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Existence and uniqueness of the solution to a class of Fredholm Integral Equations related to Difference Equations

Delphine Bresch-Pietri and Jean Auriol

Abstract—This paper addresses the existence and uniqueness of the solution to a class of Fredholm integral equations associated with scalar Linear Integral Delay Equations (LIDEs). Based on an operator-theoretic framework involving transport partial differential equations, we provide general conditions that guarantee well-posedness of these equations. Leveraging this result, we introduce a constructive approach for the numerical computation of the Lyapunov matrix corresponding to scalar LIDEs with commensurate delays. Specifically, the Lyapunov matrix equations are reformulated as the Fredholm integral equation under consideration, for which the proposed conditions are satisfied under exponential stability of the LIDE. The resulting integral equation can then be discretized, and the linear system efficiently solved to compute the Delay Lyapunov matrix. Numerical examples illustrate the effectiveness and applicability of this methodology.

Index Terms—Fredholm integral equations, Integral Difference Equations, Lyapunov matrix, Time-Delay Systems

I. INTRODUCTION

LINEAR Integral Delay Equations (LIDEs) form a class of linear difference equations involving both pointwise and distributed delays. They naturally arise in the modeling of engineering and biological systems subject to transport, communication, or measurement delays [22]. Relevant examples include sampled-data systems [13], population dynamics, and biomedical applications such as epidemic models [9]. Moreover, linear first-order hyperbolic Partial Differential Equations (PDEs), which are extensively used to describe systems governed by balance laws [7], can also be recast as LIDEs [5], [15].

Difference equations have received relatively little attention in the literature, where they are often regarded as a particular subclass of neutral time-delay systems [14]. To the best of our knowledge, only a few works have specifically addressed difference systems, such as [8], [20], which investigate controllability properties and establish spectral criteria, or [11], [12], [21], which focus on deriving necessary conditions for stability. It is only quite recently that necessary Lyapunov conditions for the exponential stability of Linear Difference Equations (LDEs) with pointwise delays were established

in [27], and that Complete-Type Lyapunov functionals for LIDEs were proposed, in line with the standard framework for time-delay systems [1], [6], [16], [23]. This line of work was subsequently extended in [4], [19], where an Input-to-State Lyapunov functional was introduced. However, such Lyapunov constructions typically rely on the so-called Delay Lyapunov Matrix. Hence, any practical application of these results relies on an effective numerical procedure for constructing the Lyapunov matrix, which may reveal challenging to obtain.

In the case of LDEs with pointwise delays (but without a distributed one), explicit constructive methods were proposed in [26] for commensurate delays. Specifically, inspired by the standard approach for linear time-delay systems [16], the Delay Lyapunov Matrix was shown to satisfy matrix difference equations, which were transformed into algebraic equations via suitable changes of variables, with the existence of a solution guaranteed by the stability of the original system. The Lyapunov matrix could then be directly computed from this solution.

The motivation of this paper is to introduce a constructive approach for the numerical computation of the Lyapunov matrix associated with scalar LIDEs with commensurate delays, both pointwise and distributed. Our approach builds upon the constructive methodology proposed in [26], [24]. By employing change of variables analogous to those in [26], we show that the Delay Lyapunov Matrix associated to a scalar LIDE can be equivalently reformulated as a Fredholm integral equation. Having a Fredholm equation instead of algebraic equations is a consequence of the presence of the distributed delay. Unfortunately, Fredholm integral equations do not always admit solutions [18], [28], and characterizing the existence of a solution represents a nontrivial challenge.

This paper thus considers a specific class of Fredholm integral equation, related to scalar LIDEs. We adopt an operator-based framework inspired by [10] to establish the existence and uniqueness of a solution for the Fredholm equation under consideration. Interestingly, the introduced operators are closely related to the ones proposed in [2], [3] for designing stabilizing controllers for LIDEs subject to both pointwise and distributed input delays. Under general conditions, we prove that the Fredholm operator under consideration is invertible. This is the main result of the paper. We then leverage this result to introduce a constructive approach for the numerical computation of the Lyapunov matrix associated with exponentially stable scalar LIDEs with commensurate delays.

Delphine Bresch-Pietri (corresponding author, delphine.bresch-pietri@minesparis.psl.eu) is with Centre Automatique et Systèmes (CAS), Mines Paris, Université PSL, 75006 Paris, France. Jean Auriol is with Université Paris-Saclay, CNRS, CentraleSupélec, Laboratoire des Signaux et Systèmes, 91190, Gif-sur-Yvette, France.

The paper is organized as follows. Section II introduces the class of Fredholm equations under consideration and states the main result, which establishes the invertibility of the associated Fredholm operator under general assumptions. Section III provides the proof of this result, relying on an operator-theoretic framework. Section IV illustrates how this result can be applied to the computation of the Lyapunov matrix for scalar LDEs with commensurate delays, and proposes a constructive numerical method, complemented with simulation results. Finally, concluding remarks are given in Section V.

Notations: For any $(a, b) \in \mathbb{R}^2$ with $a < b$ and $p \in \mathbb{R}^p$, we denote $\mathcal{L}_2(a, b; \mathbb{R}^p)$ the set of square-integrable functions from (a, b) to \mathbb{R}^p . We also denote $H^1(a, b; \mathbb{R}^p)$ the Sobolev space of functions in $L^2(a, b; \mathbb{R}^p)$ whose weak derivative of order one belongs to $L^2(a, b; \mathbb{R}^p)$. Similarly, we denote $C_{pw}^1([a, b], \mathbb{R})$ the set of functions f from $[a, b]$ into \mathbb{R} which are piecewise continuously differentiable, that is, such that $[a, b]$ can be partitioned by a finite number of points $(t_i)_{0 \leq i \leq N}$ so that f is continuously differentiable on each subinterval (t_{i-1}, t_i) and f, f' admit finite right-hand and left-hand limits at each t_i .

II. PROBLEM STATEMENT AND MAIN RESULT

A. Problem statement and motivation

In this paper, we consider the following Fredholm integral equation in $\mathcal{L}_2(0, h; \mathbb{R}^{2m})$,

$$w(x) = Aw(x) + \int_0^h K(x - \xi)w(\xi)d\xi + g(x) \quad (1)$$

where $h > 0$ is real, $m \in \mathbb{N}$ is an integer and $g \in \mathcal{L}_2(0, h; \mathbb{R}^{2m})$ a given square integrable function. The matrix $A \in \mathbb{R}^{2m \times 2m}$ is under the form

$$A = \begin{pmatrix} V & L \\ L^T & V^T \end{pmatrix} \quad (2)$$

with

$$V = \begin{pmatrix} 0 & a_1 & a_2 & \cdots & a_{m-1} \\ 0 & 0 & a_1 & \cdots & a_{m-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & a_1 \\ 0 & \cdots & \cdots & 0 & 0 \end{pmatrix} \quad (3)$$

$$L = \begin{pmatrix} a_m & 0 & \cdots & \cdots & 0 \\ a_{m-1} & a_m & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_2 & \cdots & a_{m-1} & a_m & 0 \\ a_1 & \cdots & \cdots & a_{m-1} & a_m \end{pmatrix} \quad (4)$$

in which $a_i \in \mathbb{R}$, $i = 1, \dots, m$. Besides, the kernel $K \in \mathcal{L}_2(-h, h; \mathbb{R}^{2m \times 2m})$ is under the form

$$K = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \text{ with } K_j \in \mathcal{L}_2(-h, h; \mathbb{R}^{m \times (2m)}) \text{ for } j = 1, 2 \quad (5)$$

and K_j ($j = 1, 2$) is such that its i^{th} ($i = 1, \dots, m$) line is $(K_j)_i = (0_{1 \times (i-1)} \quad k_j \quad 0_{1 \times (m-i)})$ with

$$k_1(s) = \begin{pmatrix} F(s)\mathbf{1}_{[-h,0]}(s), F(s-h) \dots \end{pmatrix} \quad (6)$$

$$\begin{pmatrix} F(s - (m-1)h), F(s - mh)\mathbf{1}_{[0,h]}(s) \\ F(-s - mh)\mathbf{1}_{[-h,0]}(s), F(-s - (m-1)h), \dots \\ F(-s - h), F(-s)\mathbf{1}_{[0,h]}(s) \end{pmatrix}. \quad (7)$$

for $F \in C_{pw}^1([0, h], \mathbb{R})$.

As emphasized in the introduction, the motivation for studying this specific class of Fredholm integral equations arises from the stability analysis of the LIDE

$$x(t) = \sum_{k=1}^m a_k x(t - kh) + \int_{-mh}^0 F(\xi) x(t + \xi) d\xi, \quad (8)$$

with $x(t) \in \mathbb{R}$ and m commensurate delays $\tau_k = kh$. More precisely, the problem of computing the Delay Lyapunov Matrix associated with (8) can be reformulated as the problem of solving (1), as detailed in Section IV. Therefore, in order to guarantee that the Delay Lyapunov Matrix can be consistently computed within this framework, it is first essential to establish the existence and uniqueness of a solution to (1).

Note that both A and K exhibit a finite impulse response (FIR)-type structure, consisting of a causal component together with a symmetric non-causal counterpart. This is not coincidental: it is intrinsically linked to the difference pattern induced by the sampling time h of the LIDE (8).

B. Structural assumptions

To guarantee that equation (1) admits a unique solution, we make the following assumptions.

Assumption 1: The matrix $I - A$ is invertible.

Assumption 2: Consider a_i , $i = 1, \dots, m$, and F the parameters and functions involved in the definitions of A and K in (2)–(7). The set $\sigma = \left\{ \lambda \in \mathbb{C} \mid 1 - \sum_{k=1}^m a_k e^{-k\lambda h} - \int_{-mh}^0 F(\xi) e^{\lambda \xi} d\xi = 0 \right\}$ of the characteristic roots of the LIDE (8) is such that $\sigma \cap \{-\sigma\} = \emptyset$.

Assumption 1 is standard in integral equation theory [28], [2] and allows one to formulate a canonical Fredholm equation of the second kind. Assumption 2, often referred to in the context of the Lyapunov Matrix as the *Lyapunov condition* [17], plays a central role. In the delay-free case, this spectral assumption concerns the spectrum of the system matrix and is both necessary and sufficient for the existence and uniqueness of a solution to the classical Lyapunov equation. It has further been shown to be sufficient in the case of retarded and first-order neutral linear time-delay systems [16]. In the present context, it is therefore natural to formulate this condition in terms of the spectrum of the LIDE (8), by direct analogy.

Our main result reads as follows.

Theorem 1: Assume A, K defined in (2)–(7) satisfy Assumptions 1–2. The Fredholm integral operator $\mathcal{T} : \mathcal{L}_2(0, h; \mathbb{R}^{2m}) \rightarrow \mathcal{L}_2(0, h; \mathbb{R}^{2m})$ defined as

$$\mathcal{T}(w) = (I - A)w - \int_0^h K(\cdot - \xi)w(\xi)d\xi \quad (9)$$

is invertible.

The following section will provide the proof of this result, while the next one will be devoted to its role and application for the computation of the Lyapunov Matrix of LIDEs.

III. PROOF OF THEOREM 1

Our approach to establish Theorem 1 consists in proving the invertibility of the adjoint operator \mathcal{T}^* , which in turns implies the invertibility of \mathcal{T} . With this aim in view, we first state a preliminary result that provides sufficient conditions to guarantee the invertibility of Fredholm operators.

A. A preliminary result on the invertibility of Fredholm operators

The following lemma was established in [10](Lemma 2.2) and reformulated in the form presented below, adapted to the present context, in [2](Lemma 9.0.2).

Lemma 1: Consider two linear operators \mathcal{A}, \mathcal{B} such that $D(\mathcal{A}) = D(\mathcal{B}) \subset \mathcal{L}_2(0, h; \mathbb{R}^{2m})$. Consider \mathcal{T}^* , the adjoint of the Fredholm integral operator \mathcal{T} defined in (9). Assume that

- 1) $\ker(\mathcal{T}^*) \subset D(\mathcal{A})$,
- 2) $\ker(\mathcal{T}^*) \subset \ker(\mathcal{B})$,
- 3) $\forall \chi \in \ker(\mathcal{T}^*) \quad \mathcal{T}^* \mathcal{A} \chi = 0$,
- 4) $\ker(s\text{Id} - \mathcal{A}) \cap \ker(\mathcal{B}) = \{0\}$ for all $s \in \mathbb{C}$.

Then, the operator \mathcal{T}^* is invertible.

In the rest of this section, we will prove that the sufficient conditions formulated in this lemma hold, thereby establishing Theorem 1.

B. Definition of the operators \mathcal{A} and \mathcal{B}

The operators \mathcal{A} and \mathcal{B} of interest are defined as

$$\mathcal{A} : D(\mathcal{A}) \rightarrow \mathcal{L}_2(0, h; \mathbb{R}^{2m}) \quad (10)$$

$$(u, v) \mapsto \begin{pmatrix} (u_x^i - F(\cdot - (m - i + 1)h)v^1(0))_{1 \leq i \leq m} \\ (v_x^i + F(\cdot - (i - 1)h)v^1(0))_{1 \leq i \leq m} \end{pmatrix}$$

$$\mathcal{B} : D(\mathcal{A}) \rightarrow \mathbb{R} \quad (11)$$

$$(u, v) \mapsto u^1(0) - a_m v^1(0)$$

with

$$D(\mathcal{A}) = \{\chi \in \mathcal{H}_1(0, h; \mathbb{R}^{2m}) \mid \chi(h) = Q\chi(0)\} \quad (12)$$

in which $Q \in \mathbb{R}^{2m \times 2m}$ is such that

$$Q_{i,j} = \begin{cases} 1, & \text{if } j = i + 1, \\ -a_{m-i}, & \text{if } i = 1, \dots, m-1 \text{ and } j = m+1, \\ a_{i-m}, & \text{if } i = m+1, \dots, 2m \text{ and } j = m+1, \\ 0, & \text{otherwise.} \end{cases} \quad (13)$$

These operators correspond to transport PDEs, which is not surprising giving that delay effects can be naturally represented through transport dynamics. Unfortunately, we cannot provide further intuition nor interpretation on how these operators were obtained. Yet, it is worth underlying that their definition can be related to the operators given in [3] in the context of the design of a stabilizing controllers for LIDE systems with pointwise

and distributed input delays. More precisely, if we consider the control problem

$$\begin{aligned} x(t) &= \sum_{k=1}^m a_k x(t - kh) + \int_{-mh}^0 F(\xi) x(t + \xi) d\xi \\ &+ \sum_{k=1}^m a_k U_0(t - (m - k)h) - U_0(t - mh) \\ &+ \int_{-mh}^0 F(\xi - mh) U_0(t + \xi) d\xi, \end{aligned} \quad (14)$$

with U_0 as the control input, then the operators \mathcal{A} and \mathcal{B} correspond to the natural extension of those given in [3, Proof of Lemma 3]. More precisely, as shown in [25, eqs. (13)–(14)], the LIDE (14) can be reformulated as an $m \times m$ PDE comparison system in which all states propagate at the same velocity, due to the commensurability of the delays. The operator \mathcal{A} can then be interpreted as the adjoint operator of this comparison system, while the operator \mathcal{B} corresponds to the adjoint of the control operator. Moreover, in this context, Assumption 2 can be interpreted as a spectral controllability condition [3].

C. Application of Lemma 1 – Proof of Theorem 1

We now proceed with the proof of Theorem 1 by showing that the operators \mathcal{A} and \mathcal{B} introduced in (10)–(13) fulfill the conditions required in Lemma 1.

Let us observe that, by definition, the adjoint operator \mathcal{T}^* is defined from $\mathcal{L}_2(0, h; \mathbb{R}^{2m})$ to $\mathcal{L}_2(0, h; \mathbb{R}^{2m})$ as $\mathcal{T}^*(w) = (I - A^T)w - \int_0^h K^T(\xi - \cdot)w(\xi)d\xi$ in which, for all $\chi = (\varphi, \psi) \in \mathbb{R}^m \times \mathbb{R}^m$, all $s \in [-h, h]$, and all $i = 1, \dots, m$,

$$\begin{aligned} (K^T(s)(\varphi, \psi))_i &= (F(s)\varphi_i + F(-s - mh)\psi_i) \mathbf{1}_{[0, h]}(s) \\ &+ \sum_{j=1}^{i-1} (F(s - jh)\varphi_{i-j} + F(-s - (m - j)h)\psi_{i-j}) \end{aligned} \quad (15)$$

$$\begin{aligned} (K^T(s)(\varphi, \psi))_{i+m} &= (F(s - mh)\varphi_i + F(-s)\psi_i) \mathbf{1}_{[-h, 0]}(s) \\ &+ \sum_{j=1}^{m-i} (F(s - (m - j)h)\varphi_{i+j} + F(-s - jh)\psi_{i+j}). \end{aligned} \quad (16)$$

a) *Proof of Properties 1) and 2):* Consider $(\varphi, \psi) \in \ker(\mathcal{T}^*)$. Due to Assumption 1 and using the regularizing effect of the integral, we immediately get that $(\varphi, \psi) \in \mathcal{H}_1(0, h, \mathbb{R}^{2m})$. Evaluating $\mathcal{T}^*(\varphi, \psi)$ at $x = h$ and $x = 0$, one obtains

$$\begin{pmatrix} 0_{1 \times 2m} \\ I_{2m} - A^T \end{pmatrix} \begin{pmatrix} \varphi(h) \\ \psi(h) \end{pmatrix} = \begin{pmatrix} I_{2m} - A^T \\ 0_{1 \times 2m} \end{pmatrix} \begin{pmatrix} \varphi(0) \\ \psi(0) \end{pmatrix} \quad (17)$$

Hence, from the first line we obtain $\varphi_1(0) = a_m \psi_1(0)$, that is, $(\varphi, \psi) \in \ker(\mathcal{B})$. Moreover, one has

$$(I - A^T) \begin{pmatrix} \varphi(h) \\ \psi(h) \end{pmatrix} = S(I - A^T) \begin{pmatrix} \varphi(0) \\ \psi(0) \end{pmatrix}, \quad (18)$$

where S denotes the forward shift matrix defined by $S_{ij} = \delta_{j, i+1}$. Furthermore, note that $Q = \begin{pmatrix} 0 & \bar{Q} \end{pmatrix}$ with $\bar{Q} \in \mathbb{R}^{2m \times (2m-1)}$. Due to Assumption 1, in order to ensure that $(\varphi, \psi) \in D(\mathcal{A})$, it only remains to prove that

$$(I - A^T)\bar{Q} = S(I - A^T)M, \quad (19)$$

where

$$M = \begin{pmatrix} 0_{1 \times (m-1)} & a_m & 0_{1 \times (m-1)} \\ & I_{2m-1} & \end{pmatrix}. \quad (20)$$

Let us decompose $I - A^T$ as $(C_1 \ C_2) = (\overline{C}_1 \ \overline{C}_2 \ \overline{C}_3 \ \overline{C}_4)$, with $C_1 \in \mathbb{R}^{2m}$, $C_2 \in \mathbb{R}^{2m \times (2m-1)}$, and $\overline{C}_1, \overline{C}_2, \overline{C}_3, \overline{C}_4$ of respective dimensions $\mathbb{R}^{2m \times (m-1)}$, \mathbb{R}^{2m} , $\mathbb{R}^{2m \times (m-1)}$, \mathbb{R}^{2m} . Straightforward algebraic manipulations then yield

$$(I - A^T)\overline{Q} = \begin{pmatrix} \overline{C}_1 & (I - A^T)C_m^{\overline{Q}} & \overline{C}_3 \end{pmatrix}, \quad (21)$$

where $C_m^{\overline{Q}}$ denotes the m^{th} column of \overline{Q} , with entries

$$((I - A^T)C_m^{\overline{Q}})_i = \begin{cases} -a_{m-i} - a_i a_m, & i = 1, \dots, m-1, \\ 1 - a_m^2, & i = m, \\ 0, & i = m+1, \dots, 2m. \end{cases} \quad (22)$$

On the other hand, it holds

$$(I - A^T)M = C_2 + \begin{pmatrix} 0_{2m \times (m-1)} & a_m C_1 & 0_{2m \times (m-1)} \end{pmatrix}. \quad (23)$$

Comparing the two expressions above, one readily concludes that

$$S(I - A^T)M = (I - A^T)\overline{Q}, \quad (24)$$

which establishes the desired result.

b) Proof of Property 3: Due to space restriction, we consider the case of a continuously differentiable function F , but the proof can easily be adjusted to the case of a finite number of discontinuities. Let $\chi \in \ker(\mathcal{T}^*)$. Taking the derivative of $\mathcal{T}^*\chi$ and performing integrations by parts, it holds, for $i = 1, \dots, m$,

$$\begin{aligned} \varphi'_i(x) &= (A^T(\varphi', \psi')(x))_i + \int_0^x (F(\xi - x)\varphi'_i(\xi) \\ &+ F(x - \xi - mh)\psi'_i(\xi))d\xi + \sum_{j=1}^{i-1} \left[\int_0^h (F(\xi - x - jh) \right. \\ &\times \varphi'_{i-j}(\xi) + F(x - \xi - (m-j)h)\psi'_{i-j}(\xi))d\xi \\ &+ F(-x - (j-1)h)(\varphi_{i-j+1}(0) - \varphi_{i-j}(h)) \\ &+ F(x - (m-j+1)h)(\psi_{i-j+1}(0) - \psi_{i-j}(h)) \left. \right] \\ &+ F(-x - (i-1)h)\varphi_1(0) + F(x - (m-i+1)h)\psi_1(0) \end{aligned} \quad (25)$$

in which $\varphi_{i-j+1}(0) - \varphi_{i-j}(h) = a_{m-i+j}\psi_1(0)$ and $\psi_{i-j+1}(0) - \psi_{i-j}(h) = -a_{i-j}\psi_1(0)$ for $j = 1, \dots, i-1$ as $(\varphi, \psi) \in \mathcal{D}(\mathcal{A})$, and $\varphi_1(0) = a_m\psi_1(0)$ as $(\varphi, \psi) \in \ker(\mathcal{B})$. In addition, observe that, for $i = 1, \dots, m$,

$$(A^T(\varphi, \psi))_i = \sum_{j=1}^{i-1} a_{i-j}\varphi_j + \sum_{j=1}^i a_{m+j-i}\psi_j. \quad (26)$$

Hence, rearranging terms, it follows that

$$\begin{aligned} (\mathcal{A}\chi)_i(x) &= (A^T\mathcal{A}\chi(x))_i + \int_0^x (F(\xi - x)\varphi'_i(\xi) \\ &+ F(x - \xi - mh)\psi'_i(\xi))d\xi + \sum_{j=1}^{i-1} \int_0^h (F(\xi - x - jh) \\ &\times \varphi'_{i-j}(\xi) + F(x - \xi - (m-j)h)\psi'_{i-j}(\xi))d\xi, \end{aligned} \quad (27)$$

Besides, it holds

$$\begin{aligned} &\int_0^x (F(\xi - x)F(\xi - (m-i+1)h) \\ &+ \sum_{j=1}^{i-1} \int_0^h F(\xi - x - jh)F(\xi - (m-i+j+1)h)d\xi \\ &= \int_0^x F(x - \xi - mh)F(-\xi - (i-1)h)d\xi \\ &+ \sum_{j=1}^{i-1} \int_0^h F(x - \xi - (m-j)h)F(-\xi - (i-j-1)h)d\xi. \end{aligned} \quad (28)$$

Indeed, the left-hand side of (28) is equal to $I_1 \triangleq \int_{-(i-1)h}^x F(\xi - x)F(\xi - (m-i+1)h)d\xi$, while the right-hand side can be shown to be equal to $I_2 \triangleq \int_{-(i-1)h}^x F(x - \xi - mh)F(-\xi - (i-1)h)d\xi$, and $I_1 = I_2$ with the change of variable $u = x - \xi - (i-1)h$. Consequently, adding the left-hand side of (28) to (27) and subtracting the right-hand side of (28), one concludes from (15) that $(\mathcal{T}^*\mathcal{A}\chi)_i = 0$ for all $i = 1, \dots, m$. The fact that $(\mathcal{T}^*\mathcal{A}\chi)_{i+m} = 0$ for all $i = 1, \dots, m$ can be established along the same lines, and is therefore omitted. This gives the desired result.

c) Proof of Property 4: Let $s \in \mathbb{C}$ and $(\varphi, \psi) \in \ker(sI - \mathcal{A}) \cap \ker(\mathcal{B})$. Then, it holds $s\varphi_i(x) = \varphi'_i(x) - F(x - (m-i+1)h)\psi_1(0)$ for all $x \in [0, h]$ and $i = 1, \dots, m$. Thus, integrating between 0 and h and using the fact that $(\varphi, \psi) \in \mathcal{D}(\mathcal{A})$, one obtains for $i > 1$

$$\begin{aligned} \varphi_i(x) &= e^{sx}\varphi_i(0) + \int_0^x e^{s(x-\xi)}F(\xi - (m-i+1)h)d\xi\psi_1(0) \\ &= e^{sx}(\varphi_{i-1}(h) + a_{m-i+1}\psi_1(0)) \\ &+ \int_{-(m-i+1)h}^{x-(m-i+1)h} e^{s(x-u-(m-i+1)h)}F(u)du\psi_1(0), \quad x \in [0, h], \end{aligned} \quad (29)$$

in which we used a change of variable. Hence, by a straightforward induction and using the fact that $(\varphi, \psi) \in \ker(\mathcal{B})$, it holds

$$\begin{aligned} \varphi_i(x) &= \psi_1(0) \left(e^{sx} \sum_{k=1}^i e^{s(k-1)h} a_{m-i+k} \right. \\ &+ \left. \int_{-mh}^{x-(m-i+1)h} e^{s(x-u-(m-i+1)h)} F(u)du \right). \end{aligned} \quad (30)$$

Using (30) for $i = m$ and $x = h$, and using the fact that $\varphi_m(h) = \psi_i(0)$ as $(\varphi, \psi) \in \mathcal{D}(\mathcal{A})$, one concludes that

$$\psi_1(0) \left(1 - \sum_{k=1}^m e^{skh} a_k - \int_{-mh}^0 e^{-su} F(u)du \right) = 0. \quad (31)$$

Similarly, it holds $s\psi_i(x) = \psi'_i(x) + F(-x - (i-1)h)\psi_1(0)$ for all $x \in [0, h]$ and $i = 1, \dots, m$, and one obtains that

$$\begin{aligned} \psi_i(x) &= \psi_1(0) \left(e^{sx} \sum_{k=i}^m e^{-s(k-i+1)h} a_k \right. \\ &+ \left. \int_{-mh}^{h-x-ih} e^{s(x+u+(i-1)h)} F(u)du \right), \end{aligned} \quad (32)$$

and, for $i = 1$ and $x = 0$, that

$$\psi_1(0) \left(1 - \sum_{k=1}^m e^{-skh} a_k - \int_{-mh}^0 e^{su} F(u) du \right) = 0. \quad (33)$$

Hence, if $\psi_1(0) \neq 0$, then $s \in \sigma \cap \{-\sigma\}$, which contradicts Assumption 2. Therefore, one must necessarily have $\psi_1(0) = 0$. By (30)–(32), this in turn implies $(\varphi, \psi) = 0$, which completes the proof.

IV. APPLICATION TO THE COMPUTATION OF THE DELAY LYAPUNOV MATRIX OF LINEAR INTEGRAL DIFFERENCE EQUATION

In this section, we ground on Theorem 1 to propose a constructive approach for the numerical computation of the Lyapunov matrix associated with scalar LIDEs with commensurate delays (8).

A. Computing Lyapunov Matrix for an Integral Difference Equation through a Fredholm integral equation

The Delay Lyapunov Matrix associated with (8), when it exists, is defined in the scalar case as

$$U(\tau) = \int_0^\infty (\mathbf{K}(t) - K_0) K(t + \tau) dt, \quad (34)$$

where

$$K_0 = \left(\sum_{k=1}^m a_k + \int_{-mh}^0 F(\xi) d\xi - \text{Id} \right)^{-1},$$

and \mathbf{K} denotes the fundamental matrix, namely the solution to the LIDE (8) with initial condition $\mathbf{K}(\theta) = K_0$ for $\theta \in [-mh, 0]$. This Lyapunov matrix exists provided that the LIDE (8) is exponentially stable. Nevertheless, this definition does not lend itself to a direct numerical computation of U , as it involves computing \mathbf{K} and thus solving (8) for all nonnegative times. To this end, we rely on the following lemma, whose proof is given in the Appendix.

Lemma 2: The Delay Lyapunov Matrix U defined in (34) satisfies

$$U(\tau) = \sum_{j=1}^m a_j U(\tau + jh) + \int_{-mh}^0 F(\xi) U(\tau - \xi) d\xi + G(\tau), \quad \tau \in [-mh, 0] \quad (35)$$

$$U(\tau) = \sum_{j=1}^m a_j U(\tau - jh) + \int_{-mh}^0 F(\xi) U(\tau + \xi) d\xi, \quad \tau \in [0, mh] \quad (36)$$

with $G(\tau) = -\tau K_0 + \left[-\sum_{j=1}^m a_j jh + \int_{-mh}^0 F(\xi) \xi d\xi \right] K_0^2$.

Relations (35)–(36) are then recast into the Fredholm integral equation (1) by introducing, for $x \in [0, h]$, the partitioned variable

$$w(x) = (U(x + (k-1-m)h))_{1 \leq k \leq 2m}, \quad (37)$$

together with $g(x) \in \mathbb{R}^{2m}$ defined as

$$g_k(x) = \begin{cases} G(x + (k-1-m)h) & \text{for } k = 1, \dots, m \\ 0 & \text{for } k = m+1, \dots, 2m \end{cases} \quad (38)$$

Theorem 1 thus allows to formulate the following result.

Corollary 1: Assume (8) is exponentially stable. Then (1) admits a unique solution w from which the Delay Lyapunov Matrix U defined in (34) is obtained as

$$U(\tau) = w_k(\tau), \tau \in [(k-m-1)h, (k-m)h], k = 1, \dots, 2m$$

Proof: Since (8) is exponentially stable, Assumption 2 is satisfied, as $\sigma \subset \mathcal{C}_-^*$ (see [14, Theorem 3.5]). Moreover, the exponential stability of (8) implies that the associated difference equation $x(t) = \sum_{k=1}^m a_k x(t - \tau_k)$, which corresponds to the principal part of (8), is also exponentially stable, i.e., $\sum_{k=1}^m |a_k| < 1$ (see [14, Corollary 6.1]). Consequently, the matrix $I - A$ is diagonally dominant, so that Assumption 1 is satisfied as well. Therefore, Theorem 1 applies, and (1) admits a unique solution, from which U follows via (37). ■

Hence, Corollary 1 paves the way for a constructive procedure to compute the Delay Lyapunov Matrix (34), by recasting it as the problem of solving the Fredholm equation (1). This can be done numerically by discretizing the integral equation (1) and solving the resulting linear system [18].

B. Numerical example

To illustrate the relevance and applicability of the proposed methodology to compute the Delay Lyapunov Matrix, we conducted numerical simulations with two sets of parameters, given as

$$h = 0.5, m = 2, a_1 = 0.5, a_2 = 0.2, F : \xi \in [-1, 0] \mapsto \xi$$

on the one hand and on the other hand

$$h = 0.25, m = 5, a_1 = a_3 = 0.1, a_2 = 0.5, a_4 = 0, \quad (41)$$

$$a_5 = 0.2, F : \xi \in [-1, 0] \mapsto \xi \sin(\xi) e^{-1.5\xi} \quad (42)$$

both resulting in an exponentially stable LIDE. Corresponding solutions are depicted in Fig. 1. We discretized the integral Fredholm equation (1), solved the resulting linear system and reconstructed the Delay Lyapunov Matrix U according to Corollary 1. The obtained variations of U are compared to the right-hand side of (35) on the interval $[-mh, 0]$ and (36) on the interval $[0, mh]$, confirming that both Eq. (35)–(36) are indeed satisfied. Besides, one can observe that U is continuous, which is expected from its definition [19].

V. CONCLUSION

In this paper, we established the existence and uniqueness of the solution to a class of Fredholm integral equations within an operator-based framework. We further demonstrated that this result can be exploited to compute the Lyapunov matrix associated with scalar LIDEs with commensurate delays.

These contributions constitute key preliminary steps in our ongoing research on LIDEs. Future work will address the case of non-commensurate delays and develop new strategies based on approximating the Lyapunov matrix by computing Lyapunov matrices of systems with commensurate-delay systems

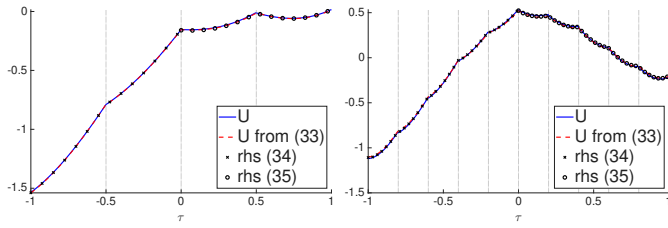


Fig. 1. Delay Lyapunov matrix of the LIDE (8) for respectively, the set of parameters (40) (left) and (41)–(42) (right). The obtained variations of U are compared to the right-hand side of (35) on the interval $[-mh, 0]$ and (36) on the interval $[0, mh]$. The times kh for $k = -m - 1, \dots, m - 1$ are depicted with vertical dotted black lines.

parameterized to converge toward the non-commensurate case, building on the promising results of [26].

Another important direction is the extension of the proposed approach to n -dimensional LIDEs, which entails dealing with a more complex class of Fredholm integral equations involving Kronecker products.

APPENDIX I PROOF OF LEMMA 2

We first formulate standard properties of the Delay-Lyapunov Matrix U . These properties can be found in [19] or directly inferred and are here formulated for a scalar LIDE with commensurate delays.

Lemma 3 ([19]): The delay Lyapunov matrix U defined in (34) satisfies the following properties:

- *Dynamic property:* for all $\tau \in [0, mh]$

$$U(\tau) = \sum_{k=1}^m a_k U(\tau - kh) + \int_{-mh}^0 F(\xi) U(\tau + \xi) d\xi. \quad (43)$$

- *Symmetry property:* for all $\tau \in [-mh, mh]$,

$$U(-\tau) = U(\tau) - \tau K_0^2 \quad (44)$$

The dynamic property (43) directly gives (36). Besides, it also implies that, for $\tau \in [0, \tau_m]$,

$$U(\tau_m - \tau) = \sum_{j=1}^m U(\tau_m - \tau_j - \tau) a_j + \int_{-\tau_M}^0 U(\tau_m - \tau + \xi) F(\xi) d\xi \quad (45)$$

and, using the symmetry property (44), one concludes that

$$U(\tau - \tau_m) = \sum_{j=1}^m a_j U(\tau + \tau_j - \tau_m) + \int_{-\tau_m}^0 F(\xi) U(\tau - \xi - \tau_m) d\xi + G(\tau - \tau_m) \quad (46)$$

in which G can be written as formulated in Lemma 2, using the definition of K_0 . This gives (35) and concludes the proof.

REFERENCES

- [1] I. Alexandrova. A finite Lyapunov matrix-based stability criterion for linear delay systems via piecewise linear approximation. *Systems & Control Letters*, 181:105624, 2023.
- [2] J. Auriol. *Contributions to the robust stabilization of networks of hyperbolic systems*. PhD thesis, Université Paris Saclay, 2024.
- [3] J. Auriol. Stabilization of integral delay equations by solving Fredholm equations. *IEEE Control Systems Letters*, 8:676–681, 2024.
- [4] J. Auriol and D. Bresch-Pietri. On input-to-state stability of linear difference equations and its characterization with a Lyapunov functional. *IFAC-PapersOnLine*, 56(2):929–934, 2023.
- [5] J. Auriol and F. Di Meglio. An explicit mapping from linear first order hyperbolic PDEs to difference systems. *Systems & Control Letters*, 123:144–150, 2019.
- [6] M. Bajodek, F. Gouaisbaut, and A. Seuret. Necessary and sufficient stability condition for time-delay systems arising from legendre approximation. *IEEE Transactions on Automatic Control*, 2022.
- [7] G. Bastin and J.-M. Coron. *Stability and boundary stabilization of 1-D hyperbolic systems*. Springer, 2016.
- [8] Y. Chitour, S. Fueyo, G. Mazanti, and M. Sigalotti. Hautus–Yamamoto criteria for approximate and exact controllability of linear difference delay equations. *Discrete and Continuous Dynamical Systems*, 43:3306–3337, 2023.
- [9] K. Cooke and J. Kaplan. A periodicity threshold theorem for epidemics and population growth. *Mathematical Biosciences*, 31(1):87–104, 1976.
- [10] J.-M. Coron, L. Hu, and G. Olive. Stabilization and controllability of first-order integro-differential hyperbolic equations. *Journal of Functional Analysis*, 271(12):3554–3587, 2016.
- [11] S. Damak, M. Di Loreto, W. Lombardi, and V. Andrieu. Exponential L^2 -stability for a class of linear systems governed by continuous-time difference equations. *Automatica*, 50(12):3299–3303, 2014.
- [12] S. Damak, M. Di Loreto, and S. Mondié. Stability of linear continuous-time difference equations with distributed delay: Constructive exponential estimates. *International Journal of Robust and Nonlinear Control*, 25(17):3195–3209, 2015.
- [13] E. Fridman. *Introduction to time-delay systems: Analysis and control*. Springer, 2014.
- [14] J.K. Hale and S.M. Verduyn Lunel. *Introduction to functional differential equations*. Springer-Verlag, 1993.
- [15] I. Karafyllis and M. Krstic. On the relation of delay equations to first-order hyperbolic partial differential equations. *ESAIM: Control, Optimisation and Calculus of Variations*, 20(3):894–923, 2014.
- [16] V. Kharitonov. *Time-delay systems: Lyapunov functionals and matrices*. Springer Science & Business Media, 2012.
- [17] Vladimir L Kharitonov. On the uniqueness of lyapunov matrices for a time-delay system. *Systems & Control Letters*, 61(3):397–402, 2012.
- [18] R. Kress. *Linear integral equations*, volume 82. Springer, 1989.
- [19] M. Lamarque, J. Auriol, and D. Bresch-Pietri. Converse Lyapunov theorem for input-to-state stability of linear integral difference equations. *Automatica*, 179:112437, 2025.
- [20] G. Mazanti. Relative controllability of linear difference equations. *SIAM Journal on Control and Optimization*, 55(5):3132–3153, 2017.
- [21] D. Melchor-Aguilar. Lyapunov functionals for linear continuous-time difference systems: A single delay case. *Systems & Control Letters*, 177:105536, 2023.
- [22] S.-I. Niculescu. *Delay effects on stability: a robust control approach*, volume 269. Springer Science & Business Media, 2001.
- [23] R. Ortiz, A. Egorov, and S. Mondié. Robust stability analysis for integral delay systems: A complete type functional approach. *IEEE Transactions on Automatic Control*, 2023.
- [24] Reynaldo Ortiz, Sabine Mondié, Saúl Del Valle, and Alexey V Egorov. Construction of delay lyapunov matrix for integral delay systems. In *2018 IEEE Conference on Decision and Control (CDC)*, pages 5439–5444. IEEE, 2018.
- [25] J. Redaud, J. Auriol, and S.-I. Niculescu. Stabilizing integral delay dynamics and hyperbolic systems using a Fredholm transformation. In *2021 60th IEEE Conference on Decision and Control (CDC)*, pages 2595–2600. IEEE, 2021.
- [26] E. Rocha, S. Mondié, and M. Di Loreto. On the Lyapunov matrix of linear delay difference equations in continuous time. *IFAC-PapersOnLine*, 50(1):6507–6512, 2017.
- [27] E. Rocha Campos, S. Mondié, and M. Di Loreto. Necessary stability conditions for linear difference equations in continuous time. *IEEE Transactions on Automatic Control*, 63(12):4405–4412, 2018.
- [28] K. Yoshida. *Lectures on differential and integral equations*, volume 10. Interscience Publishers, 1960.