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# EXACT ENUMERATION OF GRAPHS AND BIPARTITE GRAPHS WITH DEGREE CONSTRAINTS

(EXTENDED ABSTRACT)

Emma Caizergues\*      Élie de Panafieu†

## Abstract

We provide a new explicit formula enumerating graphs with constraints on their degrees, such as regular graphs, and extend it to bipartite graphs. It relies on generating function manipulations and Hadamard products.

**Keywords.** regular graphs, exact enumeration, D-finite, differentially finite

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**Related work.** The most famous graphs with degree constraints are  $k$ -regular graphs, where all vertices have degree  $k$ . There are two natural generalizations: graphs with a given degree sequence, and graphs where all vertices have their degree in a given set. In this article, we consider the later. There is a large literature on the asymptotic enumeration [1, 22, 7] and typical structure of graphs with degree constraints [25, 18, 15, 14, 3, 16, 8]. We focus on exact enumeration. The main result in this field is that the generating function of graphs with their degrees in a given finite set is D-finite, meaning that it is solution of a differential equation with polynomial coefficients. The previous proofs relied on a symmetric function approach [13, 12, 23, 24]. It starts by considering the infinite product  $\prod_{1 \leq i < j} (1 + x_i x_j)$  representing graphs where the degree of vertex  $i$  is the power of  $x_i$ . Arguments on the D-finiteness of the scalar product of symmetric functions are then applied. In contrast, we obtain a formula (**Theorem 1**) for the generating function of those graphs that is *explicit* and uses only a *finite* number of variables (assuming the degrees are bounded). Our approach relies on direct translation of combinatorial properties into

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generating function equations (symbolic method [2, 11]), and manipulation of those equations, in particular using Hadamard products. Works of similar spirit include [4, Chapters 3,4,5,7] and [7]. Our expression provides a new proof of D-finiteness, as D-finite series are stable by Hadamard product [21, 27] and evaluation [28]. Although effective algorithms exist [5] to compute the differential equation characterizing the generating function of graphs with degree constraints, they are computationally costly and the differential equation is only known up to  $k = 4$  for  $k$ -regular graphs. We hope our new formula will allow the computation of differential equations for  $k$ -regular graphs with  $k \geq 5$  and fast enumeration of those graphs [19]. Our results extend to bipartite graphs with different degree sets for the left and right vertices. For bipartite graphs, we used a multidimensional version of the Hadamard product, that has been well studied in the literature [9, 20, 26]. To our knowledge, the asymptotic structure of those graphs has not been investigated [10], and we hope our work will be a step in that direction.

**Structure.** We enumerate successively several graph-like families with degree constraints: *weighted multigraphs*, *loopless weighted multigraphs*, *weighted graphs* and finally *graphs* in **Theorem 1**. They are all depicted in **Figure 1**. The generating function of the first family is expressed directly. Then, to go from each family  $\mathcal{A}$  to the next family  $\mathcal{B}$ , we first express the generating function of  $\mathcal{A}$  using the generating function of  $\mathcal{B}$ , then invert this relation. The extension to bipartite graphs is presented in **Theorem 2**.

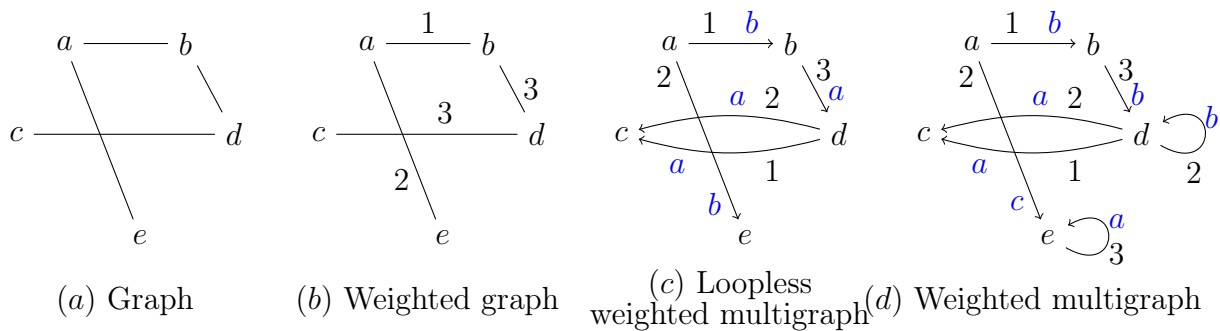


Figure 1: Steps of a possible transformation of a simple graph into a weighted multigraph. Labels are represented by letters, while weights are represented by integers.

**Notations.** The  $n$ th coefficient of a formal power series is denoted by  $[z^n] \sum_m a_m z^m = a_n$ . The *exponential Hadamard product* ([17, Theorem 3], [2, Section 2.1]) is defined as

$$\left( \sum_n a_n \frac{z^n}{n!} \right) \odot_z \left( \sum_n b_n \frac{z^n}{n!} \right) = \sum_n a_n b_n \frac{z^n}{n!}.$$

We denote by  $\odot_{z=1}$  the exponential Hadamard product followed by the evaluation at  $z = 1$ . Throughout this article, the variable  $\delta_d$  marks vertices of degree  $d$ . We denote by  $\delta$  the

infinite vector  $(\delta_0, \delta_1, \delta_2, \dots)$ . The same bold convention extends to other letters. An interesting particular case is, given a set  $\mathcal{D}$  of nonnegative integers, to set  $\delta_d = 1$  if  $d \in \mathcal{D}$ , and  $\delta_d = 0$  otherwise. That way, only graphs with all their vertices having degree in  $\mathcal{D}$  are counted. We associate to those variables the generating function

$$\Delta(z, \boldsymbol{\delta}) = \sum_{d \geq 0} \delta_d \frac{z^d}{d!}.$$

In the following, we consider graphs with vertices of degree at most  $D$  for some  $D \geq 0$ , so  $\delta_j = 0$  whenever  $j > D$  and  $\Delta(z, \boldsymbol{\delta})$  is a polynomial.

## 1 Weighted multigraphs

**Definition.** A weighted multigraph  $G$  is a finite sequence

$$G = (V(G), E_1(G), E_2(G), \dots)$$

where  $V(G) = \{1, 2, \dots, n(G)\}$  is the set of  $n(G)$  vertices, and for all  $j \geq 1$ ,  $E_j(G)$  is the finite set of  $m_j(G)$  edges of weight  $j$

$$E_j(G) = \{(u_{j,1}, v_{j,1}, 1), \dots, (u_{j,m_j(G)}, v_{j,m_j(G)}, m_j(G))\}$$

where each  $u_{i,j}$  and  $v_{k,\ell}$  belongs to  $V(G)$ . Thus, vertices are labeled, edges of weight  $j$  are labeled (and have their own independent label set), and all edges are oriented. Furthermore, loops and multiple edges are allowed. The degree  $\deg(u)$  of a vertex  $u$  is defined as the sum of the weights of all edges adjacent to it, counted twice if they are loops. For examples, in [Figure 1](#) (d), vertex  $e$  has degree 8.

**Generating function.** We use the variable  $z$  to mark the vertices, and for all  $j \geq 1$ , we use  $w_j/2$  to mark each edge of weight  $j$ . This factor  $1/2$  is here for historical reasons only [6, Section 2.3]. Additionally, for each  $d \geq 0$ , the variable  $\delta_d$  is introduced to mark vertices of degree  $d$ . The generating function of weighted multigraphs  $\text{WMG}(z, \mathbf{w}, \boldsymbol{\delta})$  is then defined as a sum over all weighted multigraphs

$$\text{WMG}(z, \mathbf{w}, \boldsymbol{\delta}) = \sum_G \left( \prod_{u=1}^{n(G)} \delta_{\deg(u)} \right) \left( \prod_{j \geq 1} \frac{(w_j/2)^{m_j(G)}}{m_j(G)!} \right) \frac{z^{n(G)}}{n(G)!}. \tag{1}$$

**Lemma 1.** Let  $P_{\text{WMG}}(\mathbf{x}, \boldsymbol{\delta})$  denote the polynomial

$$P_{\text{WMG}}(\mathbf{x}, \boldsymbol{\delta}) = \Delta(y, \boldsymbol{\delta}) \odot_{y=1} e^{\sum_{j=1}^D x_j y^j},$$

then the generating function of weighted multigraphs is equal to

$$\text{WMG}(z, \mathbf{w}, \boldsymbol{\delta}) = e^{\sum_{j=1}^D w_j x_j^2/2} \odot_{x_1=1} \cdots \odot_{x_D=1} e^{z P_{\text{WMG}}(\mathbf{x}, \boldsymbol{\delta})}.$$

*Proof.* Any weighted multigraph decomposes uniquely as a set of labeled vertices, each attached to a set of labeled half-edges of weight  $j$ , for  $j \geq 1$ . If the vertex has degree  $d$ , then the sum of the weights of the half-edges attached to it should be  $d$ . Then, using the variable  $x_j$  to mark the half-edges of weight  $j$ , the generating function of such sets is  $P_{\text{WMG}}(\mathbf{x}, \boldsymbol{\delta})$ . If the multigraph contains  $m_j$  edges of weight  $j$ , then after cutting them in two, we are left with  $2m_j$  half-edges of weight  $j$ . The symbolic method [11] implies

$$\text{WMG}(z, \mathbf{w}, \boldsymbol{\delta}) = \sum_{\mathbf{m}} (2\mathbf{m})! [\mathbf{x}^{2\mathbf{m}}] e^{z P_{\text{WMG}}(\mathbf{x}, \boldsymbol{\delta})} \prod_{j \geq 1} \frac{(w_j/2)^{m_j}}{m_j!}.$$

This expression is simplified using Hadamard products with the function

$$\sum_{m \geq 0} (2m)! \frac{(w/2)^m}{m!} \frac{x^{2m}}{(2m)!} = e^{wx^2/2}.$$

□

A multigraph is *loopless* if it has no edge containing twice the same vertex. The generating function  $\text{LWMG}(z, \mathbf{w}, \boldsymbol{\delta})$  of those weighted multigraphs is defined by restricting the sum from Equation (1) to them.

**Lemma 2.** Let  $P_{\text{LWMG}}(\mathbf{x}, \mathbf{w}, \boldsymbol{\delta})$  denote the polynomial

$$P_{\text{LWMG}}(\mathbf{x}, \mathbf{w}, \boldsymbol{\delta}) = \Delta(y, \boldsymbol{\delta}) \odot_{y=1} e^{\sum_{j=1}^D x_j y^j - \sum_{j=1}^D w_j y^{2j}/2},$$

then the generating function of loopless weighted multigraphs is equal to

$$\text{LWMG}(z, \mathbf{w}, \boldsymbol{\delta}) = e^{\sum_{j=1}^D w_j x_j^2/2} \odot_{x_1=1} \cdots \odot_{x_D=1} e^{z P_{\text{LWMG}}(\mathbf{x}, \mathbf{w}, \boldsymbol{\delta})}.$$

*Proof.* Any weighted multigraph is uniquely obtained by adding a set of loops on each vertex of a loopless weighted multigraph. During this operation, any vertex of degree  $d$  becomes a vertex of degree  $d + 2k$ , for some nonnegative integer  $k$ , and a set of weighted loops whose weights sum to  $k$ . This corresponds to replacing  $\delta_d$  with

$$\delta_d(\boldsymbol{\eta}, \mathbf{w}) = \sum_{k \geq 0} \eta_{d+2k} [y^k] e^{\frac{1}{2} \sum_{d \geq 1} w_d y^d} = \sum_{j \geq 0} \eta_j [y^j] y^d e^{\frac{1}{2} \sum_{d \geq 1} w_d y^{2d}}. \tag{2}$$

Thus, the generating functions of weighted multigraphs and loopless weighted multigraphs are linked by the relation

$$\text{WMG}(z, \mathbf{w}, \boldsymbol{\eta}) = \text{LWMG}(z, \mathbf{w}, \boldsymbol{\delta}(\boldsymbol{\eta}, \mathbf{w}))$$

Inverting Equation (2), we obtain

$$\eta_d(\boldsymbol{\delta}, \mathbf{w}) = \sum_{j \geq 0} \delta_j [y^j] y^d e^{-\frac{1}{2} \sum_{i \geq 1} w_i y^{2i}}$$

so

$$\Delta(z, \boldsymbol{\eta}(\boldsymbol{\delta}, \boldsymbol{w})) = \sum_{d=0}^D \sum_{j \geq 0} \delta_j [y^j] y^d e^{-\sum_{i=1}^D w_i y^{2i}/2} \frac{z^d}{d!} = \Delta(y, \boldsymbol{\delta}) \odot_{y=1} e^{yz - \sum_{i=1}^D w_i y^{2i}/2}.$$

Given the expression of  $\text{WMG}(z, \boldsymbol{w}, \boldsymbol{\delta})$  from [Lemma 1](#), we deduce

$$\begin{aligned} \text{LWMG}(z, \boldsymbol{w}, \boldsymbol{\delta}) &= \text{WMG}(z, \boldsymbol{w}, \boldsymbol{\eta}(\boldsymbol{\delta}, \boldsymbol{w})) \\ &= e^{\sum_{j=1}^D w_j x_j^2/2} \odot_{x_1=1} \cdots \odot_{x_D=1} e^{z\Delta(y, \boldsymbol{\eta}) \odot_{y=1} \exp(\sum_{j=1}^D x_j y^j)}. \end{aligned}$$

Finally, the properties  $F(xy) \odot_x G(x) = F(x) \odot_x G(xy)$  and  $F(x) \odot e^x = F(x)$  of the exponential Hadamard product imply

$$\begin{aligned} \Delta(y, \boldsymbol{\eta}) \odot_{y=1} e^{\sum_{j=1}^D x_j y^j} &= \Delta(y, \boldsymbol{\delta}) \odot_{y=1} e^{yz - \sum_{i=1}^D w_i y^{2i}/2} \odot_{y=1} e^{\sum_{j=1}^D x_j y^j} \\ &= \Delta(y, \boldsymbol{\delta}) \odot_{y=1} e^{\sum_{j=1}^D x_j y^j - \sum_{j=1}^D w_j y^{2j}/2} = P_{\text{LWMG}}(\boldsymbol{x}, \boldsymbol{w}, \boldsymbol{\delta}). \end{aligned}$$

□

## 2 Graphs

**Definitions.** A weighted graph  $G$  is a finite sequence

$$G = (V(G), E_1(G), E_2(G), \dots)$$

where  $V(G) = \{1, 2, \dots, n(G)\}$  is the set of  $n(G)$  vertices, and  $\bigcup_j E_j(G)$  is a finite set of edges, which are unordered pairs of distinct vertices.  $E_j(G)$  denotes the set of edges of weight  $j$  and its cardinality is  $m_j(G)$ . Thus, edges are unoriented and unlabeled, loops and multiple edges are forbidden. The degree of a vertex is still defined as the sum of the weights of its adjacent edges.

Unweighted graphs correspond to the case  $G = (V(G), E_1(G))$ , so  $E_j = \emptyset$  for all  $j \geq 2$ .

**Generating function.** The generating function of weighted graphs  $\text{WG}(z, \boldsymbol{w}, \boldsymbol{\delta})$  is defined as a sum over all weighted graphs

$$\text{WG}(z, \boldsymbol{w}, \boldsymbol{\delta}) = \sum_G \left( \prod_{u=1}^{n(G)} \delta_{\deg(u)} \right) \left( \prod_{j \geq 1} w_j^{m_j(G)} \right) \frac{z^{n(G)}}{n(G)!}.$$

This generating function is exponential with respect to the variable  $z$  marking the vertices, and ordinary with respect to the variables  $(w_j)_{j \geq 1}$  marking the edges. Note that the generating function of an edge of weight  $j$  is  $w_j$  (for weighted multigraphs, we used the convention  $w_j/2$ , linked to the fact that edges were oriented).

**Lemma 3.** For  $j \geq 0$ , let  $v_j(\mathbf{w})$  and  $P_{\text{WG}}(\mathbf{x}, \mathbf{w}, \boldsymbol{\delta})$  denote the polynomials

$$v_j(\mathbf{w}) = [y^j] \log \left( 1 + \sum_{j \geq 1} w_j y^j \right), \quad P_{\text{WG}}(\mathbf{x}, \mathbf{w}, \boldsymbol{\delta}) = \Delta(y, \boldsymbol{\delta}) \odot_{y=1} \frac{e^{\sum_{j=1}^D x_j y^j}}{\sqrt{1 + \sum_{j=1}^D w_j y^{2j}}},$$

then the generating function of weighted graphs is equal to

$$\text{WG}(z, \mathbf{w}, \boldsymbol{\delta}) = e^{\sum_{j=1}^D v_j(\mathbf{w}) x_j^2 / 2} \odot_{x_1=1} \cdots \odot_{x_D=1} e^{z P_{\text{WG}}(\mathbf{x}, \mathbf{w}, \boldsymbol{\delta})}.$$

*Proof.* Consider a weighted graph  $G$  with  $m_d$  edge of weight  $d$  for each  $d \geq 1$ . Then there exist  $\prod_d 2^{m_d} m_d!$  ways to orient and label those edges to turn  $G$  into a weighted multigraph. Thus, the generating function of loopless weighted multigraphs that contain no multiple edge is equal to the generating function of weighted graphs. To construct a loopless weighted multigraph  $G$  from a loopless weighted multigraph  $H$  without multiple edges, one replace each edge of  $H$  of weight  $d$  with a set of edges linking the same vertices and whose weights sum to  $d$ . For each  $d \geq 1$ , let us use the variable  $v_d$  to mark edges of weight  $d$  in loopless weighted multigraphs, and the variable  $w_d$  to mark edges of weight  $d$  in weighted graphs. The previous construction corresponds to replacing the variable  $w_d$  with

$$w_d(\mathbf{v}) = [y^d] (e^{\sum_{j \geq 1} v_j y^j} - 1).$$

Thus, the generating functions of weighted graphs and of loopless weighted multigraphs are linked by the relation

$$\text{WG}(z, \mathbf{w}(\mathbf{v}), \boldsymbol{\delta}) = \text{LWMG}(z, \mathbf{v}, \boldsymbol{\delta}).$$

Inverting this relation, we obtain

$$v_d(\mathbf{w}) = [y^d] \log \left( 1 + \sum_{j \geq 1} w_j y^j \right)$$

and

$$\text{WG}(z, \mathbf{w}, \boldsymbol{\delta}) = \text{LWMG}(z, \mathbf{v}(\mathbf{w}), \boldsymbol{\delta}).$$

Injecting the expression of  $\text{LWMG}(z, \mathbf{w}, \boldsymbol{\delta})$  from [Lemma 2](#) concludes the proof.  $\square$

In particular, for  $\mathbf{w} = (w, 0, 0, \dots, 0)$ , we recover the case of classical graphs, and more specifically the case of  $k$ -regular graphs, by setting  $\delta_d = 1$  if  $d = k$ , and  $\delta_d = 0$  otherwise.

**Theorem 1.** Let  $P_G(\mathbf{x}, w, \boldsymbol{\delta})$  denote the polynomial

$$P_G(\mathbf{x}, w, \boldsymbol{\delta}) = \Delta(y, \boldsymbol{\delta}) \odot_{y=1} \frac{e^{\sum_{j=1}^D x_j y^j}}{\sqrt{1 + w y^2}},$$

then the generating function of graphs is equal to

$$G(z, w, \boldsymbol{\delta}) = e^{-\sum_{j=1}^D (-w)^j x_j^2 / (2j)} \odot_{x_1=1} \cdots \odot_{x_D=1} e^{z P_G(\mathbf{x}, w, \boldsymbol{\delta})}.$$

In particular, the number of  $k$ -regular graphs with  $n$  vertices is

$$e^{\sum_{j=1}^k (-1)^{j+1} x_j^2 / (2j)} \odot_{x_1=1} \cdots \odot_{x_k=1} \left( [y^k] \frac{e^{\sum_{j=1}^k x_j y^j}}{\sqrt{1 + y^2}} \right)^n.$$

### 3 Bipartite graphs

**Definition.** A bipartite graph  $G$  is a triplet  $(V(G), \tilde{V}(G), E(G))$  with  $V(G)$  (resp  $\tilde{V}(G)$ ) the set of labeled left-vertices (resp. right-vertices) and  $E(G) \subset V(G) \times \tilde{V}(G)$  the set of edges.

**Generating function.** The generating function  $\text{BG}(z, \tilde{z}, w, \boldsymbol{\delta}, \tilde{\boldsymbol{\delta}})$  of bipartite graphs with degree at most  $D$  is defined as a sum over bipartite graphs

$$\text{BG}(z, \tilde{z}, w, \boldsymbol{\delta}, \tilde{\boldsymbol{\delta}}) = \sum_G \left( \prod_{u=1}^{|V(G)|} \delta_{\deg(u)} \right) \left( \prod_{u=1}^{|\tilde{V}(G)|} \tilde{\delta}_{\deg(u)} \right) w^{|E(G)|} \times \frac{z^{|V(G)|}}{|V(G)|!} \times \frac{\tilde{z}^{|\tilde{V}(G)|}}{|\tilde{V}(G)|!}$$

This generating function is exponential with respect to the variable  $z$  (resp.  $\tilde{z}$ ) marking the left vertices (resp. right vertices), and ordinary with respect to the variable  $w$  marking the edges. For all  $d$ ,  $\delta_d$  (resp.  $\tilde{\delta}_d$ ) marks the left vertices (resp. right vertices) of degree  $d$ .

**Notation.** The *multivariate exponential Hadamard product* is defined as

$$\sum_m A_m \frac{z^m}{m!} \odot_m \sum_m B_m \frac{z^m}{m!} = \sum_m A_m B_m \frac{z^m}{m!}.$$

This extension is compatible with the univariate product in the sense

$$A(z_1)B(z_2) \odot_{z_1 z_2} C(z_1, z_2) = A(z_1) \odot_{z_1} (B(z_2) \odot_{z_2} C(z_1, z_2)).$$

**Theorem 2.** Let  $\mathbf{v} = ((-1)^{d+1}w^d/d)_{1 \leq d \leq D}$ ,  $\Delta(y, \boldsymbol{\delta}) = \sum_{d=0}^D \delta_d \frac{y^d}{d!}$  and let  $P_{\text{BG}}(\mathbf{w}, \boldsymbol{\delta})$  denote the polynomial

$$P_{\text{BG}}(\mathbf{w}, \boldsymbol{\delta}) = \Delta(y, \boldsymbol{\delta}) \odot_{y=1} e^{\sum_{d=1}^D w_d y^d}.$$

Then the generating function of bipartite graphs with degree constraints is equal to

$$\text{BG}(z, \tilde{z}, w, \boldsymbol{\delta}, \tilde{\boldsymbol{\delta}}) = e^{z P_{\text{BG}}(\mathbf{w}, \boldsymbol{\delta})} \odot_{\mathbf{w}=\mathbf{v}} e^{\tilde{z} P_{\text{BG}}(\mathbf{w}, \tilde{\boldsymbol{\delta}})}.$$

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