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CONVERGENCE OF HERMITE EXPANSIONS IN MODULATION SPACES

PHILIPPE JAMING AND MICHAEL SPECKBACHER

ABSTRACT. The aim of this paper is to give an elementary proof that Hermite expansions of a function f in the modulation space $M^p(\mathbb{R})$ converges to f in $M^p(\mathbb{R})$ when $1 < p < \infty$ and may diverge when $p = 1, \infty$. The result was previously established for $1 < p < +\infty$ by Garling and Wojtaszczyk [10] and for $p = 1, \infty$ by Lusky [17] in an equivalent setting of Fock spaces by different methods. Higher dimensional results are also considered.

In an appendix, we also establish upper bounds for the Zak transform of Hermite functions.

1. INTRODUCTION

The aim of this paper is to investigate the convergence of Hermite series in the modulation spaces $M^p(\mathbb{R}^d)$, $1 \leq p < \infty$. Let us first recall that the m -th Hermite function is defined as

$$h_n(t) = C_n e^{\pi t^2} \frac{d^n}{dt^n} \left(e^{-2\pi t^2} \right), \quad t \in \mathbb{R}, \quad n \in \mathbb{N}_0,$$

where $C_n = (-1)^n 2^{1/4} (\sqrt{n!} (2\pi)^n)^{-1/2}$ is chosen so that $\|h_n\|_{L^2(\mathbb{R})} = 1$. It is then well known that $(h_n)_{n \in \mathbb{N}_0}$ is an orthonormal basis of eigenfunctions of both the harmonic oscillator and the Fourier transform and thus play an important role in time-frequency analysis.

The focus of this paper is to investigate to which extent one can use Hermite functions as a generating system in time-frequency analysis. To be more precise, from the orthonormality of the family $(h_n)_{n \in \mathbb{N}_0}$, it follows that any $f \in L^2(\mathbb{R})$ can be written as a series of Hermite functions

$$f = \sum_{n=0}^{\infty} \langle f, h_n \rangle h_n$$

where this series is L^2 -convergent, that is

$$\lim_{N \rightarrow \infty} \left\| f - \sum_{n=0}^N \langle f, h_n \rangle h_n \right\|_{L^2(\mathbb{R})} = 0. \quad (1.1)$$

It is then natural to ask whether other modes of convergence hold. In this direction, Uspensky [22] established the pointwise convergence of the series already a century ago. Skipping forward half a century, Askey and Wainger [1] proved that in (1.1) one may replace the L^2 -norm by L^p -norms if and only if $4/3 < p < 4$. Bochner–Riesz summability of these series outside the range $4/3 < p < 4$ was subsequently obtained by Thangavelu [19, 20]; we refer to his monograph [21] for a comprehensive account of the subject up to the late 1980s.

Further advances have been made since then. For instance, the first author, together with Karoui and Spektor [15], showed that Hermite expansions approximate almost time- and band-limited functions nearly as well as the prolate spheroidal wave functions. As a consequence, approximation rates for Sobolev functions by Hermite expansions were also established. Moreover, one may associate Sobolev

spaces to the harmonic oscillator (see [21]); in this framework, Bongionanni and Torrea [4] proved convergence of Hermite series in such spaces.

Here we consider another natural family of function spaces in time-frequency analysis, namely the modulation spaces which were introduced by Feichtinger in [6, 7]. When $z = (x, \omega) \in \mathbb{R}^{2d}$, and $f \in L^2(\mathbb{R}^d)$ we write $\pi(z)f(t) = e^{2i\pi\omega \cdot t} f(t - x)$ for the time-frequency shift of f . Denoting by $\varphi(t) = 2^{d/4} e^{-\pi|t|^2}$, $t \in \mathbb{R}^d$, the standard d -dimensional Gaussian and $\mathcal{S}'(\mathbb{R}^d)$ the space of tempered distributions, the modulation spaces are then defined as

$$M^p(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{M^p(\mathbb{R}^d)}^p := \int_{\mathbb{R}^{2d}} |\langle f, \pi(z)\varphi \rangle|^p dz < \infty \right\}.$$

When $p = 1$ this gives the Feichtinger Algebra [6]. For properties and applications of $M^p(\mathbb{R}^d)$ we refer to the surveys [8, 2] as well as the book [12] and the references therein. Let us first reduce our analysis to $p = d = 1$. As $M^1(\mathbb{R}) \subset L^2(\mathbb{R})$, if $f \in M^1(\mathbb{R})$ then the corresponding Hermite series converges in $L^2(\mathbb{R})$, i.e., (1.1) holds, but it seems unknown so far whether this convergence also holds in the M^1 -norm. To our knowledge, the only related result is due to Janssen who considered the spaces

$$C(\gamma) = \left\{ f \in L^2(\mathbb{R}) : \|f\|_{C(\gamma)} := \sum_{n=0}^{\infty} |\langle f, h_n \rangle| (1+n)^\gamma < \infty \right\}$$

and showed that $C(1/4) \subset M^1(\mathbb{R}) \subset C(-1/4)$ but the space on the right hand side can not be made smaller and the one on the left larger, at least within the $C(\gamma)$ scale. More precisely, he showed the following.

Theorem 1.1 (Janssen [16]). *There exists A_γ such that, for every $f \in C(\gamma)$, $\|f\|_{M^1(\mathbb{R})} \leq A_\gamma \|f\|_{C(\gamma)}$ if and only if $\gamma \geq 1/4$.*

There exists $B_\gamma > 0$ such that, for every $f \in M^1(\mathbb{R})$, $B_\gamma \|f\|_{C(\gamma)} \leq B_\gamma \|f\|_{M^1(\mathbb{R})}$ if and only if $\gamma \leq -1/4$.

Moreover, $M^1(\mathbb{R}) \not\subset C(\gamma)$ when $\gamma > -1/4$, and $C(\gamma) \not\subset M^1(\mathbb{R})$ when $\gamma < 1/4$.

As a consequence, this shows that convergence in $M^1(\mathbb{R})$ cannot be fully described in terms of the coefficients in the Hermite basis. Here we prove that Hermite series do not converge in $M^1(\mathbb{R}^d)$ and therefore, by duality, also not in $M^\infty(\mathbb{R})$. However, when p is in the intermediate range, $1 < p < \infty$, the series converges in $M^p(\mathbb{R}^d)$.

Let us define the partial Hermite expansion operator via

$$S_N f := \sum_{n=0}^N \langle f, h_n \rangle h_n.$$

The main result of this paper is the following:

Theorem 1.2. *Let $1 \leq p \leq \infty$. There exists a constant $C = C_p$ such that for every $N \in \mathbb{N}$*

$$\|S_N f\|_{M^p(\mathbb{R})} \leq C \|f\|_{M^p(\mathbb{R})}, \quad f \in M^p(\mathbb{R}), \quad (1.2)$$

if and only if $1 < p < \infty$. For $p = 1$, there exist constants $c, C > 1$ such that

$$\|S_N\|_{M^1(\mathbb{R}) \rightarrow M^1(\mathbb{R})} \geq C \log(cN), \quad (1.3)$$

In particular,

- (1) *for every $f \in M^p(\mathbb{R})$, $1 < p < \infty$, $\lim_{N \rightarrow \infty} S_N f = f$ in $M^p(\mathbb{R}^d)$*
- (2) *there exists $f \in M^1(\mathbb{R})$ such that $\|S_N f\|_{M^2(\mathbb{R})} \rightarrow \infty$ so that $S_N f$ does not converge to f in $M^1(\mathbb{R})$.*

We may reinterpret Theorem 1.2 as a result about the convergence of the Taylor expansion for certain spaces of holomorphic functions. To make that connection, we first recall that the *Bargmann transform* of a function f is given by

$$\mathcal{B}f(z) = 2^{1/4} \int_{\mathbb{R}} f(t) e^{2\pi z t - \pi t^2 + \pi z^2/2} dt, \quad z \in \mathbb{C},$$

and note that the Bargmann transform is an isometric isomorphism between the modulation spaces $M^p(\mathbb{R})$ and the (*Bargmann-*)*Fock spaces*

$$\mathcal{F}^p(\mathbb{C}) := \left\{ F : \mathbb{C} \rightarrow \mathbb{C} \text{ holomorphic: } \|F\|_{\mathcal{F}^p(\mathbb{C})} := \int_{\mathbb{C}} |F(z)|^p e^{-\pi p|z|^2/2} dz < \infty \right\}.$$

The subsequent corollary then follows directly from Theorem 1.2 once we note that

$$\mathcal{B}h_n(z) = \left(\frac{\pi}{n!}\right)^{n/2} z^n.$$

Corollary 1.3 (Garling-Wojtaszczyk [10], Lusky [17]). *Let $1 \leq p < \infty$. The partial Taylor expansion*

$$T_N F(z) = \sum_{n=0}^N \frac{F^{(n)}(0)}{n!} z^n$$

converges for every $F \in \mathcal{F}^p(\mathbb{C})$ with respect to $\|\cdot\|_{\mathcal{F}^p(\mathbb{C})}$ if and only if $1 < p < \infty$

This corollary is not new and is of course equivalent to Theorem 1.2. Actually, the case $1 < p < \infty$ can be found in [10, Proposition 7] as well as [17, Theorem 2.1] while the case $p = 1$ is [17, Theorem 2.3], though it is written in a slightly different language and a more general form in [17]. However, these results appear to have remained largely unnoticed within the time–frequency community¹. Beyond closing this knowledge gap, we also provide an alternative proof that is more direct and uses only elementary tools. For the range $1 < p < \infty$, our proof of (1.2) relies on [3, Proposition 4.3] where a connection between boundedness of Hermite multipliers in $M^p(\mathbb{R}^d)$ and Fourier multipliers on $L^p(\mathbb{T}^d)$ was shown. This idea is somewhat implicit in [10]. As a complementary result, this also allows us to study Bochner-Riesz means of Hermite series.

Proposition 1.4. *Let $1 \leq p < \infty$ and $f \in M^p(\mathbb{R})$. If $\alpha > 0$, then the Bochner-Riesz means*

$$B_N^\alpha f := \sum_{n=0}^N \left(1 - \frac{|n|^2}{N^2}\right)^\alpha \langle f, h_n \rangle h_n$$

converge to f in $M^p(\mathbb{R})$ as $N \rightarrow \infty$.

In this introduction, we restricted our analysis to modulation spaces of functions on the real line. However, our result extends in a straightforward way to expansions in terms of tensor products of Hermite functions in arbitrary dimensions. We will discuss this in Section 3.

The remaining of the paper is organized as follows: Section 2 is devoted to the proof of the main result while Section 3 is devoted to higher dimensional extensions.

For this arxiv version, we also include a bound on the Zak transform in Appendix A which answers a question originally asked by H. Bahouri to one of us. This is an elementary consequence of well known results and techniques in Gabor analysis. The same result has since been proved by different methods by H. Bahouri and V. Fischer in an upcoming paper.

¹For instance, during discussions at a Quantum Harmonic Analysis workshop in Hannover in 2024 and at SampTA 2025 in Vienna, several colleagues mentioned this as an open problem.

2. PROOF OF MAIN RESULTS

Our aim here is to prove Theorem 1.2. Before we give the proof, we need to establish an auxiliary result on the L^1 -norm of certain trigonometric polynomials that might be of independent interest.

Let us define

$$P_N(t, \phi) = \sum_{n=0}^N \frac{t^n}{n!} e^{i\phi n} \quad t > 0, \phi \in [0, 2\pi].$$

Theorem 2.1. *For every $N \in \mathbb{N}_0$ and $t \geq 1$ it holds*

$$\frac{\log(t) t^N}{N!} - \frac{30 e^t}{\sqrt{t}} \leq \int_0^{2\pi} |P_N(t, \phi)| \, d\phi \leq \frac{\pi \log(\pi^2 t) t^N}{2 N!} + \frac{30 e^t}{\sqrt{t}}. \quad (2.1)$$

Proof. Note that (2.1) holds trivially for $N = 0$. We may therefore assume from now on that $N \geq 1$. We divide the proof into several simple steps. We start with a preliminary step containing bounds that we will need.

Step 1. Two simple bounds.

We will use the classical pointwise Stirling bound on the factorial (see, e.g., [18])

$$\sqrt{2\pi n} \left(\frac{n}{e}\right)^n \leq n! \leq e\sqrt{n} \left(\frac{n}{e}\right)^n, \quad (2.2)$$

valid for $n \geq 1$.

We will also need the following elementary geometric bound

$$\frac{2}{\pi} \phi \leq |1 - e^{i\phi}| = 2 \sin\left(\frac{\phi}{2}\right) \leq \phi, \quad \phi \in [0, \pi].$$

Equivalently, for $\phi \in (0, \pi]$,

$$\frac{1}{\phi} \leq \frac{1}{|1 - e^{i\phi}|} \leq \frac{\pi}{2} \frac{1}{\phi}. \quad (2.3)$$

Step 2. Reformulation of the result.

Let us write

$$p_n(t) := \frac{e^{-t} t^n}{n!}, \quad n \in \mathbb{N}_0,$$

so that $\{p_n(t)\}_{n \geq 0}$ are the Poisson(t) probabilities and

$$e^{-t} P_N(t, \phi) = \sum_{n=0}^N p_n(t) e^{in\phi}.$$

Thus e^t may be factored out in (2.1) and it suffices to prove the inequality

$$\frac{\log(t) t^N e^{-t}}{N!} - \frac{30}{\sqrt{t}} \leq \int_0^{2\pi} e^{-t} |P_N(t, \phi)| \, d\phi \leq \frac{\pi \log(\pi^2 t) t^N e^{-t}}{2 N!} + \frac{30}{\sqrt{t}}. \quad (2.4)$$

Step 3. Re-summation.

For any finite sequence q_0, q_1, \dots, q_N and $z \neq 1$ one has

$$(1 - z)^2 \sum_{n=0}^N q_n z^n = \sum_{m=0}^{N+2} r_m z^m, \quad (2.5)$$

with

$$\begin{cases} r_0 := q_0, \\ r_1 := q_1 - 2q_0, \\ r_m := q_m - 2q_{m-1} + q_{m-2}, \quad \text{for } 2 \leq m \leq N, \\ r_{N+1} := -2q_N + q_{N-1}, \\ r_{N+2} := q_N. \end{cases} \quad (2.6)$$

Applying and rearranging (2.5) with $q_n = p_n(t)$ and $z = e^{i\phi}$ one obtains

$$\sum_{n=0}^N p_n(t) e^{in\phi} = \frac{1}{(1 - e^{i\phi})^2} \sum_{n=0}^{N+2} r_n(t) e^{in\phi}, \quad \phi \notin 2\pi\mathbb{Z}. \quad (2.7)$$

Next, we define $\phi_0 := 1/\sqrt{t}$, and

$$I_N(t) := \int_{\phi_0}^{\pi} \left| \sum_{n=0}^N p_n(t) e^{in\phi} \right| d\phi = \int_{\phi_0}^{\pi} \frac{1}{|1 - e^{i\phi}|^2} \left| \sum_{n=0}^{N+2} r_n(t) e^{in\phi} \right| d\phi.$$

Since

$$\left| \sum_{n=0}^N p_n(t) e^{in\phi} \right| \leq \sum_{n=0}^N |p_n(t)| \leq 1,$$

it follows immediately that

$$2I_N(t) \leq \int_0^{2\pi} \left| \sum_{n=0}^N p_n(t) e^{in\phi} \right| d\phi \leq 2I_N(t) + \frac{2}{\sqrt{t}}.$$

It is thus enough to bound I_N from below and above. To do so, we will now decompose the sum $\sum_{n=0}^{N+2} r_n(t) e^{in\phi}$ into two parts: the terms $n = N+1, N+2$ which will be shown to be dominating and the remaining sum over $0 \leq n \leq N$ which turns out to be a remainder term.

Step 4. Estimation of the dominating term.

We first write

$$\begin{aligned} r_{N+1}(t) + e^{i\phi} r_{N+2}(t) &= p_{N-1}(t) - 2p_N(t) + p_N(t) e^{i\phi} \\ &= (e^{i\phi} - 1)p_N(t) + p_{N-1}(t) - p_N(t) \\ &= (e^{i\phi} - 1)p_N(t) + \frac{N-t}{t} p_N(t). \end{aligned}$$

The absolute value of the second term can be bounded as follows

$$p_N(t) \frac{|N-t|}{t} \leq \frac{1}{t} \cdot \sup_{x>0} \{p_N(x)|N-x|\} \leq \frac{1}{t}, \quad t > 0. \quad (2.8)$$

To prove this bound, first note that the supremum is attained at one of the two local maxima $x_{\pm} = N + \frac{1}{2} \pm \sqrt{N + \frac{1}{4}}$. With the help of (2.2) we may bound

$$\begin{aligned} \frac{e^{-x+x_+^N}}{N!} |N - x_+| &\leq e^{-1/2 - \sqrt{N+1/4}} \left(1 + \frac{1}{N} \left(\frac{1}{2} + \sqrt{N + \frac{1}{4}} \right) \right)^N \frac{\sqrt{N+1/4} + 1/2}{\sqrt{2\pi N}} \\ &\leq e^{-1/2 - \sqrt{N+1/4} + N \log(1+\delta)}, \end{aligned}$$

where $\delta = (1/2 + \sqrt{N+1/4})/N$ and we used that $(\sqrt{N+1/4} + 1/2)/\sqrt{2\pi N} \leq 1$ for $N \geq 1$. Since $\log(1+\delta) \leq \delta$ for every $\delta > -1$, we infer that

$$\frac{e^{-x+x_+^N}}{N!} |N - x_+| \leq 1.$$

For x_- we argue similarly, only that this time we set $\delta = (1/2 - \sqrt{N+1/4})/N > -1$.

Using the lower and upper triangular inequalities then leads to the bounds

$$\left|1 - e^{i\phi} p_N(t) - \frac{1}{t}\right| \leq |r_{N+1}(t) + e^{i\phi} r_{N+2}(t)| \leq \left|1 - e^{i\phi} p_N(t) + \frac{1}{t}\right|. \quad (2.9)$$

Step 5. Estimation of the remainder term.

Note that, for $2 \leq n \leq N$, $r_n(t)$ is the second finite difference of the Poisson probabilities $p_n(t) = e^{-t} t^n / n!$. A short computation yields for $2 \leq n \leq N$

$$r_n(t) := p_n(t) - 2p_{n-1}(t) + p_{n-2}(t) = p_n(t) \frac{(n-t)^2 - n}{t^2}.$$

Hence

$$\sum_{n=2}^N |r_n(t)| \leq \frac{1}{t^2} \sum_{n \geq 0} p_n(t) |(n-t)^2 - n| = \frac{1}{t^2} \mathbb{E}[|(X-t)^2 - X|],$$

where $X \sim \text{Poisson}(t)$. Using the triangle inequality and the moments of X we get the crude bound

$$\mathbb{E}[|(X-t)^2 - X|] \leq \mathbb{E}[(X-t)^2 + X] = \text{Var}(X) + \mathbb{E}[X] = t + t = 2t.$$

Therefore we obtain the simple bound

$$\sum_{n=2}^N |r_n(t)| \leq \frac{2}{t}. \quad (2.10)$$

On the other hand

$$|r_0(t) + e^{i\phi} r_1(t)| \leq |r_0(t)| + |r_1(t)| \leq 3|p_0(t)| + |p_1(t)| \leq (3+t)e^{-t},$$

which is easily seen to be less than $2/t$ for $t \geq 1$. Together with (2.10), this implies that

$$\sum_{n=2}^N |r_n(t)| + |r_0(t) + e^{i\phi} r_1(t)| \leq \frac{4}{t}. \quad (2.11)$$

Step 6. The bounds in (2.4).

From (2.3), (2.7), (2.9), (2.11) and the reverse triangle inequality we obtain

$$\begin{aligned} & \left| \sum_{n=0}^N p_n(t) e^{in\phi} \right| \\ & \geq \frac{1}{|1 - e^{i\phi}|^2} \left(|r_{N+1}(t) + e^{i\phi} r_{N+2}(t)| - \sum_{n=2}^N |r_n(t)| - |r_0(t) + e^{i\phi} r_1(t)| \right) \\ & \geq \frac{1}{|1 - e^{i\phi}|^2} p_N(t) - \frac{1}{|1 - e^{i\phi}|^2} \frac{5}{t} \geq \frac{1}{\phi} p_N(t) - \frac{5\pi^2}{4\phi^2 t}, \quad \phi_0 \leq \phi \leq \pi. \end{aligned} \quad (2.12)$$

Similarly, applying the triangle inequality instead of the reverse triangle inequality and the upper instead of the lower bound in (2.3) yields

$$\left| \sum_{n=0}^N p_n(t) e^{in\phi} \right| \leq \frac{\pi}{2\phi} p_N(t) + \frac{5\pi^2}{4\phi^2 t}, \quad \phi_0 \leq \phi \leq \pi.$$

Integrating over ϕ thus gives

$$I_N(t) \geq \int_{\phi_0}^{\pi} \left[\frac{1}{\phi} p_N(t) - \frac{5\pi^2}{4\phi^2 t} \right] d\phi = (\log(\pi) - \log(\phi_0)) p_N(t) - \frac{5\pi^2}{4t} \left(\frac{1}{\phi_0} - \frac{1}{\pi} \right)$$

$$\geq \frac{1}{2} \log(t) p_N(t) - \frac{5\pi^2}{4\sqrt{t}} \geq \frac{1}{2} \log(t) p_N(t) - \frac{15}{\sqrt{t}},$$

where we substituted $\phi_0 = 1/\sqrt{t}$ in the final step and removed useless positive terms. Moreover,

$$\begin{aligned} I_N(t) &\leq \int_{\phi_0}^{\pi} \left[\frac{\pi}{2\phi} p_N(t) + \frac{5\pi^2}{4\phi^2 t} \right] d\phi = \frac{\pi}{2} (\log(\pi) - \log(\phi_0)) p_N(t) + \frac{5\pi^2}{4t} \left(\frac{1}{\phi_0} - \frac{1}{\pi} \right) \\ &\leq \frac{\pi}{4} \log(\pi^2 t) p_N(t) + \frac{5\pi^2}{4\sqrt{t}}, \end{aligned}$$

which completes the proof once we note that $5\pi^2/2 + 2 \leq 30$. \square

To prove Theorem 1.2 in the range $1 < p < \infty$ we need the following result which connects boundedness of Hermite multipliers on $M^p(\mathbb{R}^d)$ with boundedness of Fourier multipliers with the same symbol on $L^p(\mathbb{T}^d)$.

Proposition 2.2. ([3, Proposition 4.3]) *Let $1 \leq p < \infty$, $m : \mathbb{Z}^d \rightarrow \mathbb{C}$ and $g : \mathbb{T}^d \rightarrow \mathbb{C}$. If the Fourier multiplier A_m given by $\widehat{A_m g}(n) := m(n)\widehat{g}(n)$, $n \in \mathbb{Z}^d$, defines a bounded operator on $L^p(\mathbb{T}^d)$, then the Hermite multiplier with symbol $m|_{\mathbb{N}_0^d}$ given by*

$$H_m f = \sum_{n \in \mathbb{N}_0^d} m(n) \langle f, h_n \rangle h_n,$$

is bounded on $M^p(\mathbb{R}^d)$ with

$$\|H_m f\|_{M^p(\mathbb{R}^d)} \lesssim \|A_m\|_{L^p(\mathbb{T}^d) \rightarrow L^p(\mathbb{T}^d)} \|f\|_{M^p(\mathbb{R}^d)}$$

We are now ready to give the proof of Theorem 1.2.

Proof of Theorem 1.2. Let us identify the phase space \mathbb{R}^2 with \mathbb{C} via $z = (x, \omega) \in \mathbb{R}^2 \sim z = x + i\omega \in \mathbb{C}$. By the Laguerre connection (see, e.g., [9, Theorem 1.104]), one has

$$\langle \pi(z)h_0, h_n \rangle = e^{\pi i x \xi - \pi |z|^2/2} \sqrt{\frac{\pi^n}{n!}} z^n, \quad z \in \mathbb{C}, \quad n \in \mathbb{N}_0.$$

Therefore, for $w = y + i\eta$

$$\sum_{n=0}^N \langle \pi(z)h_0, h_n \rangle \langle h_n, \pi(w)h_0 \rangle = e^{\pi i(x\xi - y\eta)} e^{-\pi(|z|^2 + |w|^2)/2} \sum_{n=0}^N \frac{(\pi z \bar{w})^n}{n!}, \quad (2.13)$$

If we set $|z| = r$ and $|w| = t$, we obtain by changing to polar coordinates

$$\begin{aligned} \|S_N(\pi(z)h_0)\|_{M^1(\mathbb{R})} &= \int_{\mathbb{C}} \left| \sum_{n=0}^N \langle \pi(z)h_0, h_n \rangle \langle h_n, \pi(w)h_0 \rangle \right| dw \\ &= \int_{\mathbb{C}} e^{-\pi(|z|^2 + |w|^2)/2} \left| \sum_{n=0}^N \frac{(\pi w \bar{z})^n}{n!} \right| dw \\ &= \int_0^\infty t e^{-\pi(r^2 + t^2)/2} \int_0^{2\pi} \left| \sum_{n=0}^N \frac{(\pi r t)^n}{n!} e^{i\phi n} \right| d\phi dt \\ &= \int_0^\infty t e^{-\pi(r^2 + t^2)/2} \int_0^{2\pi} |P_N(\pi r t, \phi)| d\phi dt. \end{aligned}$$

Now we set $z_N = \sqrt{N/\pi}$ and assume $N \geq 3$. Applying the change of variables $\rho = \sqrt{\pi} t$ and Theorem 2.1 then yields

$$\|S_N(\pi(z_N)h_0)\|_{M^1(\mathbb{R})} = \frac{1}{\pi} \int_0^\infty \rho e^{-(\rho^2 + N)/2} \int_0^{2\pi} |P_N(\sqrt{N}\rho, \phi)| d\phi d\rho$$

$$\begin{aligned}
&\geq \frac{1}{\pi} \int_{\sqrt{N-1}}^{\sqrt{N+1}} \rho e^{-(\rho^2+N)/2} \int_0^{2\pi} |P_N(\sqrt{N}\rho, \phi)| \, d\phi \, d\rho \\
&\geq \frac{1}{\pi} \int_{\sqrt{N-1}}^{\sqrt{N+1}} \rho e^{-(\rho^2+N)/2} \left(\frac{\log(\rho\sqrt{N})(\rho\sqrt{N})^N}{N!} - \frac{30 e^{\rho\sqrt{N}}}{(\rho\sqrt{N})^{1/2}} \right) d\rho.
\end{aligned}$$

Note that we were allowed to apply Theorem 2.1 since $\rho\sqrt{N} \geq 1$ for every $\rho \in [\sqrt{N-1}, \sqrt{N+1}]$ and $N \geq 3$.

Next we prove the following auxiliary inequality: There exists $C > 0$ such that for every $N \geq 3$

$$\frac{e^{-t} t^{N+1/2}}{N!} \geq \frac{1}{4e}, \quad |t - N| \leq \sqrt{N}. \quad (2.14)$$

Note that the map $t \mapsto e^{-t} t^{N+1/2}$ has only one local maximum at $t^* = N + 1/2$. Therefore, to verify (2.14) it suffices to check the inequality for $t_{\pm} = N \pm \sqrt{N}$. Using Stirling's approximation (2.2) we thus deduce for t_+

$$\frac{e^{-N-\sqrt{N}}(N+\sqrt{N})^{N+1/2}}{N!} \geq e^{-\sqrt{N}-1} \left(1 + 1/\sqrt{N}\right)^{N+1/2} \geq \frac{1}{2e}.$$

Similarly, for t_-

$$\frac{e^{-N+\sqrt{N}}(N-\sqrt{N})^{N+1/2}}{N!} \geq e^{\sqrt{N}-1} \left(1 - 1/\sqrt{N}\right)^{N+1/2} \geq \frac{1}{4e}.$$

Consequently,

$$\begin{aligned}
\|S_N(\pi(z_N)h_0)\|_{M^1(\mathbb{R})} &\geq \frac{1}{\pi} \int_{\sqrt{N-1}}^{\sqrt{N+1}} \rho e^{-(\rho^2+N)/2} \left(\frac{\log(\rho\sqrt{N})}{4e} - 30 \right) \frac{e^{\rho\sqrt{N}}}{(\rho\sqrt{N})^{1/2}} d\rho \\
&\geq \frac{1}{4e\pi} \int_{\sqrt{N-1}}^{\sqrt{N+1}} \sqrt{\frac{\rho}{\sqrt{N}}} e^{-(\rho-\sqrt{N})^2/2} \log\left(\frac{\rho\sqrt{N}}{120e}\right) d\rho \\
&\geq \frac{1}{4e\pi} \sqrt{1 - \frac{1}{\sqrt{N}}} \log\left(\frac{N-\sqrt{N}}{120e}\right) \int_{-1}^1 e^{-t^2/2} dt \\
&\geq \frac{1}{4e^{3/2}\pi} \log\left(\frac{N}{360e}\right) \gtrsim \log(cN),
\end{aligned}$$

where we used that $N - \sqrt{N} \geq N/3$ for $N \geq 3$. Since $\|\pi(z)h_0\|_{M^1(\mathbb{R})} = \|h_0\|_{M^1(\mathbb{R})}$ for every $z \in \mathbb{C}$ it follows that

$$\|S_N\|_{M^1(\mathbb{R}) \rightarrow M^1(\mathbb{R})} \gtrsim \log(cN),$$

which concludes the proof of (1.3). The existence of an element $f \in M^1(\mathbb{R})$ whose Hermite expansion does not converge in $M^1(\mathbb{R})$ then follows from the Banach-Steinhaus theorem.

To prove the uniform boundedness of S_N on $M^p(\mathbb{R})$, $1 < p < \infty$, we first observe that the partial Fourier sum

$$\sum_{-N \leq n \leq N} \widehat{f}(n) e^{2\pi i \xi \cdot n}, \quad \xi \in [0, 1], \quad (2.15)$$

can be written as a Fourier multiplier A_{m_N} with symbol

$$m_N(n) = \chi_{[-N, N]}(n), \quad n \in \mathbb{Z}.$$

It is well-known by the Marcel Riesz inequality that the partial Fourier sums in (2.15) are uniformly bounded on $L^p(\mathbb{T})$ which implies by Proposition 2.2 that the family of Hermite multipliers H_{m_N} are uniformly norm bounded on $M^p(\mathbb{R})$. But H_{m_N} is nothing else than S_N , which concludes the proof. \square

3. CONVERGENCE IN HIGHER DIMENSIONS

In higher dimensions we consider the tensor product Hermite functions, defined by

$$h_n(t) = h_{(n_1, n_2, \dots, n_d)}(t_1, t_2, \dots, t_d) = \prod_{j=1}^d h_{n_j}(t_j), \quad n \in \mathbb{N}_0^d, t \in \mathbb{R}^d.$$

Let us define the partial Hermite expansion operator via

$$S_N f := \sum_{0 \leq n_1, \dots, n_d \leq N} \langle f, h_n \rangle h_n.$$

Corollary 3.1. *Let $1 \leq p \leq \infty$. There exists a constant $C = C_{p,d}$ such that for every $N \in \mathbb{N}$*

$$\|S_N f\|_{M^p(\mathbb{R}^d)} \leq C \|f\|_{M^p(\mathbb{R}^d)}, \quad f \in M^p(\mathbb{R}^d), \quad (3.1)$$

if and only if $1 < p < \infty$. For $p = 1$, there exists a constant $c > 1$ such that

$$\|S_N\|_{M^1(\mathbb{R}^d) \rightarrow M^1(\mathbb{R}^d)} \gtrsim_d \log(cN)^d, \quad (3.2)$$

In particular, for every $f \in M^p(\mathbb{R}^d)$, $1 < p < \infty$, $\lim_{N \rightarrow \infty} S_N f = f$ in $M^p(\mathbb{R}^d)$ and there exists $f \in M^1(\mathbb{R}^d)$ such that $\lim_{N \rightarrow \infty} S_N f \neq f$ in $M^1(\mathbb{R}^d)$.

Proof. For $d \geq 2$ one can simply consider the sequence $z_N = (\sqrt{N}, \dots, \sqrt{N}, 0, \dots, 0) \in \mathbb{R}^{2d}$ and observe that

$$\begin{aligned} S_N(\pi(z_N)h_{(0, \dots, 0)}) &= \sum_{0 \leq n_1, \dots, n_d \leq N} \prod_{k=1}^d \langle \pi(\sqrt{N}, 0)h_0, h_{n_k} \rangle h_{(n_1, \dots, n_d)} \\ &= \bigotimes_{k=1}^d S_N(\pi(\sqrt{N}, 0)h_0). \end{aligned}$$

Therefore, we obtain (1.3) as

$$\begin{aligned} \|S_N(\pi(z_N)h_{(0, \dots, 0)})\|_{M^1(\mathbb{R}^d)} &= \|S_N(\pi(\sqrt{N}, 0)h_0)\|_{M^1(\mathbb{R})}^d \\ &\gtrsim \log(CN)^d \|h_0\|_{M^1(\mathbb{R})}^d = \log(CN)^d \|h_{(0, \dots, 0)}\|_{M^1(\mathbb{R}^d)}. \end{aligned}$$

To prove the uniform boundedness of S_N on $M^p(\mathbb{R}^d)$, $1 < p < \infty$, we first observe that the partial Fourier sum

$$\sum_{-N \leq n_1, \dots, n_d \leq N} \widehat{f}(n) e^{2\pi i \xi \cdot n}, \quad \xi \in [0, 1]^d, \quad (3.3)$$

can be written as a Fourier multiplier A_{m_N} with symbol

$$m_N(n) = \chi_{[-N, N]}(n_1) \cdots \chi_{[-N, N]}(n_d), \quad n \in \mathbb{Z}^d.$$

It is well-known that the partial Fourier sums in (3.3) are uniformly bounded on $L^p(\mathbb{T}^d)$ (see, e.g., [11, Section 4.1]) which implies by Proposition 2.2 that the family of Hermite multipliers H_{m_N} are uniformly norm bounded on $M^p(\mathbb{R}^d)$. But H_{m_N} is nothing else than S_N , which concludes the proof. \square

Our proof of (1.2) relies on [3, Proposition 4.3] where a connection between boundedness of Hermite multipliers in $M^p(\mathbb{R}^d)$ and Fourier multipliers on $L^p(\mathbb{T}^d)$ was established. This allows us to also study Bochner-Riesz means of Hermite series.

Proposition 1.4 is a special case of the subsequent result.

Proposition 3.2. *Let $1 \leq p < \infty$, $f \in M^p(\mathbb{R}^d)$ and $(x)_+ := x \cdot \chi_{[0, \infty)}(x)$. If $\alpha > (d-1)\left|\frac{1}{2} - \frac{1}{p}\right|$, then the Bochner-Riesz means*

$$B_R^\alpha f := \sum_{n \in \mathbb{N}_0^d} \left(1 - \frac{|n|^2}{R^2}\right)_+^\alpha \langle f, h_n \rangle h_n$$

converge to f in $M^p(\mathbb{R}^d)$ as $R \rightarrow \infty$.

Proof. Since the family of Fourier multipliers A_{m_R} with $m_R(n) = (1 - |n|^2/R^2)_+^\alpha$ is uniformly bounded on $L^p(\mathbb{T}^d)$ whenever $\alpha > (d-1)\left|\frac{1}{2} - \frac{1}{p}\right|$ (see, e.g., [11, Proposition 4.1.9]), the result follows immediately from Proposition 2.2. \square

Remark 3.3. Since [3, Proposition 4.3] provides only a sufficient condition for the boundedness of Hermite multipliers, it remains an open problem whether certain symbols that generate unbounded Fourier multipliers also yield unbounded Hermite multipliers. A natural example to consider is Fefferman's famous counterexample for the ball multiplier [5]. This leads to the following question: for which range of p does the sequence of spherical partial sums in the Hermite expansion,

$$\overset{\circ}{S}_N f := \sum_{n \in \mathbb{N}_0^d, |n| \leq N} \langle f, h_n \rangle h_n$$

converge in $M^p(\mathbb{R}^d)$?

APPENDIX A. UPPER FRAME AND BESSEL BOUNDS OF GABOR SYSTEMS ASSOCIATED TO HERMITE FUNCTIONS

This appendix is not for final publication.

In this appendix, we answer a question raised to one of us by H. Bahouri, namely an upper bound of the Zak transform of Hermite functions. While this result seems not written as such, it follows from standard results and techniques in Gabor analysis. We have decided to write this appendix for the arxiv version of the paper for eventual reference.

A.1. Statement of results.

Notation A.1. For $\Lambda \subset \mathbb{R}^2$ a lattice, we denote by

$$\text{rel}(\Lambda) = \max_{x \in \mathbb{R}^2} |\Lambda \cap (x + [0, 1]^2)|$$

the largest number of points of Λ in a unit square. Note that $\text{rel}(\mathbb{Z}^2) = 1$.

The Gabor system generated by h_n and Λ is then defined by

$$\mathcal{G}(h_n, \Lambda) = \{\pi(\lambda)h_n : \lambda \in \Lambda\}.$$

The aim of this section is to obtain an estimate of the upper frame bound and the upper Bessel bounds of $\mathcal{G}(h_n, \Lambda)$.

Theorem A.2. *There exists a universal constant C such that, for every lattice $\Lambda \subset \mathbb{R}^2$ and every $n \in \mathbb{N}_0$,*

(1) *for every $f \in L^2(\mathbb{R})$*

$$\sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda)h_n \rangle|^2 \leq C \text{rel}(\Lambda)(n+1)^{1/2} \|f\|_2^2;$$

(2) for every $a = (a_\lambda)_{\lambda \in \Lambda} \in \ell^2(\Lambda)$,

$$\left\| \sum_{\lambda \in \Lambda} a_\lambda \pi(\lambda) h_n \right\|_{L^2(\mathbb{R})}^2 \leq C \operatorname{rel}(\Lambda) (n+1)^{1/2} \|a\|_2^2.$$

Recall that the Zak transform of a function $f \in L^2(\mathbb{R})$ is defined as

$$Zf(x, y) = \sum_{k \in \mathbb{Z}} f(x+k) e^{2i\pi kx}.$$

It is well-known that upper and lower bounds for the Zak transform of the window function g are related to the frame bounds of the corresponding Gabor system $\mathcal{G}(g, \mathbb{Z}^2)$. In particular, from [12, Corollary 8.3.2]

$$\sup_{\|f\|=1} \sum_{\lambda \in \mathbb{Z}^2} |\langle f, \pi(\lambda)g \rangle|^2 = \|Zg\|_{L^\infty([0,1]^2)},$$

we may directly deduce the following bound²:

Corollary A.3. *We have*

$$\|Zh_n\|_{L^\infty([0,1]^2)} \lesssim (n+1)^{1/4}.$$

A.2. The proof of Theorem A.2. We will use the basic properties of the Bargman transform that we recalled at the end of the introduction. The key element of the proof is the following simple estimate of the $M^1(\mathbb{R})$ -norm of Hermite functions that was already proved by A.E.J.M. Janssen:

Lemma A.4. *When $n \rightarrow \infty$,*

$$\|h_n\|_{M^1(\mathbb{R})} = (2^3 \pi)^{1/4} n^{1/4} (1 + O(n^{-1})).$$

Proof. Using properties of the Gabor transform and integration in polar coordinates, we have

$$\begin{aligned} \|h_n\|_{M^1(\mathbb{R})} &= \int_{\mathbb{C}} |\mathcal{B}[h_n](z)| e^{-\pi|z|^2/2} dz = \frac{1}{\sqrt{n!}} \int_{\mathbb{C}} (\pi^{1/2}|z|)^n e^{-\pi|z|^2/2} dz \\ &= \frac{2\pi}{\sqrt{n!}} \int_0^\infty (\pi^{1/2}r)^n e^{-\pi r^2/2} r dr \\ &= 2 \frac{2^{\frac{n}{2}}}{\sqrt{n!}} \int_0^\infty s^{\frac{n}{2}+1} e^{-s} \frac{ds}{s} = 2 \frac{2^{\frac{n}{2}} \Gamma(\frac{n}{2} + 1)}{\Gamma(n+1)^{\frac{1}{2}}} \end{aligned}$$

with the change of variable $r = \sqrt{\frac{2}{\pi}}s$. Next, we apply Stirling's formula

$$\Gamma(x+1) = \sqrt{2\pi} e^{-x} x^{x+\frac{1}{2}} (1 + O(x^{-1}))$$

to obtain

$$\|h_n\|_{M^1(\mathbb{R})} = 2 \left(\frac{\pi}{2}n\right)^{1/4} (1 + O(n^{-1}))$$

as claimed. \square

Theorem A.2 is then a direct consequence of the following lemma [14, Lemma 2.1]:

Lemma A.5 (Gröchenig-Lyubarskii). *If $\|\varphi\|_{M^1} < \infty$ then*

²This answers a question posed by H. Bahouri to one of us, and we thank her for sparking our interest in the problem.

(1) for every $f \in L^2(\mathbb{R})$

$$\sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda)\varphi \rangle|^2 \leq C \operatorname{rel}(\Lambda) \|\varphi\|_{M^1}^2 \|f\|_{L^2(\mathbb{R})}^2;$$

(2) for every $a = (a_\lambda)_{\lambda \in \Lambda} \in \ell^2(\Lambda)$,

$$\left\| \sum_{\lambda \in \Lambda} a_\lambda \pi(\lambda)\varphi \right\|_{L^2(\mathbb{R})}^2 \leq C \operatorname{rel}(\Lambda) \|\varphi\|_{M^1}^2 \|a\|_2^2.$$

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