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Geometric flow of Bloch sphere manifolds and Quantum error corrections

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It is well known that quantum states, particularly qubits and BECs qubits, can be represented by a spherical structure known as the Bloch sphere. Several error models describe errors that effectively shrink the Bloch sphere, such as the depolarizing channel and particle loss, which can be mitigated through redundancy error correction models. In contrast, it may be possible to realize error correction by directly deforming the Bloch sphere itself. Geometric flows, particularly the Ricci flow, describe the evolution of a metric over time. In the Ricci flow equation, the metric contracts along directions of higher curvature, leading to a uniformization of curvature across the manifold. In our approach, we utilize such geometric flows to achieve error correction by allowing the metric of the Bloch sphere naturally defined by geometric quantities such as the Fubini-Study or Bures metric to evolve. To employ these geometric quantities and equations, it is necessary to endow the Bloch sphere with a differentiable manifold structure, which can indeed be done. Moreover, our error correction scheme is distinctive in that it does not rely on redundancy.

I. INTRODUCTION

Quantum error correction is an essential component for large-scale quantum computers. However, quantum systems are highly sensitive to their environment, leading to the occurrence of errors. Various forms of error models exist, and it is necessary to address them; many of these models manifest as deformations or shrinks of the Bloch sphere. Dealing with quantum errors, redundancy is often employed, and a variety of codes have been proposed, including the Shor code[1], Steane code[2], and surface codes[3]. In addition, particle loss errors can be particularly detrimental in multi-particle systems. Under such errors, it may not sufficient to simply replace the number of loss particles from the environment. Then quantum error correction continues to play a crucial role in multi-particle systems, and codes such as the cat code[4], GKP code[5], and binomial code[6] have been proposed to correct for particle loss and related errors. However, these error-correcting codes cannot correct an arbitrarily large number of particle losses. In practice, as the number of lost particles increases, issues arise such as deviations of the state from the code space and the inability to recover the logical state. This issue may potentially be overcome by moving beyond the framework of quantum mechanics described by unitary time evolution, for example, by employing approaches that involve non-Hermitian systems[7–9], non-Unitary systems[10] or open quantum systems.

In this way, the contraction of the Bloch sphere caused by errors can be understood within a geometric framework. Indeed, several theoretical frameworks have been proposed for the geometric treatment of the Bloch sphere[11–13]. In modern geometry, manifolds constitute a major area of geometry that has been extensively studied for their mathematical significance. The spherical surface $S^2 = \{(x, y, z) \in R^3 | x^2 + y^2 + z^2 = 1\}$ is a type of manifold in Riemannian geometry, characterized by charts and an atlas. While S^2 represents a unit sphere of radius one, we also consider $NS^2 = \{(x, y, z) \in$

$R^3 | x^2 + y^2 + z^2 = N\}$, a sphere of radius N . It should be noted that S^2 and NS^2 are topologically equivalent (homeomorphic). In addition, a differential structure can be introduced on a manifold, which allows one to naturally define metrics such as the Fubini-Study metric[14] and the Bures metric[11, 15, 16] on the Bloch sphere. However, it should also be noted that operations such as exterior differentiation lie outside the framework of unitary time evolution.

From a geometric perspective, particularly in our case where we focus on spherical surfaces, the Ricci flow equation serves as a time-evolution equation. The Ricci flow equation was introduced by Hamilton as an analogy to the heat diffusion equation[17]. According to the Ricci flow, the metric evolves in time so as to smooth out the curvature of the sphere and make it more uniform. Moreover, the normalized Ricci flow is a geometric flow that preserves the total area, and it is known that, after sufficient time, the surface approaches a sphere with a radius determined by its Gaussian curvature. There, time is treated not as physical time, but as a parameter.

Then, could Hamiltons idea be applied to the metric of a quantum system as well? While such an approach may fall outside the framework of unitary time evolution and thus requires caution, it also offers certain advantages. Giving to theoretical frameworks that go beyond conventional quantum computation, namely those that depart from unitary time evolution governed by a Hermitian Hamiltonian such as non-Hermitian systems[7–9] and non-unitary dynamics[10]. Errors like particle loss are difficult to fully correct within the framework of unitary evolution. Then, to address such errors, it may be necessary to invoke theories that transcend the conventional unitary framework. In our approach, error correction is achieved by allowing the metric of the Bloch sphere to evolve under Ricci flow, independently of the quantum system performing the computation. This scheme requires no redundancy and, in principle, can correct an large number of particle losses.

II. QUBIT BLOCH SPHERE AND BECS BLOCH SPHERE MANIFOLDS

A. Definition of Topological and Smooth Manifolds

Definition (Topological Manifold): A set M is called an n -dimensional *topological manifold* if it is equipped with a topology such that:

1. M is a Hausdorff space.
2. M satisfies the second countability axiom.
3. For every point $p \in M$, there exists an open neighborhood $U \subset M$ and an open set $V \subset \mathbb{R}^n$ together with a homeomorphism

$$\varphi : U \rightarrow V.$$

Each pair (U, φ) is called a *chart*, and a collection of charts covering M is called an *atlas*.

Definition (Smooth Manifold): A topological manifold M is called a *smooth* (or C^∞) manifold if it is equipped with an atlas \mathcal{A} such that for any two charts

$$(U, \varphi), (U', \psi) \in \mathcal{A},$$

whenever $U \cap U' \neq \emptyset$, the transition map

$$\psi \circ \varphi^{-1} : \varphi(U \cap U') \rightarrow \psi(U \cap U')$$

is a C^∞ map between open subsets of \mathbb{R}^n . This condition is called *smooth compatibility*.

B. Qubit Bloch sphere manifolds

Any density matrix ρ of a single qubit can be uniquely expressed in terms of the Pauli matrices $\sigma_x, \sigma_y, \sigma_z$ as

$$\rho = \frac{1}{2} (I + \vec{r} \cdot \vec{\sigma}), \quad (1)$$

where

- I is the 2×2 identity matrix,
- $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$,
- $\vec{r} = (x, y, z) \in \mathbb{R}^3$ is the Bloch vector,
- $\vec{r} \cdot \vec{\sigma} = x\sigma_x + y\sigma_y + z\sigma_z$.

A pure state on the Bloch sphere can be written in spherical coordinates (θ, ϕ) as

$$|\psi\rangle = \cos \frac{\theta}{2} |0\rangle + e^{-i\phi} \sin \frac{\theta}{2} |1\rangle \quad (2)$$

where $0 \leq \theta \leq \pi$, and $0 \leq \phi < 2\pi$. The corresponding Bloch vector is given by

$$\vec{r} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \quad (3)$$

These interpret following coordinates

$$\begin{aligned} \langle \sigma_x \rangle &= \sin \theta \cos \phi, \\ \langle \sigma_y \rangle &= \sin \theta \sin \phi, \\ \langle \sigma_z \rangle &= \cos \theta, \end{aligned} \quad (4)$$

which parametrizes points on the unit sphere S^2 . On S^2 , one can construct a C^∞ structure by covering the north and south poles with different charts, using the transition map between the chart expressed in position coordinates and the chart expressed in spherical coordinates. Furthermore, since the global phase has no physical significance in a qubit system, the state space, modulo the equivalence relation $|\psi\rangle = e^{i\phi} |\psi\rangle$ is homeomorphic to the complex projective space CP^1 . Once the Bloch sphere is endowed with a smooth manifold structure, one can naturally introduce additional geometric objects such as tangent spaces, differential forms, and exterior derivatives. Moreover, treating the Bloch sphere as a manifold allows us to define the Fubini-Study metric, which plays the role of a Riemannian metric on CP^1 , in a natural manner. We regard the Bloch sphere as CP^1 equipped with a smooth Riemannian metric. Since the exterior derivative is well defined in this setting, the differential form $d|\psi\rangle \equiv |d\psi\rangle$ also acquires a precise meaning. Using this framework, the Fubini-Study metric as

$$\begin{aligned} ds^2 &= \langle d\psi | d\psi \rangle - \langle \psi | d\psi \rangle^2 \\ &= \frac{1}{4} (d\theta^2 + \sin^2 \theta d\phi^2). \end{aligned} \quad (5)$$

where,

$$\begin{aligned} |d\psi\rangle &= -\frac{1}{2} \sin \frac{\theta}{2} d\theta |0\rangle + e^{-i\phi} (-i \sin \frac{\theta}{2} d\phi \\ &\quad + \frac{1}{2} \cos \frac{\theta}{2} d\theta) |1\rangle, \end{aligned} \quad (6)$$

and $d : \Omega^0(S^2) \rightarrow \Omega^1(S^2)$. When the quantum system is in a mixed state, the metric on the Bloch sphere is given by the Bures metric. The Bures metric is

$$ds_{Bures}^2 = \frac{1}{4} (d\vec{r} \cdot d\vec{r} + \frac{(\vec{r} \cdot d\vec{r})^2}{1 - r^2}). \quad (7)$$

It should be noted that the operation of taking the exterior derivative lies outside the scope of unitary transformations. Also, in the case of pure states, the Bures metric coincides with the Fubini-Study metric.

C. BECs Bloch sphere manifolds

The BECs coherent state was introduced by Byrnes et al[18], together with the Bloch sphere structure. We first verify that the $S^2 \cong CP^1$ structure is also present on this BECs Bloch sphere same as the qubit case. To begin with, the BECs coherent state and the Stokes operators

are defined as

$$\begin{aligned}
|\cos \frac{\theta}{2}, e^{-i\phi} \sin \frac{\theta}{2}\rangle &\equiv \frac{1}{\sqrt{N!}} (\cos \frac{\theta}{2} a^\dagger + e^{-i\phi} \sin \frac{\theta}{2} b^\dagger)^N |\text{vac}\rangle, \\
S^x &= a^\dagger b + b^\dagger a, \\
S^y &= -ia^\dagger b + ib^\dagger a, \\
S^z &= a^\dagger a - b^\dagger b, \\
N &= a^\dagger a + b^\dagger b
\end{aligned} \tag{8}$$

where a^\dagger, b^\dagger obey commutation relations $[a, a^\dagger] = [b, b^\dagger] = 1$ and $|\text{vac}\rangle$ is a vacuum state. The radius of the BECs Bloch sphere can be expressed in terms of the sum of expectation value of the Stokes operators.

$$\begin{aligned}
\langle S^x \rangle &= N \sin \theta \cos \phi, \\
\langle S^y \rangle &= N \sin \theta \sin \phi, \\
\langle S^z \rangle &= N \cos \theta.
\end{aligned} \tag{9}$$

Sum is

$$\langle S^x \rangle^2 + \langle S^y \rangle^2 + \langle S^z \rangle^2 = N^2. \tag{10}$$

In this way, by choosing the polar coordinate chart

$$\vec{r} = (N \sin \theta \cos \phi, N \sin \theta \sin \phi, N \cos \theta). \tag{11}$$

In the case of the BECs Bloch sphere manifold, just as in the qubit Bloch sphere, one can introduce separate charts around the north and south poles. Then, the NS^2 structure is naturally induced, allowing us to treat it as a manifold. The key difference from the qubit case is that the radius is determined by the particle number, and therefore takes discrete values. It should be noted that, in the qubit case, including the interior of the sphere yields the space B^3 , whereas in the BECs case the angular directions remain smooth but the radial direction is discretized. As a result, the space is no longer a smooth manifold in the usual sense; instead, it may be regarded as a manifold with a partially broken differentiable structure. We denote this space by $M = \{r_n\}_{n \in \mathbb{Z}} \times NS^2 \equiv S_{r_n}^2$. Here, S^2 denotes the usual radius N , C^∞ smooth sphere, while $\{r_n\}$ has discrete values. Then (N, θ, ϕ) with $N \in \{1, 2, 3, \dots\}$, $(\theta, \phi) \in S^2$. The essential point is that the $S_{r_n}^2 \cong S^2$ exists, and that the space as a whole is characterized by a stacked-sphere structure, in which spheres are piled up in a stepwise manner along the radial direction. In this notation, the tangent bundle vanishes in the radial direction, where only a difference operator can be defined. In other words, the structure of the BECs Bloch sphere decomposes into a two-dimensional continuous part (θ, ϕ) and a one-dimensional discrete part N . Therefore, the directions θ and ϕ remain smooth, and both the Fubini-Study metric and the Bures metric can be defined on S^2 . In the case of pure states, these two metrics coincide and reduce to

$$ds^2 = \frac{N^2}{4} (d\theta^2 + \sin^2 \theta d\phi^2). \tag{12}$$

. Here, we have used the fact that it is

$$\begin{aligned}
d|\cos \frac{\theta}{2}, e^{-i\phi} \sin \frac{\theta}{2}\rangle &= \frac{1}{\sqrt{N!}} (-\frac{1}{2} \sin \frac{\theta}{2} d\theta a^\dagger + e^{-i\phi} (-i \sin \frac{\theta}{2} d\phi + \frac{1}{2} \cos \frac{\theta}{2} d\theta) b^\dagger) \\
&\times (\cos \frac{\theta}{2} a^\dagger + e^{-i\phi} \sin \frac{\theta}{2} b^\dagger)^{N-1} |\text{vac}\rangle.
\end{aligned} \tag{13}$$

Moreover we can calculate inner product using boson-particle commutation relations

$$\langle d\psi | d\psi \rangle = \frac{N^2}{4} (d\theta^2 + 4 \sin^2 \frac{\theta}{2} d\phi^2), \tag{14}$$

and

$$\langle \psi | d\psi \rangle = N \sin^2 \frac{\theta}{2} d\phi. \tag{15}$$

Then we can get

$$\begin{aligned}
ds^2 &= \langle d\psi | d\psi \rangle - |\langle \psi | d\psi \rangle|^2 \\
&= \frac{N^2}{4} (d\theta^2 + \sin^2 \theta d\phi^2).
\end{aligned} \tag{16}$$

From these results, it follows that the BECs Bloch sphere possesses the metric of a sphere with radius N .

III. ERROR MODELS

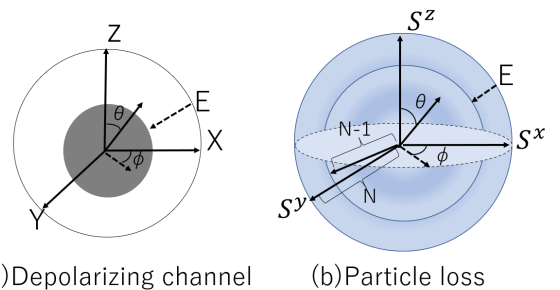


FIG. 1: (a) The depolarizing channel uniformly shrinks the radius of the qubit Bloch sphere by a factor of $(1 - 2p)$. (b) The particle loss channel shrinks the radius of the BEC Bloch sphere to $N - N'$. This figure corresponds to the case of $N' = 1$.

In this section, we examine the error model under consideration and its geometric representation. The type of errors we focus on are those that shrink the Bloch sphere. In contrast, errors that modify the phase of the Bloch vector or deform of the Bloch sphere are excluded from our analysis. A quantum error process can be described using Kraus operators as follows: $\mathcal{E}(\rho) \rightarrow \sum_i E_i \rho E_i^\dagger$. This mapping can be expressed in this form regardless of whether *an error actually occurs or not*. In fact, this form assumes that p represents the

average over multiple trials when the experimental setup is fixed. Although the weight p indicates the probability of an error occurring, it is noteworthy that it also has a geometric aspect as it determines the radius of the Bloch sphere.

A. Error models of qubits

A representative example of shrinking the Bloch sphere error models in a qubit system is the depolarizing channel. In this case, an X, Y, Z errors occur uniformly with probability $\frac{p}{3}$, and correspondingly, the radius of the Bloch sphere is shrunk along the all directions (Fig.1(a)). This error model is described by Kraus operators constructed from the Pauli operators, and is given by

$$\mathcal{E}(\rho) = (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z). \quad (17)$$

This error acts on the Bloch vector by shifting it as in $(x, y, z) \rightarrow ((1-p)x, (1-p)y, (1-p)z)$. Let us determine the metric of the Bloch sphere under the action of a depolarizing channel. First, the exterior derivative of the Bloch vector is given by

$$\begin{aligned} d\vec{r} = & ((1-p)(\cos\theta \cos\phi d\theta - \sin\theta \sin\phi d\phi), \\ & (1-p)(\cos\theta \sin\phi d\theta + \sin\theta \cos\phi d\phi), \\ & - (1-p)\sin\theta d\theta), \end{aligned} \quad (18)$$

and since the inner product between Bloch vectors is independent of θ and ϕ , it also yields $\vec{r} \cdot d\vec{r} = 0$, because $\vec{r} \cdot \vec{r} = \text{const} \rightarrow \vec{r} \cdot d\vec{r} = 0$. Consequently, the second term in the Bures metric vanishes. The Bures metric becomes

$$ds^2 = \frac{(1-p)^2}{4}(d\theta^2 + \sin^2\theta d\phi^2). \quad (19)$$

which is simply the metric of the sphere scaled by a factor of $(1-p)$. This result is consistent with the fact that the depolarizing channel uniformly contracts the sphere.

B. Error models of BECs qubits

Particle loss is an unavoidable error in many-body systems, and various error correction codes have been proposed[4–6], and [19]. We propose a particle-loss error channel model, whose Kraus operators are defined as

$$E = e^{-iH_{1,2}t}, \quad (20)$$

where

$$H_{1,2} \equiv (a_1^\dagger a_2 + b_1^\dagger b_2 + a_2^\dagger a_1 + b_2^\dagger b_1)/N. \quad (21)$$

Considering the commutation relation

$$[H_{1,2}, \alpha a_2^\dagger + \beta b_2^\dagger] = (\alpha a_1^\dagger + \beta b_1^\dagger)/N, \quad (22)$$

this Kraus operator functions by introducing system 1, which counts the particle number, and coupling it to system 2. The particle-loss channel can be expressed as

$$\mathcal{E}(\rho) = (1-p)\rho + pE\rho E^\dagger. \quad (23)$$

We can readily identify the space in which particle loss has occurred by measuring the auxiliary system. For example, if auxiliary system 1 yields a measurement outcome of the particle number as 1, the system used for quantum computation will have $N-1$ particles. the system, after appropriate renormalization, becomes

$$\rho_{\text{loss}} = \left| \cos\frac{\theta}{2}, e^{-i\phi} \sin\frac{\theta}{2} \right\rangle_2 \langle \langle \cos\frac{\theta}{2}, e^{-i\phi} \sin\frac{\theta}{2} | \quad (24)$$

In this case, The action of the error channel in particle loss reduces the particle number after measuring auxiliary system as $N' = N-1$, that is, it effectively shrinks the radius of the BECs Bloch sphere (Fig.1(b)). Since this state is pure state, we can calculate the Fubini-Study metric. A metric is

$$ds^2 = \frac{(N-1)^2}{4}(d\theta^2 + \sin^2\theta d\phi^2). \quad (25)$$

IV. RICCI-TYPE GEOMETRIC FLOWS AND QUANTUM ERROR CORRECTIONS

A. Ricci Flows

The Ricci flow equation was proposed by Hamilton in 1982[17]. He focused on the fact that the metric on a Riemannian manifold can be evolved by considering an equation analogous to the heat diffusion equation. According to Ref.[17], the Ricci Flow equation is defined as

$$\frac{\partial g_{ij}}{\partial t} = -2Ric_{i,j}. \quad (26)$$

where, $g_{i,j}$ denotes the Riemannian metric, and $Ric_{i,j}$ denotes the Ricci tensor. It is noteworthy that in two dimensions, the Ricci tensor can be expressed as in $Ric_{i,j} = Kg_{i,j}$ using the Gaussian curvature K . Then, the Ricci flow equation is

$$\frac{\partial g_{ij}}{\partial t} = -2Kg_{i,j}, \quad (27)$$

When the curvature is positive, the time evolution under the Ricci flow causes the sphere to gradually shrink and eventually collapse to a point. To avoid this, Hamilton considers the normalized Ricci flow, which is appropriately scaled and constrained to preserve the area.

$$\frac{\partial g_{ij}}{\partial t} = -2Ric_{i,j} + \frac{2}{n}\bar{K}g_{i,j}, \quad (28)$$

where $\bar{K} = \frac{1}{\text{Area}(M,g)} \int_M K d\mu$, usually on S^2 , $\bar{K} = \frac{4\pi}{4\pi R^2} = \frac{1}{R^2}$. Eq(27) becomes

$$\frac{\partial g}{\partial t} = (-2K + 2\bar{K})g. \quad (29)$$

It is known that this equation converges to $K = \bar{K} = \frac{1}{R_\infty^2}$ as $t \rightarrow \infty$ [17].

B. A Quantum error correction using the Ricci flow on the qubit Bloch sphere

We investigate quantum error correction based on the normalized Ricci flow equation in the qubit system. Our scheme is as follows. First, we define the code space as the set of states on the Bloch sphere that possess Gaussian curvature as a normalization term before any error occurs. When an error occurs, the radius of the Bloch sphere contracts with probability p . We then let the errored state evolve under the normalized Ricci flow so that it asymptotically approaches the pre-error Gaussian curvature by varying the metric. To realize our scheme, we need to compute the Gaussian curvature. Since all the metrics we deal with take the form shown in $ds^2 = A(p(t))d\theta^2 + B(p(t), \theta)d\phi^2$, the Gaussian curvature can be easily calculated using the following formula

$$K(p(t)) = -\frac{1}{2\sqrt{A(p(t))B(p(t), \theta)}} \times \frac{d}{d\theta} \left(\frac{1}{\sqrt{A(p(t))B(p(t), \theta)}} \frac{d}{d\theta} B(p(t), \theta) \right). \quad (30)$$

The Gaussian curvature after the error is expressed as

$$K = \frac{1}{(1-p(t))^2}. \quad (31)$$

Here, we take the probability $p(t)$ as the time factor associated with the time evolution of the Ricci flow. Originally, p represents the fixed error probability that occurs in a given experimental setup. However, we reinterpret $p(t)$ as describing the probabilistic flow of the entire experimental system, and consider the deformation of the Bloch sphere radius through $p(t)$ under the Ricci flow equation. By taking a sufficiently large t , the condition $K = \bar{K}$, namely $\frac{1}{(1-p(t))^2} = \frac{1}{1}$ is achieved, meaning that $p(t)$ decreases over time, the radius of Bloch sphere becomes original radius 1, then eventually settles to the result $p(t) \rightarrow 0$. Here, note that the time-evolving Fubini-Study metric, while it appears to be expressed in terms of the inner product of qubit states, is in fact defined independently from the original state vectors acting to unitary time evolutions, since the metric is a geometric quantity. The quantum state space CP^1 is first introduced, and then, as an additional geometric structure, the Fubini-Study metric is incorporated to provide a natural geometric interpretation for comparing distances between quantum states. Moreover, there are two independent types of time evolution: the original state vectors evolve unitarily, whereas the geometric quantities evolve non-unitarily. Since these two types of evolution concern independently defined quantities, the geometric structures, such as those related to quantum informational phases, remain intact under time evolution.

C. A Quantum error correction using the Ricci flow on the BECs Bloch sphere

We investigate a quantum error correction scheme for BECs systems based on the Ricci flow equation. The discussion proceeds in analogy with the qubit case; however, an important point is that the radius $N(t)$ takes discrete values, which means that continuous time derivatives cannot be defined. In other words, the Ricci flow equation must be reformulated in a discrete form using finite differences. In this case, while the Ricci flow equation becomes analytically simpler, there is no guarantee that K converges to \bar{K} as $t \rightarrow \infty$. Nevertheless, we show that even in the discrete Ricci flow equation, the radius of BECs Bloch sphere still converges to the original N in the limit of large t . In the following discussion, we work within the space $\{r_n\}_{n \in \mathbb{Z}} \times NS^2 \equiv S_{r_n}^2$. The meaning of this space is that it possesses a discrete, $(t \in \mathbb{R})$ -dependent radial coordinate $N(t)$ in the r -direction, while the space S^2 retains a continuous differentiable structure. First, the Gaussian curvature of Eq.(12) and Eq.(25) in general value $N(t)$ is given by

$$K = \frac{1}{N(t)^2}. \quad (32)$$

The discrete Ricci flow equation is defined as

$$\frac{ds^2(t+1) - ds^2(t)}{\Delta t} = -(K - \bar{K})ds^2(t). \quad (33)$$

This becomes

$$\frac{N^2(t+1) - N^2(t)}{\Delta t} = -(K - \bar{K})N^2(t). \quad (34)$$

We set the Gaussian curvature $\bar{K} = \frac{1}{N^2}$ as the code space, where N represents the radius of the BECs Bloch sphere before the error occurs. Since particle loss causes a decrease in the particle number, that is, $N^2(t+1) < N^2(t)$, we reverse the sign of equation on the right-hand side. With these considerations, the discrete Ricci flow equation as

$$N^2(t+1) = N^2(t) + \Delta t \left(1 - \frac{N^2(t)}{N^2}\right). \quad (35)$$

We introduce a perturbation into this equation to study its behavior. Assuming the following form, we perform a linearization: $x_t = N^2(t) - N^2$. Then, the discrete Ricci flow equation as

$$x_{t+1} = \left(1 - \frac{\Delta t}{N^2}\right)x_t. \quad (36)$$

This sequence converges to $x_{t=\infty} = 0$, then $N(\infty) = N$ when the condition $|1 - \frac{\Delta t}{N^2}| < 1$ namely, $0 < \Delta t < 2N^2$ is satisfied. Therefore, in the limit of large time, the radius of the Bloch sphere can be expanded, completing the quantum error correction.

V. CONCLUSION AND DISCUSSION

We have shown that error correction can be achieved by varying the radius of the Bloch sphere; alternatively, this can also be interpreted as a suppression of the error probability. From this perspective, the time evolution can be viewed not only as an error-correcting process but also as an experimental scheme for error mitigation. In the case of qubit systems, where the radius of the Bloch sphere is a continuous parameter, the error probability $p(t)$ takes continuous values. In contrast, for BECs qubits, the radius is discrete and directly associated with the number of particles. In qubit systems, other types of errors that deform the Bloch sphere, such as the phase flip channel, bit flip channel, and bit-phase flip channel, must also be considered. The Bures metric of the states after these errors is expected to be equivalent to that of a sphere in Riemann space; however, for example phase flip channel, due to the presence of the $\vec{r} \cdot d\vec{r}$ term, the equivalence does not hold exactly. This may indicate that the quantum system possesses certain intrinsic effects that are reflected in the metric structure. On the other hand, in BECs qubits, the radius of the Bloch sphere is determined by the number of particles and thus takes only discrete natural-number values, leav-

ing no room for such deformations, only shrinking. This fact may suggest that the nature of errors in BECs qubits is fundamentally different from that in conventional qubit systems. We also note that, since our scheme does not employ logical qubits, it is difficult to apply the Knill-Laflamme conditions or to determine an error threshold.

Regarding experimental feasibility, our error correction approach lies outside the framework of unitary time evolution, making it challenging to treat as a conventional quantum system. However, considering that the Ricci flow equation was originally introduced as an analogy to the heat conduction equation, a path toward experimental implementation may be conceivable. Specifically, it might be possible to realize such a system by coupling it to a thermal bath, allowing the metric to evolve over time while particles are quasi-statically diffused and supplied from the bath.

As an open problem, it remains to be explored whether geometric structures and geometric flows can be applied to other quantum systems, particularly multi-particle systems such as continuous-variable qubits and optical coherent states. In these systems as well, particle loss may have pathological effects. Clarifying the geometric picture underlying such systems constitutes an important direction for future research.

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