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Optimal stabilization rate for the wave equation with hyperbolic boundary condition*

Hugo Parada[†] Nicolas Vanspranghe^{‡§}

December 19, 2025

Abstract

We show that the energy of classical solutions to the wave equation with hyperbolic boundary condition (i.e., dynamic Wentzell boundary condition) and damping on the boundary decays like $1/t$. In fact we allow mixed boundary conditions: a possibly empty, disjoint part of the boundary may be kept at rest provided that the dynamic part satisfies the geometric control condition. We also prove that this decay rate is sharp. Our results follow from resolvent estimates, which we establish by studying high-frequency quasimodes.

Keywords. Wave equation, dynamic boundary conditions, stabilization, non-uniform stability, semiclassical analysis.

AMS subject classifications. 35L05, 35L20, 35B35, 35B40, 93C20.

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1 Introduction

1.1 Main result

Let Ω be a smooth bounded domain in \mathbb{R}^d , $d \geq 2$. Consider a nonempty relatively open subset Γ_0 of Γ such that, letting $\Gamma_1 \triangleq \Gamma \setminus \Gamma_0$, $\Gamma_0 \cap \Gamma_1 = \emptyset$. In other words, either Γ_1 is empty, or Γ is made of two disjoint components Γ_0 and Γ_1 . In this paper we are interested in the initial boundary value problem

$$(\partial_t^2 - \Delta)u = 0 \quad \text{in } \Omega \times (0, +\infty), \quad (1.1a)$$

$$(\partial_t^2 + \partial_t - \Delta_\Gamma)u = -\partial_\nu u \quad \text{on } \Gamma_0 \times (0, +\infty), \quad (1.1b)$$

$$u = 0 \quad \text{on } \Gamma_1 \times (0, +\infty), \quad (1.1c)$$

$$(u, \partial_t u)|_{t=0} = (u_0, u_1) \quad \text{in } \Omega, \quad (1.1d)$$

where ∂_ν is the outward normal derivative, Δ is the usual Laplacian, and Δ_Γ is the Laplace–Beltrami operator on Γ equipped with the canonical Riemannian metric induced by \mathbb{R}^d ; see Section 2.1 for a definition. Equation (1.1b) represents a *hyperbolic* boundary condition, or *dynamic* Wentzell¹ boundary condition, subject to viscous damping or dissipative feedback action. The *energy* of solutions u to (1.1) is given by

$$E(u, t) \triangleq \frac{1}{2} \int_\Omega |\nabla u|^2 + |\partial_t u|^2 dx + \frac{1}{2} \int_{\Gamma_0} |\nabla_\Gamma u|^2 + |\partial_t u|^2 d\sigma, \quad t \geq 0, \quad (1.2)$$

where dx denotes the standard Lebesgue measure on \mathbb{R}^d , $d\sigma$ denotes the induced surface measure on Γ , and ∇_Γ is the Riemannian gradient on Γ ; see Section 2.1 for further details. Solutions u to (1.1) satisfy, at least formally, the *energy identity*

$$E(u, t)|_0^T = - \iint_{\Gamma_0 \times (0, T)} |\partial_t u|^2 d\sigma dt, \quad T \geq 0, \quad (1.3)$$

and in particular the energy is nonincreasing. In fact, a standard argument based on unique continuation will show that $E(u, t) \rightarrow 0$ as $t \rightarrow +\infty$ for all finite-energy solutions u to (1.1); see Section 2.2. We are interested in *quantified* rates of decay, and our main result reads as follows.

Assumption 1.1 (Dynamic hypotheses). No *generalized bicharacteristic* (see Appendix B) has contact of infinite order with $\Gamma \times (0, +\infty)$, as stated in Assumption B.3, and Γ_0 satisfies the *geometric control condition* [BLR92], as in Definition B.6.

Theorem 1.2 (Energy decay rate). *Suppose that Assumption 1.1 holds. Let $(u_0, u_1) \in H^2(\Omega) \times H^1(\Omega)$ be such that $(u_0, u_1)|_{\Gamma_0} \in H^2(\Gamma_0) \times H^1(\Gamma_0)$ and $(u_0, u_1)|_{\Gamma_1} = 0$, and let u the unique solution of (1.1) with initial data (u_0, u_1) . Then,*

$$E(u, t) = o(t^{-1}), \quad t \rightarrow +\infty. \quad (1.4)$$

(The geometric control condition is automatically satisfied when Γ_1 is empty; on the other hand the finite-order contact condition is always satisfied when Γ is locally real-analytic.) The decay rate of Theorem 1.2, which is restated in more detail as Theorem 3.1 below, cannot be improved, at least in the polynomial scale.

Theorem 1.3 (Lower bound on decay, disk case). *Suppose that Ω is the open unit disk $\mathbb{D} \triangleq \{x \in \mathbb{R}^2 : |x| < 1\}$, in which case $\Gamma = \Gamma_0 = \partial\mathbb{D}$. Then, for all $\varepsilon > 0$ there exist initial data $(u_0, u_1) \in H^2(\Omega) \times H^1(\Omega)$ with $(u_0, u_1)|_\Gamma \in H^2(\Gamma) \times H^1(\Gamma)$ such that the corresponding solution u to (1.1) satisfies*

$$\sup_{t \geq 0} t^{1+\varepsilon} E(u, t) = +\infty. \quad (1.5)$$

In particular Theorem 1.3 shows that the stability of (1.1) is *non-uniform* in general,² since any rate that is valid for *all* finite-energy solutions would lead to exponential decay of the energy. Note that, in the setting of Theorem 1.3, Assumption 1.1 is satisfied: here, the obstruction to uniform stabilization lies within the *coupling* between interior and boundary dynamics rather than a failure of the geometric control condition. This stands in contrast with many works dealing with non-uniform energy decay of waves, for instance [Leb96, AL14], and constitutes the key feature which motivates our analysis.

1.2 Background and literature review

Generally speaking, dynamic (or kinetic) boundary conditions such as (1.1b) arise in problems where the boundary carries its own momentum and may model displacements of, e.g., point masses at the endpoints of one-dimensional media or thick membranes in the multidimensional setting.

Notably, *acoustic* boundary conditions for wave equations give rise to coupled systems that are quite close to (1.1). Such boundary conditions were introduced (in a time-harmonic form) by physicists Morse

¹Also spelled “Ventcel”, “Ventsel”, etc.

²We conjecture that the conclusion of Theorem 1.3 holds regardless of the geometry.

and Ingard in [MI68] for small, irrotational perturbations of acoustic flows. Later, Beale and Rosencrans gave in [BR74, Bea76] a more precise formulation of the associated initial boundary value problems: in a bounded medium delimited by an impenetrable (non-porous) membrane, the velocity potential ϕ solves the wave equation $(\partial_t^2 - c^2 \Delta)\phi = 0$ and is subject to the boundary condition $\partial_t \delta = \partial_\nu \phi$, where the normal displacement δ satisfies $(m\partial_t^2 + d\partial_t + k)\delta = \rho\partial_t \phi$ if tangential oscillations are neglected (“locally reacting” surface in the terminology of [MI68]) or $(m\partial_t^2 + d\partial_t + k\Delta_\Gamma)\delta = \rho\partial_t \phi$ if transverse tension is taken into account. Here c is the speed of sound in the medium, m , d and k are surface mass, resistivity and stiffness, and ρ is the density of the gas at rest. Spectral properties of the corresponding semigroup generators were investigated in [Bea76], whereas local energy decay results along with a scattering theory for a related exterior problem were given in [Bea77]. Especially relevant to our study are the findings of Littman and Liu in [LL98]. For the locally reacting acoustic model posed in an infinite strip, they proved that the energy of solutions *cannot* decay uniformly and established polynomial decay rates for smoother solutions. Their approach relies on a careful examination of the spectrum of the damped generator combined with a suitable spectral decomposition of solutions. Still in the locally reacting case (that is, without the Laplace–Beltrami operator in the boundary dynamics) but for bounded domains, polynomial stabilization rates were established in [MnRQ03, AN15], either under *a priori* knowledge of a resolvent estimate for the static Neumann feedback problem, or under differential multiplier (“Lions”) geometric conditions; see also [Tou13, NPS25]. In contrast, when the damping acts in the interior *and* on the boundary, uniform stabilization is possible: in the locally reacting case, see for instance [GLX18] (damping region covering the boundary) or [CDCFV20] (geometric control condition); in the “non-locally reacting” case, see also [VFL16] (differential multiplier condition).

In the one-dimensional setting, early investigations of dynamic boundary conditions mostly concern Euler–Bernoulli beam models coupled with rigid body ordinary differential equations [LM88a, LM88b], for instance when the beam is clamped at one end and attached to a tip mass at the other end [Rao95, CM98b]; see also [ZW14, FP25] for related later works. Similarly as for acoustic models and while mere *asymptotic* stability is preserved, it was observed that the addition of such boundary dynamics to an otherwise uniformly exponentially stable system may result in the existence of arbitrary slow finite-energy solutions. On the other hand, one-dimensional wave equations with dynamic boundary conditions have been studied in different contexts, including control of gantry crane systems [CM98a, dC00] and reduction of drilling torsional vibrations [RBPC⁺18, VFP22]; see also [CNR24], the references therein, and the recent work [DCNR25] which has the particularity of considering stability in L^p -based state spaces.

Motivated by diffusion processes, Wentzell introduced in [Ven59] (non-dynamic) boundary conditions of the form $-\Delta_\Gamma u +$ “lower-order” terms $= -\partial_\nu u$. These also arise as approximations in transmission problems involving a thin layer around the domain, in the small thinness limit; see for instance [BL96] and more specifically [Buf17, Appendix A]. Elliptic and parabolic problems with such boundary conditions were investigated in [FGGR02, FGG⁺10, BNDHV10] to cite only a few; see also the survey paper [AN00]. As for hyperbolic problems, there has been abundant research on the wave equation with static (i.e., without second-order *time* derivatives) or dynamic (such as (1.1b)) Wentzell boundary conditions. It was shown in [Hem00] by means of spectral analysis that the wave equation posed in a square with damped, static Wentzell boundary condition on two adjacent sides (and homogeneous Dirichlet boundary condition elsewhere) is *not* uniformly stable on its natural energy space. Most interestingly, here the lack of uniform stability is not related to a violation of the geometric control condition, nor does it come from boundary dynamics. A similar configuration was considered in [NL10], with the difference that only one side of the square is subject to the Wentzell boundary condition: there, it was proved that the energy of classical solutions decays like $1/t$, although in that case it is not clear what conclusions to draw since geometric control fails. Localized interior damping, however, can achieve uniform stabilization of the model with static Wentzell boundary condition even in the absence of boundary damping; see [CCFT09, CLT12], with the caveat that both references are restricted to geometries where multiplier techniques are effective. Concerning waves with dynamic Wentzell boundary as in (1.1), the aforementioned negative results constitute strong evidence that one cannot expect uniform stability via damping acting on the boundary only, which is confirmed by our Theorem 1.3.

In [Buf17], Buffe considered the wave equation with both static and dynamic Wentzell boundary condition and *localized* damping on the boundary; that is, the damping is of the form $b(x)\partial_t u$, the control function b being nonnegative and sufficiently smooth. When the damping region (i.e., the support of b) is nontrivial, he proved by means of Carleman estimates that the energy of classical solutions decays like $1/\log(2+t)^2$. This result generalizes to Wentzell boundary conditions the *a priori*, “worst-case scenario” logarithmic decay rates of [Leb96, LR97], which were established for interior damping and Neumann boundary damping; see also the logarithmic local energy decay rate of [Bur98] for exterior problems. Our setup is slightly different since we consider mixed boundary conditions and our damping function is constant, still the result of [Buf17] is applicable when $\Gamma_1 = \emptyset$ and yields logarithmic energy decay for classical solutions to (1.1).

In the present paper we complement Buffe’s analysis of the problem with hyperbolic boundary condition by exhibiting the best possible energy decay rate in general domains under additional dynamical hypotheses, namely the geometric control condition. To arrive at these results there are a number of

difficulties that differentiate our work from the literature discussed above and warrant new technical developments, some of which we highlight here.

- We tackle the stabilization problem in the presence of both time and tangential second-order derivatives on the boundary, which is arguably the most difficult case. To capture the coupling between interior and boundary dynamics, we need precise normal and tangential trace estimates, which we will establish by means of microlocal techniques (Section 3.2.2).
- In order to exploit the geometric control condition, we implement a special decoupling argument (Section 3.2.3) which makes it possible to use, as a black box, Bardos, Lebeau and Rauch’s boundary observability estimate for waves with homogeneous Dirichlet data [BLR92]. In particular we avoid a delicate propagation analysis for waves with second-order boundary conditions.
- Our investigation of traces on the dynamic boundary hints at the presence of a stability bottleneck in the microlocal regime where tangential derivatives are negligible. This motivates, in the special case of the disk, a spectral investigation of *focusing modes*, which exhibit high-frequency oscillations in the radial direction, in order to assess the sharpness of our decay rate (Section 3.3).

We should also point out that the energy decay rate $1/t$ for (1.1) was claimed in the recent work [LLX23] but only under *ad-hoc* differential multiplier conditions that severely restricts the geometry of the domain, even when the whole boundary is damped; furthermore, there seem to be gaps in their proof, namely the use of incorrect trace inequalities for Sobolev spaces of half-integer order.

Let us conclude this section with some additional references that are farther away from the topics of stability and stabilization, but are nonetheless relevant to our work. The recent work [BDEM22], which we discuss in more depth in Section 4 below, deals with observability of waves posed in an two-dimensional annulus, where the inner boundary is subject to a hyperbolic boundary condition and the outer boundary is left at rest. In [Vit17], an extensive study of well-posedness is carried out for a nonlinear model generalizing (1.1); there, nonlinear (but *subcritical*) potential terms are considered, in addition to nonlinear, monotone damping. Related questions were also explored in [GSH12]. In [GL14], the authors investigate smoothing properties of waves with a hyperbolic boundary condition and *strong* (Kelvin–Voigt) interior or boundary damping. They show that the resulting semigroup is analytic when the strong damping acts in the interior and the boundary, and that it is of Gevrey class (which is a weaker property) if only the interior is damped. Finally, a physical derivation for waves with dynamic boundary conditions (without tangential terms) can be found in [Gol06].

2 Preliminaries and operator model

2.1 The Laplace–Beltrami operator on the boundary

In this preliminary section, we recall for the reader’s convenience some basic elements of differential geometry and give the proper definition of the Laplace–Beltrami operator Δ_Γ . We furthermore set up, near Γ_0 , *normal geodesic coordinates* that will be useful in the microlocal analysis of Section 3.2.2. Our notation and presentation borrow from [LRLR22a, LRLR22b], which we refer the reader to for additional details.

2.1.1 Elements of differential geometry

The boundary Γ of Ω is a smooth $(d-1)$ -dimensional submanifold of the Euclidean space \mathbb{R}^d . Given $x \in \Gamma$ and a *local chart* (O, κ) around x , the tangent space $T_x\Gamma$ is the set of linear maps $v : \mathcal{C}^\infty(\Gamma) \rightarrow \mathbb{R}$ of the form

$$v(f) = \sum_{i=1}^{d-1} a^i \frac{\partial(f \circ \kappa^{-1})}{\partial y_i}, \quad f \in \mathcal{C}^\infty(\Gamma), \quad (a^1, \dots, a^{d-1}) \in \mathbb{R}^{d-1}. \quad (2.1)$$

Each $T_x\Gamma$ is a $(d-1)$ -dimensional vector space that does not depend on the particular choice of local chart around x . Let $n \in \mathbb{N}$. We shall identify each $T_x\mathbb{R}^n$ with \mathbb{R}^n . Given a smooth function $F = (F_1, \dots, F_n)$ from Γ to \mathbb{R}^n , the tangent map (or differential) of F at $x \in \Gamma$ is the linear map $T_x F : T_x\Gamma \rightarrow \mathbb{R}^n$ defined by

$$T_x F(v) = (v(F_1), \dots, v(F_n)), \quad v \in T_x\Gamma. \quad (2.2)$$

Note that for $f \in \mathcal{C}^\infty(\Gamma)$, $T_x f(v) = v(f)$. Since Γ is a submanifold of \mathbb{R}^d , the tangent map $T_x i$ of the inclusion $i : \Gamma \rightarrow \mathbb{R}^d$ is injective for $x \in \Gamma$, and we may canonically identify $T_x\Gamma$ with its image under $T_x i$. Then, Γ is equipped with the (smooth) Riemannian *metric* induced by the scalar product of \mathbb{R}^d :

$$g_x(u, v) \triangleq T_x i(v) \cdot T_x i(w), \quad v, w \in T_x\Gamma, \quad x \in \Gamma. \quad (2.3)$$

Let $x \in \Gamma$ and $C = (O, \kappa)$ be a local chart around x . We write $y = \kappa(x)$. Since g_x is a scalar product on $T_x\Gamma$ and using the basis given by (2.1), there exists a unique symmetric positive definite matrix

$(g_{ij}^C(y))_{1 \leq i, j \leq d-1}$ such that

$$g_x(v, w) = \sum_{i, j=1}^{d-1} g_{ij}^C(y) v^i w^j, \quad v, w \in T_x \Gamma, \quad (2.4)$$

where w^i and v^i , $i = 1, \dots, d-1$, are the coefficients of w and v in (2.1), respectively. The components $y \in \kappa(O) \mapsto g_{ij}^C(y) \in \mathbb{R}$ are all smooth. The Riemannian *gradient* $\nabla_g f(x)$ at $x \in \Gamma$ of a function $f \in C^\infty(\Gamma)$ is the unique element of $T_x \Gamma$ satisfying

$$g_x(\nabla_g f(x), v) = v(f), \quad v \in T_x \Gamma. \quad (2.5)$$

The Riemannian gradient $\nabla_g f$ defines a smooth vector field on Γ . If we choose a local chart $C = (O, \kappa)$ around $x = \kappa^{-1}(y) \in \Gamma$, then the coordinates of $\nabla_g f(x)$ in the associated basis (2.1) are

$$(\nabla_g f)^{C, i}(y) = \sum_{j=1}^{d-1} g^{C, ij}(y) \frac{\partial(f \circ \kappa^{-1})}{\partial y_j} \Big|_y, \quad (2.6)$$

where $(g^{C, ij}(y))_{1 \leq i, j \leq d-1}$ is the inverse of the matrix $(g_{ij}^C(y))_{1 \leq i, j \leq d-1}$ introduced above.

Since Γ is the boundary of the smooth bounded domain Ω , Γ can be oriented by means of an outward-pointing vector field [LRLR22b, Section 16.1.3], and there exists a canonical Riemannian *volume form* dV_g ; see, e.g., [LRLR22b, Section 17.1.4]. For our purpose, it is enough to define the *integral* of functions $f \in \mathcal{C}(\Gamma)$ with respect to dV_g . The manifold Γ is compact and thus we may find local charts $C_n = (O_n, \kappa_n)$, $n = 1, \dots, N$ such that the O_n constitute an open covering of Γ . Let $\theta_1, \dots, \theta_N$ form a *partition of unity* subordinate to that covering (see, e.g., [LRLR22b, Section 15.2.3], [Bre11, Lemma 9.3], and also the next subsection). Then,

$$\int_\Gamma f dV_g = \sum_{n=1}^N \int_{\kappa_n(O_n)} \theta_n(\kappa_n^{-1}(y)) f(\kappa_n^{-1}(y)) \det(g^{C_n}(y))^{1/2} dy, \quad f \in \mathcal{C}(\Gamma), \quad (2.7)$$

where dy denotes the Lebesgue measure on \mathbb{R}^{d-1} . The integral in (2.7) does not depend on the choice of local charts. With that in hand, we define the *divergence* $\operatorname{div}_g u$ of a smooth vector field u on Γ as the unique element in $C^\infty(\Gamma)$ satisfying

$$\int_\Gamma \varphi \operatorname{div}_g u dV_g = - \int_\Gamma g(\nabla_g \varphi, u) dV_g, \quad \varphi \in C^\infty(\Gamma). \quad (2.8)$$

Finally, the Laplace–Beltrami operator Δ_g on Γ is defined by

$$\Delta_g f = \operatorname{div}_g \nabla_g f, \quad f \in C^\infty(\Gamma). \quad (2.9)$$

Given a local chart $C = (O, \kappa)$ around $x \in \Gamma$ with $y = \kappa(x)$, we have

$$\Delta_g u(x) = \det(g^C(y))^{-1/2} \sum_{i, j=1}^{d-1} \frac{\partial}{\partial y_i} \left(\det(g^C(y))^{1/2} g^{C, ij}(y) \frac{\partial(u \circ \kappa^{-1})}{\partial y_j} \right) \Big|_y. \quad (2.10)$$

Here, the differential operators ∇_g , div_g and Δ_g act on smooth functions and vector fields.

By Riesz' representation theorem, (2.7) defines a unique Radon measure on Γ , the Riemannian measure. In fact, as Γ is the boundary of Ω , this is precisely the *surface measure* $d\sigma$ induced by the Lebesgue measure; see also [LRLR22b, Proposition 17.2]. The Riemannian measure gives a natural meaning to L^p -spaces on Γ , and in particular the (complex³) Hilbert space $L^2(\Gamma)$ is the space of all (equivalence classes of) measurable functions $f : \Gamma \rightarrow \mathbb{C}$ such that

$$\int_\Gamma |f|^2 dV_g = \int_\Gamma |f|^2 d\sigma < +\infty; \quad (2.11)$$

equivalently, it is the completion of the space of smooth functions on Γ with respect to the norm defined by (2.11). Similarly, the Sobolev space $H^1(\Gamma)$ is the completion of smooth functions with respect to

$$\|f\|_{H^1(\Gamma)}^2 \triangleq \int_\Gamma g(\nabla_g f, \overline{\nabla_g f}) dV_g. \quad (2.12)$$

At each $x \in \Gamma$, the map $T_x i$ embeds the tangent space $T_x \Gamma$ into \mathbb{R}^d , and by letting $\nabla_\Gamma f(x) \triangleq T_x i(\nabla_g f(x))$ we may simply rewrite (2.12) as

$$\|f\|_{H^1(\Gamma)}^2 = \int_\Gamma \nabla_\Gamma f \cdot \overline{\nabla_\Gamma f} d\sigma = \int_\Gamma |\nabla_\Gamma f|^2 d\sigma. \quad (2.13)$$

³We consider suitable complexifications of the previous objects.

In the context of (1.1), it is particularly important to note that, if f is the trace on Γ of a sufficiently smooth function u defined in Ω , then $\nabla_\Gamma f$ coincides with the *tangential gradient* (or surface gradient) of u : in other words, if $x \in \Gamma$ and $\nu(x)$ is the unit outward normal vector at x , then $\nabla_\Gamma f(x)$ is tangent to Γ at x and

$$\nabla u(x) = \nabla_\Gamma f(x) + (\nabla u(x) \cdot \nu(x))\nu(x). \quad (2.14)$$

With that in mind and when there is no risk of confusion, we will directly write $\nabla_\Gamma u$ instead of $\nabla_\Gamma f$ or $\nabla_\Gamma(u|_\Gamma)$. We shall also use fractional Sobolev spaces $H^s(\Gamma)$, $s \in \mathbb{R}$, which are modeled after $H^s(\mathbb{R}^{d-1})$ via local charts; see for instance [Tay96, Chapter 4, Section 3]. Since Γ is a manifold without boundary, note that $H^s(\Gamma)$ and $H^{-s}(\Gamma)$ are dual. Similarly as in the Euclidian case, differential operators on Γ have natural extensions to Sobolev spaces and even *distributions*; see, e.g., [LRLR22b, Sections 16.3.3 and 18.2.1]. In particular, Δ_g , which we will now write Δ_Γ , extends to a bounded linear operator from $H^2(\Gamma)$ into $L^2(\Gamma)$ and from $H^1(\Gamma)$ into $H^{-1}(\Gamma)$; see also [LRLR22b, Section 17.3].

2.1.2 Local setting near the boundary

Throughout the manuscript we will use the following notation:

$$D_{y_i} = -i\partial_{y_i}, \quad i = 1, \dots, d. \quad (2.15)$$

Let $x^0 \in \Gamma$. By [LRLR22a, Theorem 9.7] there exists a local chart $C = (O, \kappa)$ (in \mathbb{R}^d) around x^0 such that, writing $\kappa = (\kappa_1, \dots, \kappa_d)$, $y = \kappa(x) = (\kappa_1(x), \dots, \kappa_d(x)) = (y_1, \dots, y_d)$:

1. $\kappa(x^0) = 0$ and

$$\kappa(\Omega \cap O) = \{y \in \kappa(O) : y_d \geq 0\}, \quad \kappa(\Gamma \cap O) = \{y \in \kappa(O) : y_d = 0\}; \quad (2.16)$$

2. There exist smooth coefficients $a_{ij} : \kappa(O) \rightarrow \mathbb{R}$, $1 \leq i, j \leq d-1$, such that

$$-\Delta^C \triangleq -\Delta(\kappa^{-1})^* = D_{y_d}^2 + \sum_{i,j=1}^{d-1} a_{ij}(y)D_{y_i}D_{y_j} \quad \text{in } \kappa(O) \quad (2.17)$$

(with pullback notation, $(\kappa^{-1})^* f = f \circ \kappa^{-1}$);

3. There exists a constant $c > 0$ such that

$$\sum_{i,j=1}^{d-1} a_{ij}(y)\xi_i\xi_j \geq c|\xi_\tau|^2, \quad \xi_\tau = (\xi_1, \dots, \xi_{d-1}) \in \mathbb{R}^{d-1}, \quad y \in \kappa(O). \quad (2.18)$$

Such local coordinates are referred to as *normal geodesic coordinates*. In fact, it is possible to choose such a local chart in a way that guarantees that the pullback of the normal derivative $\partial_\nu = \nabla \cdot \nu$ coincides, on the boundary, with the derivative with respect to the normal variable y_d :

$$\partial_\nu^C \triangleq \partial_\nu(\kappa^{-1})^* = \partial_{y_d} = iD_{y_d}; \quad (2.19)$$

see the proof of [LRLR22a, Theorem 9.7] and in particular [LRLR22a, Equation (9.4.5)].

Since Γ_0 is compact, there exists an open covering of Γ_0 made of such charts $C_n = (O_n, \kappa_n)$, $n = 1, \dots, N$. We use a partition of unity: by, e.g., [Bre11, Lemma 9.3] there exist smooth functions $\theta_n : \mathbb{R}^d \rightarrow [0, 1]$, $n = 0, \dots, N$ such that:

- 1.

$$\sum_{n=0}^N \theta_n(x) = 1, \quad x \in \mathbb{R}^d; \quad (2.20)$$

2. For $n = 1, \dots, N$, $\theta_n|_{O_n} \in \mathcal{D}(O_n)$;
3. The support of θ_0 is contained in $\mathbb{R}^d \setminus \Gamma_0$ and $\theta_0|_\Omega \in \mathcal{D}(\Omega)$.

Therefore, for each $n = 1, \dots, N$, there exist smooth coefficients $a_{ij}^n : \kappa_n(O_n) \rightarrow \mathbb{R}$, $1 \leq i, j \leq d-1$ such that $-\Delta^{C_n}$ has the form (2.17). Now, pick $\theta'_n \in \mathcal{D}(O_n)$ such that $\theta'_n = 1$ near the support of θ_n , and let

$$\tilde{a}_{ij}^n \triangleq (\theta'_n \circ \kappa_n^{-1})a_{ij}^n, \quad 1 \leq i, j \leq d-1; \quad (2.21)$$

then the \tilde{a}_{ij}^n trivially extend to smooth coefficients in \mathbb{R}^d that are compactly supported and coincide with a_{ij}^n near $\kappa_n(\text{supp}(\theta_n))$, the image of the support of θ_n under κ_n . The differential operator on \mathbb{R}^d defined by (2.17) with a_{ij}^n replaced by \tilde{a}_{ij}^n is written $-\tilde{\Delta}^{C_n}$. On the other hand, by restricting each κ_n to $O_n \cap \Gamma_0$ we see that C_n also defines a local chart in Γ_0 as a $(d-1)$ -dimensional submanifold of \mathbb{R}^d , and (2.10) provides smooth coefficients $b_{ij}^n : \kappa_n(\Gamma_0 \cap O_n) \rightarrow \mathbb{R}$ such that

$$-\Delta_\Gamma^{C_n} = \sum_{i,j=1}^{d-1} b_{ij}^n(y_\tau)D_{y_i}D_{y_j} \quad \text{in } \kappa_n(\Gamma_0 \cap O_n), \quad (2.22)$$

with, for some $c > 0$, the ellipticity condition

$$\sum_{i,j}^{d-1} b_{ij}^n(y_\tau) \xi_i \xi_j \geq c |\xi_\tau|^2, \quad y_\tau \in \kappa(O_n \cap \Gamma_0), \quad \xi_\tau = (\xi_1, \dots, \xi_{d-1}) \in \mathbb{R}^{d-1}. \quad (2.23)$$

Exactly as before, using the cutoffs θ'_n we define localized coefficients \tilde{b}_{ij}^n and differential operators $-\tilde{\Delta}_{\Gamma}^{C_n}$ on \mathbb{R}^{d-1} . Since Γ_0 is covered by a finite number of those charts C_n , we may choose an ellipticity constant $c > 0$ in (2.18) and (2.23) that does not depend on n .

2.2 Well-posedness and asymptotic stability

In this section, we establish well-posedness of (1.1) and asymptotic convergence to zero of the energy of solutions. What we present here is either standard or relatively straightforward, and there might be some overlap with, e.g., [Buf17, Vit17]. Still, we go over most details for the sake of clarity and self-containedness.

2.2.1 Functional setting and semigroup generation

By multiplying the wave equation (1.1a) by a smooth function φ of the space variable and using Green's formula along with the boundary conditions (1.1b) and (1.1c), we (formally) derive a weak formulation of (1.1): for all sufficiently smooth φ with $\varphi|_{\Gamma_1} = 0$, solutions u to (1.1) satisfy

$$\frac{d^2}{dt^2} \left(\int_{\Omega} u \varphi \, dx + \int_{\Gamma_0} u \varphi \, d\sigma \right) + \frac{d}{dt} \int_{\Gamma_0} u \varphi \, d\sigma + \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx + \int_{\Gamma_0} \nabla_{\Gamma} u \cdot \nabla_{\Gamma} \varphi \, d\sigma = 0. \quad (2.24)$$

Equation (2.24) suggests working with the following spaces:

$$H \triangleq L^2(\Omega) \times L^2(\Gamma_0), \quad (2.25a)$$

$$V \triangleq \{(u, w) \in H^1(\Omega) \times H^1(\Gamma_0) : u|_{\Gamma_0} = w\} \simeq \{u \in H^1(\Omega) : u|_{\Gamma_0} \in H^1(\Gamma_0)\}. \quad (2.25b)$$

Both spaces are Hilbert spaces if equipped with their natural product structures, but whenever $\Gamma_1 \neq \emptyset$ we equip V with the (equivalent, due to a standard Poincaré-type inequality for functions vanishing on Γ_1) norm $\|\cdot\|_V \triangleq \int_{\Omega} |\nabla \cdot|^2 \, dx + \int_{\Gamma_0} |\nabla_{\Gamma} \cdot|^2 \, d\sigma$. The identification in (2.25b) is straightforward, and we shall use it without further comment. Furthermore, V as a subspace of H is dense, the embedding of V into H is clearly continuous. As a result, we may consider a standard pivot duality

$$V \hookrightarrow H \hookrightarrow V^*, \quad (2.26)$$

where V^* stands for the topological antidual of V endowed with the dual norm, and H is identified with its antidual by means of Riesz' theorem. Recall also that on smooth compact manifolds the Rellich–Kondrachov theorem provides compact Sobolev embeddings [Tay11, Chapter 3, Proposition 4.4], and thus the embeddings in (2.26) are in fact compact. Next, we define a duality map $A \in \mathcal{L}(V, V^*)$ by

$$\langle Au, v \rangle_{V^*, V} \triangleq \int_{\Omega} \nabla u \cdot \overline{\nabla v} \, dx + \int_{\Gamma_0} \nabla_{\Gamma} u \cdot \overline{\nabla_{\Gamma} v} \, d\sigma, \quad u, v \in V, \quad (2.27)$$

as well as a control operator $B \in \mathcal{L}(L^2(\Gamma_0), H)$ by $Be = (0, e)$ for all $e \in L^2(\Gamma_0)$. Then, (1.1) takes the form of the abstract evolution equation

$$u'' + BB^*u' + Au = 0, \quad (2.28)$$

where the adjoint $B^* \in \mathcal{L}(H, L^2(\Gamma_0))$ of B is given by $B^*(v, w) = w$ for $(v, w) \in H$; in particular, $B^*v = v|_{\Gamma_0}$ whenever $v \in V$. Now, in view of (2.26) we may see A as an unbounded operator on H by letting

$$\text{dom}(A) \triangleq \{u \in V : Au \in H\}, \quad (2.29)$$

so that $\langle Au, v \rangle_{V^*, V} = \langle Au, v \rangle_H$ for all $u \in \text{dom}(A)$ and $v \in V$. This H -realization of A is a nonnegative (strictly positive if $\Gamma_1 \neq \emptyset$) self-adjoint operator with compact resolvent, and here we are able to give an explicit characterization of its domain.

Proposition 2.1 (Characterization of the domain). *We have*

$$\text{dom}(A) = \{u \in H^2(\Omega) : u|_{\Gamma_0} \in H^2(\Gamma_0), u|_{\Gamma_1} = 0\}, \quad (2.30a)$$

$$Au = (-\Delta u, -\Delta_{\Gamma} u + \partial_{\nu} u), \quad u \in \text{dom}(A), \quad (2.30b)$$

and the norm $\|\cdot\|_{H^2(\Omega)} + \|\cdot\|_{H^2(\Gamma_0)}$ is equivalent to the graph norm of A .

Proof. Let $u \in V$. Then $u \in \text{dom}(A)$ if and only if there exists $(f, e) \in H$ such that

$$\int_{\Omega} \nabla u \cdot \overline{\nabla v} dx + \int_{\Gamma_0} \nabla_{\Gamma} u \cdot \overline{\nabla_{\Gamma} v} d\sigma = \int_{\Omega} f \overline{v} dx + \int_{\Gamma_0} e \overline{v} d\sigma, \quad v \in V. \quad (2.31)$$

If u belongs to the set at the right-hand side of Proposition 2.1 then we see, using Green's formula, that (2.31) holds with $(f, e) = (-\Delta u, -\Delta_{\Gamma} u + \partial_{\nu} u) \in H$. Conversely, if $u \in \text{dom}(A)$ then (2.31) with v taken in $\mathcal{D}(\Omega)$ shows that $-\Delta u = f \in L^2(\Omega)$, where $-\Delta u$ was defined *a priori* in $\mathcal{D}(\Omega)$. Since $u \in H^1(\Omega)$ and $\Delta u \in L^2(\Omega)$, recall that $\partial_{\nu} u$ is (continuously) well-defined in $H^{-1/2}(\Gamma)$ by

$$\langle \partial_{\nu} u, \psi \rangle = \int_{\Omega} \nabla u \cdot \nabla \varphi + \Delta u \varphi dx = \int_{\Omega} \nabla u \cdot \nabla \varphi - f \varphi dx, \quad \psi \in H^{1/2}(\Gamma), \quad (2.32)$$

where φ is any element of $H^1(\Omega)$ such that $\varphi|_{\Gamma} = \psi$ (such a function always exists), and the brackets denote the bilinear (as opposed to sesquilinear in, e.g., (2.27)) duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. Plugging (2.32) into (2.31) with $\varphi = \overline{v}$, we then see that

$$\int_{\Gamma_0} \nabla_{\Gamma} u \cdot \overline{\nabla_{\Gamma} v} d\sigma = -\langle \partial_{\nu} u, \overline{v} \rangle + \int_{\Gamma_0} e \overline{v} d\sigma, \quad v \in V. \quad (2.33)$$

In the case $\Gamma_1 = \emptyset$ we infer from (2.33) that $-\Delta_{\Gamma} u = -\partial_{\nu} u + e$ in the sense of distributions on $\Gamma = \Gamma_0$; indeed, V contains (say) $H^2(\Omega)$, so that the image of V under the trace map contains the entirety of $H^{3/2}(\Gamma)$, including $\mathcal{D}(\Gamma)$. If $\Gamma_1 \neq \emptyset$, since anyway, $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$ we can find a function $\rho \in C^{\infty}(\overline{\Omega})$ such that $\rho = 0$ near Γ_1 and $\rho = 1$ near Γ_0 . Then, $\{\rho v, v \in H^2(\Omega)\} \subset V$ and similarly we arrive at the conclusion that, in the sense of distributions, $-\Delta_{\Gamma} u = -\partial_{\nu} u + e$ on Γ_0 . In any case, elliptic regularity for the Laplace–Beltrami operator on the boundaryless manifold Γ_0 leads to $u|_{\Gamma_0} \in H^{3/2}(\Gamma_0)$; use for instance [Tay11, Chapter 5, Theorem 1.3] and interpolation. Because $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$ and $u = 0$ on Γ_1 , it follows in fact that $u|_{\Gamma} \in H^{3/2}(\Gamma)$. Now, recalling that $\Delta u \in L^2(\Omega)$, by elliptic regularity for the Laplacian with non-homogeneous boundary data we deduce that $u \in H^2(\Omega)$; see, e.g., [LM68, Preface or Chapter 2, Theorem 5.2]. Then trace theory yields $\partial_{\nu} u \in H^{1/2}(\Gamma) \subset L^2(\Gamma)$ and using again elliptic regularity for Δ_{Γ} leads to $u|_{\Gamma_0} \in H^2(\Gamma_0)$, which completes the proof of the inclusion from left to right in (2.30a). Equation (2.30b) follows from an application of Green's formula in (2.31). We omit the detailed proof of the last statement of Proposition 2.1: simply keep track of the norms in the above arguments. \square

Note that the special second-order structure (2.28) is now standard in the literature and in particular corresponds to the setting of [AL14, Section 3]; see also, e.g., [AT01, AN14, CPS⁺23]. As we know, the natural phase space for (1.1) is

$$\mathcal{H} \triangleq V \times H \quad (2.34)$$

equipped with its product Hilbertian structure. We define an unbounded operator \mathcal{A} on \mathcal{H} by

$$\mathcal{A} \triangleq \begin{pmatrix} 0 & -1 \\ A & BB^* \end{pmatrix}, \quad \text{dom}(\mathcal{A}) \triangleq \text{dom}(A) \times V. \quad (2.35)$$

With Proposition 2.1 we immediately see that

$$\text{dom}(\mathcal{A}) = \{(u_0, u_1) \in H^2(\Omega) \times H^1(\Omega) : (u_0, u_1)|_{\Gamma_0} \in H^2(\Gamma_0) \times H^1(\Gamma_0), (u_0, u_1)|_{\Gamma_1} = 0\} \quad (2.36)$$

Well-posedness of (1.1), through its abstract formulation (2.28), is guaranteed by the following proposition.

Proposition 2.2 (Semigroup generation). *The operator $-\mathcal{A}$ is the infinitesimal generator of a strongly continuous semigroup $\{e^{-t\mathcal{A}}\}_{t \geq 0}$ of linear bounded operators on \mathcal{H} . If $\Gamma_1 \neq \emptyset$ then $\{e^{-t\mathcal{A}}\}_{t \geq 0}$ is in fact a contraction semigroup.*

Proof. Up to a bounded perturbation if $\Gamma_1 = \emptyset$, this is a standard application of the Lumer–Phillips theorem on contraction semigroups: use the special structure of (2.28) along with, for instance, [ENB00, Chapter II, Corollary 3.20 and Proposition 3.23] and [ENB00, Chapter III, Theorem 1.3]. \square

Remark 2.3. In fact, $\{e^{-t\mathcal{A}}\}_{t \geq 0}$ is a strongly continuous group.

Therefore, the unique *weak* solution u (in the sense of (2.24) and (2.28)) to (1.1) with initial data $(u_0, u_1) \in \mathcal{H}$ is given by $(u(\cdot, t), \partial_t u(\cdot, t)) = e^{-t\mathcal{A}}(u_0, u_1)$ for all $t \geq 0$, so that

$$u \in \mathcal{C}([0, +\infty), V) \cap \mathcal{C}^1([0, +\infty), H). \quad (2.37)$$

In addition, when $(u_0, u_1) \in \text{dom}(\mathcal{A}) = \text{dom}(A) \times V$, then $u \in \mathcal{C}([0, +\infty), \text{dom}(A)) \cap \mathcal{C}^1([0, +\infty), V)$ is a *classical* solution in the semigroup sense and satisfies (1.1) in a pointwise a.e. fashion. We conclude this section with an easy but important fact concerning the operator \mathcal{A} .

Lemma 2.4. *The operator \mathcal{A} has compact resolvent and in particular its spectrum $\sigma(\mathcal{A})$ is made entirely of isolated eigenvalues with finite multiplicity. Also, $\ker(\mathcal{A}) = \ker(A) \times \{0\}$.*

Proof. The first statement readily follows from compactness of the resolvent of A and the Riesz–Schauder theory for compact operators; see, e.g., [Yos65, Section X.5, Theorems 1 and 2] or [ENB00, Chapter IV, Corollary 1.19]. The second statement is clear. \square

2.2.2 Energy and stability

Clearly, the energy E defined in (1.2) as well as the energy identity (1.3) make sense for all weak solutions to (1.1). In the case $\Gamma_1 \neq \emptyset$, recalling our normalization of V , for all solutions u to (1.1) with initial data $(u_0, u_1) \in \mathcal{H}$ we have

$$E(u, t) = \frac{1}{2} \|(u(\cdot, t), \partial_t u(\cdot, t))\|_{\mathcal{H}}^2 = \frac{1}{2} \|e^{-t\mathcal{A}}(u_0, u_1)\|_{\mathcal{H}}^2, \quad t \geq 0. \quad (2.38)$$

On the other hand, if $\Gamma_1 = \emptyset$ then the energy only defines a seminorm on \mathcal{H} . Following [AL14, Section 4] we consider a Riesz projector $\Pi \in \mathcal{L}(\mathcal{H})$ defined by

$$\Pi = \frac{1}{2i\pi} \int_{\gamma} (\mathcal{A} + z)^{-1} dz, \quad (2.39)$$

where γ is a positively oriented circle in \mathbb{C} centered on 0 and of radius small enough so that by Lemma 2.4 the intersection of its interior with the spectrum of \mathcal{A} is either reduced to $\{0\}$ if Γ_1 is empty, or else empty, in which case $\Pi = 0$. Then, $\Pi\mathcal{H} = \ker(\mathcal{A})$, $\dot{\mathcal{H}} \triangleq (1 - \Pi)\mathcal{H}$ equipped with the norm

$$\|(u_0, u_1)\|_{\dot{\mathcal{H}}}^2 \triangleq \langle Au_0, u_0 \rangle_{V^*, V} + \|u_1\|_H^2, \quad (u_0, u_1) \in \dot{\mathcal{H}}, \quad (2.40)$$

is a Hilbert space and, as a closed subspace of \mathcal{H} , is in topological direct sum with $\ker(\mathcal{A})$. Furthermore, the operator $\dot{\mathcal{A}} \triangleq \mathcal{A}|_{\dot{\mathcal{H}}}$ with domain $\text{dom}(\dot{\mathcal{A}}) \triangleq \text{dom}(\mathcal{A}) \cap \dot{\mathcal{H}}$ generates a contraction semigroup $\{e^{-t\dot{\mathcal{A}}}\}_{t \geq 0}$ on $\dot{\mathcal{H}}$ and

$$e^{-t\dot{\mathcal{A}}}(u_0, u_1) = e^{-t\mathcal{A}}(1 - \Pi)(u_0, u_1) + \Pi(u_0, u_1), \quad t \geq 0, \quad (u_0, u_1) \in \mathcal{H}; \quad (2.41)$$

see [AL14, Lemma 4.3]. Note that $\sigma(\dot{\mathcal{A}}) = \sigma(\mathcal{A}) \setminus \{0\}$. As a consequence of (2.40) and (2.41), for all solutions u to (1.1) with initial data $(u_0, u_1) \in \mathcal{H}$,

$$E(u, t) = \frac{1}{2} \|e^{-t\dot{\mathcal{A}}}(1 - \Pi)(u_0, u_1)\|_{\dot{\mathcal{H}}}^2 = \frac{1}{2} \|(1 - \Pi)e^{-t\mathcal{A}}(u_0, u_1)\|_{\dot{\mathcal{H}}}^2, \quad t \geq 0. \quad (2.42)$$

(In the case $\Gamma_1 \neq \emptyset$, (2.38) and (2.42) are the same.) It is also convenient to define for smoother solutions a higher-order energy E_1 as follows: for (classical) solutions u to (1.1) with initial data $(u_0, u_1) \in \text{dom}(A) \times V$, and for $t \geq 0$,

$$E_1(u, t) \triangleq \frac{1}{2} \int_{\Omega} |\nabla u|^2 + |\Delta u|^2 + |\partial_t u|^2 + |\nabla \partial_t u|^2 dx + \frac{1}{2} \int_{\Gamma_0} |\Delta_{\Gamma} u|^2 + |\partial_t u|^2 + |\nabla_{\Gamma} \partial_t u|^2 d\sigma. \quad (2.43)$$

Note that (2.43) makes sense for such solutions due to Proposition 2.1; in fact, $E_1(u, \cdot)$ is a continuous and bounded function of the time variable. The following lemmas relate E_1 to the graph norm of $\dot{\mathcal{A}}$

Lemma 2.5. *There exists $K > 0$ such that*

$$\frac{1}{2K} \|Au\|_H^2 \leq \left(\int_{\Omega} |\Delta u|^2 + |\nabla u|^2 dx + \int_{\Gamma_0} |\Delta_{\Gamma} u|^2 d\sigma \right) \leq \frac{K}{2} \|Au\|_H^2, \quad u \in \text{dom}(A). \quad (2.44)$$

Proof. Given $u \in \text{dom}(A)$, define \tilde{u} a.e. by

$$\tilde{u} \triangleq \begin{cases} u - \frac{1}{|\Omega|} \int_{\Omega} u dx & \text{if } \Gamma_1 = \emptyset, \\ u & \text{if } \Gamma_1 \neq \emptyset, \end{cases} \quad (2.45)$$

where $|\Omega|$ denotes the Lebesgue measure of Ω ; then, $\tilde{u} \in \text{dom}(A)$. In what follows, $K, K' \dots$ denote positive constants that may change from line to line but are independent of the choice of $u \in \text{dom}(A)$. First, by Proposition 2.1,

$$\|Au\|_H = \|A\tilde{u}\|_H \leq K (\|\tilde{u}\|_{H^2(\Omega)} + \|\tilde{u}\|_{H^2(\Gamma_0)}) \leq K' \|Au\|_H, \quad u \in \text{dom}(A). \quad (2.46)$$

This already gives the right inequality in (2.44). Furthermore, elliptic theory [LM68] yields

$$\|\tilde{u}\|_{H^2(\Omega)} \leq K \left(\|\Delta \tilde{u}\|_{L^2(\Omega)} + \|\tilde{u}\|_{L^2(\Omega)} + \|\tilde{u}\|_{H^{3/2}(\Gamma_0)} \right), \quad u \in \text{dom}(A). \quad (2.47)$$

By the Poincaré–Wirtinger inequality (if $\Gamma_1 = \emptyset$) or the Poincaré-type inequality for functions vanishing on Γ_1 (if $\Gamma_1 \neq \emptyset$),

$$\|\tilde{u}\|_{L^2(\Omega)} \leq K \|\nabla \tilde{u}\|_{L^2(\Omega)^d} = K \|\nabla u\|_{L^2(\Omega)^d}, \quad u \in \text{dom}(A). \quad (2.48)$$

On the other hand, with elliptic theory on Γ_0 [Tay11] and trace continuity we obtain

$$\|\tilde{u}\|_{H^2(\Gamma_0)} \leq K (\|\Delta_{\Gamma} \tilde{u}\|_{L^2(\Gamma_0)} + \|\tilde{u}\|_{L^2(\Gamma_0)}) \leq K \|\Delta_{\Gamma} u\|_{L^2(\Gamma_0)} + K' \|\nabla u\|_{L^2(\Omega)^d}, \quad u \in \text{dom}(A). \quad (2.49)$$

The left part of (2.44) follows at once. \square

Lemma 2.6. *There exists $K > 0$ such that, for all (classical) solutions u to (1.1) with initial data $(u_0, u_1) \in \text{dom}(A) \times V$,*

$$\frac{1}{2K} \|\dot{\mathcal{A}}(1 - \Pi)e^{-tA}(u_0, u_1)\|_{\mathcal{H}}^2 \leq E_1(u, t) \leq \frac{K}{2} \|\dot{\mathcal{A}}(1 - \Pi)e^{-tA}(u_0, u_1)\|_{\mathcal{H}}^2, \quad t \geq 0. \quad (2.50)$$

Proof. Recall that solutions to (1.1) with data in $\text{dom}(A) \times V$ remain in $\text{dom}(A) \times V$; as a result it suffices to prove (2.50) for $t = 0$. For all $(u_0, u_1) \in \text{dom}(A) \times V$, we have

$$\begin{aligned} \frac{1}{2} \|\dot{\mathcal{A}}(1 - \Pi)(u_0, u_1)\|_{\mathcal{H}}^2 &= \frac{1}{2} \|(1 - \Pi)\mathcal{A}(u_0, u_1)\|_{\mathcal{H}}^2 \\ &= \frac{1}{2} \langle Au_1, u_1 \rangle_{V^*, V} + \frac{1}{2} \|Au_0 + BB^*u_1\|_H^2. \end{aligned} \quad (2.51)$$

Since $BB^* \in \mathcal{L}(H)$, we readily deduce from (2.51) that there exists a constant $K > 0$ such that

$$\frac{1}{K} \|\dot{\mathcal{A}}(1 - \Pi)(u_0, u_1)\|_{\mathcal{H}}^2 \leq \langle Au_1, u_1 \rangle_{V^*, V} + \|u_1\|_H^2 + \|Au_0\|_H^2 \leq K \|\dot{\mathcal{A}}(1 - \Pi)(u_0, u_1)\|_{\mathcal{H}}^2 \quad (2.52)$$

for all $(u_0, u_1) \in \text{dom}(A) \times V$. We complete the proof by applying Lemma 2.5 to u_0 . \square

Finally, we introduce a quadratic operator pencil P defined by

$$P(z) \triangleq A + zBB^* + z^2, \quad z \in \mathbb{C}, \quad (2.53)$$

which is related to the Laplace transform of the abstract evolution equation (2.28). For $z \in \mathbb{C}$, $P(z)$ is well-defined as a map in $\mathcal{L}(V, V^*)$ but also as an unbounded operator on H with $\text{dom}(P(z)) = \text{dom}(A)$. We are now ready to state and prove a first stability result.

Proposition 2.7 (Semi-uniform stability). *We have $\sigma(\dot{\mathcal{A}}) \cap i\mathbb{R} = \emptyset$ and*

$$\|e^{-t\dot{\mathcal{A}}}\dot{\mathcal{A}}^{-1}\|_{\mathcal{L}(\dot{\mathcal{H}})} \rightarrow 0, \quad t \rightarrow +\infty. \quad (2.54)$$

Thus, there exists a rate $r : [0, +\infty) \rightarrow [0, +\infty)$ with $r(t) \rightarrow 0$ as $t \rightarrow +\infty$ such that, for (classical) solutions u to (1.1) with initial data $(u_0, u_1) \in \text{dom}(A) \times V$,

$$E(u, t) \leq r(t)E_1(u, 0), \quad t \geq 0. \quad (2.55)$$

Furthermore, for (weak) solutions u to (1.1) with initial data $(u_0, u_1) \in \mathcal{H}$,

$$E(u, t) \rightarrow 0, \quad t \rightarrow +\infty. \quad (2.56)$$

Proof. Recall that the semigroup $\{e^{-t\dot{\mathcal{A}}}\}_{t \geq 0}$ is contractive and in particular uniformly bounded. By virtue of [BD08, Theorem 1], the property that $\sigma(\dot{\mathcal{A}}) \cap i\mathbb{R} = \emptyset$ and (2.54) are actually equivalent, so it suffices to prove the latter. Keeping in mind that the spectrum of $\dot{\mathcal{A}}$ is made of eigenvalues only, it is easy to see that, given $\lambda \in \mathbb{R} \setminus \{0\}$, $i\lambda \in \sigma(\dot{\mathcal{A}})$ if and only if there exists $u \in \text{dom}(A)$, $u \neq 0$, such that $P(i\lambda)u = 0$; see for instance the proof of [AL14, Lemma 4.2]. For such u ,

$$0 = \langle P(i\lambda)u, u \rangle_H = \langle Au, u \rangle_H + i\lambda \|u\|_{L^2(\Gamma_0)}^2 - \lambda^2 \|u\|_H^2, \quad (2.57)$$

which leads to $u = 0$ on Γ_0 after taking the imaginary part (A is self-adjoint). On the other hand, u solves the stationary boundary value problem

$$(-\Delta - \lambda^2)u = 0 \quad \text{in } \Omega, \quad (2.58a)$$

$$(-\Delta_\Gamma + i\lambda - \lambda^2)u = -\partial_\nu u \quad \text{on } \Gamma_0, \quad (2.58b)$$

and (2.58b) reduces to $\partial_\nu u = 0$ on Γ_0 . Therefore, u solves (2.58a), an elliptic equation with analytic coefficients, and has vanishing Cauchy data on Γ_0 ; an application of the John–Holmgren theorem on unique continuation across non-characteristic hypersurfaces (see, e.g., [Joh82, Uniqueness Theorem, Section 5]) finally yields $u = 0$, which proves the desired spectral condition. Now we must turn the operator convergence property (2.54) into energy decay (2.55). For all $(u_0, u_1) \in \text{dom}(A) \times V = \text{dom}(\dot{\mathcal{A}})$, $(1 - \Pi)(u_0, u_1) \in \text{dom}(\dot{\mathcal{A}})$, i.e., $\dot{\mathcal{A}}(1 - \Pi)(u_0, u_1) \in \mathcal{H}$: denoting by u the solution to (1.1) originating from (u_0, u_1) , with (2.42) we have

$$E(u, t) = \frac{1}{2} \|e^{-t\dot{\mathcal{A}}}\dot{\mathcal{A}}^{-1}\dot{\mathcal{A}}(1 - \Pi)(u_0, u_1)\|_{\mathcal{H}}^2 \leq \frac{1}{2} \|e^{-t\dot{\mathcal{A}}}\dot{\mathcal{A}}^{-1}\|_{\mathcal{L}(\dot{\mathcal{H}})}^2 \|\dot{\mathcal{A}}(1 - \Pi)(u_0, u_1)\|_{\mathcal{H}}^2, \quad t \geq 0. \quad (2.59)$$

By Lemma 2.6,

$$E(u, t) \leq K \|e^{-t\dot{\mathcal{A}}}\dot{\mathcal{A}}^{-1}\|_{\mathcal{L}(\dot{\mathcal{H}})}^2 E_1(u, 0), \quad t \geq 0, \quad (2.60)$$

so that letting

$$r(t) \triangleq K \|e^{-t\dot{\mathcal{A}}}\dot{\mathcal{A}}^{-1}\|_{\mathcal{L}(\dot{\mathcal{H}})}^2 \rightarrow 0, \quad t \rightarrow +\infty, \quad (2.61)$$

gives (2.55). In particular, $E(u, t) \rightarrow 0$ for a set of initial data that is dense in \mathcal{H} , and thus (2.56) follows from a standard density argument along with (2.42). \square

Remark 2.8 (*A priori logarithmic decay rate*). As mentioned in the introduction, the results, and analysis of [Buf17] suggest that, regardless of geometry and dynamical hypotheses, the rate r in Proposition 2.7 can be taken as

$$r(t) = \frac{K}{\log(2+t)^2}, \quad t \geq 0, \quad (2.62)$$

for some positive constant K . In fact, if $\Gamma_1 = \emptyset$ then (2.62) is precisely given by [Buf17, Theorem 1.2].

Remark 2.9. From Proposition 2.7 and its proof we see that $P(i\lambda)^{-1}$ is well-defined in $\mathcal{L}(H)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$.

3 Energy decay rate and analysis of quasimodes

3.1 Statement of the main result

We now state the full version of Theorem 1.2.

Theorem 3.1 (Polynomial stability). *Suppose that Assumption 1.1 holds. Then,*

$$\|e^{-t\dot{A}}\dot{A}^{-1}\|_{\mathcal{L}(\dot{H})} = O(t^{-1/2}), \quad t \rightarrow +\infty. \quad (3.1)$$

Thus, there exists $K > 0$ such that, for all (classical) solutions u to (1.1) with initial data $(u_0, u_1) \in \text{dom}(A) \times V$,

$$E(u, t) \leq K(1+t)^{-1}E_1(u, 0), \quad t \geq 0. \quad (3.2)$$

Furthermore, for fixed data $(u_0, u_1) \in \text{dom}(A) \times V$,

$$E(u, t) = o(t^{-1}), \quad t \rightarrow +\infty. \quad (3.3)$$

As it is now standard, the asymptotic theory of strongly continuous semigroups on Hilbert spaces [BT10] and the special structure (2.28) of (1.1) [AL14] will allow us to deduce Theorem 3.1 from the following resolvent estimate.

Proposition 3.2 (Second-order resolvent estimate). *Under Assumption 1.1,*

$$\|P(i\lambda)^{-1}\|_{\mathcal{L}(H)} = O(|\lambda|), \quad \lambda \rightarrow \pm\infty. \quad (3.4)$$

Proof of Theorem 3.1 assuming Proposition 3.2. To deduce (3.1) from (3.4), use, e.g., [AL14, Proposition 2.4]. Passing from (3.1) to (3.2) is done exactly as in the proof of Proposition 2.7. Finally, the o -improvement (3.3) for individual orbits is contained in [BT10, Theorem 2.4], which we apply to the contraction semigroup generated by $-\dot{A}$. \square

Proposition 3.2 is in fact the main achievement of this paper, and is proved in Section 3.2 below.

3.2 Analysis of quasimodes

After *reductio ad absurdum*, establishing energy decay rates for (1.1) through high-frequency asymptotics for $P(i\lambda)^{-1}$ amounts to analyzing the limit behavior of so-called *quasimodes* at certain semiclassical scales. Here we give a definition of quasimodes for (1.1) inspired by that of [AL14].

Definition 3.3 (Quasimodes). Let $\delta \geq 0$. A sequence $\{(h_n, u_n)\}_{n \in \mathbb{N}}$ of positive reals h_n with $h_n \rightarrow 0$, $n \rightarrow +\infty$, and elements $u_n \in \text{dom}(A)$ such that

$$\|u_n\|_H = 1, \quad n \in \mathbb{N}, \quad (3.5a)$$

$$\|(h_n^2 A + ih_n BB^* - 1)u_n\|_H = o(h_n^{1+\delta}), \quad n \rightarrow +\infty, \quad (3.5b)$$

is called a sequence of (normalized) $o(h_n^{1+\delta})$ -quasimodes.

In the sequel, subscripts indicating the dependence on n and h_n are omitted: with a slight abuse in notation we will speak of $o(h^{1+\delta})$ -quasimodes u and consider the limit $h \rightarrow 0$. Furthermore, δ will always denote a nonnegative parameter and, when considering quasimodes u , we will write

$$(f, e) \triangleq (h^2 A + ih BB^* - 1)u \in H. \quad (3.6)$$

With that in mind, we see that $o(h^{1+\delta})$ -quasimodes u solve the stationary boundary value problem

$$(-h^2 \Delta - 1)u = f \quad \text{in } \Omega, \quad (3.7a)$$

$$(-h^2 \Delta_\Gamma + ih - 1)u = -h^2 \partial_\nu u + e \quad \text{on } \Gamma_0, \quad (3.7b)$$

$$u = 0 \quad \text{on } \Gamma_1, \quad (3.7c)$$

with

$$\|u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Gamma_0)}^2 = 1, \quad \|f\|_{L^2(\Omega)} = o(h^{1+\delta}), \quad \|e\|_{L^2(\Gamma_0)} = o(h^{1+\delta}), \quad h \rightarrow 0. \quad (3.8)$$

(Compare (3.7) with (2.58), which we studied to rule out purely imaginary modes for (1.1).)

Remark 3.4. Since we aim at establishing a non-uniform energy decay rate $o(t^{-1})$, it suffices to study $o(h^3)$ -quasimodes, which ultimately we prove *do not* exist under Assumption 1.1. When possible, we nevertheless consider more general $o(h^{1+\delta})$ -quasimodes in an attempt to better highlight the obstruction to uniform stabilization and the role of the geometric control condition. In this regard, we point out that uniform (exponential) energy decay would correspond to the case $\delta = 0$.

3.2.1 Preliminary estimates

Our first estimates are immediate consequences of the abstract dissipative second-order structure (2.28).

Lemma 3.5. *For $o(h^{1+\delta})$ -quasimodes u ,*

$$h\|u\|_V = O(1), \quad \|u\|_{L^2(\Gamma_0)} = o(h^{\delta/2}), \quad h \rightarrow 0. \quad (3.9)$$

Proof. Since $(h^2A + ihBB^* - 1)u = (f, e)$, taking the scalar product with u in H yields

$$h^2\langle Au, u \rangle_{V^*,V} + ih\|u\|_{L^2(\Gamma_0)}^2 - \|u\|_H^2 = \langle (f, e), u \rangle_H. \quad (3.10)$$

Using the real part of (3.10) and the Cauchy–Schwarz inequality leads to

$$h^2\langle Au, u \rangle_{V^*,V} = 1 + \operatorname{Re}\langle (f, e), u \rangle_H \leq 1 + \|(f, e)\|_H = O(1), \quad h \rightarrow 0. \quad (3.11)$$

Since $\langle A\cdot, \cdot \rangle_{V^*,V} + \|\cdot\|_H$ is equivalent to the squared V -norm, (3.11) implies the first part of (3.9). On the other hand, by taking the imaginary part of (3.10) we see that

$$h\|u\|_{L^2(\Gamma_0)}^2 = \operatorname{Im}\langle (f, e), u \rangle_H \leq \|(f, e)\|_H = o(h^{1+\delta}), \quad h \rightarrow 0, \quad (3.12)$$

which completes the proof of (3.9). \square

Next, we give a differential multiplier identity that will be used at different points of our analysis; see, e.g., [Spe14, Lemma 3.7] for a proof.

Lemma 3.6 (Rellich identity). *For all $h > 0$, for all $u \in H^1(\Omega)$ with $\Delta u \in L^2(\Omega)$, $\partial_\nu u \in L^2(\Gamma)$ and $u \in H^1(\Gamma)$, and for all real-valued vector fields $q = (q_1, \dots, q_d) \in C^1(\bar{\Omega})^d$,*

$$\begin{aligned} \int_{\Omega} 2 \operatorname{Re}(\overline{q \cdot \nabla u} (h^2 \Delta u + u)) - (\operatorname{div} q)(h^2 |\nabla u|^2 - |u|^2) + 2h^2 \operatorname{Re} \sum_{i=1}^d \sum_{j=1}^d \frac{\partial q_j}{\partial x_i} \frac{\partial u}{\partial x_i} \frac{\partial \bar{u}}{\partial x_j} \, dx \\ = \int_{\Gamma} 2h^2 \operatorname{Re} \left(\overline{q \cdot \nabla u} \frac{\partial u}{\partial \nu} \right) + (|u|^2 - h^2 |\nabla u|^2)(q \cdot \nu) \, d\sigma. \end{aligned} \quad (3.13)$$

With Lemma 3.6 we can give an *a priori* estimate of the Neumann trace of quasimodes on Γ_0 .

Lemma 3.7. *For $o(h^{1+\delta})$ -quasimodes u ,*

$$h\|\partial_\nu u\|_{L^2(\Gamma_0)} = O(1), \quad h \rightarrow 0. \quad (3.14)$$

Proof. Since Ω is smooth and $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$, by [Kom94, Lemma 2.1] we may find a smooth vector field $q : \bar{\Omega} \rightarrow \mathbb{R}^d$ such that $q = \nu$ on Γ_0 and $q = 0$ on Γ_1 . With this choice of q , the Rellich identity from Lemma 3.6 applied to $o(h^{1+\delta})$ -quasimodes u yields

$$\begin{aligned} \int_{\Omega} -2 \operatorname{Re}(\overline{q \cdot \nabla u} f) - (\operatorname{div} q)(h^2 |\nabla u|^2 - |u|^2) + 2h^2 \operatorname{Re} \sum_{i=1}^d \sum_{j=1}^d \frac{\partial q_j}{\partial x_i} \frac{\partial u}{\partial x_i} \frac{\partial \bar{u}}{\partial x_j} \, dx \\ = \int_{\Gamma_0} h^2 \left| \frac{\partial u}{\partial \nu} \right|^2 + (|u|^2 - h^2 |\nabla_\Gamma u|^2) \, d\sigma, \end{aligned} \quad (3.15)$$

where we used that $|\nabla u|^2 = |\partial_\nu u|^2 + |\nabla_\Gamma u|^2$ on Γ_0 . From (3.15) it is easy to deduce that

$$h^2 \|\partial_\nu u\|_{L^2(\Gamma_0)}^2 = O(\|u\|_{H^1(\Omega)} \|f\|_{L^2(\Omega)} + h^2 \|u\|_{H^1(\Omega)}^2 + \|u\|_{L^2(\Gamma_0)}^2 + h^2 \|u\|_{H^1(\Gamma_0)}^2), \quad h \rightarrow 0, \quad (3.16)$$

which in turn leads to the desired estimate (3.14) due to (3.8) and (3.9). \square

We can now estimate tangential derivatives of quasimodes on Γ_0 .

Lemma 3.8. *For $o(h^{1+\delta})$ -quasimodes u ,*

$$h\|\nabla_\Gamma u\|_{L^2(\Gamma_0)^{d-1}} = o(h^{\delta/2} + h^{(2+\delta)/4}), \quad h \rightarrow 0. \quad (3.17)$$

Proof. Multiply (3.7b) by \bar{u} and integrate by parts over Γ_0 to see that, after taking real parts,

$$h^2 \int_{\Gamma_0} |\nabla_\Gamma u|^2 \, d\sigma = \int_{\Gamma_0} |u|^2 \, d\sigma - \operatorname{Re} \int_{\Gamma_0} h^2 \partial_\nu u \bar{u} - e \bar{u} \, d\sigma. \quad (3.18)$$

Apply Lemmas 3.5 and 3.7 to complete the proof. \square

3.2.2 Microlocalized estimates near the boundary

The purpose of this section is to investigate the behavior of quasimodes u near Γ_0 and, specifically for $o(h^3)$ -quasimodes, improve the normal derivative estimate of Lemma 3.7. To do so, we will carry out a more involved analysis in two separate microlocal regimes, namely, at low and high *tangential* frequencies.

We shall use the local setting of Section 2.1.2. First, let us recall and introduce some notation. Let $\mathbb{R}_+^d \triangleq \mathbb{R}^{d-1} \times (0, +\infty)$ be the open half-space made of points $y = (y_1, \dots, y_{d-1}, y_d) = (y_\tau, y_d) \in \mathbb{R}^d$ with $y_d > 0$. We will refer to y_τ and y_d as the tangential and normal components of $y \in \mathbb{R}^d$, respectively. Similarly, $\xi_\tau = (\xi_1, \dots, \xi_{d-1}) \in \mathbb{R}^{d-1}$ indicates the tangential Fourier variable. We recall that

$$D_{y_i} = -i\partial_{y_i}, \quad i = 1, \dots, d-1. \quad (3.19)$$

In the subsequent analysis of quasimodes, it will be convenient to use semiclassical norms for H^1 -spaces. If X denotes Ω , \mathbb{R}_+^d or \mathbb{R}^{d-1} , then for $h > 0$ we let $H_h^1(X)$ be the space $H^1(X)$ equipped with the h -dependent norm

$$\|w\|_{H_h^1(X)}^2 \triangleq \int_X h^2 |\nabla w|^2 + |w|^2 dx, \quad w \in H^1(X), \quad (3.20)$$

where ∇ and dx are the ambient gradient and Lebesgue measure. Similarly, if M is Γ or Γ_0 , $H_h^1(M)$ is $H^1(M)$ with the norm

$$\|w\|_{H_h^1(M)}^2 \triangleq \int_M h^2 |\nabla_\Gamma w|^2 + |w|^2 d\sigma, \quad w \in H^1(M). \quad (3.21)$$

Now, given a function w of Ω , we will write

$$w^{C_n} \triangleq (\kappa_n^{-1})^* w = w \circ \kappa_n^{-1}, \quad \tilde{w}^{C_n} \triangleq (\kappa_n^{-1})^* (\theta_n w), \quad n = 0, \dots, N, \quad (3.22)$$

so that w^{C_n} , defined in $\kappa_n(O_n \cap \Omega)$, is the pullback of w by κ_n , whereas \tilde{w}^{C_n} is the pullback of its localized version $\theta_n w$ and trivially extends to a function of \mathbb{R}_+^d if $n = 1, \dots, N$ or of \mathbb{R}^d if $n = 0$. On the other hand, if $n = 1, \dots, N$ and w is instead a function of Γ_0 , we use the same notation, so that the localized pullback \tilde{w}^{C_n} becomes a function of \mathbb{R}^{d-1} . In what follows we will not write the dependence on $n = 1, \dots, N$, and now $C = (O, \kappa)$ is *any* of the N local charts defined in Section 2.1.2.

Consider now $o(h^{1+\delta})$ -quasimodes u . Then, localizing (3.7) via multiplication by θ and taking pullbacks by κ , we see that \tilde{u}^C solves

$$(-h^2 \Delta^C - 1)\tilde{u}^C = \tilde{f}^C - [\theta, h^2 \Delta]^C u^C \quad \text{in } \kappa(O \cap \Omega), \quad (3.23a)$$

$$(-h^2 \Delta_\Gamma^C + ih - 1)\tilde{u}^C = -ih^2 \theta^C D_{y_d} u^C + \tilde{e}^C - [\theta, h^2 \Delta_\Gamma]^C u^C \quad \text{in } \kappa(O \cap \Gamma_0). \quad (3.23b)$$

Here and in the sequel brackets $[X, Y] = XY - YX$ denote commutators. Since \tilde{u}^C and all the terms at the right-hand sides of (3.23) vanish outside of $\kappa(\text{supp}(\theta))$, we may rewrite (3.23) as

$$(-h^2 \tilde{\Delta}^C - 1)\tilde{u}^C = \tilde{f}^C - [\theta^C, h^2 \tilde{\Delta}^C] u^C \quad \text{in } \mathbb{R}_+^d, \quad (3.24a)$$

$$(-h^2 \tilde{\Delta}_\Gamma^C + ih - 1)\tilde{u}^C = -ih^2 \theta^C D_{y_d} u^C + \tilde{e}^C - [\theta^C, h^2 \tilde{\Delta}_\Gamma^C] u^C \quad \text{in } \mathbb{R}^{d-1}, \quad (3.24b)$$

where we recall from Section 2.1.2 that $\tilde{\Delta}^C$ and $\tilde{\Delta}_\Gamma^C$ coincide with their “tilteless” counterparts near $\kappa(\text{supp}(\theta))$ but have tangential coefficients that vanish away from $\kappa(O)$. By the way, we may pick $\theta'' \in \mathcal{D}(O)$ taking values in $[0, 1]$ such that $\text{supp}(\theta'')$ is contained in an open subset of $\{x \in O : \theta'(x) = 1\}$ and $\theta'' = 1$ near $\text{supp}(\theta)$.⁴

Next, choose a smooth and (strictly) increasing cutoff $\rho : [0, +\infty) \rightarrow [0, 1]$ such that $\rho(r) = 0$ for $r^2 \leq 4/c$ and $\rho(r) = 1$ for (say) $r^2 \geq 8/c$, where $c > 0$ is the ellipticity constant from (2.18). Define a (h -independent) symbol χ as follows:

$$\chi(y, \xi_\tau) \triangleq \theta''(\kappa^{-1}(y))\rho(|\xi_\tau|), \quad y \in \mathbb{R}_+^d, \quad \xi_\tau \in \mathbb{R}^{d-1}. \quad (3.25)$$

Then, χ belongs to the class of tangential symbols $S_{\tau, h}^0$ defined in Appendix A. Note that χ has compact spatial support. Define a zeroth-order tangential semiclassical pseudodifferential operator $X \in \Psi_{\tau, h}^0$ by

$$X \triangleq \text{Op}_h^\tau(\chi), \quad (3.26)$$

where the space $\Psi_{\tau, h}^0$ and the quantization procedure Op_h^τ are described in Appendix A, which also contains some additional notation and facts concerning pseudodifferential operators on \mathbb{R}^{d-1} .

The first result of this section is an estimate for quasimodes in the regime where tangential derivatives are dominant.

Proposition 3.9 (High-frequency regime). *For $o(h^{1+\delta})$ -quasimodes u , letting $v \triangleq X\tilde{u}^C$, we have*

$$\|v\|_{H_h^1(\mathbb{R}_+^d)} + h\|D_{y_d} v|_{y_d=0}\|_{L^2(\mathbb{R}^{d-1})} = o(1), \quad h \rightarrow 0. \quad (3.27)$$

⁴Here the prime symbols do not denote derivatives.

Proof. It follows from (3.24a) that v solves

$$(-h^2 \tilde{\Delta}^C - 1)v = X\tilde{f}^C - X[\theta^C, h^2 \tilde{\Delta}^C]u^C - [X, h^2 \tilde{\Delta}^C]\tilde{u}^C \quad \text{in } \mathbb{R}_+^d. \quad (3.28)$$

Recall from Section 2.1.2 that $-h^2 \tilde{\Delta}^C$ has the form

$$-h^2 \tilde{\Delta}^C = h^2 D_{y_d}^2 + h^2 \sum_{i,j=1}^{d-1} \tilde{a}_{ij}(y) D_{y_i} D_{y_j}. \quad (3.29)$$

We denote by P_τ the second term at the right-hand side of (3.29), that is, $P_\tau = P_\tau(h)$ is the tangential part of $-h^2 \tilde{\Delta}^C$, so that $-h^2 \tilde{\Delta}^C = h^2 D_{y_d}^2 + P_\tau$. We also denote by r the right-hand side of (3.28). Equation (3.28) in this notation reads as follows:

$$(h^2 D_{y_d}^2 + P_\tau - 1)v = r \quad \text{in } \mathbb{R}_+^d. \quad (3.30)$$

The proof is divided into several steps.

Step 1: elliptic estimates. Multiplying (3.30) by \bar{v} and integrating over \mathbb{R}_+^d lead to

$$\iint_{\mathbb{R}_+^d} h^2 D_{y_d}^2 v \bar{v} dy_d dy_\tau + \iint_{\mathbb{R}_+^d} (P_\tau - 1)v \bar{v} dy_d dy_\tau = \int_{\mathbb{R}^{d-1}} \int_0^{+\infty} r \bar{v} dy_d dy_\tau. \quad (3.31)$$

We start with the first term in (3.31): integrating by part we obtain

$$\iint_{\mathbb{R}_+^d} h^2 D_{y_d}^2 v \bar{v} dy_d dy_\tau = h^2 \iint_{\mathbb{R}_+^d} |D_{y_d} v|^2 dy_d dy_\tau - h^2 \int_{\mathbb{R}^{d-1}} D_{y_d} v \bar{v}|_{y_d=0} dy_\tau, \quad (3.32)$$

where we used that v vanish for large y_d since \tilde{u}^C is compactly supported and X is nonlocal only in the tangential variable. To estimate the second term in (3.31), we shall now employ an argument based on Gårding's inequality. The (semiclassical) principal symbol of the tangential differential operator $P_\tau - 1$ is given by

$$\sigma_h(P_\tau - 1)(y, \xi_\tau) = \sum_{i,j=1}^{d-1} \tilde{a}_{ij}(y) \xi_i \xi_j - 1, \quad y \in \mathbb{R}_+^d, \quad \xi_\tau = (\xi_1, \dots, \xi_{d-1}) \in \mathbb{R}^{d-1}. \quad (3.33)$$

We will now use the properties the position and frequency cutoffs θ'' and ρ involved in the definition (3.25) of our microlocalization symbol χ . On the one hand, in some open neighborhood $N \subset \kappa(O) \cap \mathbb{R}_+^d$ of $\text{supp}(\theta'' \circ \kappa^{-1})$ the modified coefficients \tilde{a}_{ij} coincide with the original ones a_{ij} and, as such, satisfy the ellipticity condition (2.18) for all $\xi_\tau \in \mathbb{R}^{d-1}$. On the other hand, $\rho(|\xi_\tau|) > 0$ if and only if $|\xi_\tau|^2 > 4/c$. Let $U \triangleq N \times \{\xi_\tau \in \mathbb{R}^{d-1} : |\xi_\tau|^2 > 2/c\}$. Then $\text{supp}(\chi) \subset U$ and

$$\sigma_h(P_\tau - 1)(y, \xi_\tau) = \sum_{i,j=1}^{d-1} a_{ij}(y) \xi_i \xi_j - 1 \geq c|\xi_\tau|^2 - 1 \geq \frac{c}{2}|\xi_\tau|^2, \quad (y, \xi_\tau) \in U. \quad (3.34)$$

As a result, the microlocal tangential Gårding inequality from Theorem A.2, which is adapted from [LRLR22a, Theorem 2.50], provides positive constants K and K' such that

$$\iint_{\mathbb{R}_+^d} (P_\tau - 1)X\varphi \overline{X\varphi} dy_d dy_\tau \geq K \left(\iint_{\mathbb{R}_+^d} |X\varphi|^2 + h^2 \sum_{i=1}^{d-1} |D_{y_i} X\varphi|^2 dy_d dy_\tau \right) - h^2 K' \iint_{\mathbb{R}_+^d} |\varphi|^2 dy_d dy_\tau \quad (3.35)$$

for all functions φ in, say, $\overline{\mathcal{S}}(\mathbb{R}_+^d)$,⁵ and all positive (small enough) h . Applying⁶ (3.35) to $o(h^{1+\delta})$ -quasimodes u (recall that $v = X\tilde{u}^C$), and plugging the result into (3.31) and (3.32), we obtain, for some constant $K > 0$ independent of quasimodes,

$$\|v\|_{H_h^1(\mathbb{R}_+^d)}^2 \leq K \left(h^2 \|\tilde{u}^C\|_{L^2(\mathbb{R}_+^d)}^2 + |\langle r, v \rangle_{L^2(\mathbb{R}_+^d)}| + h^2 |\langle D_{y_d} v, v \rangle_{L^2(\mathbb{R}^{d-1})}| \right). \quad (3.36)$$

Here and in the sequel, for the ease of notation we might omit explicit indications that boundary traces are taken at $y_d = 0$ when writing norms or scalar products on \mathbb{R}^{d-1} . The remainder of the proof consists in estimating the terms at the right-hand side of (3.36).

⁵Following [LRLR22a], $\overline{\mathcal{S}}(\mathbb{R}_+^d)$ denotes the space of restrictions to \mathbb{R}_+^d of Schwartz functions on \mathbb{R}^d .

⁶That is, after a suitable density argument, which we omit.

Step 2: normal derivative. Here we carry out a simple differential multiplier argument, not unrelated to that of Lemma 3.7. It is therefore convenient to put P_τ in divergence form and write

$$P_\tau = h^2 \sum_{i,j=1}^{d-1} D_{y_i}(\tilde{a}_{ij}(y)D_{y_j}) + R_\tau, \quad (3.37)$$

with $R_\tau \in h \text{Diff}_{\tau,h}^1$. Pick a smooth cutoff $\psi : [0, +\infty) \rightarrow [0, 1]$ such that $\psi = 1$ near 0 and $\psi = 0$ away from zero. We shall multiply (3.30) by $\psi(y_d)\overline{D_{y_d}v}$ and integrate over \mathbb{R}_+^d . First, note that

$$h^2 \text{Im} \iint_{\mathbb{R}_+^d} \psi D_{y_d}^2 v \overline{D_{y_d}v} dy_d dy_\tau = \frac{h^2}{2} \int_{\mathbb{R}^{d-1}} |D_{y_d}v|^2 dy_\tau \Big|_{y_d=0} + \frac{h^2}{2} \iint_{\mathbb{R}_+^d} \psi' |D_{y_d}v|^2 dy_d dy_\tau, \quad (3.38)$$

where $\psi' = (d/dy_d)\psi$. Also,

$$\begin{aligned} h^2 \text{Im} \iint_{\mathbb{R}_+^d} \sum_{i,j=1}^{d-1} D_{y_i}(\tilde{a}_{ij}(y)D_{y_j}v)\psi(y_d)\overline{D_{y_d}v} dy_d dy_\tau &= -\frac{h^2}{2} \int_{\mathbb{R}^{d-1}} \sum_{i,j=1}^{d-1} \tilde{a}_{ij}(y)D_{y_j}v \overline{D_{y_i}v} dy_\tau \Big|_{y_d=0} \\ &\quad - \frac{h^2}{2} \iint_{\mathbb{R}_+^d} \sum_{i,j=1}^{d-1} D_{y_d}(\psi(y_d)\tilde{a}_{ij}(y))D_{y_j}v \overline{D_{y_i}v} dy_d dy_\tau. \end{aligned} \quad (3.39)$$

With (3.38) and (3.39) we readily deduce that

$$\begin{aligned} &h^2 \|D_{y_d}v\|_{L^2(\mathbb{R}^{d-1})}^2 \\ &\leq K' \left(\|r\|_{L^2(\mathbb{R}_+^d)} \|D_{y_d}v\|_{L^2(\mathbb{R}_+^d)} + \|(R_\tau - 1)v\|_{L^2(\mathbb{R}_+^d)} \|D_{y_d}v\|_{L^2(\mathbb{R}_+^d)} + \|v\|_{H_h^1(\mathbb{R}_+^d)}^2 + \|v\|_{H_h^1(\mathbb{R}^{d-1})}^2 \right). \end{aligned} \quad (3.40)$$

for some constant positive constant K' (again, independent of quasimodes).

Step 3: tangential derivatives. Recall that, for $\varphi \in \mathcal{S}(\mathbb{R}_+^d)$, $\text{Op}_h^r(\chi)\varphi|_{y_d=0} = \text{Op}_h(\chi|_{y_d=0})(\varphi|_{y_d=0})$, where Op_h indicates standard semiclassical quantization on \mathbb{R}^{d-1} . Note also that since χ belongs to the class of tangential symbols $S_{\tau,h}^0 = S_{\tau,h}^0(\mathbb{R}_+^d \times \mathbb{R}^{d-1})$ then its boundary trace $\chi|_{y_d=0}$ is in $S_h^0(\mathbb{R}^{d-1} \times \mathbb{R}^{d-1})$; see [LRLR22a, Remark 2.40]. In particular X is bounded uniformly in h on the spaces $L^2(\mathbb{R}^{d-1})$ and $H_h^1(\mathbb{R}^{d-1})$; see, e.g., [DZ19, Proposition E.19]. With this in mind, since $v = X\tilde{u}^C$ we obtain

$$\|v\|_{H_h^1(\mathbb{R}^{d-1})} = O(\|\tilde{u}^C\|_{H_h^1(\mathbb{R}^{d-1})}) = O(\|u\|_{H_h^1(\Gamma_0)}) = o(1), \quad h \rightarrow 0, \quad (3.41)$$

where we also used Lemmas 3.5 and 3.8.

Step 4: commutators and lower-order terms. We start by stating further boundedness properties of X . Since its symbol χ is in the tangential class $S_{\tau,h}^0$, X is bounded uniformly in h on the spaces $L^2(\mathbb{R}_+^d)$ and also $L^2(0, +\infty; H_h^1(\mathbb{R}^{d-1}))$ by [LRLR22a, Theorem 2.47]. Furthermore $[X, hD_{y_d}] = h \text{Op}_h^r(D_{y_d}\chi) \in h\Psi_{\tau,h}^0(\mathbb{R}_+^d)$ (see [LRLR22a, Remark 2.46]), and thus X is bounded on $H_h^1(\mathbb{R}_+^d)$ as well. We are now ready to deal with the rest r from (3.28) and (3.30) and also the term involving R_τ in (3.40). By construction $R_\tau \in h \text{Diff}_{\tau,h}^1$ and

$$\|(R_\tau - 1)v\|_{L^2(\mathbb{R}_+^d)} \|D_{y_d}v\|_{L^2(\mathbb{R}_+^d)} = O(\|v\|_{H_h^1(\mathbb{R}_+^d)}^2), \quad h \rightarrow 0. \quad (3.42)$$

Also,

$$\|X\tilde{f}^C\|_{L^2(\mathbb{R}_+^d)} = O(\|\tilde{f}^C\|_{L^2(\mathbb{R}_+^d)}) = O(\|f\|_{L^2(\Omega)}) = o(h^{1+\delta}), \quad h \rightarrow 0. \quad (3.43)$$

It remains to estimate the two commutator terms. First, $[\theta^C, h^2\tilde{\Delta}^C] \in h \text{Diff}_{\tau,h}^1$ with coefficients vanishing of a compact subset of $\kappa(O)$; thus, using also Lemma 3.5 we see that

$$\|X[\theta, h^2\tilde{\Delta}^C]u^C\|_{L^2(\mathbb{R}_+^d)} = O(\|[\theta, h^2\tilde{\Delta}^C]u^C\|_{L^2(\mathbb{R}_+^d)}) = O(h\|u\|_{H_h^1(\Omega)}) = O(h), \quad h \rightarrow 0. \quad (3.44)$$

Next we look at the commutator $[X, h^2\tilde{\Delta}^C]$, which we may write as (minus) the sum of $[X, h^2D_{y_d}^2]$ and $[X, P_\tau]$. Observe on the one hand that $[X, h^2D_{y_d}^2] = h^2[X, D_{y_d}]D_{y_d} + h^2D_{y_d}[X, D_{y_d}]$: therefore, using the fact that $[X, D_{y_d}] \in \Psi_{\tau,h}^0$ combined with continuity properties of X , we obtain

$$\|[X, h^2D_{y_d}^2]\tilde{u}^C\|_{L^2(\mathbb{R}_+^d)} = O(h\|\tilde{u}^C\|_{H_h^1(\mathbb{R}_+^d)}) = O(h\|u\|_{H_h^1(\Omega)}) = O(h), \quad h \rightarrow 0. \quad (3.45)$$

On the other hand, the Poisson bracket (in the tangential variables) between the principal symbols of X and P_τ is given by

$$\{\chi, \sigma_h(P_\tau)\}(y, \xi_\tau) = \sum_{k=1}^{d-1} \frac{\partial \chi}{\partial \xi_k}(y, \xi_\tau) \sum_{i,j=1}^{d-1} \frac{\partial \tilde{a}_{ij}}{\partial y_k}(y) \xi_i \xi_j - \frac{\partial \chi}{\partial y_k}(y, \xi_\tau) \left(2 \sum_{i=1}^{d-1} \tilde{a}_{ki}(y) \xi_i \right) \quad (3.46)$$

for all $y \in \mathbb{R}_+^d$ and $\xi_\tau = (\xi_1, \dots, \xi_{d-1}) \in \mathbb{R}^{d-1}$. After noting that $\partial_{\xi_\tau} \chi$ vanishes for large $|\xi_\tau|$ (uniformly in y), we infer from (3.46) and [LRLR22a, Corollary 2.45] that $[X, P_\tau] \in h\Psi_{\tau, h}^1$, and thus

$$\|[X, P_\tau] \tilde{u}^C\|_{L^2(\mathbb{R}_+^d)} = O(h \|\tilde{u}^C\|_{H_h^1(\mathbb{R}_+^d)}) = O(h), \quad h \rightarrow 0. \quad (3.47)$$

Summing up, we have shown that the term r in (3.30) decays like $O(h)$ in L^2 -norm as $h \rightarrow 0$.

Step 5: conclusion. Since X is bounded on $H_h^1(\mathbb{R}_+^d)$ uniformly in h , it is clear that

$$\|v\|_{H_h^1(\mathbb{R}_+^d)} = O(1), \quad h \rightarrow 0. \quad (3.48)$$

In light of the above developments, we may now deduce from (3.40) that

$$h \|D_{y_d} v\|_{L^2(\mathbb{R}^{d-1})} = O(1), \quad h \rightarrow 0, \quad (3.49)$$

and at this point, with the Cauchy–Schwarz inequality and the various previous estimates, it readily follows from (3.36) that, in fact,

$$\|v\|_{H_h^1(\mathbb{R}_+^d)} = O(h^{1/2}), \quad h \rightarrow 0. \quad (3.50)$$

With this improved estimate in hand we may return to (3.40) and finally deduce that

$$h \|D_{y_d} v\|_{L^2(\mathbb{R}^{d-1})} = O(h^{1/4}), \quad h \rightarrow 0. \quad (3.51)$$

The proof is now complete. \square

For the estimate at low tangential frequencies, we consider $o(h^3)$ -quasimodes only.

Proposition 3.10 (Low-frequency regime). *For $o(h^3)$ -quasimodes u , letting $w \triangleq (1 - X)\tilde{u}^C$, we have*

$$h \|D_{y_d} w|_{y_d=0}\|_{L^2(\mathbb{R}^{d-1})} = o(1), \quad h \rightarrow 0. \quad (3.52)$$

Proof. We shall keep in mind the remarks, properties and notation convention from the proof of Proposition 3.9. As a result of (3.24b) we see that w solves

$$(-h^2 \tilde{\Delta}_\Gamma^C + ih - 1)w = -ih^2(1 - X)\theta^C D_{y_d} u^C + (1 - X)\tilde{e}^C - [\theta, h^2 \Delta_\Gamma]^C u^C + [X, h^2 \tilde{\Delta}^C] \tilde{u}^C \quad \text{in } \mathbb{R}^{d-1}. \quad (3.53)$$

The proof is divided into two steps.

Step 1: normal derivative. We start by noting that

$$\begin{aligned} (1 - X)\theta^C D_{y_d} u^C &= (1 - X)D_{y_d} \tilde{u}^C - (1 - X)(D_{y_d} \theta^C) u^C \\ &= D_{y_d} w + [X, D_{y_d}] \tilde{u}^C - (1 - X)(D_{y_d} \theta^C) u^C, \end{aligned} \quad (3.54)$$

and thus, in view of the desired estimate (3.52), we may rewrite (3.53) as

$$ih^2 D_{y_d} w = -(-h^2 \tilde{\Delta}_\Gamma^C + ih - 1)(1 - X)\tilde{u}^C + r \quad \text{in } \mathbb{R}^{d-1}, \quad (3.55)$$

where the rest r is made of terms left over from (3.53) and (3.54). On the boundary \mathbb{R}^{d-1} , $1 - X = \text{Op}_h((1 - \chi)|_{y_d=0})$ and the (smooth, bounded) symbol $(1 - \chi)|_{y_d=0}$ vanishes for large $|\xi_\tau|$ (uniformly in y_τ). In particular, it belongs to the class $S_h^{-2}(\mathbb{R}^{d-1} \times \mathbb{R}^{d-1})$. Therefore, by [DZ19, Proposition E.19], $1 - X$ continuously (and uniformly in h) maps $L^2(\mathbb{R}^{d-1})$ into $H_h^2(\mathbb{R}^{d-1})$, which is simply $H^2(\mathbb{R}^{d-1})$ with the semiclassical norm

$$\|\varphi\|_{H_h^2(\mathbb{R}^{d-1})}^2 \triangleq \int_{\mathbb{R}^{d-1}} \sum_{\substack{\alpha \in \mathbb{N}^{d-1} \\ |\alpha| \leq 2}} h^{2|\alpha|} |D^\alpha \varphi|^2 dy_\tau, \quad \varphi \in H^2(\mathbb{R}^{d-1}). \quad (3.56)$$

As an immediate consequence,

$$\|(-h^2 \tilde{\Delta}_\Gamma^C + ih - 1)(1 - X)\tilde{u}^C\|_{L^2(\mathbb{R}^{d-1})} = O(\|(1 - X)\tilde{u}^C\|_{H_h^2(\mathbb{R}^{d-1})}) = O(\|\tilde{u}^C\|_{L^2(\mathbb{R}^{d-1})}), \quad h \rightarrow 0. \quad (3.57)$$

Here we specifically consider $o(h^3)$ -quasimodes: recall from Lemma 3.5 that

$$\|\tilde{u}^C\|_{L^2(\mathbb{R}^{d-1})} = O(\|u\|_{L^2(\Gamma_0)}) = o(h), \quad h \rightarrow 0. \quad (3.58)$$

Then, it follows from (3.55) and the above observations that

$$h \|D_{y_d} w\|_{L^2(\mathbb{R}^{d-1})} = h^{-1} \|r\|_{L^2(\mathbb{R}^{d-1})} + o(1), \quad h \rightarrow 0. \quad (3.59)$$

Step 2: left-over terms. Now it remains to handle the term r , which we must prove is of size $o(h)$ as $h \rightarrow 0$. The operator $1 - X$ is bounded on $L^2(\mathbb{R}^{d-1})$ uniformly in h and thus

$$\|(1 - X)\tilde{e}^C\|_{L^2(\mathbb{R}^{d-1})} = O(\|e\|_{L^2(\Gamma_0)}) = o(h^3), \quad h \rightarrow 0. \quad (3.60)$$

Similarly,

$$\|(1 - X)(D_{y_d}\theta^C)u^C\|_{L^2(\mathbb{R}^{d-1})} = O(\|u\|_{L^2(\Gamma_0)}) = o(h), \quad h \rightarrow 0. \quad (3.61)$$

Next we deal with the commutators. Noting that $[\theta, h^2\Delta_\Gamma]^C \in h \text{Diff}_h^1(\mathbb{R}^{d-1})$ we get

$$\|[\theta, h^2\Delta_\Gamma]^C u^C\|_{L^2(\mathbb{R}^{d-1})} = O(h\|u\|_{H_h^1(\Gamma_0)}) = o(h^2), \quad h \rightarrow 0, \quad (3.62)$$

where we also used Lemma 3.8. Also, just as in the proof of Proposition 3.9,

$$\|[X, D_{y_d}]\tilde{u}^C\|_{L^2(\mathbb{R}^{d-1})} = O(\|\tilde{u}^C\|_{L^2(\mathbb{R}^{d-1})}) = o(h), \quad h \rightarrow 0. \quad (3.63)$$

The last term to estimate is given by $[X, h^2\tilde{\Delta}^C]\tilde{u}^C$: by a Poisson bracket argument almost identical to that of the previous proof, together with, e.g., [DZ19, Proposition E.8], we have $[X, h^2\tilde{\Delta}^C] \in h\Psi_h^1(\mathbb{R}^{d-1})$. Then, finally,

$$\|[X, h^2\tilde{\Delta}^C]\tilde{u}^C\|_{L^2(\mathbb{R}^{d-1})} = O(h^2\|u\|_{H^1(\Gamma_0)} + h\|u\|_{L^2(\Gamma_0)}) = o(h^2), \quad h \rightarrow 0, \quad (3.64)$$

and the proof is complete. \square

We conclude this section by gluing together the estimates of Propositions 3.9 and 3.10.

Proposition 3.11 (Improved normal derivative estimate). *For $o(h^3)$ -quasimodes u ,*

$$h\|\partial_\nu u\|_{L^2(\Gamma_0)} = o(1), \quad h \rightarrow 0. \quad (3.65)$$

Proof. We indicate again dependence on the particular local chart $C_n = (O_n, \kappa_n)$ from Section 2.1.2 with subscripts n . With the partition of unity that we defined previously, we have $\partial_\nu u = \sum_{n=1}^N \partial_\nu(\theta_n u)$ on Γ_0 , and thus

$$\|\partial_\nu u\|_{L^2(\Gamma_0)} \leq \sum_{n=1}^N \|\partial_\nu(\theta_n u)\|_{L^2(\Gamma_0)} \leq K \sum_{n=1}^N \|D_{y_d}\tilde{u}_n^C\|_{L^2(\mathbb{R}^{d-1})}, \quad (3.66)$$

where the positive constant K comes from changes of variables by means of the diffeomorphisms κ_n and is independent of quasimodes. For each $n = 1, \dots, N$,

$$\|D_{y_d}\tilde{u}_n^C\|_{L^2(\mathbb{R}^{d-1})} \leq \|D_{y_d}X_n\tilde{u}_n^C\|_{L^2(\mathbb{R}^{d-1})} + \|D_{y_d}(1 - X_n)\tilde{u}_n^C\|_{L^2(\mathbb{R}^{d-1})}, \quad (3.67)$$

and the result immediately follows from Propositions 3.9 and 3.10. \square

3.2.3 Decoupling arguments and conclusion

At this point, we know from Lemma 3.8 and Proposition 3.11 that $o(h^3)$ -quasimodes u satisfy

$$\|u\|_{H_h^1(\Gamma_0)} + h\|\partial_\nu u\|_{L^2(\Gamma_0)} = o(1), \quad h \rightarrow 0. \quad (3.68)$$

Thus, we have suitable control over boundary traces on Γ_0 in both tangential and normal directions. Our next goal is to take advantage of the geometric control condition to propagate our boundary estimates (3.68) over the whole domain Ω . To that end, we shall use a splitting $u = u_1 + u_2$ of quasimodes in the following manner.

1. Define u_1 as a suitable lifting of the boundary data $u|_{\Gamma_0}$. Here we exploit *well-posedness* of the Dirichlet problem for the wave equation to derive interior and boundary estimates of u_1 .
2. Recover u_2 in terms of its Neumann trace $\partial_\nu u_2 = \partial_\nu(u - u_1)$ on Γ_0 . This step relies on *exact observability* of the wave equation with homogeneous Dirichlet boundary condition.

To implement the first step of our strategy, we need the following lemma.

Lemma 3.12. *Let $h > 0$. For every $g \in H^{1/2}(\Gamma)$ there exists a unique $w \in H^1(\Omega)$ such that*

$$(-h^2\Delta + 2ih - 1)w = 0 \quad \text{in } \Omega, \quad (3.69a)$$

$$w = g \quad \text{on } \Gamma. \quad (3.69b)$$

Proof. Let $D \in \mathcal{L}(H^{1/2}(\Gamma), H^1(\Omega))$ denote the Dirichlet *harmonic extension* map, which is defined as follows: $w = Dg$ if and only if $\Delta w = 0$ in Ω and $w = g$ on Γ ; see, e.g., [LM68, Chapter 2] for the existence and continuity of such a map. The Dirichlet Laplacian Δ_D is defined as an unbounded (negative self-adjoint) operator on $L^2(\Omega)$ by $\text{dom}(\Delta_D) \triangleq \{w_0 \in H_0^1(\Omega) : \Delta w \in L^2(\Omega)\} = H^2(\Omega) \cap H_0^1(\Omega)$ and $\Delta_D w = \Delta w$ for $w \in \text{dom}(\Delta_D)$; it is negative self-adjoint and also naturally extends to isomorphisms from $H_0^1(\Omega) = \text{dom}((-\Delta_D)^{1/2})$ onto $H^{-1}(\Omega)$ ⁷ and from $L^2(\Omega)$ onto $\text{dom}(\Delta_D)^*$. With this in mind, we see that, given $g \in H^{1/2}(\Gamma)$, $w \in H^1(\Omega)$ satisfies (3.69) if and only if $w - Dg \in H_0^1(\Omega)$ and $-h^2\Delta_D(w - h^{-2}Dg) + 2ihw - w = 0$ in $H^{-1}(\Omega)$. On the other hand, since the operator $-h^2\Delta_D$ is positive, $1 - 2ih \notin \mathbb{R}$ lies in its resolvent set, and we may therefore let $w \triangleq -(-h^2\Delta_D + 2ih - 1)^{-1}\Delta_D Dg$. The candidate w is *a priori* defined in $L^2(\Omega)$; following the above observations, it is immediate to check that w belongs to $H^1(\Omega)$ and is indeed the unique solution to (3.69). \square

⁷Strictly speaking, here $H^{-1}(\Omega)$ is seen as a space of *antilinear* forms.

Next, we give estimates for solutions w in Lemma 3.12 that are uniform in the small parameter h .

Proposition 3.13 (Dirichlet-to-Neumann bounds). *If $g \in H^1(\Gamma)$ then the corresponding solution w to (3.69), as given by Lemma 3.12, satisfies $\partial_\nu w \in L^2(\Gamma)$. In fact, there exists $K > 0$ such that,*

$$\|w\|_{H_h^1(\Omega)} + h\|\partial_\nu w\|_{L^2(\Gamma)} \leq K\|g\|_{H_h^1(\Gamma)}, \quad g \in H^1(\Gamma), \quad 0 < h \leq 1. \quad (3.70)$$

Proof. The property that $\partial_\nu w$ (*a priori* defined in $H^{-1/2}(\Gamma)$ as in (2.32)) belongs to $L^2(\Gamma)$ when $g \in H^1(\Gamma)$ is already contained in [Neč12, Theorem 1.1, Chapter 5]; alternatively we may establish (3.70) for sufficiently smooth functions first and recover boundary regularity via density arguments. Now, it follows from (3.69a) that

$$\int_\Omega h^2|\nabla w|^2 dx + (2ih - 1)|w|^2 dx = h^2 \int_\Gamma \partial_\nu w \bar{g} d\sigma, \quad g \in H^1(\Gamma), \quad h > 0. \quad (3.71)$$

By successively taking the imaginary and real parts of (3.71) and using Young's inequality, we deduce that, for some constant $K > 0$,

$$\int_\Omega h^2|\nabla w|^2 + |w|^2 dx \leq \frac{\varepsilon h^2 K}{2} \int_\Gamma |\partial_\nu w|^2 d\sigma + \frac{K}{2\varepsilon} \int_\Gamma |g|^2 d\sigma, \quad \varepsilon > 0, \quad g \in H^1(\Gamma), \quad 0 < h \leq 1. \quad (3.72)$$

Similarly as in the proof of Lemma 3.7, we may use the multiplier identity (3.13) from Lemma 3.6 with a vector field q that satisfies $q \cdot \nu = 1$ on Γ . After a couple of Cauchy–Schwarz and Young inequalities we then see that there exists a constant $K' > 0$ such that

$$h^2 \int_\Gamma |\partial_\nu w|^2 d\sigma \leq K \left(\int_\Omega |w|^2 + h^2|\nabla w|^2 dx + \int_\Gamma |g|^2 + h^2|\nabla_\Gamma g|^2 d\sigma \right), \quad g \in H^1(\Gamma), \quad h > 0. \quad (3.73)$$

(Here w satisfies (3.15) with f replaced by $-2ihw$ and Γ_0 replaced by Γ .) To complete the proof, simply make the sum of (3.72) and (3.73) with ε chosen sufficiently small. \square

Remark 3.14. In (3.69) the additional ‘‘damping’’ term $2ihw$ allows us to stay away from the spectrum of the Dirichlet Laplacian, so that w is uniquely determined by its boundary data g . For the pure Helmholtz equation $(-h^2\Delta - 1)w = 0$, Dirichlet-to-Neumann bounds such as (3.70) must take the form of *a priori estimates*; see for instance [Spe14, BSW16].

We are ready to introduce our splitting of quasimodes u . Let u_1 be the unique solution to (3.69) with Dirichlet data $g = u|_\Gamma$, and let $u_2 \triangleq u - u_1$.

Corollary 3.15. *For $o(h^{1+\delta})$ -quasimodes u ,*

$$\|u_1\|_{H_h^1(\Omega)} + h\|\partial_\nu u_1\|_{L^2(\Gamma)} = o(1), \quad h \rightarrow 0. \quad (3.74)$$

Proof. This is an immediate consequence of Lemma 3.8 and Proposition 3.13. Here we also use that $\|u\|_{H_h^1(\Gamma)} = \|u\|_{H_h^1(\Gamma_0)}$ since $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$ and $u|_{\Gamma_1} = 0$. \square

Now we deal with the remaining part u_2 . This is the final step towards the conclusion that there are actually no $o(h^3)$ -quasimodes under the geometric control condition.

Proposition 3.16 (Observation from the boundary). *Suppose that Assumption 1.1 holds. For $o(h^3)$ -quasimodes u ,*

$$\|u_2\|_{H_h^1(\Omega)} = o(1), \quad h \rightarrow 0. \quad (3.75)$$

Proof. Under the geometric control condition and by virtue of [BLR92, Theorem 3.8], the wave equation with homogeneous Dirichlet boundary condition is exactly observable through the Neumann trace on Γ_0 . Lemma B.8 expresses this fact in resolvent form as a *Hautus test* and yields $M, m > 0$ such that

$$\|w\|_{H_h^1(\Omega)} \leq h^{-1}M\|(-h^2\Delta - 1)w\|_{L^2(\Omega)} + mh\|\partial_\nu w\|_{L^2(\Gamma_0)}, \quad w \in \text{dom}(\Delta_D), \quad h > 0. \quad (3.76)$$

(Here Δ_D denotes the Dirichlet Laplacian, introduced in the proof of Lemma 3.12.) By construction $u_2 = u - u_1$ solves

$$(-h^2\Delta - 1)u_2 = f + 2ihu_1 \quad \text{in } \Omega, \quad (3.77a)$$

$$u_2 = 0 \quad \text{on } \Gamma. \quad (3.77b)$$

In particular $u_2 \in \text{dom}(\Delta_D)$ and (3.76) gives, with the triangle inequality,

$$\|u_2\|_{H_h^1(\Omega)} \leq h^{-1}M(\|f\|_{L^2(\Omega)} + 2h\|u_1\|_{L^2(\Omega)}) + mh(\|\partial_\nu u\|_{L^2(\Gamma_0)} + \|\partial_\nu u_1\|_{L^2(\Gamma_0)}) \quad (3.78)$$

Thus the result follows from Proposition 3.11 and Corollary 3.15. \square

Corollary 3.17. *Under Assumption 1.1 there are in fact no $o(h^3)$ -quasimodes.*

Proof. By Corollary 3.15 and Proposition 3.16, $o(h^3)$ -quasimodes u must satisfy $\|u\|_{L^2(\Omega)} = o(1)$ as $h \rightarrow 0$. Since $\|u\|_{L^2(\Gamma_0)} \rightarrow 0$ as well, this contradicts $\|u\|_H = 1$. \square

Our study of quasimodes is now complete, and at last we can write down the proof of Proposition 3.2.

Proof of Proposition 3.2. Observe first that

$$\|P^{-1}(i\lambda)\|_{\mathcal{L}(H)} = \|P^{-1}(-i\lambda)^*\|_{\mathcal{L}(H)} = \|P^{-1}(-i\lambda)\|_{\mathcal{L}(H)}, \quad \lambda \in \mathbb{R}, \quad \lambda \neq 0. \quad (3.79)$$

Thus it suffices to prove that (3.4) holds for $\lambda \rightarrow +\infty$. Arguing by contradiction, suppose that for all integers $n \geq 1$ there exists a real $\lambda_n > n$ and a vector $(f_n, e_n) \in H$ such that

$$\lambda_n^{-1} \|P(i\lambda_n)^{-1}(f_n, e_n)\|_H > n \|(f_n, e_n)\|_H. \quad (3.80)$$

Letting $v_n \triangleq P(i\lambda_n)^{-1}(f_n, e_n) \in \text{dom}(A) \setminus \{0\}$ and $u_n \triangleq v_n / \|v_n\|_H$, we have $\|u_n\|_H = 1$ and

$$\|(-A + i\lambda_n BB^* - \lambda_n^2)u_n\|_H < \frac{1}{n\lambda_n}, \quad n \geq 1. \quad (3.81)$$

Writing $h_n \triangleq \lambda_n^{-1}$ and multiplying (3.81) by h_n^2 we immediately deduce that $\{(h_n, u_n)\}_{n \geq 1}$ is a sequence of normalized $o(h_n^3)$ -quasimodes in the sense of Definition 3.3, which is impossible by Corollary 3.17. \square

3.3 Focusing modes in the disk and optimality of the decay rate

Now we are in a position to prove Theorem 1.3, which guarantees that the energy decay rate of Theorem 1.2 is sharp. To do so we investigate the *spectrum* $\sigma(P)$ of the quadratic operator pencil P in the case where Ω is the unit disk $\mathbb{D} \triangleq \{x \in \mathbb{R}^2 : |x| < 1\}$, with $\Gamma = \Gamma_0 = \partial\mathbb{D}$. By spectrum we mean the set of complex numbers z such that $P(z)$ does not have an inverse in $\mathcal{L}(H)$; as we have seen in Section 2.2.1, it is made of *eigenvalues* z , for which the equation $P(z)u = 0$ has nontrivial solutions u which we refer to as *eigenvectors*. In Proposition 3.20 below, we exhibit a sequence of eigenvalues $\{z_k\}_{k \gg 1}$ such that

$$\text{Im } z_k \rightarrow +\infty, \quad \text{Re } z_k < 0, \quad \text{Re } z_k = O((\text{Im } z_k)^{-2}), \quad k \rightarrow +\infty. \quad (3.82)$$

This will show that the resolvent estimate of Proposition 3.2 cannot be improved (see Corollary 3.21) and yield Theorem 1.3 as a consequence of semigroup theory.

Analyzing the spectrum of P amounts to investigating a quadratic eigenvalue problem for the Laplacian with a special, z -dependent boundary condition. In the case of the disk, we shall therefore make heavy use of *Bessel functions* and their properties; see the monograph [Wat95] for extensive details. Recall that the Bessel function J_n of the first kind and of order $n \in \mathbb{N}$ is given by

$$J_n(z) = \sum_{m=0}^{+\infty} \frac{(-1)^m}{m!(m+k)!} \left(\frac{z}{2}\right)^{2m+n}, \quad z \in \mathbb{C}. \quad (3.83)$$

Each J_n is an entire function that solves the Bessel differential equation

$$z^2 \frac{d^2 J_n}{dz^2} + z \frac{dJ_n}{dz} + (z^2 - n^2)J_n = 0. \quad (3.84)$$

As we will see in a moment, (3.84) typically arises when expressing the Laplacian in *polar* coordinates (r, θ) :

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}. \quad (3.85)$$

Note that the Laplace–Beltrami operator Δ_Γ and the normal derivative ∂_ν on $\Gamma = \partial\mathbb{D}$ simply read as

$$\Delta_\Gamma = \frac{\partial^2}{\partial \theta^2}, \quad \frac{\partial}{\partial \nu} = \frac{\partial}{\partial r}. \quad (3.86)$$

Here and in the sequel, with a slight abuse in notation, we write polar coordinates freely: for instance, for a function u of \mathbb{D} , $u(x_1, x_2) = u(r, \theta)$, $r = (x_1^2 + x_2^2)^{1/2}$, $0 \leq r \leq 1$, $x_1 = r \cos \theta$, $x_2 = r \sin \theta$, $\theta \in \mathbb{R}/2\pi$. That is, we omit change of variable maps.

Lemma 3.18. *Let $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$. If λ is nonzero and satisfies*

$$(n^2 + i\lambda - \lambda^2)J_n(\lambda) + \lambda J'_n(\lambda) = 0, \quad (3.87)$$

then $i\lambda \in \sigma(P)$ ⁸ and a corresponding eigenvalue $u \in \text{dom}(A)$ is given by

$$u(r, \theta) = J_n(\lambda r) e^{in\theta}, \quad 0 \leq r \leq 1, \quad \theta \in \mathbb{R} \setminus 2\pi. \quad (3.88)$$

⁸Here we let λ be complex, in contrast with the previous sections.

Proof. Since J_n is entire, $J_n(\lambda r)$ cannot vanish for all $0 \leq r \leq 1$ unless $\lambda = 0$; as a result the function u given by (3.88) is nontrivial. Now, using (3.85) and then (3.84) we see that

$$-\Delta u(r, \theta) = -\frac{1}{r^2}(\lambda^2 r^2 J_n''(\lambda r) + \lambda r J_n'(\lambda r) - n^2 J_n(\lambda r))e^{in\theta} = \lambda^2 J_n(\lambda r)e^{in\theta} = \lambda^2 u(r, \theta) \quad (3.89)$$

for all $0 < r < 1$ and $\theta \in \mathbb{R}/2\pi$. On the other hand,

$$(-\Delta_\Gamma + i\lambda + \partial_\nu)u(\theta) = (n^2 + i\lambda)J_n(\lambda)e^{in\theta} + \lambda J_n'(\lambda)e^{in\theta} = \lambda^2 J_n(\lambda)e^{in\theta} = \lambda^2 u(1, \theta) \quad (3.90)$$

for all $\theta \in \mathbb{R}/2\pi$ due to (3.87). Recalling the explicit form of P , we immediately deduce from (3.89) and (3.90) that $P(i\lambda)u = 0$, as required. \square

Remark 3.19. We are not concerned with the multiplicity of eigenvalues or the description of all eigenvectors.

Since we look for eigenvalues z of P in the high-frequency regime, we shall investigate complex roots λ of (3.87) with large real part and small imaginary part. For fixed $n \in \mathbb{N}$, after dividing (3.87) by $n^2 + i\lambda - \lambda^2$, we expect that roots satisfy

$$J_n(\lambda) \simeq 0, \quad \operatorname{Re} \lambda \gg 1. \quad (3.91)$$

This suggests seeking our λ as perturbations of *real* large roots α of

$$J_n(\alpha) = 0, \quad (3.92)$$

which, in terms of Laplacian eigenvalues, corresponds to the well-known case of homogeneous Dirichlet boundary condition. Positive roots of (3.92) are discrete and denoted by α_{nk} , $n, k \in \mathbb{N}$, where the sequence is increasing in k for fixed n , and corresponding eigenmodes with zero Dirichlet data are spanned by

$$u_{nk}^\pm(r, \theta) = J_n(\alpha_{nk}r)e^{\pm in\theta}, \quad 0 \leq r \leq 1, \quad \theta \in \mathbb{R} \setminus 2\pi. \quad (3.93)$$

In particular, for fixed n , the limit $k \rightarrow +\infty$ yields *focusing modes* that are highly oscillating in the radial direction and concentrate around the origin: see [NG13, GN13] for an in-depth investigation of their localization properties. Our subsequent analysis will shed some light on how the associated eigenvalues behave upon replacing the homogeneous Dirichlet boundary condition with our dissipative dynamic boundary condition (1.1b). We consider the case $n = 0$ only.

Proposition 3.20 (Root asymptotics). *There exists a sequence $\{\lambda_k\}_{k \gg 1}$ of complex roots of*

$$(i\lambda - \lambda^2)J_0(\lambda) + \lambda J_0'(\lambda) = 0 \quad (3.94)$$

that satisfies

$$\lambda_k = k\pi - \frac{\pi}{4} + \frac{1}{8(k\pi - \frac{\pi}{4})} + \frac{k\pi - \frac{\pi}{4}}{1 + (k\pi - \frac{\pi}{4})^2} + \frac{i}{1 + (k\pi - \frac{\pi}{4})^2} + O\left(\frac{1}{k^3}\right), \quad k \rightarrow +\infty. \quad (3.95)$$

Proof. Let us first recall from [Wat95] some further facts regarding Bessel functions. For all $n \in \mathbb{N}$ and $z \in \mathbb{C}$, we have the bound $|J_n(z)| \leq e^{\operatorname{Im} z}$ and the recurrence relation

$$\frac{d}{dz}(z^n J_n(z)) = z^n J_{n-1}. \quad (3.96)$$

In particular, $J_0' = -J_1$, and both J_0 and J_0' remain bounded in any horizontal strip of finite width. Furthermore, $J_0''(z) = -J_1'(z) = -(J_0(z) + J_1(z)/z)$, and thus J_0' is uniformly Lipschitz on, say the region $\{z \in \mathbb{C} : \operatorname{Re} z \geq 1, -1 \leq \operatorname{Im} z \leq 1\}$. We shall use the Hankel expansion [Wat95, Section 7.21], which reads as

$$J_0(z) = \sqrt{\frac{2}{\pi z}} \left(p(z) \cos\left(z - \frac{\pi}{4}\right) - q(z) \sin\left(z - \frac{\pi}{4}\right) \right), \quad -\pi < \arg z < \pi, \quad |z| \rightarrow +\infty, \quad (3.97)$$

where we take the principal branch of the square root, and

$$p(z) = 1 + O\left(\frac{1}{|z|^2}\right), \quad q(z) = -\frac{1}{8z} + O\left(\frac{1}{|z|^3}\right), \quad |z| \rightarrow +\infty. \quad (3.98)$$

Similarly,

$$J_0'(z) = -\sqrt{\frac{2}{\pi z}} \left(f(z) \sin\left(z - \frac{\pi}{4}\right) + g(z) \cos\left(z - \frac{\pi}{4}\right) \right), \quad -\pi < \arg z < \pi, \quad |z| \rightarrow +\infty, \quad (3.99)$$

where

$$f(z) = 1 + O\left(\frac{1}{|z|^2}\right), \quad g(z) = \frac{3}{8z} + O\left(\frac{1}{|z|^3}\right), \quad |z| \rightarrow +\infty. \quad (3.100)$$

Furthermore, zeroes α_{0k} of J_0 satisfy the McMahon expansion [Wat95, Section 15.52]

$$\alpha_{0k} = k\pi - \frac{\pi}{4} + \frac{1}{8(k\pi - \frac{\pi}{4})} + O\left(\frac{1}{k^3}\right), \quad k \rightarrow +\infty. \quad (3.101)$$

With that in hand, we may proceed to the proof, which is divided into three steps. In what follows, since we consider the Bessel function J_0 only, we drop the subscript indicating the order $n = 0$.

Step 1: lower bound away from zeroes. Fix $\varepsilon \in \mathbb{C}$ with $|\varepsilon| \leq 1$. Then, according to (3.97),

$$H_k(\varepsilon) \triangleq \sqrt{\frac{\pi(\alpha_k + \varepsilon)}{2}} J(\alpha_k + \varepsilon) = \cos\left(\alpha_k + \varepsilon - \frac{\pi}{4}\right) + O\left(\frac{1}{k}\right), \quad k \rightarrow +\infty \quad (3.102)$$

Using (3.101) at leading order, we see that

$$\alpha_k + \varepsilon - \frac{\pi}{4} = k\pi - \frac{\pi}{2} + \varepsilon + O\left(\frac{1}{k}\right), \quad k \rightarrow +\infty, \quad (3.103)$$

and thus, with a trigonometric formula,

$$\cos\left(\alpha_k + \varepsilon - \frac{\pi}{4}\right) = (-1)^k \sin\left(\varepsilon + O\left(\frac{1}{k}\right)\right), \quad k \rightarrow +\infty. \quad (3.104)$$

Now recall that the only complex root of $\sin z = 0$ in the open unit disk is 0. As a result, there surely exists $c > 0$ such that $|\sin z| \geq c$ if $1/2 \leq |z| \leq 1$. In particular, if $|\varepsilon| = 3/4$ (say), then it follows from (3.102) and (3.104) that, whenever k is sufficiently large,

$$|H_k(\varepsilon)| = \left| \sqrt{\frac{\pi(\alpha_k + \varepsilon)}{2}} J(\alpha_k + \varepsilon) \right| \geq \frac{c}{2}. \quad (3.105)$$

A priori (3.105) holds for integers k greater than some quantity that depends on ε . Let us check that, actually, a uniform lower bound holds. First,

$$H'_k(\varepsilon) = \frac{d}{d\varepsilon} \sqrt{\frac{\pi(\alpha_k + \varepsilon)}{2}} J(\alpha_k + \varepsilon) = \sqrt{\frac{\pi}{2}} \left(-\frac{J(\alpha_k + \varepsilon)}{2\sqrt{\alpha_k + \varepsilon}} + \sqrt{\alpha_k + \varepsilon} J'(\alpha_k + \varepsilon) \right). \quad (3.106)$$

Equations (3.99) and (3.101) show that $\sqrt{\alpha_k} J'(\alpha_k) = O(1)$ as $k \rightarrow \infty$, and since, for large k , the $\alpha_k + \varepsilon$ live in a region where J' is uniformly Lipschitz continuous, we see that $\sqrt{\alpha_k + \varepsilon} J'(\alpha_k + \varepsilon)$, and thus $H'_k(\varepsilon)$ by (3.106), remains bounded uniformly in $|\varepsilon| \leq 1$ and k . Thus, when k is sufficiently large, H_k is Lipschitz continuous on the unit disk with a constant that we may choose independently of k . Therefore, since the circle of radius $3/4$ is compact, with a simple covering argument we may deduce from (3.105) that there exists $K \in \mathbb{N}$ such that $|H_k(\varepsilon)| \geq c/4$ for all $\varepsilon \in \mathbb{C}$ with $|\varepsilon| = 3/4$ and $k \geq K$. In particular, using (3.101) we deduce that there exists $c' > 0$ such that, taking K larger if needed,

$$\inf_{\varepsilon \in \mathbb{C}, |\varepsilon| = \frac{3}{4}} |J(\alpha_k + \varepsilon)| \geq \frac{c'}{\sqrt{k}}, \quad k \geq K. \quad (3.107)$$

Step 2: existence of complex roots. Our argument here is based on Rouché's theorem. We consider roots λ with large real part and small imaginary part, and thus both λ and $i\lambda - \lambda^2$ may be assumed nonzero. Therefore, (3.94) is equivalent to

$$J(\lambda) + \frac{1}{i - \lambda} J'(\lambda) = 0. \quad (3.108)$$

As mentioned above, we are looking for roots of the form $\lambda_k = \alpha_k + \varepsilon_k$, where ε_k is a complex number of small modulus. Recall that $\alpha_k \rightarrow +\infty$ as $k \rightarrow +\infty$, and thus, by taking k sufficiently large, we may assume that $|i - \alpha_k - \varepsilon| \geq 1$ for all complex numbers ε with $|\varepsilon| \leq 1$. Then, we define functions F_k and G_k of the variable ε by

$$F_k(\varepsilon) \triangleq J(\alpha_k + \varepsilon), \quad G_k(\varepsilon) \triangleq J(\alpha_k + \varepsilon) + \frac{1}{i - \alpha_k - \varepsilon} J'(\alpha_k + \varepsilon). \quad (3.109)$$

These are holomorphic on the open unit disk, where each F_k has a single simple zero, 0. Let

$$a_k \triangleq \sup_{\varepsilon \in \mathbb{C}, |\varepsilon| = \frac{3}{4}} \left| \frac{1}{i - \alpha_k - \varepsilon} J'(\alpha_k + \varepsilon) \right| = \sup_{\varepsilon \in \mathbb{C}, |\varepsilon| = \frac{3}{4}} |G_k(\varepsilon) - F_k(\varepsilon)|, \quad b_k \triangleq \inf_{\varepsilon \in \mathbb{C}, |\varepsilon| = \frac{3}{4}} |F_k(\varepsilon)|. \quad (3.110)$$

Then, using (3.101), boundedness of J' near the real axis and the lower bound (3.107) established at the previous step, we see that $a_k/b_k = O(1/\sqrt{k}) \rightarrow 0$ as $k \rightarrow +\infty$. As a result, there exists $K \in \mathbb{N}$ such that

$$|F_k(\varepsilon) - G_k(\varepsilon)| < |G_k(\varepsilon)|, \quad |\varepsilon| = \frac{3}{4}, \quad k \geq K. \quad (3.111)$$

Then, Rouché's theorem guarantees that, for all $k \geq K$, F_k possesses a unique zero ε_k , which is simple, in the open disk of radius $3/4$; by (3.108), $\lambda_k \triangleq \alpha_k + \varepsilon_k$ is indeed a complex root of (3.94).

Step 3: asymptotic behavior of roots. Let us now derive finer asymptotics for the sequence $\{\lambda_k\}_{k \geq 1}$. By (3.97) and (3.99), since $\lambda_k = \alpha_k + \varepsilon_k$ are roots of (3.94) they must satisfy

$$\begin{aligned} p(\alpha_k + \varepsilon_k) \cos\left(\alpha_k + \varepsilon_k - \frac{\pi}{4}\right) - q(\alpha_k + \varepsilon_k) \sin\left(\alpha_k + \varepsilon_k - \frac{\pi}{4}\right) \\ = \frac{1}{i - \alpha_k - \varepsilon_k} \left(f(\alpha_k + \varepsilon_k) \sin\left(\alpha_k + \varepsilon_k - \frac{\pi}{4}\right) + g(\alpha_k + \varepsilon_k) \cos\left(\alpha_k + \varepsilon_k - \frac{\pi}{4}\right) \right). \end{aligned} \quad (3.112)$$

To ease the notation, let $\varphi_k \triangleq k\pi - \pi/4$. In what follows we shall use McMahon's expansion (3.101) at the order $O(1/k^3)$. First, similarly as in (3.104), when $k \rightarrow +\infty$,

$$\cos\left(\alpha_k + \varepsilon_k - \frac{\pi}{4}\right) = \cos\left(k\pi - \frac{\pi}{2} + \frac{1}{8\varphi_k} + \varepsilon_k + O\left(\frac{1}{k^3}\right)\right) = (-1)^k \sin\left(\varepsilon_k + \frac{1}{8\varphi_k} + O\left(\frac{1}{k^3}\right)\right), \quad (3.113)$$

We claim that $\varepsilon_k \rightarrow 0$ as $k \rightarrow +\infty$. Indeed, by (3.98),

$$\text{Left-hand side of (3.112)} = \cos\left(\alpha_k + \varepsilon_k - \frac{\pi}{4}\right) + O\left(\frac{1}{k}\right), \quad k \rightarrow +\infty. \quad (3.114)$$

But we also see that the right-hand side of (3.112) converges to 0. Since for large k , the argument of the sinus in (3.113) lies within the unit disk, that convergence is possible only if $\varepsilon_k \rightarrow 0$ as $k \rightarrow +\infty$, which proves our claim. In fact, the right-hand side of (3.112) is of size $O(1/k)$; and since Taylor expansions in ε_k are now possible, (3.113) reveals in turn that $\varepsilon_k = O(1/k)$ as well. Then,

$$\sin\left(\varepsilon_k + \frac{1}{8\varphi_k} + O\left(\frac{1}{k^3}\right)\right) = \varepsilon_k + \frac{1}{8\varphi_k} + O\left(\frac{1}{k^3}\right), \quad k \rightarrow +\infty. \quad (3.115)$$

On the other hand

$$\sin\left(\alpha_k + \varepsilon_k - \frac{\pi}{4}\right) = (-1)^{k+1} \cos\left(\varepsilon_k + \frac{1}{8\varphi_k} + O\left(\frac{1}{k^3}\right)\right) = (-1)^{k+1} + O\left(\frac{1}{k^2}\right), \quad k \rightarrow +\infty, \quad (3.116)$$

and

$$\frac{1}{i - \alpha_k - \varepsilon_k} = \frac{1}{i - \varphi_k} + O\left(\frac{1}{k^3}\right), \quad k \rightarrow +\infty. \quad (3.117)$$

Our goal now is to expand (3.112) at the order $O(1/k^3)$. Keeping in mind the order of each term in our previous developments, we observe that the following asymptotic estimates, which result from (3.98) and (3.100), are good enough: as $k \rightarrow +\infty$,

$$p(\alpha_k + \varepsilon_k) = 1 + O\left(\frac{1}{k^2}\right), \quad q(\alpha_k + \varepsilon_k) = -\frac{1}{8\varphi_k} + O\left(\frac{1}{k^3}\right), \quad (3.118a)$$

$$f(\alpha_k + \varepsilon_k) = 1 + O\left(\frac{1}{k^2}\right), \quad g(\alpha_k + \varepsilon_k) = O\left(\frac{1}{k}\right). \quad (3.118b)$$

By carefully plugging the above computations into (3.112) we finally get

$$(-1)^k \varepsilon_k = \frac{(-1)^{k+1}}{i - \varphi_k} + O\left(\frac{1}{k^3}\right), \quad k \rightarrow +\infty, \quad (3.119)$$

which shows that

$$\varepsilon_k = -\frac{1}{i - \varphi_k} + O\left(\frac{1}{k^3}\right) = \frac{\varphi_k}{1 + \varphi_k^2} + \frac{i}{1 + \varphi_k^2} + O\left(\frac{1}{k^3}\right), \quad k \rightarrow +\infty, \quad (3.120)$$

as required. \square

Corollary 3.21. *For all sufficiently large integers k , let λ_k be the complex root of (3.94) given by Proposition 3.20. Then $z_k \triangleq i\lambda_k$ is an eigenvalue of P . Furthermore, writing $\omega_k \triangleq \text{Im } z_k$, and taking any sequence of normalized eigenvectors u_k associated with z_k , we have*

$$\|P(i\omega_k)u_k\|_{\mathcal{L}(H)} = O(\omega_k^{-1}), \quad k \rightarrow +\infty. \quad (3.121)$$

Proof. The first statement immediately follows from Lemma 3.18. We have $\omega_k = \text{Re } \lambda_k > 0$ and $\sigma_k \triangleq \text{Re } z_k = -\text{Im } \lambda_k < 0$. Then, taking normalized eigenvectors u_k ,

$$0 = P(z_k)u_k = (A + z_k BB^* + z_k^2)u_k = (A + i\omega_k BB^* - \omega_k^2)u_k + (\sigma_k BB^* + \sigma_k^2 + 2i\sigma_k \omega_k)u_k, \quad (3.122)$$

and recognizing here the expression of $P(i\omega_k)$ we deduce that

$$\|P(i\omega_k)u_k\|_H = O(|\sigma_k|) + O(\sigma_k^2) + O(|\sigma_k|\omega_k), \quad k \rightarrow +\infty. \quad (3.123)$$

Owing to the asymptotics of Proposition 3.20, $\sigma_k = O(1/k^2)$ and $\omega_k \sim k\pi$, and the result follows. \square

Proof of Theorem 1.3. If for some $\varepsilon > 0$, the supremum in (1.5) were finite for all data $(u_0, u_1) \in \text{dom}(A) \times V$ then [AL14, Proposition 2.4], based on [BT10, Theorem 2.4], would give

$$\|P(i\omega)^{-1}\|_{\mathcal{L}(H)} = O(|\omega|^{1-\varepsilon}), \quad \omega \rightarrow \pm\infty, \quad (3.124)$$

which contradicts Corollary 3.21. \square

4 Concluding remarks

We conclude our article by giving some comments and perspectives.

- In our work the whole dynamic boundary is damped. As a possible direction for future work one could investigate *localized* damping, of the form $b(x)\partial_t u$ in (1.1), similarly as in [Buf17]. With a (nonnegative, sufficiently smooth) control function b supported on some subset of Γ_0 that geometrically controls Ω , it is reasonable to expect that the energy decay rate $1/t$ of Theorem 3.1 remains valid.
- We have shown that the geometric control condition is essentially a sufficient condition for the stabilization rate $1/t$. With our approach we should also be able to establish slower rates when the geometric control conditions fails to hold but *non-uniform* Hautus tests are available nonetheless, for instance when the Schrödinger equation in Ω with homogeneous Dirichlet data is exactly observable from its Neumann trace on Γ_0 . In this more general setting however, there are reasons to believe that our black box observability argument would lead to non-optimal results; see the brief discussion regarding the result of [NL10] in our introduction. Whether or not the geometric control condition is necessary for achieving the energy decay rate $1/t$ remains an open question.
- An alternative approach to establishing Theorem 3.1 would be to prove that the Schrödinger equation with first-order (in time) Wentzell boundary condition

$$(i\partial_t - \Delta)\Psi = 0 \quad \text{in } \Omega \times (0, T), \quad (4.1a)$$

$$(i\partial_t - \Delta_\Gamma)\Psi = -\partial_\nu \Psi \quad \text{on } \Gamma_0 \times (0, T), \quad (4.1b)$$

$$\Psi = 0 \quad \text{on } \Gamma_1 \times (0, T), \quad (4.1c)$$

is exactly observable for some $T > 0$ (with respect to the $L^2(\Omega) \times L^2(\Gamma_0)$ -energy) from its Dirichlet trace on Γ_0 . We do not know if this property is equivalent to the stabilization rate $1/t$ for (1.1).

- Theorem 1.3 also provides a non-controllability result in the disk case $\Omega = \mathbb{D}$. Let us consider the “open-loop” system associated with (1.1):

$$(\partial_t^2 - \Delta)u = 0 \quad \text{in } \Omega \times (0, T), \quad (4.2a)$$

$$(\partial_t^2 - \Delta_\Gamma)u = -\partial_\nu u + f \quad \text{on } \Gamma \times (0, T), \quad (4.2b)$$

where the control f is taken in $L^2(\Gamma \times (0, T))$ and the state space is still \mathcal{H} . Without control ($f = 0$), (4.2) is conservative: the energy $E(u, t)$ of each solution remains constant. Even when T is taken large, this system fails to be *exactly* controllable; otherwise, following Lions’ Hilbert uniqueness method, the observability inequality

$$E(u, 0) \leq C \int_0^T \int_\Gamma |\partial_t u|^2 d\sigma dt, \quad (4.3)$$

would hold for some $C > 0$ and all initial data. In the context of second-order systems of the form (2.28), it is known that such a uniform observability estimate for the uncontrolled problem implies exponential stability of the feedback system (1.1),⁹ which is indeed not possible by Theorem 1.3.

- On the other hand, our analysis does not reveal any *a priori* obstruction to exact controllability or uniform stabilization by means of control or damping on the non-dynamic boundary Γ_1 . For instance, consider the particular case of a two-dimensional annular domain $\Omega = \{x \in \mathbb{R}^2 : R_1 < |x| < R_2\}$, $0 < R_1 < R_2$, with inner and outer boundaries $\Gamma_0 = \{|x| = R_1\}$ and $\Gamma_1 = \{|x| = R_2\}$. In this particular geometry, the authors of [BDEM22] investigated the following system:

$$(\partial_t^2 - \Delta)u = 0 \quad \text{in } \Omega \times (0, T), \quad (4.4a)$$

$$(\alpha\partial_t^2 - \beta\Delta_\Gamma)u = -\partial_\nu u \quad \text{on } \Gamma_0 \times (0, T), \quad (4.4b)$$

$$u = 0 \quad \text{on } \Gamma_1 \times (0, T), \quad (4.4c)$$

where α and β are positive parameters. Here the outer boundary Γ_1 carries a homogeneous Dirichlet boundary condition and satisfies the geometric control condition. They proved [BDEM22, Theorems 2.2 and 2.4] that exact observability of solutions to (4.4) via their velocity traces on the *static* boundary Γ_1 holds when $0 < \alpha < \beta$ but *not* when $0 < \beta < \alpha$. The case $\alpha = \beta$ is open. These findings constitute further evidence that, in models such as (1.1) and (4.4), the dynamic nature of the boundary coupling is the main limiting factor for stabilization and observation.

⁹This is sometimes referred to as *Russell’s principle*; see also [Har89, AT01, CPS⁺23].

A Tangential semiclassical pseudodifferential operators

In this section we define a class of *tangential* pseudodifferential operators on $\mathbb{R}_+^d \triangleq \mathbb{R}^{d-1} \times (0, +\infty)$ with a small parameter $h > 0$. We will mostly present or adapt results from [Zwo12, DZ19, LRLR22a]. We also wish to point out that supplementary material from [Buf17, Fil24] has been helpful to the authors.

Here $y = (y_1, \dots, y_{d-1}, y_d) = (y_\tau, y_d) \in \mathbb{R}_+^d$ is the space variable, y_τ and y_d being the tangential and normal components, respectively, and $\xi_\tau = (\xi_1, \dots, \xi_{d-1}) \in \mathbb{R}^{d-1}$ is the tangential dual variable. We fix a positive constant h_0 and consider a small parameter $0 < h < h_0$.

First we introduce some semiclassical symbols and pseudodifferential operators on \mathbb{R}^{d-1} . For $m \in \mathbb{Z}$, the symbol class $S_h^m(\mathbb{R}^{d-1} \times \mathbb{R}^{d-1})$ is the set of smooth functions $b : \mathbb{R}^{d-1} \times \mathbb{R}^{d-1} \times (0, h_0) \rightarrow \mathbb{C}$ satisfying, for all multiindices $(\alpha, \beta) \in \mathbb{N}^{d-1} \times \mathbb{N}^{d-1}$,

$$\sup_{0 < h < h_0} \sup_{(y_\tau, \xi_\tau) \in \mathbb{R}^{d-1} \times \mathbb{R}^{d-1}} (1 + |\xi_\tau|^2)^{(|\beta| - m)/2} |\partial_{y_\tau}^\alpha \partial_{\xi_\tau}^\beta b(y_\tau, \xi_\tau; h)| < +\infty. \quad (\text{A.1})$$

We may associate to each symbol $b \in S_h(\mathbb{R}^{d-1} \times \mathbb{R}^{d-1})$ a semiclassical pseudodifferential operator $\text{Op}_h(b)$ via the standard quantization procedure:

$$(\text{Op}_h(b)\varphi)(y_\tau) \triangleq \frac{1}{(2\pi h)^{d-1}} \iint_{\mathbb{R}^{d-1} \times \mathbb{R}^{d-1}} e^{ih^{-1}\langle y_\tau - y'_\tau, \xi_\tau \rangle} b(y_\tau, \xi_\tau; h) \varphi(y'_\tau) dy'_\tau d\xi_\tau, \quad y_\tau \in \mathbb{R}^{d-1}, \quad (\text{A.2})$$

where φ is taken in the Schwartz class $\mathcal{S}(\mathbb{R}^{d-1})$. Such operators are linear and continuous on $\mathcal{S}(\mathbb{R}^{d-1})$ and also $\mathcal{S}'(\mathbb{R}^{d-1})$ (tempered distributions); see [Zwo12, Theorem 4.16]. In (A.2) the integral may be viewed as an *oscillatory integral*, or rewritten as $(\text{Op}_h(b)\varphi)(y_\tau) = \mathcal{F}_h^{-1}(b(y_\tau, \cdot; h)\mathcal{F}_h\varphi)(y_\tau)$, where \mathcal{F}_h denotes the semiclassical Fourier transform; for details regarding convergence, the reader is referred to [Zwo12, Chapters 3 and 4] or [LRLR22a, Section 2.4]. The resulting space of semiclassical pseudodifferential operators is denoted by $\Psi_h^m(\mathbb{R}^{d-1})$.

Now, the class of tangential symbols $S_{\tau, h}^m(\overline{\mathbb{R}_+^d} \times \mathbb{R}^{d-1})$, $m \in \mathbb{Z}$, is the set of smooth functions $a : \overline{\mathbb{R}_+^d} \times \mathbb{R}^{d-1} \times (0, h_0) \rightarrow \mathbb{C}$ satisfying, for all multiindices $(\alpha, \beta) \in \mathbb{N}^d \times \mathbb{N}^{d-1}$,

$$\sup_{0 < h < h_0} \sup_{(y, \xi_\tau) \in \overline{\mathbb{R}_+^d} \times \mathbb{R}^{d-1}} (1 + |\xi_\tau|^2)^{(|\beta| - m)/2} |\partial_y^\alpha \partial_{\xi_\tau}^\beta a(y, \xi_\tau; h)| < +\infty. \quad (\text{A.3})$$

We then introduce a quantization procedure Op_h^τ for symbols in $S_{\tau, h}^m(\overline{\mathbb{R}_+^d} \times \mathbb{R}^{d-1})$. Let $\overline{\mathcal{S}}(\overline{\mathbb{R}_+^d})$ be the space of restrictions to $\overline{\mathbb{R}_+^d}$ of functions in $\mathcal{S}(\mathbb{R}^d)$. Given $a \in S_{\tau, h}^m(\overline{\mathbb{R}_+^d} \times \mathbb{R}^{d-1})$ and $\varphi \in \overline{\mathcal{S}}(\overline{\mathbb{R}_+^d})$, we let

$$(\text{Op}_h^\tau(a)\varphi)(y) \triangleq \frac{1}{(2\pi h)^{d-1}} \iint_{\mathbb{R}^{d-1} \times \mathbb{R}^{d-1}} e^{ih^{-1}\langle y_\tau - y'_\tau, \xi_\tau \rangle} a(y, \xi_\tau; h) \varphi(y'_\tau, y_d) dy'_\tau d\xi_\tau \quad (\text{A.4})$$

for all $y = (y_\tau, y_d) \in \overline{\mathbb{R}_+^d}$. Since φ is taken in $\overline{\mathcal{S}}(\overline{\mathbb{R}_+^d})$, the integrand in (A.4) as well as all its derivatives with respect to y are absolutely integrable, and thus $\text{Op}_h^\tau(a)\varphi$ is well-defined in $C^\infty(\overline{\mathbb{R}_+^d})$ by standard Lebesgue integration theory. These tangential operators constitute a space which we denote by $\Psi_{\tau, h}^m(\overline{\mathbb{R}_+^d})$. Additional results can be found in [LRLR22a, Section 2.10].

The subspace of $\Psi_{\tau, h}^m(\overline{\mathbb{R}_+^d})$ (resp. $\Psi_h^m(\mathbb{R}^{d-1})$) for which a (resp. b) is a polynomial of order at most $m \geq 0$ in ξ_τ with coefficients varying in y (resp. y_τ) is denoted by $\text{Diff}_{\tau, h}^m(\overline{\mathbb{R}_+^d})$ (resp. $\text{Diff}_h^m(\mathbb{R}^{d-1})$); those are spaces of (semiclassical) *differential operators*.

All these operators, defined here for smooth functions, possess continuous extension on some natural Sobolev spaces. For the reader's convenience, such properties are given precise statements when needed throughout the manuscript; again, see also [Zwo12, DZ19, LRLR22a]. When there is no risk of confusion, we shall drop \mathbb{R}_+^d and \mathbb{R}^{d-1} from our notation and simply write $S_{\tau, h}^m$, $\Psi_{\tau, h}$, and so on.

Remark A.1. We use a small parameter $h > 0$ as in [Zwo12, DZ19] whereas [LRLR22a] uses a large parameter $\lambda = 1/h$. The operator classes under consideration are essentially the same but the scaling and notation are different. For instance, if Δ_τ is the Laplacian on \mathbb{R}^{d-1} then, as a differential operator with large parameter in the terminology of [LRLR22a], $-\Delta_\tau - \lambda^2$ has principal symbol $|\xi_\tau|^2 - \lambda^2$; it is related to the semiclassical operator $-h^2\Delta_\tau - 1$ by

$$-\Delta_\tau - \lambda^2 = \text{Op}(|\xi_\tau|^2 - \lambda^2) = \lambda^2 \text{Op}_h(|\xi_\tau|^2 - 1) = \lambda^2(-h^2\Delta_\tau - 1), \quad (\text{A.5})$$

where Op denotes the non-semiclassical standard quantization on \mathbb{R}^{d-1} , that is, (A.2) with $h = 1$. The reader is referred to [LRLR22a, Section 2.11] for further details regarding the correspondence.

We conclude this section with a positivity inequality adapted from [LRLR22a, Theorem 2.50].

Theorem A.2 (Microlocal tangential Gårding inequality). *Consider*

- A tangential semiclassical differential operator $P = \text{Op}_h(p) \in \text{Diff}_{\tau, h}^m$, where $m \geq 0$ is even;
- A compact subset C of $\overline{\mathbb{R}_+^d}$ and an open subset U of $C \times \mathbb{R}^{d-1}$;
- A tangential symbol $\chi \in S_{\tau, h}^0$ that is h -independent and satisfies $\text{supp}(\chi) \subset U$.

Suppose that there exists $c > 0$ such that

$$\operatorname{Re} p(y, \xi_\tau) \geq c |\xi_\tau|^m, \quad (y, \xi_\tau) \in U. \quad (\text{A.6})$$

Then, there exist $K, K' > 0$ and $0 < h_1 < h_0$ such that, writing $X \triangleq \operatorname{Op}_h^\tau(\chi)$,

$$\iint_{\mathbb{R}_+^d} P X \varphi \overline{X \varphi} \, dy_d \, dy_\tau \geq K \iint_{\mathbb{R}_+^d} \sum_{\substack{\alpha \in \mathbb{N}^{d-1} \\ |\alpha| \leq m/2}} h^{2|\alpha|} |D^{(\alpha, 0)} X \varphi|^2 \, dy_d \, dy_\tau - K' h^m \iint_{\mathbb{R}_+^d} |\varphi|^2 \, dy_d \, dy_\tau \quad (\text{A.7})$$

for all $\varphi \in \overline{\mathcal{S}}(\mathbb{R}_+^d)$ and all $0 < h < h_1$.

Proof. This is a particular case of [LRLR22a, Theorem 2.50] reformulated in the setting of a small parameter h instead of a large parameter λ ; see Remark A.1. Note that the homogeneity hypotheses of that theorem are automatically satisfied due to P being a differential operator and χ being h -independent. \square

B The geometric control condition

In this section we will give a precise statement of the *geometric control condition* (“GCC”) and briefly discuss its central role in observation and control of wave equations. The exposition here is inspired by [MS78, BLR92, LRLTT17, Kle25].

Wave equations possess solutions that are localized near some particular curves in space-time called *rays*. Rays follow the laws of geometric optics and in the case of a one-dimensional spatial domain they coincide with characteristic curves. To effectively observe or control such solutions, it is necessary that every ray intersects the observation or damping region. The use of rays for the stability analysis of damped hyperbolic systems originates in the work of Rauch and Taylor [RTP74], which considers boundaryless manifolds and builds upon Hörmander’s theorem on the propagation of singularities. In the case of boundary observation and damping, the theory was subsequently developed by Bardos, Lebeau, and Rauch [BLR92].

Recall that Ω is a smooth bounded domain of \mathbb{R}^d and let $M \triangleq \mathbb{R} \times \overline{\Omega}$ be the associated space-time manifold. For simplicity of exposition and since we are interested in the pure wave equation (1.1a), we will only consider the wave operator $P = \partial_t^2 - \Delta$, whose principal symbol is $p = -\omega^2 + |\xi|^2$; here, $p = p(t, x, \omega, \xi)$ is a smooth real function of the cotangent bundle T^*M . The *Hamiltonian vector field* on T^*M associated with p is $H_p = -2\omega \partial_t + 2\xi \cdot \partial_x$. A *bicharacteristic* of P is an integral curve of the Hamiltonian vector field H_p in T^*M that is contained in the characteristic set

$$\operatorname{char}(p) \triangleq \{(t, x, \omega, \xi) \in T^*M : p(t, x, \omega, \xi) = 0, (\omega, \xi) \neq 0\}. \quad (\text{B.1})$$

Equivalently, it is a maximal solution $s \mapsto (t(s), x(s), \omega(s), \xi(s))$ of the Hamilton–Jacobi system

$$\frac{d}{ds}(t, x, \omega, \xi) = (\partial_\omega p, \partial_\xi p, -\partial_t p, -\partial_x p), \quad (\text{B.2})$$

which in our case reduces to

$$\frac{d}{ds}(t, x, \omega, \xi) = (-2\omega, 2\xi, 0, 0). \quad (\text{B.3})$$

Hence, for the wave operator, ω and ξ are constant along the flow while t and x evolve linearly. A ray is the projection of a bicharacteristic onto the (t, x) -space.

To define generalized bicharacteristics we follow [MS78]; see also [Hör07, Chapter 24], [LRLTT17, Section 1C1] and [Kle25, Appendix A.1]. Near the boundary $\Gamma = \partial\Omega$, using normal geodesic coordinates (x_τ, x_d) , the principal symbol of $-\Delta$ reads as $\xi_d^2 + \ell(x, \xi_\tau)$, where ℓ is a suitable elliptic tangential symbol; see Section 2.1.2. In these local coordinates $(t, x, \omega, \xi) = (t, x_\tau, x_d, \omega, \xi_\tau, \xi_d)$, the principal symbol p of the wave operator $\partial_t^2 - \Delta$ takes the form $p = -\omega^2 + \xi_d^2 + \ell(x, \xi_\tau)$, and the associated Hamiltonian vector field H_p reads as

$$H_p = -2\omega \partial_t + 2\xi_d \partial_{x_d} + (\partial_{\xi_\tau} \ell) \cdot \partial_{x_\tau} - (\partial_x \ell) \cdot \partial_\xi. \quad (\text{B.4})$$

We will need to define the *order of contact* of points at the boundary. Note that in local coordinates the boundary ∂T^*M of the cotangent bundle T^*M is described by $\{y_n = 0\}$, and let $\Sigma \triangleq \operatorname{char}(p) \cap \partial T^*M$.

Definition B.1 (Order of contact with the boundary). Let $q = (t, x, \omega, \xi) \in \Sigma$, and let f be the (smooth, boundary-defining) function defined in local coordinates near (t, x) by y_d . The *order of contact* of bicharacteristics with the boundary ∂M at q is the smallest integer $k \geq 1$ such that

$$(H_p^j f)(q) = 0 \quad \text{for } j = 0, \dots, k-1, \quad \text{and } (H_p^k f)(q) \neq 0. \quad (\text{B.5})$$

If $(H_p^j f)(q) = 0$ for all j , the contact is said to be of *infinite order*.

In what follows we make repeated use of the previous local coordinates. First, define a symbol r on ∂T^*M by

$$r(t, x_\tau, \omega, \xi_\tau) = -\omega^2 + \ell(0, x_\tau, \xi_\tau), \quad (\text{B.6})$$

so that r represents the tangential part of p restricted to ∂T^*M . The associated Hamiltonian vector field H_r on ∂T^*M is given by

$$H_r = -2\omega\partial_t + (\partial_{\xi_\tau}\ell) \cdot \partial_{x_\tau} - (\partial_{x_\tau}\ell) \cdot \partial_{\xi_\tau}. \quad (\text{B.7})$$

We define the *glancing set* $\mathcal{G} \subset \Sigma$ by

$$\mathcal{G} = \{(t, x_\tau, 0, \omega, \xi_\tau, \xi_d) \in \Sigma : \xi_d = r(t, x_\tau, \omega, \xi_\tau) = 0\}. \quad (\text{B.8})$$

The *hyperbolic set* $\mathcal{H} \triangleq \Sigma \setminus \mathcal{G}$ is the complement in Σ of the glancing set. Given $q = (t, x_\tau, 0, \omega, \xi_\tau, \xi_d) \in \Sigma$, observe that if $q \in \mathcal{H}$ then $r(t, x_\tau, \omega, \xi_\tau) = -\xi_d^2$; as a result, it is easy to see that

$$q \in \mathcal{H} \text{ if and only if } r(t, x_\tau, \omega, \xi_\tau) < 0, \quad q \in \mathcal{G} \text{ if and only if } r(t, x_\tau, \omega, \xi_\tau) = 0. \quad (\text{B.9})$$

We may decompose the glancing set as $\mathcal{G} = \mathcal{G}_2 \supset \mathcal{G}_3 \supset \dots \supset \mathcal{G}^\infty$, where $1 \leq k \leq +\infty$ indicates the order of contact with the boundary. The construction of generalized bicharacteristics requires that the glancing set $\mathcal{G}^2 \setminus \mathcal{G}^3$ of order exactly 2 be decomposed into the union of a *diffractive set* \mathcal{G}_d^2 and a *gliding set* \mathcal{G}_g^2 :

$$\mathcal{G}_d^2 \triangleq \{q \in \mathcal{G}^2 \setminus \mathcal{G}^3 : (H_p^2 f)(q) > 0\}, \quad \mathcal{G}_g^2 \triangleq \{q \in \mathcal{G}^2 \setminus \mathcal{G}^3 : (H_p^2 f)(q) < 0\}, \quad (\text{B.10})$$

where f is as in Definition B.1. More generally, we consider glancing sets $\mathcal{G}^{2l} \setminus \mathcal{G}^{2l+1}$ of even order $2l$, $l \geq 1$, and we let

$$\mathcal{G}_d^{2l} \triangleq \{q \in \mathcal{G}^{2l} \setminus \mathcal{G}^{2l+1} : (H_p^{2l} f)(q) > 0\}, \quad \mathcal{G}_g^{2l} \triangleq \{q \in \mathcal{G}^{2l} \setminus \mathcal{G}^{2l+1} : (H_p^{2l} f)(q) < 0\}, \quad (\text{B.11})$$

and, just as in [LRLTT17], we define the set of *diffractive points* as

$$\mathcal{G}_d \triangleq \bigcup_{l \geq 0} \mathcal{G}_d^{2l}. \quad (\text{B.12})$$

We are in a position to present a definition of generalized bicharacteristics.

Definition B.2 (Generalized bicharacteristic). A *generalized bicharacteristic* of p is a map

$$\gamma : \text{dom}(\gamma) \subset \mathbb{R} \rightarrow (\text{char}(p) \setminus \partial T^*M) \cup \mathcal{G}, \quad (\text{B.13})$$

where $\text{dom}(\gamma)$ is open, that is differentiable and satisfies the following properties:

1. For all $s \in \text{dom}(\gamma)$, $\dot{\gamma}(s) = H_p(\gamma(s))$ if $\gamma(s) \in \text{char}(p) \setminus \partial T^*M$ or $\gamma(s) \in \mathcal{G}_d^2$;
2. For all $s \in \text{dom}(\gamma)$, $\dot{\gamma}(s) = H_r(\gamma(s))$ if $\gamma(s) \in \mathcal{G} \setminus \mathcal{G}_d^2$;
3. For every $s_0 \in \mathbb{R} \setminus \text{dom}(\gamma)$, there exists $\delta > 0$ such that $s \in \text{dom}(\gamma)$ if $|s - s_0| < \delta$ and

$$\gamma(s) \in \text{char}(p) \setminus \partial T^*M, \quad 0 < |s - s_0| < \delta; \quad (\text{B.14})$$

furthermore, γ has left and right limits q^\pm at s_0 that satisfy, in local coordinates,

$$x_d^- = x_d^+ = 0, \quad (t^-, x_\tau^-) = (t^+, x_\tau^+), \quad (\omega^-, \xi_\tau^-) = (\omega^+, \xi_\tau^+), \quad \xi_d^- = -\xi_d^+. \quad (\text{B.15})$$

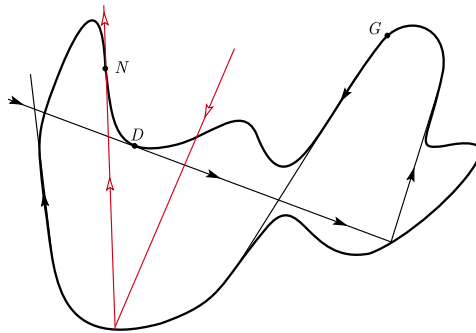


Figure 1: Two generalized bicharacteristics (projected onto the x -space), with different type of points of contact: nondiffractive (N), diffractive (D), gliding (G). Adapted from [Mil02, Figure 1.2].

Figure 1 illustrates projections of generalized bicharacteristics onto the x -space and different type of contacts at the boundary. Next, given a point (t, x) near ∂M , let ${}^bT_{(t,x)}M$ be the tangent space spanned by $\partial_t, \partial_{x_1}, \dots, \partial_{x_{d-1}}$, and $x_d \partial_{x_d}$. The *compressed cotangent bundle* is

$${}^bT^*M \triangleq \bigcup_{(t,x) \in M} ({}^bT_{(t,x)}M)^* \quad (\text{B.16})$$

and the *compression map* $j : T^*M \rightarrow {}^bT^*M$ is defined, using local coordinates, by

$$j(t, x_\tau, x_d, \omega, \xi_\tau, \xi_d) = j(t, x_\tau, x_d, \omega, \xi_\tau, x_d \xi_d). \quad (\text{B.17})$$

A *compressed generalized bicharacteristic* is the image under j of a generalized bicharacteristic of p . We will use the following standard hypothesis on points of contact with the boundary.

Assumption B.3 (No infinite-order contact). We have $\mathcal{G}^\infty = \emptyset$.

Proposition B.4 ([MS78]). *Under Assumption B.3, compressed generalized bicharacteristic are uniquely determined by any of their points.*

Remark B.5. One can show that Assumption B.3 is always verified when Γ is real-analytic.

Assumption B.3 guarantees that every bicharacteristic admits a unique continuation as a compressed generalized bicharacteristic. Now we have all the tools to give a proper definition of the geometric control condition.

Definition B.6 (Geometric control condition). Suppose that Assumption B.3 holds. We say that Γ_0 satisfies the geometric control condition if there exists $T > 0$ such that, for every compressed generalized bicharacteristic ${}^b\gamma$ of p , there exists $s \in \mathbb{R}$ such that, writing ${}^b\gamma(s) = (t(s), x(s), \omega(s), \xi(s)) \in {}^bT^*M$, we have $0 < t(s) < T$, $x(s) \in \Gamma_0$, and ${}^b\gamma(s) \in j(\mathcal{H} \cup (\mathcal{G} \setminus \mathcal{G}_d))$.

Remark B.7. In the absence of infinite-order contact with the boundary (Assumption B.3), the geometric control condition *characterizes* exact observability of the wave equation with homogeneous Dirichlet boundary condition via the Neumann trace on Γ_0 ; see, in a more general setting, [BLR92] for the sufficiency, and [BG97] for the necessity.

Another important tool used in this work is the so-called *Hautus test*, a resolvent-type condition that characterizes exact observability by means of frequency-domain estimates involving the system's generator; see [BZ04, Mil05] and also [TW09, Chapter 6]. The next lemma gives the exact observability inequality from [BLR92] in the form of a Hautus test that is most convenient for our analysis.

Lemma B.8 (Hautus test). *Suppose that Assumption 1.1 holds, i.e., Assumption B.3 is valid and Γ_0 satisfies the geometric control condition. Then there exist constants $M, m > 0$ such that*

$$\|w\|_{H_h^1(\Omega)} \leq h^{-1} M \|(-h^2 \Delta - 1)w\|_{L^2(\Omega)} + m h \|\partial_\nu w\|_{L^2(\Gamma_0)}, \quad w \in \text{dom}(\Delta_D), \quad 0 < h \leq 1. \quad (\text{B.18})$$

Proof. Let $\mathcal{H}_0 \triangleq H_0^1(\Omega) \times L^2(\Omega) = \text{dom}((-\Delta_D)^{1/2} \times L^2(\Omega))$ endowed with its natural energy inner product. The operator $-\mathcal{A}_0$, where

$$\mathcal{A}_0 \triangleq \begin{pmatrix} 0 & -1 \\ -\Delta_D & 0 \end{pmatrix}, \quad \text{dom}(\mathcal{A}_0) = \text{dom}(\Delta_D) \times H_0^1(\Omega) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega), \quad (\text{B.19})$$

is skew-adjoint and generates a strongly continuous group $\{e^{-t\mathcal{A}_0}\}_{t \in \mathbb{R}}$ on \mathcal{H}_0 which is unitary and describes all finite-energy solutions to the wave equation with homogeneous Dirichlet conditions. Let $\mathcal{C} : \text{dom}(\mathcal{A}_0) \rightarrow L^2(\Gamma_0)$ be the observation operator given by

$$\mathcal{C} \triangleq (\partial_\nu \quad 0). \quad (\text{B.20})$$

A standard differential multiplier argument shows that $\partial_\nu u$ is well-defined in $L^2(\Gamma \times (0, T))$, $T > 0$, for all finite-energy solutions to the wave equation with zero Dirichlet data; see [Kom94, Theorem 2.2] or, alternatively, use Lemma 3.6. This means that \mathcal{C} is *admissible* for $-\mathcal{A}_0$ in the sense of [TW09, Definition 4.3.1]. On the other hand, [BLR92, Theorem 3.8], translated into system-theoretic language, asserts that the pair $(-\mathcal{A}_0, \mathcal{C})$ is exactly observable. Therefore, [TW09, Theorem 6.6.1] provides constants $M, m > 0$ such that

$$\|z\|_{\mathcal{H}_0}^2 \leq M^2 \|(\mathrm{i}\lambda + \mathcal{A}_0)z\|_{\mathcal{H}_0}^2 + m^2 \|\mathcal{C}z\|_{L^2(\Gamma_0)}^2, \quad z = (u_0, u_1) \in \text{dom}(\mathcal{A}_0), \quad \lambda \in \mathbb{R}. \quad (\text{B.21})$$

Now, choosing z of the form $z = (w, \mathrm{i}\lambda w)$ in (B.21) yields

$$\|\nabla w\|_{L^2(\Omega)^d}^2 + \lambda^2 \|w\|_{L^2(\Omega)}^2 \leq M^2 \|(-\Delta - \lambda^2)w\|_{L^2(\Omega)}^2 + m^2 \|\partial_\nu w\|_{L^2(\Gamma_0)}^2, \quad w \in \text{dom}(\Delta_D), \quad \lambda \in \mathbb{R}. \quad (\text{B.22})$$

In the semiclassical setting we take $\lambda = 1/h$ with $0 < h \leq 1$, and we rewrite (B.22) as

$$\|\nabla w\|_{L^2(\Omega)^d}^2 + h^{-2} \|w\|_{L^2(\Omega)}^2 \leq M^2 h^{-4} \|(-h^2 \Delta - 1)w\|_{L^2(\Omega)}^2 + m^2 \|\partial_\nu w\|_{L^2(\Gamma_0)}^2, \quad (\text{B.23})$$

Finally, we multiply (B.23) by h^2 , recall from (3.20) the definition of the semiclassical norm $\|\cdot\|_{H_h^1(\Omega)}$ and readily get (B.18) with a slight modification of the constants $m, M > 0$. \square

Remark B.9. If one allows infinite-order contact with the boundary, which can occur if Γ is merely C^∞ , uniqueness of compressed generalized bicharacteristics is lost and the propagation of singularities becomes more delicate. In this case, [BLR92, Theorem 3.4] provides exact observability but under stronger geometric conditions, for which, consequently, our Theorem 1.2 remains valid.

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