manifold with boundary, $n \in \mathbb{N}$. We consider singular Riemannian metrics g on X that are smooth and non-degenerate in the interior of X and have a certain critical singularity near the boundary.

For a suitable choice of coordinates near the boundary, where x is a transverse coordinate to the boundary, we can identify the manifold X near its boundary with $[0, 1] \times M$, where M is a smooth manifold of dimension n that corresponds to the boundary manifold. Here $\{0\} \times M$ is identified with the boundary ∂X . We will also use, for any $\varepsilon > 0$, the notation X_ε for the subset of X that is identified with $[0, \varepsilon] \times M$.

Definition of the singular Riemannian metric g . For any family $\{g_1(x)\}_{x \in [0,1]}$ of smooth non-degenerate Riemannian metrics $g_1(x)$ on M that depend continuously on the parameter $x \in [0, 1]$, we consider singular Riemannian metrics g on X that are smooth and non-singular inside X , and are of the form

$$(1) \quad g = dx^2 + x^{-\beta} g_1(x) \quad \text{on } X_1 \cong [0, 1] \times M.$$

Here $\beta \in (0, \infty)$ is a parameter modelling the singularity of g near the boundary. In this paper, we will focus on the case of a critical singularity, that is, $\beta = 2/n$.

We denote the Laplace-Beltrami operator with respect to g on X with Dirichlet boundary conditions by Δ_g . We use the sign convention that Δ_g is a non-negative operator.

Δ_g as a β -Grushin-type operator plus a potential. The operator Δ_g is a model example for an operator in sub-Riemannian geometry. More precisely, Δ_g can be seen as a β -Grushin-type operator plus a potential: For $\varepsilon_0 > 0$ small, the metric g is quasi-isometric with a constant close to 1 depending on ε_0 to a metric \tilde{g} on X , where \tilde{g} satisfies

$$(2) \quad \tilde{g} = dx^2 + x^{-\beta} g_1(0) \quad \text{on } X_{\varepsilon_0} \cong [0, \varepsilon_0] \times M.$$

2010 *Mathematics Subject Classification.* Primary: 58C40; Secondary: 53C17, 35P20.

Key words and phrases. Grushin, sub-Riemannian geometry, Weyl's law, singular Riemannian metric, gas giants.

On X_{ε_0} , we have

$$(3) \quad \Delta_{\tilde{g}} = -\partial_x^2 + \frac{C_\beta}{x^2} + x^\beta \Delta_{g_1(0)} \quad \text{with } C_\beta = \frac{\beta n}{4} \left(\frac{\beta n}{4} + 1 \right),$$

where $\Delta_{g_1(0)}$ denotes the Laplace-Beltrami operator with respect to $g_1(0)$ on M , see [7]. The case $\beta = 2$ corresponds to the classical Grushin operator

$$(4) \quad (\partial_x)^2 + \sum_{k=1}^n (x \partial_{y_k})^2$$

plus a potential $-\frac{C_\beta}{x^2}$. Thus, $-\Delta_g$ can be seen as a β -Grushin operator plus a potential, see also [2, (1.9)], where the case $n = 1$, $\beta = 2$ is considered.

Spectral properties of Δ_g in the literature. The spectral properties of Δ_g , and more generally Laplace-Beltrami operators of singular Riemannian metrics or operators from sub-Riemannian geometry, have been considered in many works in the literature, see for example [5]. In [17], the authors prove Weyl asymptotics of classical pseudo-differential operators on smooth manifolds with a principal symbol vanishing exactly to second order on a smooth symplectic submanifold, which corresponds to the Grushin case. Their asymptotics also include the critical case, where a logarithmic term appears, compare with (6) below. In [3], the Weyl asymptotics for the Grushin cylinder were computed using an explicit computation. We would also like to mention [20] where weighted Weyl laws are derived, and logarithmic terms appear. For the Grushin case, we also refer to [1, 15]. In [6], the authors study a general class of singular metrics, inspired by the Grushin model.

For the study of the small time asymptotics of sub-Riemannian heat kernels, we refer to [8, 9, 10, 4]. To treat singular boundaries, Melrose developed the b-calculus [16], see also [14] for an introduction. Concerning the localisation of eigenfunctions, in [13], the authors consider the magnetic Robin Laplacian and show exponential localisation of eigenfunctions near the boundary.

In the following, we give a detailed overview of the results in [7, 12] as they motivate and provide the context for the main result in the present paper.

For any $\lambda > 0$, we denote by $N(\lambda)$ the number of eigenvalues of Δ_g below λ . In [7, Theorem 1], the authors determined the leading-order Weyl asymptotics for $N(\lambda)$. As was pointed out after the completion of [7], equivalent results can be found much earlier in the literature in [18, 22], see also [21].

For $\beta < 2/n$, which is referred to as the subcritical case, the leading-order Weyl asymptotics of $N(\lambda)$ agree with the asymptotics one would expect if g was a non-singular Riemannian metric on X :

$$(5) \quad N(\lambda) = C_{n+1}^W \text{vol}(X, g) \lambda^{\frac{n+1}{2}} + o\left(\lambda^{\frac{n+1}{2}}\right) \quad \text{as } \lambda \rightarrow \infty.$$

Here $C_{n+1}^W > 0$ is the classical constant in Weyl's law that only depends on the dimension $n + 1$.

In the critical case $\beta = 2/n$, [7, Theorem 1] states that

$$(6) \quad N(\lambda) = C_n \text{vol}(M, g_1(0)) \lambda^{\frac{n+1}{2}} \log(\lambda) + o\left(\lambda^{\frac{n+1}{2}} \log(\lambda)\right) \quad \text{as } \lambda \rightarrow \infty.$$

Here, $C_n > 0$ is an explicit constant only depending on n . In a slightly different, but in a way more general setting, (6) was also shown in [6, Theorem 5.1].

In the supercritical case $\beta > 2/n$, using the notation $d := n(1 + \beta/2)$, we have

$$(7) \quad N(\lambda) = C_{n,\beta} \operatorname{vol}(M, g_1(0)) \lambda^{\frac{d}{2}} + o\left(\lambda^{\frac{d}{2}}\right) \quad \text{as } \lambda \rightarrow \infty,$$

where $C_{n,\beta} > 0$ is a constant only depending on n and β .

Note that both in the critical and supercritical case, the leading-order term in the Weyl asymptotics involves $\operatorname{vol}(M, g_1(0))$. Put differently, only the behaviour of the metric g near the boundary matters for the leading-order behaviour. This suggests an interesting behaviour near the boundary.

Indeed, in [7, Theorem 2], it is shown that in the critical and supercritical case, the average density of eigenfunctions accumulates at the boundary, see also [6, Theorem 6.1]. Denote by $\{\Phi_j\}_{j \in \mathbb{N}}$ an orthonormal basis in $L^2(X, \operatorname{vol}_g)$ of eigenfunctions of Δ_g with eigenvalues $\{\lambda_j\}_{j \in \mathbb{N}}$. More precisely, [7, Theorem 2] states that the average density of eigenfunctions with eigenvalues below λ

$$(8) \quad \frac{1}{N(\lambda)} \sum_{\lambda_j < \lambda} |\Phi_j|^2 \, d \operatorname{vol}_g$$

converges weakly to the uniform distribution on the boundary $\partial X \cong \{0\} \times M$ with respect to $g_1(0)$ as $\lambda \rightarrow \infty$.

It is a natural question to ask if more can be said about the average density of eigenfunctions defined in (8). In particular, one might ask at which scale near the boundary this average density typically lives. For the supercritical case $\beta > 2/n$, this was answered in [12, Theorem 1], namely the average density of eigenfunctions with eigenvalues less than λ is located at a length-scale $\lambda^{-1/2}$ near the boundary ∂X . Furthermore, when zooming in at that scale and letting $\lambda \rightarrow \infty$, it converges to an explicitly known profile B . More precisely, identifying $X_1 \cong [0, 1] \times M$, according to [12, Theorem 1], we have for $\beta > 2/n$ for any continuous and bounded test function $f : [0, \infty) \times M \rightarrow \mathbb{R}$

$$(9) \quad \begin{aligned} & \lim_{\lambda \rightarrow \infty} \frac{1}{N(\lambda)} \sum_{\lambda_j < \lambda} \int_{[0,1] \times M} f(\sqrt{\lambda}x, y) |\Phi_j(x, y)|^2 \, d \operatorname{vol}_g(x, y) \\ &= \int_{[0,\infty) \times M} f(z, y) B(z, y) \, dz \frac{d \operatorname{vol}_{g_1(0)}(y)}{\operatorname{vol}_{g_1(0)}(M)}. \end{aligned}$$

Main result. An open question that remained was at which scale the average density of eigenfunctions accumulates at the boundary of X in the critical case $\beta = 2/n$. The answer to this question is given in Theorem 1 below. It states that there is no such scale at which the eigenfunctions concentrate near the boundary. The average density of eigenfunctions is distributed over all length scales between $\lambda^{-1/2}$ and 1: For all $\gamma \in [-1/2, 0]$, the integral of the average density of eigenfunctions with eigenvalues below λ integrated over X_{λ^γ} converges to $2(1/2 + \gamma)$ as $\lambda \rightarrow \infty$.

Theorem 1. *Let $\gamma \in [-1/2, 0]$. Then*

$$(10) \quad \lim_{\lambda \rightarrow \infty} \frac{1}{N(\lambda)} \sum_{\lambda_j < \lambda} \int_{X_{\lambda^\gamma}} |\Phi_j|^2 \, d \operatorname{vol}_g = 2 \left(\frac{1}{2} + \gamma \right).$$

Theorem 1 states that for any $\gamma \in [-1/2, 0]$, the proportion of eigenfunctions with eigenvalue less than λ located in the neighbourhood of the boundary of X of size λ^γ is approximately $2(1/2 + \gamma)$ for λ large.

In particular, for $\gamma = -1/2$, this proportion is equal to zero. This is in sharp contrast to the corresponding result in the supercritical case [12, Theorem 1], where “most of the eigenfunctions live on a scale $\lambda^{-1/2}$ near the boundary”.

More precisely, due to the linear behaviour in $\gamma \in [-1/2, 0]$ of the right-hand side in (10), we can also say that the eigenfunctions are localised at scale λ^γ , uniformly distributed in $\gamma \in [-1/2, 0]$ for $\lambda \rightarrow \infty$.

In order to facilitate the comparison with [12, Theorem 1], see also (9) above, we present another version of Theorem 1 involving a test function f , where we also take the distribution along the boundary variable into account:

Theorem 2. *Let $f : (-\infty, 0] \times M \rightarrow \mathbb{R}$ be continuous and bounded. Then*

$$(11) \quad \begin{aligned} & \lim_{\lambda \rightarrow \infty} \frac{1}{N(\lambda)} \sum_{\lambda_j < \lambda} \int_{[0,1] \times M} f\left(\frac{\log(x)}{\log(\lambda)}, y\right) |\Phi_j(x, y)|^2 \, d \operatorname{vol}_g(x, y) \\ &= 2 \int_{[-1/2, 0] \times M} f(\tilde{\gamma}, y) \, d\tilde{\gamma} \frac{d \operatorname{vol}_{g_1(0)}(y)}{\operatorname{vol}_{g_1(0)}(M)}. \end{aligned}$$

Structure of the paper. In Section 2, we prove the key ingredient of the proof of Theorem 1, which are some trace asymptotics, detailed in Proposition 3 below. Then, in Section 3, we prove Theorem 1 using Proposition 3. We also explain the proof of Theorem 2.

Notation. For any self-adjoint operator H , we denote by $\operatorname{Tr}(H)_-$ the sum of negative eigenvalues of H if this quantity is finite. Otherwise, we set $\operatorname{Tr}(H)_- = -\infty$. In particular, we use the sign convention that we always have $\operatorname{Tr}(H)_- \leq 0$.

Furthermore, we denote by $N(H, \lambda)$ the number of eigenvalues of H that are below $\lambda \in \mathbb{R}$ counted with multiplicity.

Acknowledgments. I would like to thank Yves Colin de Verdière and Emmanuel Trélat for introducing me to the topic of sub-Riemannian Laplacians, for many very helpful discussions, and for remarks on the literature. I would also like to thank Larry Read for helpful discussions and remarks. I would like to thank Phan Thành Nam and Laure Saint-Raymond for their support and for making possible several visits to Institut des Hautes Etudes Scientifiques, where I started working on this topic. I acknowledge the support from the European Research Council via the ERC CoG RAMBAS (Project-Nr. 10104424).

2. TRACE ASYMPTOTICS

In this section, we prove the key ingredient of the proof of Theorem 1, which is the following:

Proposition 3. *Let $\gamma \in [-1/2, 0]$. Then, uniformly in $\delta \in [-1/2, 1/2]$,*

$$(12) \quad \begin{aligned} & \frac{1}{\lambda N(\lambda)} (|\operatorname{Tr}(\Delta_g - \lambda)_-| - |\operatorname{Tr}(\Delta_g + \delta \lambda 1_{X^{\lambda^\gamma}} - \lambda)_-|) \\ &= \frac{2}{n+3} \left(1 - (1 - \delta)^{\frac{n+3}{2}}\right) 2(1/2 + \gamma) + o(1) \quad \text{as } \lambda \rightarrow \infty. \end{aligned}$$

Here $1_{X_{\lambda^\gamma}}$ denotes the multiplication operator with the indicator function of X_{λ^γ} .

Proof. As we have seen in [7, 11], the problem can be reduced to a simpler one involving the one-dimensional operator

$$(13) \quad P_\mu := -\partial_x^2 + \frac{3}{4x^2} + \mu x^\beta \quad \text{on } [0, \varepsilon_0],$$

where $\varepsilon_0 > 0$ is small, but fixed. Compare also with (3). We put Dirichlet boundary conditions at $x = 0$ and Dirichlet or Neumann boundary conditions at $x = \varepsilon_0$. As in [11, Section 3.13], the leading order asymptotics of

$$(14) \quad \left| \text{Tr} (\Delta_g - \lambda)_- \right| - \left| \text{Tr} (\Delta_g + \delta \lambda 1_{X_{\lambda^\gamma}} - \lambda)_- \right|,$$

up to an explicit constant, is given by the leading order asymptotics of

$$(15) \quad \sum_{j=\lambda^{n/2}}^{E\lambda^{(n+1)/2}} \left(\left| \text{Tr}_{[0, \varepsilon_0]}^{D, D/N} (P_{j^{2/n}} - \lambda)_- \right| - \left| \text{Tr}_{[0, \varepsilon_0]}^{D, D/N} (P_{j^{2/n}} + \delta \lambda 1_{[0, \lambda^\gamma]} - \lambda)_- \right| \right)$$

for $E > 0$ fixed, but chosen arbitrarily small. Later, after letting $\lambda \rightarrow \infty$, we will let $E \rightarrow 0$. Here the superscript $D, D/N$ denotes the Dirichlet boundary conditions at $x = 0$ and the Dirichlet or Neumann boundary conditions at $x = \varepsilon_0$.

Similarly, the leading order asymptotics of $N(\lambda)$ is given, up to the same explicit constant, by

$$(16) \quad \sum_{j=\lambda^{n/2}}^{E\lambda^{(n+1)/2}} N(P_{j^{2/n}}, \lambda).$$

By [11, Section 3.13], we have

$$(17) \quad \lim_{E \rightarrow 0} \lim_{\lambda \rightarrow \infty} \left(\lambda^{\frac{n+1}{2}} \log(\lambda) \right)^{-1} \sum_{j=\lambda^{n/2}}^{E\lambda^{(n+1)/2}} N(P_{j^{2/n}}, \lambda) = \frac{1}{2} A,$$

where $A > 0$ was defined in [11, (3.32)] by

$$(18) \quad A := \frac{1}{\pi} \int_0^1 \sqrt{1 - z^\beta} dz.$$

Let us now turn to the computation of the leading-order asymptotics of (15). First note that by Dirichlet-Neumann bracketing, we have for any $\delta \in [-1/2, 1/2]$ (and in particular also for $\delta = 0$)

$$(19) \quad \left| \text{Tr}_{[0, \varepsilon_0]}^{D, D/N} (P_{j^{2/n}} + \delta \lambda 1_{[0, \lambda^\gamma]} - \lambda)_- \right| \geq \left| \text{Tr}_{[0, \lambda^\gamma]}^{D, D} (P_{j^{2/n}} - (1 - \delta)\lambda)_- \right| + \left| \text{Tr}_{[\lambda^\gamma, \varepsilon_0]}^{D, D/N} (P_{j^{2/n}} - \lambda)_- \right|$$

and

$$(20) \quad \left| \text{Tr}_{[0, \varepsilon_0]}^{D, D/N} (P_{j^{2/n}} + \delta \lambda 1_{[0, \lambda^\gamma]} - \lambda)_- \right| \leq \left| \text{Tr}_{[0, \lambda^\gamma]}^{D, N} (P_{j^{2/n}} - (1 - \delta)\lambda)_- \right| + \left| \text{Tr}_{[\lambda^\gamma, \varepsilon_0]}^{N, D/N} (P_{j^{2/n}} - \lambda)_- \right|$$

As in [11, Section 3.13], in order to compute the leading order asymptotics of the terms on the right-hand side of (19) and (20), one can decompose the interval $[0, \lambda^\gamma]$ or $[\lambda^\gamma, \varepsilon_0]$ into smaller subintervals using Dirichlet-Neumann bracketing. In particular, the Dirichlet or

Neumann boundary conditions at $x = \lambda^\gamma$ do not make a difference for the leading-order asymptotics.

Using (19) and (20) for δ and $\delta = 0$, in order to compute the leading order asymptotics of (15), it suffices to compute the leading-order asymptotics of

$$(21) \quad \sum_{j=\lambda^{n/2}}^{E\lambda^{(n+1)/2}} \left(\left| \operatorname{Tr}_{[0,\lambda^\gamma]}^{D,D/N} (P_{j^{2/n}} - \lambda)_- \right| - \left| \operatorname{Tr}_{[0,\lambda^\gamma]}^{D,D/N} (P_{j^{2/n}} - (1-\delta)\lambda)_- \right| \right).$$

Recall from [7, Proposition 5] that for any $\mu, a > 0$, the operator P_μ on $[0, a]$ is unitarily equivalent to

$$(22) \quad \mu^{2/(2+\beta)} P_1 \quad \text{on} \quad [0, \mu^{1/(2+\beta)} a].$$

Using $\beta = 2/n$ and taking $\mu = j^{2/n}$, we obtain

$$(23) \quad \left| \operatorname{Tr}_{[0,\lambda^\gamma]}^{D,D/N} (P_{j^{2/n}} - (1-\delta)\lambda)_- \right| = j^{\frac{2}{n+1}} \left| \operatorname{Tr}_{[0,j^{1/(n+1)}\lambda^\gamma]}^{D,D/N} (P_1 - (1-\delta)\lambda j^{-\frac{2}{n+1}})_- \right|.$$

Note that if $j \geq 2^{1/\beta} \lambda^{1/\beta-\gamma}$, then for all $\delta \in [-1/2, 1/2]$,

$$(24) \quad \left(j^{\frac{1}{n+1}} \lambda^\gamma \right)^\beta \geq (1-\delta) \lambda j^{-\frac{2}{n+1}}.$$

We use the notation

$$(25) \quad \omega_\delta := (1-\delta) \lambda j^{-\frac{2}{n+1}}.$$

For all $j \geq 2^{1/\beta} \lambda^{1/\beta-\gamma}$, we know by (24) that $[0, j^{1/(n+1)}\lambda^\gamma] \supset [0, \omega_\delta^{1/\beta}]$ and therefore,

$$(26) \quad \left| \operatorname{Tr}_{[0,j^{1/(n+1)}\lambda^\gamma]}^{D,D/N} (P_1 - \omega_\delta)_- \right| \geq \left| \operatorname{Tr}_{[0,\omega_\delta^{1/\beta}]}^{D,D} (P_1 - \omega_\delta)_- \right|.$$

Since $P_1 \geq \omega_\delta$ on $[\omega_\delta^{1/\beta}, \infty)$, we also have

$$(27) \quad \left| \operatorname{Tr}_{[0,j^{1/(n+1)}\lambda^\gamma]}^{D,D/N} (P_1 - \omega_\delta)_- \right| \leq \left| \operatorname{Tr}_{[0,\omega_\delta^{1/\beta}]}^{D,N} (P_1 - \omega_\delta)_- \right| + \left| \operatorname{Tr}_{[\omega_\delta^{1/\beta}, j^{1/(n+1)}\lambda^\gamma]}^{N,D/N} (P_1 - \omega_\delta)_- \right| \\ = \left| \operatorname{Tr}_{[0,\omega_\delta^{1/\beta}]}^{D,N} (P_1 - \omega_\delta)_- \right|.$$

Thus, in view of (23), we would like to compute the leading-order asymptotics of

$$(28) \quad \sum_{j=\lambda^{n/2}}^{E\lambda^{(n+1)/2}} j^{\frac{2}{n+1}} \left(\left| \operatorname{Tr}_{[0,\omega_0^{1/\beta}]}^{D,D/N} (P_1 - \omega_0)_- \right| - \left| \operatorname{Tr}_{[0,\omega_\delta^{1/\beta}]}^{D,D/N} (P_1 - \omega_\delta)_- \right| \right).$$

Uniformly in $\delta \in [-1/2, 1/2]$, recall from [11, (3.31)] that for every $\eta > 0$, there exists $E = E(\eta) > 0$ small enough such that for every $j \leq E\lambda^{\frac{n+1}{2}}$, we have

$$(29) \quad N_{[0,\omega_\delta^{1/\beta}]}^{D,D/N} (P_1, \omega_\delta) \in \left[(1+\eta)^{-1} A \omega_\delta^{\frac{1}{2}+\frac{1}{\beta}}, (1+\eta) A \omega_\delta^{\frac{1}{2}+\frac{1}{\beta}} \right],$$

where $A > 0$ was defined in (18). Similarly, for the traces, we have

$$(30) \quad \left| \operatorname{Tr}_{[0,\omega_\delta^{1/\beta}]}^{D,D/N} (P_1 - \omega_\delta)_- \right| \in \left[(1+\eta)^{-1} \frac{2}{n+3} A \omega_\delta^{\frac{3}{2}+\frac{1}{\beta}}, (1+\eta) \frac{2}{n+3} A \omega_\delta^{\frac{3}{2}+\frac{1}{\beta}} \right].$$

Here we used that

$$(31) \quad \frac{2}{3\pi} \int_0^1 (1 - z^\beta)^{\frac{3}{2}} dz = \frac{2}{n+3} A.$$

Note that

$$(32) \quad \begin{aligned} & \sum_{j=2^{1/\beta} \lambda^{1/\beta-\gamma}}^{E\lambda^{(n+1)/2}} j^{\frac{2}{n+1}} \left(\omega_\delta^{\frac{n+3}{2}} - \omega_0^{\frac{n+3}{2}} \right) = \left(1 - (1-\delta)^{\frac{n+3}{2}} \right) \lambda^{\frac{n+3}{2}} \sum_{j=2^{1/\beta} \lambda^{1/\beta-\gamma}}^{E\lambda^{(n+1)/2}} j^{-1} \\ & = \left(1 - (1-\delta)^{\frac{n+3}{2}} \right) \lambda^{\frac{n+3}{2}} \left(\log \left(E\lambda^{\frac{n+1}{2}} \right) - \log \left(2^{-1/\beta} \lambda^{1/\beta-\gamma} \right) \right) + o \left(\lambda^{\frac{n+3}{2}} \log(\lambda) \right) \\ & = \left((1-\delta)^{\frac{n+3}{2}} - 1 \right) \left(\frac{1}{2} + \gamma \right) \lambda^{\frac{n+3}{2}} \log(\lambda) + o \left(\lambda^{\frac{n+3}{2}} \log(\lambda) \right) \end{aligned}$$

as $\lambda \rightarrow \infty$. Here we used that $\beta = 2/n$ in the last step.

Furthermore, the sum of the terms corresponding to $\lambda^{n/2} \leq j \leq 2^{1/\beta} \lambda^{1/\beta-\gamma}$ is of sub-leading order. Indeed, for all $j \in \mathbb{N}$ and all $\delta \in [-1/2, 1/2]$,

$$(33) \quad \left| \text{Tr}_{[0, \lambda^\gamma]}^{D, D/N} (P_{j^{2/n}} - (1-\delta)\lambda)_- \right| \leq \left| \text{Tr}_{[0, \lambda^\gamma]}^{D, D/N} (-\partial_x^2 - (1-\delta)\lambda)_- \right| \leq 1 + \frac{2}{3\pi} \lambda^\gamma ((1-\delta)\lambda)^{\frac{3}{2}}.$$

By (33), we obtain for all $\lambda \geq 1$

$$(34) \quad \sum_{j=\lambda^{n/2}}^{2^{1/\beta} \lambda^{1/\beta-\gamma}} \left| \text{Tr}_{[0, \lambda^\gamma]}^{D, D/N} (P_{j^{2/n}} - (1-\delta)\lambda)_- \right| \leq C \lambda^{\frac{1}{\beta}-\gamma} \lambda^{\gamma+\frac{3}{2}} = C \lambda^{\frac{n+3}{2}}$$

where the constant $C > 0$ is uniform in $\lambda \geq 1$ and $\delta \in [-1/2, 1/2]$.

Combining (30), (32) and (34), we obtain

$$(35) \quad \begin{aligned} & \lim_{E \rightarrow 0} \lim_{\lambda \rightarrow \infty} \left(\lambda^{\frac{n+3}{2}} \log(\lambda) \right)^{-1} \sum_{j=\lambda^{n/2}}^{E\lambda^{(n+1)/2}} j^{\frac{2}{n+1}} \left(\left| \text{Tr}_{[0, \omega_0^{1/\beta}]}^{D, D/N} (P_1 - \omega_0)_- \right| - \left| \text{Tr}_{[0, \omega_\delta^{1/\beta}]}^{D, D/N} (P_1 - \omega_\delta)_- \right| \right) \\ & = \left((1-\delta)^{\frac{n+3}{2}} - 1 \right) \left(\frac{1}{2} + \gamma \right) \frac{2}{n+3} A. \end{aligned}$$

Finally, from (17), (28) and (35), we obtain (12). \square

3. PROOFS OF THEOREM 1 AND THEOREM 2

In this section, we prove Theorem 1 using Proposition 3. Then we explain how to prove Theorem 2 using Theorem 1.

Proof of Theorem 1. We define the projection operator

$$(36) \quad \Gamma_\lambda := \sum_{\lambda_j < \lambda} |\Phi_j\rangle \langle \Phi_j|.$$

Then by the linearity of the trace, we have for all $\delta \in [-1/2, 1/2] \setminus \{0\}$

$$\begin{aligned}
(37) \quad & \sum_{\lambda_j < \lambda} \int_{X_{\lambda^\gamma}} |\Phi_j|^2 \, d \operatorname{vol}_g = \operatorname{Tr} (\Gamma_\lambda 1_{X_{\lambda^\gamma}}) \\
& = \frac{1}{\delta \lambda} (\operatorname{Tr} (\Gamma_\lambda (\Delta_g + \delta \lambda 1_{X_{\lambda^\gamma}} - \lambda)) - \operatorname{Tr} (\Gamma_\lambda (\Delta_g - \lambda))) \\
& = \frac{1}{\delta \lambda} (\operatorname{Tr} (\Gamma_\lambda (\Delta_g + \delta \lambda 1_{X_{\lambda^\gamma}} - \lambda)) - \operatorname{Tr} (\Delta_g - \lambda)_-),
\end{aligned}$$

where we used in the last step that Γ_λ is the projection on the negative spectral subspace of $\Delta_g - \lambda$. By the variational principle, we have

$$(38) \quad \operatorname{Tr} (\Gamma_\lambda (\Delta_g + \delta \lambda 1_{X_{\lambda^\gamma}} - \lambda)) \geq \operatorname{Tr} (\Delta_g + \delta \lambda 1_{X_{\lambda^\gamma}} - \lambda)_-.$$

Thus, for $0 < \delta < 1/2$, we get

$$\begin{aligned}
(39) \quad & \sum_{\lambda_j < \lambda} \int_{X_{\lambda^\gamma}} |\Phi_j|^2 \, d \operatorname{vol}_g \geq \frac{1}{\delta \lambda} (\operatorname{Tr} (\Delta_g + \delta \lambda 1_{X_{\lambda^\gamma}} - \lambda)_- - \operatorname{Tr} (\Delta_g - \lambda)_-) \\
& = \frac{1}{\delta \lambda} (|\operatorname{Tr} (\Delta_g - \lambda)_-| - |\operatorname{Tr} (\Delta_g + \delta \lambda 1_{X_{\lambda^\gamma}} - \lambda)_-|).
\end{aligned}$$

and similarly, for $-1/2 < \delta < 0$, we have

$$(40) \quad \lim_{\lambda \rightarrow \infty} \frac{1}{N(\lambda)} \sum_{\lambda_j < \lambda} \int_{X_{\lambda^\gamma}} |\Phi_j|^2 \, d \operatorname{vol}_g \leq \frac{1}{\delta \lambda} (|\operatorname{Tr} (\Delta_g - \lambda)_-| - |\operatorname{Tr} (\Delta_g + \delta \lambda 1_{X_{\lambda^\gamma}} - \lambda)_-|).$$

Now, using (39) and Proposition 3, we get for all $0 < \delta < 1/2$

$$\begin{aligned}
(41) \quad & \liminf_{\lambda \rightarrow \infty} \frac{1}{N(\lambda)} \sum_{\lambda_j < \lambda} \int_{X_{\lambda^\gamma}} |\Phi_j|^2 \, d \operatorname{vol}_g \\
& \geq \liminf_{\lambda \rightarrow \infty} \frac{1}{N(\lambda)} \frac{1}{\delta \lambda} (|\operatorname{Tr} (\Delta_g - \lambda)_-| - |\operatorname{Tr} (\Delta_g + \delta \lambda 1_{X_{\lambda^\gamma}} - \lambda)_-|) \\
& = \frac{1}{\delta} \frac{2}{n+3} \left(1 - (1-\delta)^{\frac{n+3}{2}}\right) 2(1/2 + \gamma).
\end{aligned}$$

In particular, since the left-hand side does not depend on δ , we can let $\delta \downarrow 0$ and obtain

$$\begin{aligned}
(42) \quad & \liminf_{\lambda \rightarrow \infty} \frac{1}{N(\lambda)} \sum_{\lambda_j < \lambda} \int_{X_{\lambda^\gamma}} |\Phi_j|^2 \, d \operatorname{vol}_g \\
& \geq \lim_{\delta \downarrow 0} \frac{1}{\delta} \frac{2}{n+3} \left(1 - (1-\delta)^{\frac{n+3}{2}}\right) 2(1/2 + \gamma) = 2(1/2 + \gamma).
\end{aligned}$$

Similarly, by (40), Proposition 3 and letting $\delta \uparrow 0$, we also have

$$(43) \quad \limsup_{\lambda \rightarrow \infty} \frac{1}{N(\lambda)} \sum_{\lambda_j < \lambda} \int_{X_{\lambda^\gamma}} |\Phi_j|^2 \, d \operatorname{vol}_g \leq 2(1/2 + \gamma).$$

Combining (42) and (43), we obtain (10). \square

At this point, we would like to remark that the idea of considering differences of traces and letting a small parameter $\delta \downarrow 0$ and $\delta \uparrow 0$, can also be found in [19, 12].

Next, we prove Theorem 2.

Proof of Theorem 2. Since f continuous and bounded, we may approximate it in L^∞ on $[-1/2, 0] \times M$ by a finite sum of indicator functions on sets that have a product structure and are of the form $(-\infty, \gamma] \times \tilde{M}$, where $\gamma \leq 0$ and \tilde{M} is an open subset of M with a piecewise smooth boundary. Due to linearity of both sides of (11), it suffices to show (11) for test functions $f(x, y) = 1_{(-\infty, \gamma] \times \tilde{M}}$. In the case $\tilde{M} = M$, this is precisely the statement of Theorem 1. For the general case, one decomposes M into a union of \tilde{M} with piecewise smooth boundaries and applies Dirichlet-Neumann bracketing for the variable $y \in M$. The proof for each \tilde{M} follows precisely as the proof of Theorem 1 with the only difference being the Dirichlet or Neumann boundary conditions at the boundary of \tilde{M} . \square

REFERENCES

- [1] Laura Abatangelo, Alberto Ferrero, and Paolo Luzzini. On solutions to a class of degenerate equations with the grushin operator. *Journal of Differential Equations*, 445:113666, 2025.
- [2] Ugo Boscain and Camille Laurent. The laplace-beltrami operator in almost-riemannian geometry. In *Annales de l'institut Fourier*, volume 63, pages 1739–1770, 2013.
- [3] Ugo Boscain, Dario Prandi, and Marcello Seri. Spectral analysis and the aharonov-bohm effect on certain almost-riemannian manifolds. *Communications in Partial Differential Equations*, 41(1):32–50, 2016.
- [4] Der-Chen Chang and Yutian Li. Heat kernel asymptotic expansions for the heisenberg sub-laplacian and the grushin operator. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 471(2175):20140943, 2015.
- [5] Jeff Cheeger. Spectral geometry of singular riemannian spaces. *Journal of differential geometry*, 18(4):575–657, 1983.
- [6] Yacine Chitour, Dario Prandi, and Luca Rizzi. Weyl’s law for singular riemannian manifolds. *Journal de Mathématiques Pures et Appliquées*, 181:113–151, 2024.
- [7] Yves Colin de Verdière, Charlotte Dietze, Maarten V de Hoop, and Emmanuel Trélat. Weyl formulae for some singular metrics with application to acoustic modes in gas giants. *arXiv preprint arXiv:2406.19734*, 2024.
- [8] Yves Colin de Verdière, Luc Hillairet, and Emmanuel Trélat. Spectral asymptotics for sub-riemannian laplacians, i: Quantum ergodicity and quantum limits in the 3-dimensional contact case. *Duke Math. J.*, 167:109–174, 2018.
- [9] Yves Colin de Verdière, Luc Hillairet, and Emmanuel Trélat. Small-time asymptotics of hypoelliptic heat kernels near the diagonal, nilpotentization and related results. *Annales Henri Lebesgue*, 4:897–971, 2021.
- [10] Yves Colin de Verdière, Luc Hillairet, and Emmanuel Trélat. Spectral asymptotics for sub-riemannian laplacians. *arXiv preprint arXiv:2212.02920*, 2022.
- [11] Charlotte Dietze. *Spectral estimates for singular systems*. PhD thesis, LMU Munich, 2025.
- [12] Charlotte Dietze and Larry Read. Concentration of eigenfunctions on singular riemannian manifolds. *arXiv preprint arXiv:2410.20563*, 2024.
- [13] Rayan Fahs, Loïc Le Treust, Nicolas Raymond, and San Vũ Ngoc. Boundary states of the robin magnetic laplacian. *Documenta Mathematica*, 29(5):1157–1200, 2024.
- [14] Daniel Grieser. Basics of the b-calculus. In *Approaches to Singular Analysis: A Volume of Advances in Partial Differential Equations*, pages 30–84. Springer, 2001.
- [15] Pier Domenico Lamberti, Paolo Luzzini, and Paolo Musolino. Shape perturbation of grushin eigenvalues. *The Journal of Geometric Analysis*, 31(11):10679–10717, 2021.
- [16] Richard B Melrose. Transformation of boundary problems. *Acta mathematica*, 147:149–236, 1981.
- [17] Arthur Menikoff and Johannes Sjöstrand. On the eigenvalues of a class of hypoelliptic operators. *Mathematische Annalen*, 235(1):55–85, 1978.
- [18] Guy Métivier. Comportement asymptotique des valeurs propres d’opérateurs elliptiques dégénérés. *Journées équations aux dérivées partielles*, pages 215–249, 1975.
- [19] Larry Read. On the asymptotic number of low-lying states in the two-dimensional confined stark effect. *arXiv preprint arXiv:2404.14363*, 2024.

- [20] Alejandro Rivera. Weighted local weyl laws for elliptic operators. In *Annales de la Faculté des sciences de Toulouse: Mathématiques*, volume 31, pages 423–490, 2022.
- [21] IA Solomešč. Eigenvalues of some degenerate elliptic equations. *Matematicheskii Sbornik*, 96(3):295–310, 1961.
- [22] IL Vulis and MZ Solomjak. Spectral asymptotic analysis for degenerate second order elliptic operators. *Izv. Akad. Nauk SSSR Ser. Mat.*, 38:1362–1392, 1974.

(Charlotte Dietze) SORBONNE UNIVERSITÉ, CNRS, LABORATOIRE JACQUES-LOUIS LIONS, 4 PLACE
JUSSIEU, 75005 PARIS, FRANCE
Email address: `Charlotte.Dietze@sorbonne-universite.fr`