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# Inhomogeneous Sobolev and Besov Spaces: Embeddings and prevalent smoothness

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## Abstract

In this article, we introduce inhomogeneous Sobolev spaces that naturally generalise the standard Sobolev-Slobodeckij spaces. The inhomogeneity of these spaces is governed by a set function  $\mu$ , referred to as an environment. In the case where  $\mu$  is an almost doubling set function, we relate these new spaces with inhomogeneous Besov spaces recently introduced by Barral-Seuret in 2023. When  $\mu$  is in addition a capacity, we also prove that prevalent elements in such spaces are multifractal (with a singularity spectrum that we determine), completing previous Baire generic results already obtained.

## 1 Introduction

The interest in multifractal properties dates back to the 1980s, when physicists sought to quantify the local variations in the velocity of a turbulent fluid. Indeed, turbulent flows are not spatially homogeneous, and the pointwise regularity of the velocity seems to differ widely from point to point. The relationship between multifractality and the global scaling properties of the velocity was first discovered by U. Frisch and G. Parisi [24]. Since then, multifractal notions have proved to be useful and relevant in various mathematical domains: geometric measure theory, real analysis [13, 29, 50], dynamical systems and ergodic theory, diophantine approximation [20, 51, 52, 53], to name a few.

This article focuses on the multifractal analysis of functions, which consists in describing the pointwise behaviours of functions  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ , computing their singularity spectrum  $\sigma_f$ , and, when possible, relating  $\sigma_f$  with the scaling function associated with  $f$ . More precisely, given  $x_0 \in \mathbb{R}^d$ ,  $H \in \mathbb{R}_+$ , a function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  belongs to  $\mathcal{C}^H(x_0)$  if there exist a polynomial  $P_{x_0}$  of degree less than  $\lfloor H \rfloor$ , a constant  $C > 0$ , and a neighbourhood  $V$  of  $x_0$  such that

$$\forall x \in V, \quad |f(x) - P_{x_0}(x - x_0)| \leq C |x - x_0|^H. \quad (1.1)$$

The *pointwise Hölder exponent* of  $f \in L_{\text{loc}}^\infty(\mathbb{R}^d)$  at  $x_0$  is then

$$h_f(x_0) = \sup\{H \in \mathbb{R}_+ : f \in \mathcal{C}^H(x_0)\}. \quad (1.2)$$

The associated *singularity* (or *multifractal*) *spectrum*  $\sigma_f$  of  $f$  is the mapping

$$\sigma_f : h \in \mathbb{R} \cup \{\infty\} \mapsto \dim_H(E_f(h)) \in [0, d] \cup \{-\infty\}, \quad \text{where } E_f(h) := \{x \in \mathbb{R}^d : h_f(x) = h\}. \quad (1.3)$$

Here,  $\dim_H$  stands for the Hausdorff dimension with the convention  $\dim_H(\emptyset) = -\infty$  (note that  $E_f(h) = \emptyset$  for  $h < 0$ ). This singularity spectrum describes the size of the level sets  $E_f(h)$  of the singularities of  $f$  in terms of Hausdorff dimension and provides a geometrical description of their distribution.

Results on the pointwise regularity of functions belonging to classical spaces such as Bessel potential spaces  $H^{s,p}(\mathbb{R}^d)$ , Sobolev-Slobodeckij  $W^{s,p}(\mathbb{R}^d)$  or Besov  $B_{p,q}^s(\mathbb{R}^d)$  spaces have been obtained in [3, 22, 30, 31, 34]. They are of two types: first, the identification of a universal upper bound for the singularity spectrum of all function in a given space, and second, when possible, the computation of the explicit value for the spectrum of a generic set of functions in the sense of prevalence or Baire categories.

Let us emphasize that for large classes of functions  $f$  (this holds similarly for Borel measures), the singularity spectrum of  $f$  also satisfies a so-called *multifractal formalism*, meaning that  $\sigma_f$  coincides with the Legendre transform  $\tau_f^*$  of a scaling function  $\tau_f : \mathbb{R} \rightarrow \mathbb{R}$  associated with  $f$ , i.e.

$$\sigma_f(h) = \tau_f^*(h) := \inf_{q \in \mathbb{R}} hq - \tau_f(q). \quad (1.4)$$

The scaling function  $\tau_f$ , defined using wavelets (see definitions below and [32, 33]), describes the statistical scaling properties of  $f$ . This relationship is key, since when properly defined,  $\tau_f$  is numerically estimable on real data, see for instance [33]. Such estimation algorithms have generated many breakthrough methods in signal and image processing [57, 58] and other scientific fields [1, 35]. When (1.4) holds, one says that the *multifractal formalism* holds for  $f$ .

A key feature of typical functions in the classical function spaces  $W^{s,p}(\mathbb{R}^d)$  and  $B_{p,q}^s(\mathbb{R}^d)$  is that their singularity spectrum is always affine increasing. This is an apparent contradistinction with the fact that the spectra estimated on real data through the multifractal formalism always have a strictly concave shape, with an increasing *and* a decreasing part.

Tackling the problem of concave shape spectra not necessarily affine for functional spaces, Jaffard [31] showed that given some *admissible* concave function  $\tau : \mathbb{R}^+ \rightarrow \mathbb{R}$ , typical functions in the sense of Baire categories in the intersection of Besov spaces  $\bigcap_{\varepsilon > 0, p > 0} B_{p,q,loc}^{(\tau(p)-\varepsilon)/p}(\mathbb{R}^d)$  have concave increasing spectra. Fraysse [21] obtained that prevalent typical functions in such spaces possess the same spectra.

In order to consider non-affine singularity spectra, one can look at the existing natural generalizations of the classical Sobolev and Besov spaces, the weighted Sobolev and Besov spaces, which have various possible definitions.

For  $h \in \mathbb{R}^d$  and  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ , consider the finite difference operator  $\Delta_h f : x \in \mathbb{R}^d \mapsto f(x+h) - f(x)$ . Then, for  $n \geq 2$ , set  $\Delta_h^n f = \Delta_h(\Delta_h^{n-1} f)$ .

Let  $\rho : \mathbb{R} \rightarrow \mathbb{R}$  be a measurable function, positive almost everywhere, called a weight. Under some hypotheses on the behaviour of  $\rho$  given in [2], weighted Besov spaces  $B_{p,q}^\rho(\mathbb{R}^d)$  consist of all measurable functions  $f : \mathbb{R}^d \rightarrow \mathbb{C}$  such that

$$\int_{|h| \leq 1} \rho(|h|)^{-q} \|\Delta_h^n f\|_{L^p(\mathbb{R}^d)}^q \frac{dh}{|h|^d} < \infty.$$

A function  $f$  belonging to such spaces presents a homogeneous pointwise behaviour as the weight in the integral depends only  $|h|$ . These spaces have a generalised version called *Besov space of generalised smoothness*  $B_{p,q}^{\gamma,N}$ , with  $\gamma, N$  some sequences, studied by Farkas and Leopold in [19]. Moura in [45] and Loosveldt and Nicolay in [41] gave alternative definition. Kreit and Nicolay linked them to generalized

Hölder-Zygmund  $\Lambda_{\sigma,N}^\gamma$  spaces [37, 38]. Such Besov space of generalised smoothness have been studied by Kleynssens and Nicolay in [36]

A second version of the weighted Besov spaces exists where the weight does not dictates the scale behaviour of the function oscillations but the local behaviour of the function itself. The second version of weighted Besov spaces [55] is defined as follows: for a weight  $\rho \in L_{\text{loc}}^1(\mathbb{R})$ , a function  $f \in L_{\text{loc}}^1(\mathbb{R}^d)$  belongs to  $B_{p,q}^s(\mathbb{R}^d, \rho)$  if

$$\int_{|h| \leq 1} |h|^{-sq} \|\rho \Delta_h^n f\|_{L^p(\mathbb{R}^d)}^q \frac{dh}{|h|^d} < \infty.$$

Even if the multifractal behaviour of functions in such spaces has not been studied yet, such spaces have a characterisation through wavelet explicitly given by Triebel in [56, Chapter 6] that can be linked to the spaces of interest in this article.

Another direction is the variable exponent Lebesgue and Sobolev, which are defined through the weighted Orlicz spaces [18, 25] as follows. Let  $p : \mathbb{R}^d \rightarrow [1, \infty)$  be a measurable bounded function, called a variable exponent. The *variable exponent Lebesgue space*  $L^{p(\cdot)}(\mathbb{R}^d)$  consist of all measurable functions  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  such that

$$\int_{\mathbb{R}^d} |f(x)|^{p(x)} dx < +\infty.$$

The *variable exponent Sobolev space*  $W^{k,p(\cdot)}(\mathbb{R}^d)$ ,  $k \in \mathbb{N}$ , consist of all functions of  $L^{p(\cdot)}(\mathbb{R}^d)$  satisfying in addition that  $f^{(k)} \in L^{p(\cdot)}(\mathbb{R}^d)$ . The variable exponent  $p$  is usually at least continuous, the usual hypothesis is that it is log-Hölder continuous. While typical functions in  $W^{k,p}(\mathbb{R}^d)$  exhibit affine spectra with no decreasing parts [31], the spaces  $W^{k,p(\cdot)}(\mathbb{R}^d)$  have not been studied yet.

This motivated the introduction by Barral and Seuret [9] of *inhomogeneous* Besov spaces, in which Baire typical functions may have as singularity spectrum any admissible continuous concave curve, prescribed in advance. For a Hölder set function  $\mu$  (which can be thought as a measure for the moment, see Definition 1.1 below), the *inhomogeneous* Besov space  $B_{p,q}^\mu(\mathbb{R}^d)$  consists of all functions  $f$  in  $L^p(\mathbb{R}^d)$  such that

$$\int_{|h| \leq 1} |h|^{-dq/p} \left\| \frac{\Delta_h^n f}{\mu(B(\cdot + nh/2, nh/2))} \right\|_{L^p(\mathbb{R}^d)}^q \frac{dh}{|h|^d} < +\infty \quad (1.5)$$

These spaces are not spatially homogeneous in the sense that, in the above integral, the oscillations  $\Delta_h^n f(x)$  of  $f$  are locally compared to the  $\mu$ -mass of a small ball of radius  $h$  close to  $x$ , and this mass may strongly depend on  $x$ . They provide a natural framework in which typical functions have multifractal behaviours similar to those observed in real data [9].

The present article has two objectives. First, we propose a generalised version of Sobolev-Slobodeckij spaces in an inhomogeneous environment governed by almost doubling set functions, and relate them to the already defined standard and inhomogeneous function spaces. Second, we obtain prevalent generic results for the multifractal properties of functions in inhomogeneous Sobolev and Besov spaces.

Let us now give precise definitions and results. The diameter of a set  $E$  is denoted by  $|E|$ .

**Definition 1.1.** *The set of Hölder set functions is*

$$\mathcal{H}(\mathbb{R}^d) = \{\mu : \mathcal{B}(\mathbb{R}^d) \rightarrow \mathbb{R}_+ \cup \{+\infty\} : \exists C, s > 0, \forall E \in \mathcal{B}(\mathbb{R}^d), \mu(E) \leq C|E|^s\}.$$

When  $\mu(E) \leq C|E|^s$  for all  $E$  with  $0 \leq |E| \leq 1$ ,  $\mu$  is said to be  $s$ -Hölder.

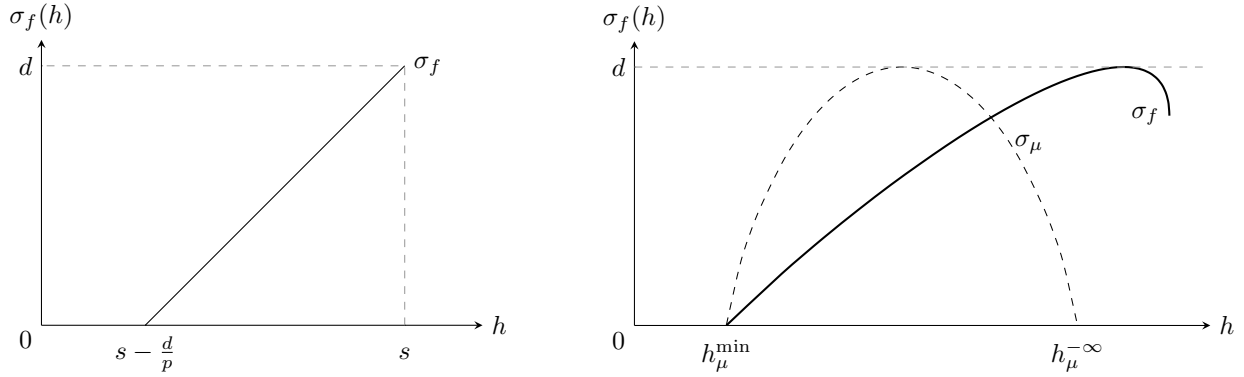


Figure 1.1: **Left:** Singularity spectrum of a typical function  $f \in B_{p,q}^s(\mathbb{R}^d)$ . **Right:** Singularity spectrum of a typical function  $f \in B_{1,q}^\mu(\mathbb{R}^d)$ . The dashed graph represents the (initial) singularity spectrum of  $\mu$ . When  $p = \infty$  and  $f$  is typical in  $B_{\infty,q}^\mu(\mathbb{R}^d)$ ,  $\sigma_f = \sigma_\mu$ .

The topological support of  $\mu \in \mathcal{H}(\mathbb{R}^d)$  is  $\text{supp}(\mu) = \{x \in [0, 1]^d : \mu(B(x, r)) > 0, \forall r > 0\}$ . A set function  $\mu$  is fully supported on a Borel set  $A$  when  $\text{supp}(\mu) = A$ .

By a small abuse of notations we will denote

$$\mathcal{H}([0, 1]^d) = \{\mu \in \mathcal{H}(\mathbb{R}^d) : \text{supp}(\mu) = [0, 1]^d\}. \quad (1.6)$$

For  $\mu \in \mathcal{H}(\mathbb{R}^d)$ ,  $s > 0$ , and  $E \subset \mathbb{R}^d$ , define

$$\mu^s(E) = \mu(E)^s \quad \text{and} \quad \mu^{(+s)}(E) = \mu(E)|E|^s,$$

and if  $\mu$  is  $s_0$ -Hölder, define for all  $s \in (0, s_0)$ ,

$$\mu^{(-s)}(E) = \begin{cases} 0 & \text{if } |E| = 0, \\ \mu(E)|E|^{-s} & \text{if } 0 < |E| < \infty. \end{cases}$$

Observe that the set functions  $\mu^s$ ,  $\mu^{(+s)}$  and  $\mu^{(-s)}$ , as defined above, still belongs to  $\mathcal{H}(\mathbb{R}^d)$ .

In the second part of the paper, we will need the additional assumptions that the set functions  $\mu \in \mathcal{H}(\mathbb{R}^d)$  are decreasing with respect to the embedding relationship, i.e. they are *capacities*. This is natural since in many situations, the set functions, that are considered, are measures or powers of measures. Hence, we define the sets of capacities

$$\begin{aligned} \mathcal{C}(\mathbb{R}^d) &= \{\mu \in \mathcal{H}(\mathbb{R}^d) : \forall E, F \in \mathcal{B}(\mathbb{R}^d), E \subset F \Rightarrow \mu(E) \leq \mu(F)\} \\ \mathcal{C}([0, 1]^d) &= \mathcal{H}([0, 1]^d) \cap \mathcal{C}(\mathbb{R}^d). \end{aligned}$$

**Definition 1.2.** A set function  $\mu \in \mathcal{H}(\mathbb{R}^d)$  satisfies property (P) if there exist  $C, s_1, s_2 > 0$  such that

$$\text{for all } r > 0, \text{ for all } x \in \text{supp}(\mu), \quad C^{-1}r^{s_2} \leq \mu(B(x, r)) \leq Cr^{s_1}. \quad (1.7)$$

In particular, such a  $\mu$  is  $s_1$ -Hölder. We will only consider set functions satisfying property (P) on  $\mathbb{R}^d$  (or  $[0, 1]^d$ ), implying that these functions will be fully supported on  $\mathbb{R}^d$  (or  $[0, 1]^d$ ).

Recall that a finite Borel measure  $\mu$  is said to be doubling if for some constant  $C_\mu \geq 1$  (Soria and

Tradacete [54] proved that necessarily  $C_\mu \geq 2$ ), for every  $x \in \mathbb{R}^d$ ,  $r > 0$ ,

$$\mu(B(x, r)) \leq \mu(B(x, 2r)) \leq C_\mu \mu(B(x, r)).$$

Since we work with set functions (for which one does not have in full generality that  $\mu(B(x, r)) \leq \mu(B(x, 2r))$ ), and not necessarily with measures, we need to adapt this notion to our situation.

**Definition 1.3.** Let  $\Phi$  be the set of non increasing functions  $\phi : (0, 1] \rightarrow \mathbb{R}_+$  such that  $\lim_{r \rightarrow 0^+} \frac{\phi(r)}{\log(r)} = 0$ . A set function  $\mu \in \mathcal{H}(\mathbb{R}^d)$  is almost doubling on a set  $A$  when there exists  $\phi \in \Phi$  such that for every  $x \in A$ ,  $r \in (0, 1]$ ,  $R \in [r, 2r]$  and  $x' \in B(x, r)$ ,

$$e^{-\phi(r)} \mu(B(x', r)) \leq \mu(B(x, R)) \leq e^{\phi(r)} \mu(B(x', r)). \quad (1.8)$$

The set function  $\mu$  is said to be doubling when (1.8) holds with a constant function  $\phi$ .

Intuitively, this formula implies that moving a bit the centre and the radius of a ball keep its  $\mu$ -mass controlled. Obviously, a doubling measure satisfies (1.8) with a constant function  $\phi$ . Examples of doubling set functions include (1-periodic) Gibbs measures  $\mu$  [11, 17, 49] and their powers  $\mu^s(E) := \mu(E)^s$  for  $s > 0$ .

We will refer to  $\mu \in \mathcal{H}(\mathbb{R}^d)$  as an *environment*, and we now describe how such environments are used to define inhomogeneous Sobolev and Besov spaces.

Recall that for an integer  $k$  and  $p \in [1, +\infty]$ , the classical Sobolev space  $W^{k,p}(\mathbb{R}^d)$  is defined as:

$$W^{k,p}(\mathbb{R}^d) = \left\{ f \in L^p(\mathbb{R}^d) : \forall |\alpha| \leq k, D^\alpha f \in L^p(\mathbb{R}^d) \right\}.$$

It is a Banach space when endowed with the norm  $\|f\|_{W^{k,p}(\mathbb{R}^d)} = \sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^p(\mathbb{R}^d)}$ . When  $s \in \mathbb{R}_+ \setminus \mathbb{N}$ , and  $p \in [1, +\infty]$ , the Sobolev-Slobodeckij space  $W^{s,p}(\mathbb{R}^d)$  is the set of functions  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  such that

$$\|f\|_{W^{s,p}(\mathbb{R}^d)} = \|f\|_{W^{\lfloor s \rfloor, p}(\mathbb{R}^d)} + \sup_{|\alpha| = \lfloor s \rfloor} \|D^\alpha f\|_{W^{s - \lfloor s \rfloor, p}(\mathbb{R}^d)} < +\infty$$

where, when  $\theta \in (0, 1)$ ,

$$|f|_{W^{\theta,p}(\mathbb{R}^d)} = \left( \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|f(x+h) - f(x)|^p}{|h|^{\theta p + d}} dx dh \right)^{1/p}. \quad (1.9)$$

We now introduce new inhomogeneous Sobolev spaces in a  $\mu$ -environment by modifying the above formula in order to obtain a class of functions with a singularity spectrum not necessarily affine. To do so, we interpret in (1.9) the term  $|h|^\theta$  in the denominator as, up to a constant, the power of the  $d$ -dimensional Lebesgue measure, denoted by  $\mathcal{L}^d$ , of the ball  $B(x, h)$ , and we replace it by the  $\mu$ -mass of this ball. This introduces spatial inhomogeneity in (1.9), including Sobolev spaces as special case.

Let  $B[x, y]$  denote the Euclidean ball of diameter  $[x, y]$ .

**Definition 1.4.** Let  $\mu \in \mathcal{H}(\mathbb{R}^d)$  satisfy property (P) with exponents  $0 < s_1 < s_2$ , and consider  $p \geq 1$  as well as an integer  $n \geq \left\lfloor s_2 + \frac{d}{p} \right\rfloor + 1$ .

A function  $f$  belongs to  $W^{\mu,p}(\mathbb{R}^d)$  if and only if  $\|f\|_{W^{\mu,p}(\mathbb{R}^d)} < +\infty$ , where  $\|f\|_{W^{\mu,p}(\mathbb{R}^d)} = \|f\|_{L^p(\mathbb{R}^d)} + |f|_{W^{\mu,p}(\mathbb{R}^d)}$  and

$$|f|_{W^{\mu,p}(\mathbb{R}^d)} := \left( \iint_{\mathbb{R}^{2d}} \frac{|\Delta_h^n f(x)|^p}{\mu(B[x, x + nh])^p |h|^{2d}} dx dh \right)^{1/p}. \quad (1.10)$$

Consider  $s, s_1, s_2 > 0$  such that  $s + d/p \notin \mathbb{N}$  and similarly for  $s_1$  and  $s_2$ . For  $\mu = (\mathcal{L}^d)^{\frac{s}{d} - \frac{1}{p}}$ , it follows from the definition that  $W^{\mu,p}(\mathbb{R}^d) = W^{s,p}(\mathbb{R}^d)$ . When  $\mu$  satisfies (P) with  $0 < s_1 < s_2$ , the following embeddings holds

$$W^{s_2 + \frac{d}{p}, p}(\mathbb{R}^d) \hookrightarrow W^{\mu,p}(\mathbb{R}^d) \hookrightarrow W^{s_1 + \frac{d}{p}, p}(\mathbb{R}^d), \quad (1.11)$$

see Remark 2.8. In particular, for an  $s$ -Hölder set function  $\mu \in \mathcal{H}(\mathbb{R}^d)$ , one has  $W^{\mu,p}(\mathbb{R}^d) \hookrightarrow W^{s + \frac{d}{p}, p}(\mathbb{R}^d)$ .

Our first purpose is to relate these new inhomogeneous Sobolev spaces to the inhomogeneous Besov spaces in a multifractal environment introduced by Barral and Seuret in [9]. To define these spaces, already mentioned in (1.5), they first extended the classical definition of  $L^p$  moduli of smoothness.

**Definition 1.5.** For every  $\mu \in \mathcal{H}(\mathbb{R}^d)$ , for every  $n \in \mathbb{N}^*$ ,  $h \in \mathbb{R}^d \setminus \{0\}$  and  $x \in \text{supp}(\mu)$ , set

$$\Delta_h^{\mu,n} f(x) = \frac{\Delta_h^n f(x)}{\mu(B[x, x + nh])}.$$

For  $p \in [1, +\infty]$ , for any open set  $\Omega \subset \mathbb{R}^d$ , the  $\mu$ -adapted  $n$ -th order  $L^p$  modulus of smoothness of  $f$  is defined for any  $t > 0$  by

$$\omega_n^\mu(f, t, \Omega)_p = \sup_{t/2 \leq h \leq t} \|\Delta_h^{\mu,n} f\|_{L^p(\Omega_{h,n})},$$

where  $\Omega_{h,n} = \{x \in \Omega : B[x, x + nh] \subset \Omega\}$ .

Observe that it may happen, when  $\mu$  has a bounded support for instance, that  $\mu(B[x, x + nh]) = 0$  even if  $x \in \text{supp}(\mu)$ . In this case we write  $\Delta_h^{\mu,n} f(x) = 0$ . Inhomogeneous Besov spaces in  $\mu$ -environment are defined in [9] as follows.

**Definition 1.6.** Let  $\mu \in \mathcal{H}(\mathbb{R}^d)$  with  $\text{supp}(\mu) = \mathbb{R}^d$  satisfying (P). Let  $1 \leq p, q \leq +\infty$ . Let  $n$  be an integer such that  $n \geq \left\lfloor s_2 + \frac{d}{p} \right\rfloor + 1$ .

The Besov space in  $\mu$ -environment  $B_{p,q}^\mu(\mathbb{R}^d)$  is the set of functions  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  satisfying

$$\|f\|_{L^p(\mathbb{R}^d)} < +\infty \quad \text{and} \quad \|f\|_{B_{p,q}^\mu(\mathbb{R}^d)} := \left\| (2^{jd/p} \omega_n^\mu(f, 2^{-j}, \mathbb{R}^d)_p)_{j \in \mathbb{N}} \right\|_{l^q(\mathbb{N})} < +\infty. \quad (1.12)$$

We will also consider the spaces

$$\tilde{B}_{p,q}^\mu(\mathbb{R}^d) := \bigcap_{0 < \varepsilon < \min(1, s_1)} B_{p,q}^{\mu^{(-\varepsilon)}}(\mathbb{R}^d). \quad (1.13)$$

If  $\text{supp}(\mu) = [0, 1]^d$ , we use  $\omega_n^\mu(f, 2^{-j}, (0, 1)^d)_p$  in equation (1.12) to define the space  $B_{p,q}^\mu([0, 1]^d)$  and  $\tilde{B}_{p,q}^\mu([0, 1]^d)$  similarly as in (1.12) and (1.13).

Taking  $\mu = (\mathcal{L}^d)^{\frac{s}{d} - \frac{1}{p}}$ , it is easily seen, see [9], that  $B_{p,q}^\mu(\mathbb{R}^d) = B_{p,q}^s(\mathbb{R}^d)$  when  $s > d/p$ .

Our first result concerns embeddings between inhomogeneous Sobolev and inhomogeneous Besov spaces.

**Theorem 1.7.** Let  $p \geq 1$ . If  $\mu \in \mathcal{H}(\mathbb{R}^d)$  with  $\text{supp}(\mu) = \mathbb{R}^d$  satisfies property (P) with exponents  $0 < s_1 < s_2$ , then

$$B_{p,p}^\mu(\mathbb{R}^d) \subset W^{\mu,p}(\mathbb{R}^d). \quad (1.14)$$

If in addition  $\mu$  is almost doubling, then

$$B_{p,p}^\mu(\mathbb{R}^d) \subset W^{\mu,p}(\mathbb{R}^d) \subset \tilde{B}_{p,p}^\mu(\mathbb{R}^d). \quad (1.15)$$

If  $\mu$  is doubling, then, for  $p \geq 1$ ,  $B_{p,p}^\mu(\mathbb{R}^d) = W^{\mu,p}(\mathbb{R}^d)$  (see Remark 3.4), which is the analog of the standard equality for usual Sobolev-Slobodeckij and Besov spaces. This property and the previous embeddings emphasise that formula (1.10) is indeed natural and generalises in an inhomogeneous way the standard function spaces.

**Remark 1.8.** Similar embeddings could be obtained when replacing  $\mathbb{R}^d$  by  $[0, 1]^d$  in (1.15).

Let us now proceed to prevalent multifractal results. In [8], Barral and Seuret identified the typical multifractal behaviour, in the Baire categories sense, of functions in  $B_{p,q}^\mu(\mathbb{R}^d)$ , when  $\mu$  belongs to a large class of set functions satisfying not only (P) but also an extra condition (P<sub>2</sub>), see Definition 2.4. This typical multifractal behaviour is intimately related to that of  $\mu$ , and the value of  $p$ . Barral and Seuret used this space to solve the so-called Frisch-Parisi conjecture, which consists in proving the existence of function spaces in which typical functions satisfy the multifractal formalism [8, 9, 30] for any possible admissible spectrum. To achieve this, they prove that the  $B_{p,q}^\mu(\mathbb{R}^d)$  spaces admit a wavelet characterization. The second part of this article will also use such a characterization to obtain the prevalent multifractal properties in  $B_{\infty,q}^\mu(\mathbb{R}^d)$ .

Write  $L^d = \{0, 1\}^d \setminus \{0^d\}$ , and consider a wavelet family  $\{\Phi, \Psi_\lambda := \Psi^l(2^j \cdot -k)\}_{\lambda=(j,k,l) \in \mathbb{Z} \times \mathbb{Z}^d \times L^d}$  generated by a family of wavelets  $(\Phi, \Psi^l)_{l \in L^d}$  (see Section 2.2 for details), such that every  $f \in L^2(\mathbb{R}^d)$  can be written

$$f \stackrel{L^2}{=} \sum_{\lambda \in \mathbb{Z} \times \mathbb{Z}^d \times L^d} c_\lambda \Psi_\lambda \quad \text{with} \quad c_\lambda = 2^{dj} \int_{\mathbb{R}^d} f(x) \Psi_\lambda(x) dx. \quad (1.16)$$

For  $k = (k_1, \dots, k_d) \in \mathbb{Z}^d$ ,  $j \in \mathbb{Z}$ , we often identify the triplet  $\lambda = (j, k, l)$  with the dyadic cube  $\prod_{i=1}^d [k_i 2^{-j}, (k_i + 1) 2^{-j})$ , and in particular we write  $\mu(\lambda)$  for  $\mu(\prod_{i=1}^d [k_i 2^{-j}, (k_i + 1) 2^{-j})$ .

Write  $\Lambda_j = \{\lambda = (j, k, l) : k \in \mathbb{Z}^d, l \in L^d\}$  for the set of the dyadic cubes of generation  $j$ .

If the function  $f$  has a support in  $[0, 1]^d$ , we will write

$$f = \langle f, \Phi \rangle \Phi + \sum_{j \geq 0} \sum_{\lambda \in \Lambda_j} c_\lambda \Psi_\lambda. \quad (1.17)$$

**Definition 1.9.** Let  $\mu \in \mathcal{H}(\mathbb{R}^d)$  with  $\text{supp}(\mu) = \mathbb{R}^d$  satisfying (P) with  $0 < s_1 < s_2$  and  $p \in [1, +\infty]$ . Consider an integer  $n \geq \lfloor s_2 + \frac{d}{p} \rfloor + 1$ . Consider wavelets  $(\Phi, \Psi^l)_{l \in L^d}$  that belong to  $C^n(\mathbb{R}^d)$  and have at least  $n$  vanishing moments (see Section 2.2 for details).

A function  $f$  belongs to  $b_{p,q}^{\mu,\Psi}(\mathbb{R}^d)$  if and only if  $f \in L^p(\mathbb{R}^d)$  and  $|f|_{\mu,p,q,\Psi} < +\infty$ , where

$$|f|_{\mu,p,q,\Psi} = \left\| (\varepsilon_j^{\mu,p})_{j \in \mathbb{N}} \right\|_{l^q(\mathbb{N})}, \quad \text{with} \quad \varepsilon_j^{\mu,p} = \left\| \left( \frac{c_\lambda}{\mu(\lambda)} \right)_{\lambda \in \Lambda_j \times L^d} \right\|_{l^p(\Lambda_j \times L^d)}. \quad (1.18)$$

Then define

$$\tilde{b}_{p,q}^{\mu,\Psi}(\mathbb{R}^d) = \bigcap_{0 < \varepsilon < \min(1, s_1)} b_{p,q}^{\mu^{(-\varepsilon)}, \Psi}(\mathbb{R}^d). \quad (1.19)$$

Similarly, when  $\text{supp}(\mu) = [0, 1]^d$ , a function  $f$  belongs to  $b_{p,q}^{\mu,\Psi}([0, 1]^d)$  if and only if  $f \in L^p([0, 1]^d)$  and  $|f|_{\mu,p,q,\Psi} < +\infty$ , with the convention that  $\frac{c_\lambda}{\mu(\lambda)} = 0$  as soon as  $\mu(\lambda) = 0$ , and  $\tilde{b}_{p,q}^{\mu,\Psi}([0, 1]^d)$  is defined by replacing  $\mathbb{R}^d$  by  $[0, 1]^d$  in (1.19).

Remark that  $b_{p,q}^{\mu,\Psi}(\mathbb{R}^d) \subset \tilde{b}_{p,q}^{\mu,\Psi}(\mathbb{R}^d)$  and similarly on  $[0, 1]^d$ . In particular, when  $p = q = \infty$ ,  $b_{\infty,\infty}^{\mu,\Psi}(\mathbb{R}^d)$  is the space of those functions  $f$  for which

$$C = \sup_{\lambda \in \mathbb{Z} \times \mathbb{Z}^d \times L^d} \frac{|c_\lambda|}{\mu(\lambda)} < +\infty.$$

This can be rewritten as  $|c_\lambda| \leq C \mu(\lambda)$ , so the space  $b_{\infty,\infty}^{\mu,\Psi}(\mathbb{R}^d)$  can be understood as a inhomogeneous generalisation of Hölder  $C^\alpha(\mathbb{R}^d)$  spaces, which are known to be characterised by the fact that  $|c_\lambda| \leq C |\lambda|^\alpha$  for every  $\lambda$ . This justifies, once again, the name "environment" used for  $\mu$ , since the set function may have various local behaviours and the constraint  $|c_\lambda| \leq C \mu(\lambda)$  is highly dependent on  $\mu$  and the location of the cube  $\lambda$ .

When  $p < \infty$ , the space  $b_{p,\infty}^{\mu,\Psi}(\mathbb{R}^d)$  can be seen as a inhomogeneous generalisation of Caldéron and Zygmund  $T_p^\alpha(\mathbb{R}^d)$  spaces.

Let us also mention that the spaces  $C^\alpha(\mathbb{R}^d)$  and  $T_p^\alpha(\mathbb{R}^d)$  have been generalised by Kreit, Loosveldt and Nicolay [37, 38, 39, 42, 43] with the spaces  $C^\gamma(\mathbb{R}^d)$  and  $T_{p,q}^\gamma(\mathbb{R}^d)$  with  $\gamma$  a sequence. The space  $C^\gamma(\mathbb{R}^d)$  is characterise by  $|c_{\lambda_j}| \leq C \gamma_j$  for every  $\lambda \in \Lambda_j$  (for a sequence  $\gamma = (\gamma_j)$  verifying, for all  $j$ ,  $C^{-1} \gamma_j \leq \gamma_{j+1} \leq C \gamma_j$ .)

**Remark 1.10.** The spaces  $\tilde{b}_{p,q}^\xi(\mathbb{R}^d)$  and  $\tilde{b}_{p,q}^\xi([0, 1]^d)$  are Fréchet space [9]. A translation-invariant complete metric that induces the same topology as the family  $\left\{ \|\cdot\|_{b_{p,q}^{\mu(-\varepsilon),\Psi}} := \|\cdot\|_{L^p} + |\cdot|_{\xi^{(-\varepsilon),p,q,\Psi}} \right\}_{0 < \varepsilon < \min(1,s_1)}$  can be defined by

$$\forall f, g \in \tilde{b}_{p,q}^{\mu,\Psi}, \quad d(f, g) = \sum_{n > \max(1, s_1^{-1})} 2^{-n} \frac{\|f - g\|_{b_{p,q}^{\mu(-1/n),\Psi}}}{1 + \|f - g\|_{b_{p,q}^{\mu(-1/n),\Psi}}}. \quad (1.20)$$

The key property, relating the two parts of the present work, is the following: Assume that the set function  $\mu \in \mathcal{H}(\mathbb{R}^d)$  is in addition a capacity, i.e. for every Borel sets  $E, F \in \mathcal{B}(\mathbb{R}^d)$  such that  $E \subset F$ ,  $\mu(E) \leq \mu(F)$ . Then, it is proved in [9] that, **provided an extra condition on the capacity  $\mu$  (condition  $(P_2)$  that we give below),  $\tilde{b}_{p,q}^{\mu,\Psi}(\mathbb{R}^d)$  and  $\tilde{B}_{p,q}^\mu(\mathbb{R}^d)$  coincide (resp.  $\tilde{b}_{p,q}^{\mu,\Psi}([0, 1]^d)$  and  $\tilde{B}_{p,q}^\mu([0, 1]^d)$ ), and in particular do not depend on the choice of  $\Psi$  satisfying the condition of Definition 1.9, see Theorem 2.7 for the precise statement. So, when working with suitable set functions  $\mu$ , the prevalent multifractal behaviour in  $B_{p,q}^\mu(\mathbb{R}^d)$  coincides with that in  $b_{p,q}^\mu(\mathbb{R}^d)$ .**

Furthermore, since multifractal properties are local, we will focus on behaviours of functions on the cube  $[0, 1]^d$ , and will consider functions written under the form (1.17). This is not a restriction, since any function  $f$  written as in (1.16) can be decomposed as a (countable) sum

$$f = \tilde{f} + \sum_{\ell \in \mathbb{Z}^d} f_\ell(\cdot + \ell),$$

where  $\tilde{f}$  is regular (corresponding to the low frequency part of  $f$ ) and  $f_\ell$  has a decomposition of the form (1.17). Then, each function  $f_\ell(\cdot + \ell)$  has an interesting behaviour only on the cube  $\ell + [0, 1]^d$ , and the multifractal properties of  $f$  will be deduced from the juxtaposition of the properties of the  $f_\ell$ .

**From now on, we focus on set functions  $\mu \in \mathcal{H}([0, 1]^d)$  and on functions belonging to  $b_{p,q}^{\mu,\Psi}([0, 1]^d)$  and  $\tilde{b}_{p,q}^{\mu,\Psi}([0, 1]^d)$  written as (1.17).**

Our goal is to obtain the prevalent multifractal behaviour in  $b_{\infty,q}^{\mu,\Psi}(\mathbb{R}^d)$  in larger classes of capacities than in [9]. Recall that the prevalence theory, proposed by Christensen [15] and Hunt, Sauer and York [27] independently, supersedes the notion of Lebesgue for universally measurable sets in any real or complex topological vector space  $E$  endowed with its Borel  $\sigma$ -algebra  $\mathcal{B}(E)$ . All Borel measures  $\vartheta$  on  $(E, \mathcal{B}(E))$  will be automatically completed, i.e. for every set  $A \subset E$  and  $B \in \mathcal{B}(E)$  such that the symmetric difference  $A\Delta B$  is included in some  $D \in \mathcal{B}(E)$  with  $\vartheta(D) = 0$ , one sets  $\vartheta(A) := \vartheta(B)$ .

**Definition 1.11.** *A set  $A \subset E$  is universally measurable if it is measurable for any (completed) Borel measure on  $E$ . A universally measurable set  $A \subset E$  is called Haar-null, or shy, if there exists a finite completed measure  $\vartheta$  that is positive on some compact subset  $K$  of  $E$  and such that*

$$\text{for every } x \in E, \quad \vartheta(A + x) = 0. \quad (1.21)$$

*The measure  $\vartheta$  used to show that some subset is Haar-null is called transverse. More generally, a set that is included in a Haar-null universally measurable set is also called Haar-null.*

*Finally, the complement in  $E$  of a Haar-null subset  $S$ , denoted  $S^c$ , is called prevalent.*

Hunt pioneered the use of prevalence in function spaces [26]. More recently, Bayart and Heurteaux [10] and Balka, Darji and Elekes [5] studied the dimension of images of continuous functions. Fraysse, Jaffard and Kahane [22, 23] and then Aubry, Maman and Seuret [4] studied, respectively, the prevalent regularity of functions and traces of functions in Besov spaces. Olsen studied the multifractal and  $L^q$ -dimension of prevalent measures on compact sets of  $\mathbb{R}^d$  [47, 48].

Let us give the last definitions needed to state our second main theorem.

**Definition 1.12.** *Let  $\mu \in \mathcal{H}(\mathbb{R}^d)$ . For  $x \in \text{supp}(\mu)$ , the lower local dimension of  $\mu$  at  $x$  is defined by*

$$\underline{h}_\mu(x) = \liminf_{r \rightarrow 0^+} \frac{\log \mu(B(x, r))}{\log(r)}. \quad (1.22)$$

*The singularity spectrum of  $\mu$  is then the Hausdorff dimension of the level sets of the local dimension*

$$\sigma_\mu : h \in \mathbb{R} \mapsto \dim_H(\underline{E}_\mu(h)) \quad \text{where} \quad \underline{E}_\mu(h) = \{x \in \text{supp}(\mu) : \underline{h}_\mu(x) = h\}.$$

*We also defined  $E_\mu(h)$  as the set of points  $x$  where the local dimension (1.22) is a limit and equals  $h$ .*

*Finally, for every integer  $j \geq 1$ , the scaling function of  $\mu \in \mathcal{H}([0, 1]^d)$  is defined by*

$$\tau_\mu(q) = \liminf_{j \rightarrow +\infty} \frac{1}{-j} \log_2 \sum_{\lambda \in \Lambda_j \cap [0, 1]^d} \mu(\lambda)^q, \quad (1.23)$$

*where  $0^q = 0$  by convention.*

When  $\text{supp}(\mu) = [0, 1]^d$ , the singularity spectrum of  $\mu$  is bounded above by the Legendre transform  $\tau_\mu^*$  of the scaling function of  $\mu$  (recall formula (1.4)) : for every  $h \geq 0$ ,  $\sigma_\mu(h) \leq \tau_\mu^*(h)$ , see [12, 31, 40].

**Definition 1.13.** *A set function  $\mu \in \mathcal{H}([0, 1]^d)$  satisfies the strong multifractal formalism (SMF) if for every  $h \geq 0$ ,  $\sigma_\mu(h) = \tau_\mu^*(h) = \dim_H(E_\mu(h))$ .*

Our main theorem regarding prevalent multifractal properties holds for capacities. This additional property on set functions to get capacities (the decreasing property) is in fact natural in the functional

setting. Indeed, looking back to formula (1.5), it is intuitive that if the oscillation of  $f$  on an interval  $I$  is controlled by  $\mu(I)$ , then the oscillations on a subinterval  $J \subset I$  shall be controlled by  $\mu(J) \leq \mu(I)$ , since the oscillation on  $J$  is necessarily smaller than that on  $I$ .

**We only consider the case  $p = \infty$ .** Indeed, the general case  $p < \infty$  would require a finer study.

**Theorem 1.14.** *Let  $\mu \in \mathcal{C}([0, 1]^d)$  be an almost doubling capacity satisfying (P) and verifying the SMF. For all  $f \in \tilde{b}_{\infty, q}^{\mu, \Psi}([0, 1]^d)$ ,*

$$\sigma_f(h) \leq \begin{cases} \sigma_\mu(h) & \text{if } h \leq \tau'_\mu(0) \\ d & \text{if } h > \tau'_\mu(0). \end{cases} \quad (1.24)$$

*In addition, there exists a prevalent set of functions  $f \in \tilde{b}_{\infty, q}^{\mu, \Psi}([0, 1]^d)$  satisfying  $\sigma_f = \sigma_\mu$ .*

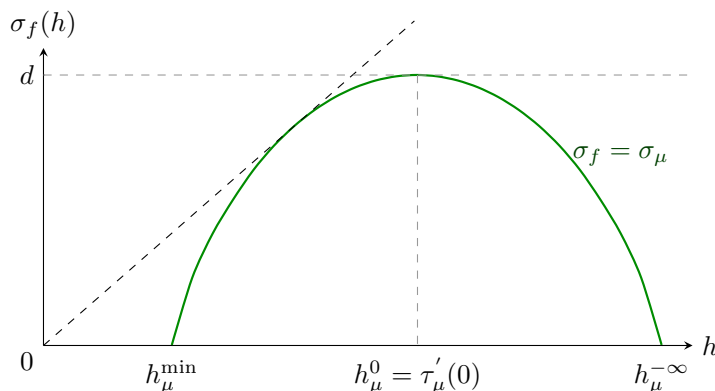


Figure 1.2: Singularity spectrum of a prevalent set of functions in  $\tilde{b}_{\infty, q}^{\mu, \Psi}([0, 1]^d)$

Remark that this result stays true for  $f \in b_{\infty, q}^{\mu, \Psi}([0, 1]^d) \subset \tilde{b}_{\infty, q}^{\mu, \Psi}([0, 1]^d)$ . In other words, the prevalent singularity spectrum in  $\tilde{b}_{\infty, q}^{\mu, \Psi}([0, 1]^d)$  or  $b_{\infty, q}^{\mu, \Psi}([0, 1]^d)$  coincides with that of  $\mu$ . One directly deduces a similar result for  $\tilde{B}_{\infty, q}^{\mu, \Psi}([0, 1]^d)$ .

**Corollary 1.15.** *If  $\mu$  satisfies condition (P<sub>2</sub>) given in Definition 2.4, then the previous theorem holds when replacing  $\tilde{b}_{\infty, q}^{\mu, \Psi}([0, 1]^d)$  by  $\tilde{B}_{\infty, q}^{\mu, \Psi}([0, 1]^d)$  and  $W^{\mu, \infty}([0, 1]^d)$ .*

This simply follows from the fact that, as said before, provided that (P<sub>2</sub>) is satisfied,  $\tilde{b}_{\infty, q}^{\mu, \Psi}([0, 1]^d)$  and  $\tilde{B}_{\infty, q}^{\mu, \Psi}([0, 1]^d)$  coincide.

We shall prove Theorem 1.14 for  $\tilde{b}_{\infty, q}^{\mu, \Psi}([0, 1]^d)$ , the proof for  $b_{\infty, q}^{\mu, \Psi}([0, 1]^d)$  is simpler, we will explain where to adapt and short-cut our proof to get the simpler case.

In [9], Barral and Seuret obtained the same singularity spectrum as in Theorem 1.14 for Baire typical functions, but only for two large classes of capacities (a class of measure with a multifractal spectrum prescribed in advance and Gibbs capacities defined as power of Gibbs measures). Here we treat a larger class of capacities, and complete their result by confirming that prevalent and Baire typical regularity coincide.

It is certainly an interesting question to extend Theorem 1.14 to  $\tilde{b}_{p, q}^{\mu, \Psi}([0, 1]^d)$  and  $W^{\mu, p}$  when  $p < +\infty$ .

## Organisation of the paper

Section 2 gives the necessary definitions and the properties of measures and capacities used to define a *multifractal environment*, then presents the wavelet multifractal formalism and the wavelet leaders formalism developed by Jaffard in [32, 33].

Section 3 contains the proof of our first main Theorem 1.7. This section also proves the case  $B_{p,p}^\mu(\mathbb{R}^d) = W^{\mu,p}(\mathbb{R}^d)$  for  $\mu$  doubling.

In Section 4, the proof of Theorem 1.14 is given. For this, using a wavelet construction, we introduce a *saturating function*  $\mathcal{G}_q$  of  $\tilde{b}_{\infty,q}^{\mu,\Psi}([0,1]^d)$  having a prescribed multifractal behaviour coinciding with that of  $\mu$ . This function will be a key object we build to show prevalent results on the spectrum of functions of  $\tilde{b}_{\infty,q}^{\mu,\Psi}([0,1]^d)$ .

## 2 Additional definitions and properties on capacities, wavelets and multifractal formalism

Let  $\Lambda = \bigcup_{j \in \mathbb{Z}} \Lambda_j$  be the set of all dyadic cubes.

For  $x \in \mathbb{R}^d$ ,  $\lambda_j(x)$  is the dyadic cubes  $\lambda$  of generation  $j$  such that  $x \in \lambda$  (with the small abuse of notation using the identification between  $\lambda = (j, k = (k_1, \dots, k_d), l) \in \mathbb{Z} \times \mathbb{Z}^d \times L^d$  and the corresponding dyadic cube  $\prod_{i=1}^d [k_i 2^{-j}, (k_i + 1) 2^{-j})$ ).

For  $j \in \mathbb{Z}$ ,  $\lambda \in \Lambda_j$  and  $N \in \mathbb{N}^*$ , let  $N\lambda$  denote the cube with the same centre as  $\lambda$  and radius equal to  $N \cdot 2^{-j-1}$  in  $(\mathbb{R}^d, \|\cdot\|_\infty)$ . For instance,  $3\lambda$  is the union of those  $\lambda' \in \Lambda_j$  such that  $\partial\lambda \cap \partial\lambda' \neq \emptyset$  where  $\partial\lambda$  is the boundary of the cube  $\lambda$ .

### 2.1 Capacity and formalism

**Notation.** For any set Borel  $A \subset \mathbb{R}^d$  and capacity  $\mu$ , we will denote  $\mu|_A = \mu(\cdot \cap A)$ .

To describe the local behaviour of  $\mu \in \mathcal{H}([0,1]^d)$ , one needs, in addition to the lower local dimension (1.22), other local dimension.

**Definition 2.1.** Let  $\mu \in \mathcal{H}([0,1]^d)$ . For  $x \in \text{supp}(\mu)$ , the upper local dimension of  $\mu$  at  $x$  is

$$\bar{h}_\mu(x) = \limsup_{r \rightarrow 0^+} \frac{\log \mu(B(x, r))}{\log(r)}.$$

Whenever  $\underline{h}_\mu(x) = \bar{h}_\mu(x)$ , the common limit is denoted by  $h_\mu(x)$ . For  $\alpha \in \mathbb{R}$ ,

$$\underline{E}_\mu(\alpha) = \{x \in \text{supp}(\mu) : \underline{h}_\mu(x) = \alpha\}, \quad \bar{E}_\mu(\alpha) = \{x \in \text{supp}(\mu) : \bar{h}_\mu(x) = \alpha\}$$

and  $E_\mu(\alpha) = \underline{E}_\mu(\alpha) \cap \bar{E}_\mu(\alpha)$ .

**Remark 2.2.** Note that the local dimensions are sometimes defined as

$$\underline{h}_\mu(x) = \liminf_{j \rightarrow +\infty} \frac{\log_2 \mu(\lambda_j(x))}{-j} \quad \text{and} \quad \bar{h}_\mu(x) = \limsup_{j \rightarrow +\infty} \frac{\log_2 \mu(\lambda_j(x))}{-j}, \quad (2.1)$$

or  $\underline{h}_\mu(x) = \liminf_{j \rightarrow +\infty} \frac{\log_2 \mu(3\lambda_j(x))}{-j} \quad \text{and} \quad \bar{h}_\mu(x) = \limsup_{j \rightarrow +\infty} \frac{\log_2 \mu(3\lambda_j(x))}{-j}$ .

However, this paper mainly considers doubling or almost doubling set functions for which all previous notions of local dimensions (1.22) and (2.1), level sets, singularity spectrum and scaling function do not depend on whether dyadic cubes or centred balls are considered.

Recall that the Legendre spectrum  $h \mapsto \tau_\mu^*(h)$  is increasing on the interval  $h \leq \tau_\mu'(0^+)$  and is decreasing on  $h \geq \tau_\mu'(0^-)$ . The following Proposition 2.3 can be deduced from [6, 7, 9, 40, 46].

**Proposition 2.3.** *Let  $\mu \in \mathcal{H}([0, 1]^d)$  with  $\text{supp}(\mu) = [0, 1]^d$ , such that  $\mu$  obeys the SMF (Definition 1.12). Then for every  $h \in \mathbb{R}^+$ :*

- (i) *One has  $\sigma_\mu(h) = \dim_H(\underline{E}_\mu(h)) = \dim_H(\overline{E}_\mu(h)) = \tau_\mu^*(h)$ .*
- (ii) *When  $h \leq \tau_\mu'(0^+)$ ,  $\dim_H(\underline{E}_\mu^{\leq}(h)) = \tau_\mu^*(h)$ .*
- (iii) *When  $h \geq \tau_\mu'(0^-)$ ,  $\dim_H(\overline{E}_\mu^{\geq}(h)) = \tau_\mu^*(h)$ .*
- (iv) *When  $\tau_\mu'(0^+) \leq h \leq \tau_\mu'(0^-)$ ,  $\sigma_\mu(h) = d$ .*

All the set functions we work with satisfy the property (P) given in Definition 1.2. Observe that (P) implies that for all  $j \in \mathbb{N}$  and  $\lambda \in \Lambda_j$ ,

$$C^{-1}2^{-js_2} \leq \mu(\lambda) \leq C2^{-js_1}. \quad (2.2)$$

Property (P) and (2.2) are not equivalent in general, but it is the case when  $\mu$  is almost doubling (with slightly different constant  $C$  and exponents  $s_1$  and  $s_2$ ), which will always be the case for us.

**Definition 2.4.** *A set function  $\mu \in \mathcal{H}(\mathbb{R}^d)$  satisfies property (P<sub>2</sub>) if there exist  $C, s_2 > 0$  and  $\phi \in \Phi$  (see Definition 1.3) such that for all  $j, j' \in \mathbb{N}$  with  $j \leq j'$ , for all  $\lambda, \tilde{\lambda} \in \Lambda_j$  such that  $\partial\lambda \cap \partial\tilde{\lambda} \neq \emptyset$ , and  $\lambda' \in \Lambda_{j'}$  such that  $\lambda' \subset \lambda$ :*

$$\mu(\tilde{\lambda}) \leq C2^{\phi(2^{-j})}2^{(j'-j)s_2}\mu(\lambda'). \quad (2.3)$$

In particular,  $\mu$  is almost doubling in this case. While the  $\mu$ -mass of each dyadic cube is controlled by  $s_1$  and  $s_2$  by (P), Property (P<sub>2</sub>) can be seen as a kind of weak quasi-Bernoulli property of  $\mu$ .

**Lemma 2.5.** *Any doubling set function  $\mu \in \mathcal{H}(\mathbb{R}^d)$  satisfies Property (P<sub>2</sub>).*

*Proof.* Let  $j, j' \in \mathbb{N}$  with  $j \leq j'$ ,  $\lambda, \tilde{\lambda} \in \Lambda_j$  such that  $\partial\lambda \cap \partial\tilde{\lambda} \neq \emptyset$ , and  $\lambda' \in \Lambda_{j'}$  such that  $\lambda' \subset \lambda$ . Calling  $y_{\lambda'}$  the centre of  $\lambda'$  and  $y_{\tilde{\lambda}}$  the one of  $\tilde{\lambda}$ , one has  $\tilde{\lambda} \subset B(y_{\lambda'}, 2 \cdot 2^{-j})$ .

Since  $\mu$  is doubling, (1.8) holds for a constant function  $\phi$ , so there exists a  $C_\mu$  such that for  $R \in [r, 2r]$  and  $x' \in B(x, r)$ ,

$$C_\mu^{-1}\mu(B(x', r)) \leq \mu(B(x, R)) \leq C_\mu\mu(B(x', r)). \quad (2.4)$$

Remark that  $|y_{\lambda'} - y_{\tilde{\lambda}}| < 3 \times 2^{-(j+1)}$ . So, applying inequality (2.4) between  $\tilde{\lambda}$  and  $B\left(\frac{y_{\lambda'} - y_{\tilde{\lambda}}}{2}, 2^{-j}\right)$ , and then from this last ball to  $B(y_{\lambda'}, 2 \cdot 2^{-j})$ , one gets  $\mu(\tilde{\lambda}) \leq C_\mu^2\mu(B(y_{\lambda'}, 2 \cdot 2^{-j}))$ . Then applying it  $j' - j + 2$  times to pass from radius  $2^{-j+1}$  to  $2^{-j'-1}$  yields finally

$$\mu(\tilde{\lambda}) \leq C_\mu^4 C_\mu^{j'-j}\mu(\lambda') = C_\mu^4 2^{(j'-j)\log_2(C_\mu)}\mu(\lambda'). \quad (2.5)$$

One concludes that  $\mu$  satisfies property (P<sub>2</sub>) with  $C = C_\mu^4$  and  $s_2 = \log_2(C_\mu)$ .  $\square$

## 2.2 Multi-resolution wavelet analysis

Let  $d \in \mathbb{N}^*$  and  $1 \leq p, q \leq +\infty$ . Recall that a function  $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$  has  $n + 1$  vanishing moments when for every multi-index  $\alpha \in \mathbb{N}^d$  such that  $\sum_{i=1}^d \alpha_i \leq n$ ,  $\int_{\mathbb{R}^d} x_1^{\alpha_1} \dots x_d^{\alpha_d} \psi(x) dx = 0$ .

For an arbitrary integer  $N \geq 1$ , there exist compactly supported functions  $\Psi^0 \in C^N(\mathbb{R})$  (called the scaling function) and  $\Psi^1 \in C^N(\mathbb{R})$  (called the mother wavelet), with  $\Psi^1$  having at least  $N + 1$  vanishing moments and such that the set of function  $\Psi_{j,k}^1 : x \mapsto \Psi^1(2^j x - k)$  for  $j \in \mathbb{Z}$  and  $k \in \mathbb{Z}$  form an orthogonal basis of  $L^2(\mathbb{R})$ . In this case, the wavelet is said to be  $N$ -regular.

Let us introduce the notations

$$0^d := (0, \dots, 0), \quad 1^d := (1, \dots, 1), \quad L^d = \{0, 1\}^d \setminus 0^d.$$

An orthogonal basis of  $L^2(\mathbb{R}^d)$  is obtained by tensorization as follows (see [44, Chapter 2 and 3] for a general construction): for  $\lambda = (j, k, l) \in \mathbb{Z} \times \mathbb{Z}^d \times L^d$ , let us define the tensorised wavelet

$$\Psi_\lambda = \prod_{i=1}^d \Psi_{j,k_i}^{l_i}$$

with  $k = (k_1, \dots, k_D)$  and  $l = (l_1, \dots, l_D) \in L^d$ , with the additional notation  $\Phi = \Psi^{0^d}$ .

Thus for every  $f \in L^2(\mathbb{R}^d)$ , one has the decomposition (1.16) (observe that we use an  $L^\infty$  normalisation for the wavelet coefficient  $(c_\lambda)_{\lambda \in \Lambda}$ .)

**Proposition 2.6.** [28, 32] *Suppose that  $\gamma > 0$  and  $N = \lfloor \gamma + 1 \rfloor$ . Suppose that  $\Phi, \Psi \in C^N(\mathbb{R})$  is at least  $N + 1$ -regular. Let  $f : [0, 1]^d \rightarrow \mathbb{R}$  be a locally bounded function with wavelet coefficients  $\{c_\lambda\}_{\lambda \in \Lambda \times L^d}$ , and let  $x \in [0, 1]^d$ .*

*If  $f \in C^\gamma(x)$ , then there exists a constant  $M < \infty$  such that for all  $\lambda = (j, k, l) \in \Lambda \times L^d$ ,*

$$|c_\lambda| \leq M 2^{-j\gamma} (1 + |2^j x - k|)^\gamma. \quad (2.6)$$

*Conversely, if (2.6) holds true and if  $f \in \bigcup_{\varepsilon > 0} C^\varepsilon([0, 1]^d)$ , then  $f \in C^{\gamma-\eta}(x)$  for every  $\eta > 0$ .*

The *inhomogeneous Besov* spaces introduced in Section 1 and those of Definition 1.9 are related thanks to the following Theorem proved in [9].

**Theorem 2.7.** *Let  $\mu \in \mathcal{C}(\mathbb{R}^d)$ ,  $0 < s_1 \leq s_2$  and  $r \geq \left\lfloor s_2 + \frac{d}{p} \right\rfloor + 1$ . Suppose that properties (P) and (P<sub>2</sub>) holds for  $\mu$  with the exponents  $(s_1, s_2)$  and that  $B_{p,q}^\mu(\mathbb{R}^d)$  has been constructed using the  $\mu$ -adapted  $n$ -th order  $L^p$  moduli of smoothness, for some  $n \geq r$ . Consider a wavelet  $\Psi$  that belong to  $C^n(\mathbb{R}^d)$  and have at least  $n$  vanishing moments.*

*For every  $\varepsilon \in (0, 1)$ , there exists a constant  $C_\varepsilon > 1$  such that for all  $f \in L^p(\mathbb{R}^d)$ ,*

$$\begin{aligned} \|f\|_{L^p(\mathbb{R}^d)} + |f|_{\mu,p,q,\Psi} &\leq C_\varepsilon (\|f\|_{L^p(\mathbb{R}^d)} + |f|_{B_{p,q}^{\mu(+\varepsilon)}(\mathbb{R}^d)}), \\ \|f\|_{L^p(\mathbb{R}^d)} + |f|_{B_{p,q}^\mu(\mathbb{R}^d)} &\leq C_\varepsilon (\|f\|_{L^p(\mathbb{R}^d)} + |f|_{\mu(+\varepsilon),p,q,\Psi}). \end{aligned}$$

*Moreover, when  $\mu$  is doubling (with Lemma 2.5),  $B_{p,q}^\mu(\mathbb{R}^d) = B_{p,q}^{\mu,\Psi}(\mathbb{R}^d)$  and the norms  $\|f\|_{L^p(\mathbb{R}^d)} + |f|_{\mu,p,q,\Psi}$  and  $\|f\|_{L^p(\mathbb{R}^d)} + |f|_{B_{p,q}^\mu(\mathbb{R}^d)}$  are equivalent. Also, this equivalence is independent of the choice of  $n \geq r$  and  $\Psi \in \mathcal{F}_r$ .*

**Remark 2.8.** [9, Remark 2.17] When  $\mu$  satisfies (P) with  $0 < s_1 < s_2$ , one has  $(\mathcal{L}^d)^{\frac{s_2}{d}} \leq \mu \leq (\mathcal{L}^d)^{\frac{s_1}{d}}$  (in the sense that for every  $B(x, r) \subset \mathbb{R}^d$ ,  $(\mathcal{L}^d)^{\frac{s_2}{d}}(B(x, r)) \leq \mu(B(x, r)) \leq (\mathcal{L}^d)^{\frac{s_1}{d}}(B(x, r))$ ) inducing the embeddings

$$B_{p,q}^{s_2+\frac{d}{p}}(\mathbb{R}^d) \hookrightarrow B_{p,q}^\mu(\mathbb{R}^d) \hookrightarrow B_{p,q}^{s_1+\frac{d}{p}}(\mathbb{R}^d). \quad (2.7)$$

When  $\alpha \geq \beta$ , one has

$$B_{p,q}^{\mu(+\alpha)}(\mathbb{R}^d) \hookrightarrow B_{p,q}^{\mu(+\beta)}(\mathbb{R}^d) \quad \text{and} \quad \widetilde{B}_{p,q}^{\mu(+\alpha)}(\mathbb{R}^d) \hookrightarrow \widetilde{B}_{p,q}^{\mu(+\beta)}(\mathbb{R}^d) \quad (2.8)$$

Similarly, we can replace  $\mathbb{R}^d$  by  $[0, 1]^d$  in all the previous embeddings.

### 3 Embeddings between inhomogeneous Sobolev-Slobodeckij and Besov spaces.

#### 3.1 Preliminary

We start by extending [9, lemma 2.3] for the  $\mu$ -adapted  $L^p$  modulus of smoothness to obtain the following lemma.

**Lemma 3.1.** *Let  $\varepsilon > 0$ ,  $p \in [1, +\infty)$ ,  $n \in \mathbb{N}^*$  and  $\mu \in \mathcal{H}(\mathbb{R}^d)$  be an almost doubling set function.*

*There exists a constant  $C = C(\varepsilon, d, n, p)$  such that for all  $f \in L^p(\mathbb{R}^d)$  and  $0 < t \leq 1$ , the following inequality holds:*

$$\omega_n^\mu(f, t, \mathbb{R}^d)_p^p \leq C t^{-d} \int_{t \leq |h| \leq 4nt} \int_{x \in \mathbb{R}^d} \left| \Delta_h^{\mu(+\varepsilon), n} f(x) \right|^p dx dh. \quad (3.1)$$

*Proof.* The approach follows the lines of the proof of [16, inequality (3.3.17)] and [9, Lemma 2.3].

For any  $h, y \in \mathbb{R}^d$ , recall the following equality (see (3.3.19) in [16]):

$$\Delta_h^n f(x) = \sum_{k=1}^n (-1)^k \binom{n}{k} [\Delta_{ky}^n f(x + kh) - \Delta_{h+ky}^n f(x)]. \quad (3.2)$$

First, one has

$$\begin{aligned} \omega_n^\mu(f, t, \mathbb{R}^d)_p^p &= \sup_{t/2 \leq |h| \leq t} \left\| \Delta_h^{\mu, n} f \right\|_{L^p(\mathbb{R}^d)}^p = \sup_{t/2 \leq |h| \leq t} \int_{x \in \mathbb{R}^d} |\Delta_h^{\mu, n} f(x)|^p dx \\ &= \sup_{t/2 \leq |h| \leq t} \int_{x \in \mathbb{R}^d} \left| \frac{\Delta_h^n f(x)}{\mu(B[x, x + nh])} \right|^p dx. \end{aligned}$$

By (3.2), for  $t/2 \leq |h| \leq t$  and  $y \in \mathbb{R}^d$ , one has

$$\begin{aligned} \int_{x \in \mathbb{R}^d} \left| \frac{\Delta_h^n f(x)}{\mu(B[x, x + nh])} \right|^p dx &= \int_{x \in \mathbb{R}^d} \left| \frac{\sum_{k=1}^n (-1)^k \binom{n}{k} [\Delta_{ky}^n f(x + kh) - \Delta_{h+ky}^n f(x)]}{\mu(B[x, x + nh])} \right|^p dx \\ &\leq C_{n,p} \sum_{k=1}^n \int_{x \in \mathbb{R}^d} \left| \frac{\Delta_{ky}^n f(x + kh)}{\mu(B[x, x + nh])} \right|^p + \left| \frac{\Delta_{h+ky}^n f(x)}{\mu(B[x, x + nh])} \right|^p dx. \end{aligned} \quad (3.3)$$

In particular, the last inequality holds for  $y \in B(0, 3t) \setminus B(0, 2t)$ .

For a fixed  $t/2 \leq |h| \leq t$  and  $y \in B(0, 3t) \setminus B(0, 2t)$ , for every  $k \in \{1, \dots, n\}$ ,  $|h| \leq |n(h + ky)| \leq 8n^2|h|$

and  $|h| \leq |k(h + ny)| \leq 8n^2|h|$ . Hence,

$$B[x, x + n(h + ky)] \subset B\left(x + \frac{nh}{2}, 8n^2|h|\right) \quad \text{and} \quad B[x + kh, x + k(h + ny)] \subset B\left(x + \frac{nh}{2}, 8n^2|h|\right). \quad (3.4)$$

**Lemma 3.2.** *There exists a constant  $C_{\varepsilon, n}$  that depends only on  $n$  and  $\varepsilon$  such that*

$$\mu^{(+\varepsilon)}(B[x, x + n(h + ky)]) \leq C_{\varepsilon, n} \mu\left(B\left(x + \frac{nh}{2}, 8n^2|h|\right)\right), \quad (3.5)$$

$$\mu^{(+\varepsilon)}(B[x + kh, x + k(h + ny)]) \leq C_{\varepsilon, n} \mu\left(B\left(x + \frac{nh}{2}, 8n^2|h|\right)\right), \quad (3.6)$$

$$\mu^{(+\varepsilon)}\left(B\left(x + \frac{nh}{2}, 8n^2|h|\right)\right) \leq C_{\varepsilon, n} \mu(B[x, x + nh]). \quad (3.7)$$

*Proof.* The key property is that all the considered balls have comparable diameters (with a constant depending on  $n$  only), and the almost doubling property of  $\mu$  ensures that their  $\mu$ -mass is comparable up to a small exponential error term.

Let us prove (3.7), and we will use the same argument to prove the rest. Remark that  $B[x, x + nh] = B\left(x + \frac{nh}{2}, \frac{n|h|}{2}\right)$  and  $B\left(x + \frac{nh}{2}, 8n^2|h|\right)$  have the same centre and  $8n^2|h| \leq 2^K \frac{n|h|}{2}$  for  $K = \lfloor \log_2(n) \rfloor + 5$ . Using the almost doubling property (1.8) applied  $K$  times, one obtains

$$\mu\left(B\left(x + \frac{nh}{2}, 8n^2|h|\right)\right) \leq e^{\sum_{k=0}^{K-1} \phi(2^k n|h|/2)} \mu\left(B\left(x + \frac{nh}{2}, \frac{n|h|}{2}\right)\right). \quad (3.8)$$

Using that  $\phi$  is non-increasing, we have

$$\mu\left(B\left(x + \frac{nh}{2}, 8n^2|h|\right)\right) \leq e^{K\phi(n|h|/2)} \mu(B[x, x + nh]).$$

Since  $\lim_{r \rightarrow 0^+} \frac{\phi(r)}{\log(r)} = 0$ , there exists an  $0 < R_\varepsilon < 1$  such that for all  $0 < r \leq R_\varepsilon$ ,  $0 \leq \phi(r) \leq -\frac{\varepsilon}{K} \log(r)$ . So depending on the value of  $|h|$ , two cases can be separated:

- If  $0 < \frac{n|h|}{2} \leq R_\varepsilon$ , then  $e^{K\phi(n|h|/2)} \leq 2^\varepsilon (n|h|)^{-\varepsilon}$ .
- If  $R_\varepsilon < \frac{n|h|}{2}$ , then, since  $\phi$  is non-increasing,  $\phi(n|h|/2) \leq \phi(R_\varepsilon)$ . So  $e^{K\phi(n|h|/2)} \leq e^{K\phi(R_\varepsilon)} \leq R_\varepsilon^{-\varepsilon} \leq \left(\frac{n}{R_\varepsilon}\right)^\varepsilon (n|h|)^{-\varepsilon}$ .

Summarising the above, for some  $C_{\varepsilon, n}$  large enough, one has

$$\mu\left(B\left(x + \frac{nh}{2}, 8n^2|h|\right)\right) \leq C_{\varepsilon, n} (16n^2|h|)^{-\varepsilon} \mu(B[x, x + nh]),$$

leading to (3.7).

The same argument gives (3.5) and (3.6), we give them quickly. Using the almost doubling property (1.8) at most  $K$  times and that  $\phi$  is non-increasing gives inequalities similar to (3.8), which in turn yields that for  $k \in \{1, \dots, n\}$ ,

$$\begin{aligned} \mu(B[x, x + n(h + ky)]) &\leq e^{K\phi(n|h+ky|/2)} \mu\left(B\left(x + \frac{nh}{2}, 8n^2|h|\right)\right), \\ \mu(B[x + kh, x + k(h + ny)]) &\leq e^{K\phi(nk|y|/2)} \mu\left(B\left(x + \frac{nh}{2}, 8n^2|h|\right)\right) \end{aligned}$$

Then, the same discussion based on the value of  $|h|$  gives

$$\begin{aligned}\mu^{(+\varepsilon)}(B[x, x + n(h + ky)]) &\leq C_{\varepsilon, n} \mu\left(B\left(x + \frac{nh}{2}, 8n^2|h|\right)\right), \\ \mu^{(+\varepsilon)}(B[x + kh, x + k(h + ny)]) &\leq C_{\varepsilon, n} \mu\left(B\left(x + \frac{nh}{2}, 8n^2|h|\right)\right).\end{aligned}$$

□

*Proof of Lemma 3.1 (continued).* The combination of (3.5) and (3.6) with (3.7) gives

$$\max\left(\mu^{(+2\varepsilon)}(B[x, x + n(h + ky)]), \mu^{(+2\varepsilon)}(B[x + kh, x + k(h + ny)])\right) \leq C_{\varepsilon, n} \mu\left(B\left(x + \frac{nh}{2}, 8n^2|h|\right)\right)$$

for some uniform constant  $C_{\varepsilon, n}$ . Using this suitably in (3.3) leads to

$$\begin{aligned}\|\Delta_h^{\mu, n} f\|_{L^p(\mathbb{R}^d)}^p &\leq C_{\varepsilon, n, p} \sum_{k=1}^n \int_{x \in \mathbb{R}^d} \left| \frac{\Delta_{ky}^n f(x + kh)}{\mu^{(+2\varepsilon)}(B[x + kh, x + k(h + ny)])} \right|^p \\ &\quad + \left| \frac{\Delta_{h+ky}^n f(x)}{\mu^{(+2\varepsilon)}(B[x, x + n(h + ky)])} \right|^p dx \\ &\leq C_{\varepsilon, n, p} \sum_{k=1}^n \int_{x \in \mathbb{R}^d} \left| \Delta_{ky}^{\mu^{(+2\varepsilon)}, n} f(x + kh) \right|^p + \left| \Delta_{h+ky}^{\mu^{(+2\varepsilon)}, n} f(x) \right|^p dx \\ &\leq C_{\varepsilon, n, p} \sum_{k=1}^n \int_{x \in \mathbb{R}^d} \left| \Delta_{ky}^{\mu^{(+2\varepsilon)}, n} f(x) \right|^p + \left| \Delta_{h+ky}^{\mu^{(+2\varepsilon)}, n} f(x) \right|^p dx.\end{aligned}$$

Then, defining  $C_d = \mathcal{L}^d(B(0, 3) \setminus B(0, 2))$ , an integration with respect to  $y$  over  $B(0, 3t) \setminus B(0, 2t)$  yields

$$C_d t^d \|\Delta_h^{\mu, n} f\|_{L^p(\mathbb{R}^d)}^p \leq C_{\varepsilon, n, p} \sum_{k=1}^n \int_{2t \leq |y| \leq 3t} \int_{x \in \mathbb{R}^d} \left| \Delta_{ky}^{\mu^{(+2\varepsilon)}, n} f(x) \right|^p + \left| \Delta_{h+ky}^{\mu^{(+2\varepsilon)}, n} f(x) \right|^p dx dy.$$

Further, operating the change of variable  $y' = ky$  in each term of the sum yields

$$C_d t^d \|\Delta_h^{\mu, n} f\|_{L^p(\mathbb{R}^d)}^p \leq C_{\varepsilon, n, p} \sum_{k=1}^n \int_{2kt \leq |y| \leq 3kt} \int_{x \in \mathbb{R}^d} \left| \Delta_y^{\mu^{(+2\varepsilon)}, n} f(x) \right|^p + \left| \Delta_{h+y}^{\mu^{(+2\varepsilon)}, n} f(x) \right|^p dx dy.$$

By observing that  $t \leq |h + y| \leq 4nt$ , it gives

$$C_d t^d \|\Delta_h^{\mu, n} f\|_{L^p(\mathbb{R}^d)}^p \leq 2n C_{\varepsilon, n, p} \int_{t \leq |y| \leq 4nt} \int_{x \in \mathbb{R}^d} \left| \Delta_y^{\mu^{(+2\varepsilon)}, n} f(x) \right|^p dx dy.$$

The previous upper bound is independent of  $h$ , so one concludes as desired by

$$\omega_n^\mu(f, t, \mathbb{R}^d)_p^p = \sup_{t/2 \leq |h| \leq t} \|\Delta_h^{\mu, n} f\|_{L^p(\mathbb{R}^d)}^p \leq \frac{2n C_{\varepsilon, n, p}}{C_d t^d} \int_{t \leq |y| \leq 4nt} \int_{x \in \mathbb{R}^d} \left| \Delta_y^{\mu^{(+2\varepsilon)}, n} f(x) \right|^p dx dy.$$

□

**Remark 3.3.** Let  $p \in [1, +\infty)$ ,  $n \in \mathbb{N}^*$ . If  $\mu \in \mathcal{H}(\mathbb{R}^d)$  is doubling, we obtain that there exists a constant  $C = C(d, n, p)$  such that for all  $f \in L^p(\mathbb{R}^d)$  and  $0 < t \leq 1$  :

$$\omega_n^\mu(f, t, \mathbb{R}^d)_p^p \leq C t^{-d} \int_{t \leq |h| \leq 4nt} \int_{x \in \mathbb{R}^d} \left| \Delta_h^{\mu, n} f(x) \right|^p dx dh. \quad (3.9)$$

The  $\varepsilon$  can be dropped in (3.1).

### 3.2 Embedding of inhomogeneous Besov spaces into inhomogeneous Sobolev spaces

Let us explain our approach to get the first embedding (1.14) of Theorem 1.7. Recall that  $B_{p,p}^\mu(\mathbb{R}^d)$  and  $W^{\mu,p}(\mathbb{R}^d)$  are defined using  $\mu$ -adapted  $n$ -th order finite difference operator and moduli of smoothness with  $n$  an integer such that  $n \geq \left\lfloor s_2 + \frac{d}{p} \right\rfloor + 1$

To prove (1.14), we first need to obtain that  $\|\cdot\|_{W^{\mu,p}(\mathbb{R}^d)} \lesssim \|\cdot\|_{B_{p,p}^\mu(\mathbb{R}^d)}$  to get  $B_{p,p}^\mu(\mathbb{R}^d) \subset W^{\mu,p}(\mathbb{R}^d)$ .

Let  $f \in B_{p,p}^\mu(\mathbb{R}^d)$ . We begin by decomposing the semi-norm defined in (1.10) as follows

$$\begin{aligned} \iint_{\mathbb{R}^{2d}} \frac{|\Delta_h^n f(x)|^p}{\mu(B[x, x+nh])^p |h|^{2d}} dx dh &\leq \sum_{j \in \mathbb{Z}} \int_{2^{-j} \leq |h| \leq 2^{-j+1}} \int_{x \in \mathbb{R}^d} \frac{|\Delta_h^n f(x)|^p}{\mu(B[x, x+nh])^p |h|^{2d}} dx dh \\ &\leq I_1 + I_2 \end{aligned}$$

where

$$\begin{aligned} I_1 &= \sum_{j \in \mathbb{N}} \int_{2^{-j} \leq |h| \leq 2^{-j+1}} \int_{x \in \mathbb{R}^d} \frac{|\Delta_h^n f(x)|^p}{\mu(B[x, x+nh])^p |h|^{2d}} dx dh \\ I_2 &= \int_{|h| \geq 2} \int_{x \in \mathbb{R}^d} \frac{|\Delta_h^n f(x)|^p}{\mu(B[x, x+nh])^p |h|^{2d}} dx dh. \end{aligned}$$

The term  $I_1$  can be bounded above as follows

$$\begin{aligned} I_1 &\leq \sum_{j \in \mathbb{N}} \int_{2^{-j} \leq |h| \leq 2^{-j+1}} \int_{x \in \mathbb{R}^d} 2^{2jd} \frac{|\Delta_h^n f(x)|^p}{\mu(B[x, x+nh])^p} dx dh \\ &\leq \sum_{j \in \mathbb{N}} \int_{2^{-j} \leq |h| \leq 2^{-j+1}} \int_{x \in \mathbb{R}^d} 2^{2jd} |\Delta_h^{\mu,n} f(x)|^p dx dh \\ &\leq \sum_{j \in \mathbb{N}} \int_{2^{-j} \leq |h| \leq 2^{-j+1}} 2^{2jd} \|\Delta_h^{\mu,n} f\|_{L^p(\mathbb{R}^d)}^p dh \\ &\leq \sum_{j \in \mathbb{N}} \int_{2^{-j} \leq |h| \leq 2^{-j+1}} 2^{2jd} \omega_n^\mu(f, 2^{-j+1}, \mathbb{R}^d)_p^p dh \\ &\leq C_d \sum_{j \in \mathbb{N}} 2^{2jd} 2^{-jd} \omega_n^\mu(f, 2^{-j+1}, \mathbb{R}^d)_p^p. \end{aligned}$$

So

$$I_1 \leq C_d \sum_{j \in \mathbb{N}} 2^{d(j-1)} \omega_n^\mu(f, 2^{-(j-1)}, \mathbb{R}^d)_p^p. \quad (3.10)$$

We next proceed with  $I_2$ . For every  $n \in \mathbb{N}^*$  and  $x, h \in \mathbb{R}^d$

$$\Delta_h^n f(x) = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} f(x+kh).$$

So the convexity of  $t \in \mathbb{R} \mapsto |t|^p$  yields

$$|\Delta_h^n f(x)|^p \leq C_{n,p} \sum_{k=0}^n \binom{n}{k} |f(x+kh)|^p.$$

Moreover, as  $\mu$  satisfy property (P) from Definition 1.2 and  $nh/2 > 1$ , one has

$$C^{-1} \leq \mu(B[x, x+nh]).$$

Further,

$$\begin{aligned}
 I_2 &\leq C_{n,p} \sum_{k=0}^n \binom{n}{k} \int_{2 \leq |h|} \frac{1}{|h|^{2d}} \int_{x \in \mathbb{R}^d} |f(x+kh)|^p dx dh \\
 &\leq C_{n,p} \sum_{k=0}^n \binom{n}{k} \int_{2 \leq |h|} \frac{1}{|h|^{2d}} \|f\|_{L^p(\mathbb{R}^d)}^p dh \\
 &\leq C_{n,p,d} \sum_{k=0}^n \binom{n}{k} \|f\|_{L^p(\mathbb{R}^d)}^p \\
 &\leq C_{n,p,d} \|f\|_{L^p(\mathbb{R}^d)}^p.
 \end{aligned} \tag{3.11}$$

Combining the previous inequalities on  $I_1$  and  $I_2$ , with the subadditivity of  $t \mapsto t^{1/p}$ , one gets

$$\begin{aligned}
 \|f\|_{W_p^{\mu,s}(\mathbb{R}^d)} &\leq C_{n,p,d}^{1/p} \|f\|_{L^p(\mathbb{R}^d)} + C_d^{1/p} \left( \sum_{j \in \mathbb{N}} 2^{d(j-1)} \omega_n^\mu(f, 2^{-(j-1)}, \mathbb{R}^d)_p \right)^{1/p} \\
 &\leq C_{n,p,d} \|f\|_{B_{p,p}^\mu(\mathbb{R}^d)}.
 \end{aligned}$$

We conclude that  $\|f\|_{W_p^{\mu,s}(\mathbb{R}^d)} \lesssim \|f\|_{B_{p,p}^\mu(\mathbb{R}^d)}$  i.e.  $B_{p,p}^\mu(\mathbb{R}^d) \subset W^{\mu,p}(\mathbb{R}^d)$ .

### 3.3 Embedding of inhomogeneous Sobolev spaces into inhomogeneous Besov spaces

We prove now the converse embedding in (1.15) with the additional assumption that  $\mu \in \mathcal{H}(\mathbb{R}^d)$  is almost doubling. To prove this result, we will show the following property:

For  $\varepsilon \in (0, 1)$ , there exist a constant  $C_\varepsilon > 1$  such that for all  $f \in L^p(\mathbb{R}^d)$ ,

$$\|f\|_{L^p(\mathbb{R}^d)} + |f|_{B_{p,p}^\mu(\mathbb{R}^d)} \leq C_\varepsilon (\|f\|_{L^p(\mathbb{R}^d)} + |f|_{W^{\mu(+\varepsilon),p}(\mathbb{R}^d)}). \tag{3.12}$$

This yields  $W^{\mu,p}(\mathbb{R}^d) \subset \bigcap_{0 < \varepsilon < \min(s_1, 1)} B_{p,p}^{\mu(-\varepsilon)}(\mathbb{R}^d)$ .

Let us prove (3.12) for a fixed  $\varepsilon \in (0, 1)$ . Let  $f \in L^p(\mathbb{R}^d)$ . One has by definition

$$\left\| (2^{jd/p} \omega_n^\mu(f, 2^{-j}, \mathbb{R}^d)_p)_{j \in \mathbb{N}} \right\|_{l^p(\mathbb{N})}^p = \sum_{j \in \mathbb{N}} 2^{jd} \omega_n^\mu(f, 2^{-j}, \mathbb{R}^d)_p^p.$$

Then, using Lemma 3.1, we obtain

$$\begin{aligned}
 \sum_{j \in \mathbb{N}} 2^{jd} \omega_n^\mu(f, 2^{-j}, \mathbb{R}^d)_p^p &\leq C_{d,n,p,\varepsilon} \sum_{j \in \mathbb{N}} 2^{2jd} \int_{2^{-j} \leq |h| \leq 4n2^{-j}} \int_{\mathbb{R}^d} \left| \Delta_h^{\mu(+\varepsilon),n} f(x) \right|^p dx dh \\
 &\leq C_{d,n,p,\varepsilon} \sum_{k=1}^{K_n} \sum_{j \in \mathbb{N}} 2^{2jd} \int_{2^{-j+k} \leq |h| \leq 2^{-j+k+1}} \int_{\mathbb{R}^d} \left| \Delta_h^{\mu(+\varepsilon),n} f(x) \right|^p dx dh \\
 &\leq C_{d,n,p,\varepsilon} \sum_{k=1}^{K_n} \sum_{j \in \mathbb{N}} 2^{2jd} \int_{2^{-j+k} \leq |h| \leq 2^{-j+k+1}} \int_{\mathbb{R}^d} \frac{|\Delta_h^n f(x)|^p}{\mu^{(+\varepsilon)}(B[x, x+nh])^p} dx dh,
 \end{aligned}$$

with  $K_n = \lfloor \log_2(4n) \rfloor = 2 + \lfloor \log_2(n) \rfloor$ .

For a fixed  $j \in \mathbb{N}$ , for all  $k \in \{1, \dots, K_n\}$ ,  $2^{j-K_n-1} \leq 2^{j-k-1}$ . Hence, for  $|h| \leq 2^{-j+k+1}$ , we have

$2^j \leq 2^{K_n+1} |h|^{-1}$ . It follows that

$$\begin{aligned} \sum_{j \in \mathbb{N}} 2^{jd} \omega_n^\mu(f, 2^{-j}, \mathbb{R}^d)_p^p &\leq 2^{K_n+1} C_{d,n,p,\varepsilon} \sum_{k=1}^{K_n} \sum_{j \in \mathbb{N}} \int_{2^{-j+k} \leq |h| \leq 2^{-j+k+1}} \int_{\mathbb{R}^d} \frac{|\Delta_h^n f(x)|^p}{\mu^{(+\varepsilon)}(B[x, x+nh])^p |h|^{2d}} dx dh \\ &\leq C_{d,n,p,\varepsilon} \int_{0 \leq |h| \leq 1} \int_{\mathbb{R}^d} \frac{|\Delta_h^n f(x)|^p}{\mu^{(+\varepsilon)}(B[x, x+nh])^p |h|^{2d}} dx dh \\ &\leq C_{d,n,p,\varepsilon} \int_{|h| \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\Delta_h^n f(x)|^p}{\mu^{(+\varepsilon)}(B[x, x+nh])^p |h|^{2d}} dx dh. \end{aligned}$$

We can conclude that

$$\|f\|_{L^p(\mathbb{R}^d)} + |f|_{B_{p,p}^\mu(\mathbb{R}^d)} \leq C_{d,n,p,\varepsilon} (\|f\|_{L^p(\mathbb{R}^d)} + |f|_{W^{\mu(+\varepsilon),p}(\mathbb{R}^d)}).$$

**Remark 3.4.** Using Remark 3.3 instead of Lemma 3.1 above, one gets that

$$\|f\|_{L^p(\mathbb{R}^d)} + |f|_{B_{p,p}^\mu(\mathbb{R}^d)} \leq C_{d,n,p,\varepsilon} (\|f\|_{L^p(\mathbb{R}^d)} + |f|_{W^{\mu,p}(\mathbb{R}^d)}).$$

So the  $\varepsilon$  can be dropped in (3.12) and we obtain that for every  $p \geq 1$ ,  $B_{p,p}^\mu(\mathbb{R}^d) = W^{\mu,p}(\mathbb{R}^d)$ , as claimed after Theorem 1.7.

## 4 Prevalent spectrum in $\tilde{b}_{\infty,q}^{\mu,\Psi}([0,1]^d)$

We fix from now on  $q \in [1, +\infty]$  and an almost doubling capacity  $\mu \in \mathcal{C}([0,1]^d)$  with  $\text{supp}(\mu) = [0,1]^d$  satisfying (P) and verifying the SMF (Definition 1.12). The local behaviour of elements of  $b_{p,q}^{\mu,\Psi}([0,1]^d)$  and  $\tilde{b}_{p,q}^{\mu,\Psi}([0,1]^d)$  is described via their pointwise Hölder exponent, whose study is based on the characterisation developed by Jaffard in [33].

Consider an integer  $n \geq \left\lfloor s_2 + \frac{d}{p} \right\rfloor + 1$  and wavelets  $(\Phi, \{\Psi^l\}_{l \in L^d})$  that belong to  $C^n(\mathbb{R}^d)$  and have at least  $n$  vanishing moments.

**Definition 4.1.** Let  $f \in L_{\text{loc}}^p(\mathbb{R}^d)$  for  $p \in [1, +\infty]$ . Let  $(c_\lambda)_{\lambda \in \Lambda \times L^d}$  be the wavelet coefficients of  $f$  associated with  $\Psi$ . The wavelet leader of  $f$  associated with  $\lambda \in \Lambda$  is defined as

$$L_\lambda^f = \sup\{|c_{\lambda'}| : \lambda' = (j, k, l) \in \Lambda \times L^d, \lambda' \subset 3\lambda\}. \quad (4.1)$$

**Notation.** Consider a function  $f \in \tilde{b}_{p,q}^{\mu,\Psi}([0,1]^d)$ . Following Definition 4.1, for  $x \in [0,1]^d$ , the wavelet leader of  $f$  at scale  $j$  at  $x$  is denoted by

$$L_j^f(x) := L_{\lambda_j^d(x)}^f,$$

with  $\lambda_j^d(x) \in \Lambda_j^d$  the unique dyadic cube of generation  $j$  containing  $x$  (recall the definitions of Section 2). The mention of  $f$  in the leaders might be omitted when the function is clear from the context.

The pointwise Hölder exponent (1.2) can be expressed as follows (see [33, Proposition 4]).

**Proposition 4.2.** If  $f \in \mathcal{C}^\varepsilon(\mathbb{R}^d)$  for some  $\varepsilon > 0$ , then for every  $x_0 \in \mathbb{R}^d$ ,  $h_f(x_0) < r$  if and only if

$$\liminf_{j \rightarrow +\infty} \frac{\log L_j^f(x_0)}{\log(2^{-j})} < r, \text{ and in this case}$$

$$h_f(x_0) = \liminf_{j \rightarrow +\infty} \frac{\log L_j^f(x_0)}{\log(2^{-j})}.$$

#### 4.1 A saturating function in $\tilde{b}_{\infty,q}^{\mu,\Psi}$

To identify the prevalent properties of functions in  $\tilde{b}_{p,q}^{\mu,\Psi}([0,1]^d)$ , the classical strategy is to perturb a countable family of prevalent sets with a *saturating function* whose multifractal structure is the one claimed to be generic. Let us consider the saturating function of [9, Section 6].

**Definition 4.3.** *Let us define, for every  $\lambda = (j, k, l) \in \Lambda_j \times L^d$ ,*

$$g_\lambda = \frac{1}{j^{\frac{2}{q}}} \mu(\lambda^d) \quad (4.2)$$

with the convention  $\frac{2}{\infty} = 0$ . The saturating function is  $\mathcal{G}_q := \sum_{\lambda \in \Lambda \times L^d} g_\lambda \Psi_\lambda$ .

From [9, Lemma 6.4], recalling the scaling function  $\tau_\mu$  (1.23), one has

**Proposition 4.4.** *The function  $\mathcal{G}_q$  belongs to  $\tilde{b}_{\infty,q}^{\mu,\Psi}([0,1]^d)$  and its singularity spectrum is*

$$\sigma_{\mathcal{G}_q} = \tau_{\mu,\infty}^* = \sigma_\mu.$$

The fact that  $\mathcal{G}_q \in \tilde{b}_{\infty,q}^{\mu,\Psi}([0,1]^d)$  is directly deduced from the definition of the norm (1.18) which gives that for every  $\varepsilon \in (0, \min(1, s_1))$ ,  $|\mathcal{G}_q|_{\mu^{(-\varepsilon)}, \infty, q, \Psi} = \sum_{j=1}^{\infty} 2^{-j\varepsilon + 2 \log_2(j)}$  that converges. The singularity spectrum is deduced through the key property that for every  $x \in (0, 1)^d$ ,  $h_{\mathcal{G}_q}(x) = h_\mu(x)$ .

#### 4.2 Lower bound of the pointwise Hölder exponent in $\tilde{b}_{\infty,q}^{\mu,\Psi}([0,1]^d)$

Let us begin with the following result.

**Lemma 4.5.** *For all  $f \in \tilde{b}_{\infty,q}^{\mu,\Psi}([0,1]^d)$ , for every  $x \in [0, 1]^d$ ,  $h_f(x) \geq \underline{h}_\mu(x)$ .*

*Proof.* Let us write the wavelet decomposition

$$f = c_{0,0^d,0^d} \Psi_{0,0^d,0^d} + \sum_{\lambda \in \Lambda \times L^d} c_\lambda^f \Psi_\lambda \quad (4.3)$$

Let  $\varepsilon \in (0, \min(1, s_1))$ . Recall Definition 1.9. Since  $f \in \tilde{b}_{\infty,q}^{\mu^{(-\varepsilon)}, \Psi}([0,1]^d)$ , for every  $j \in \mathbb{N}$ ,  $\varepsilon_j^{\mu^{(-\varepsilon)}, \infty} < +\infty$  and one has

$$\sup_{j \in \mathbb{N}} \sup_{\lambda \in \Lambda_j \times L^d} \left| \frac{c_\lambda^f}{\mu(\lambda) 2^{j\varepsilon}} \right| \leq C.$$

For a fixed  $x \in [0, 1]^d$ , one has

$$L_j^f(x) = \sup_{\lambda \subset 3\lambda_j^d(x)} |c_\lambda^f| \leq C \sup_{\lambda_{j'} \subset 3\lambda_j^d(x)} |\mu(\lambda_{j'})| 2^{j'\varepsilon}.$$

Recall that  $\mu$  satisfy property (P) with  $0 < s_1 < s_2$ . Hence, setting  $M = \left\lfloor \frac{s_2}{s_1} \right\rfloor + 1$ , (2.2) gives that for every  $j' > Mj$ , if  $\lambda_{j'} \subset 3\lambda_j^d(x)$

$$\mu(\lambda_{j'}) 2^{j'\varepsilon} \leq \tilde{C} 2^{-j'(s_1-\varepsilon)} \leq \tilde{C} 2^{-j(s_2-\varepsilon)} \leq \tilde{C}^2 \mu(\lambda_j^d(x)) 2^{j\varepsilon}.$$

For  $j \leq j' \leq Mj$ , using that  $\mu$  is a capacity, one has

$$\mu(\lambda_{j'}) 2^{j'\varepsilon} \leq \mu(\lambda_{j'}) 2^{Mj\varepsilon} \leq \mu(3\lambda_j^d(x)) 2^{Mj\varepsilon}.$$

The almost-doubling property of  $\mu$  gives that there exist  $\phi_\mu : [0, 1) \rightarrow \mathbb{R}_+$ , such that  $\lim_{j \rightarrow +\infty} \frac{\phi_\mu(r)}{\log(r)} = 0$ , for which (1.8) applies. Considering  $r = 2^{-j}$ , one has  $\mu(3\lambda_j^d(x)) \leq e^{\phi_\mu(2^{-j})} \mu(\lambda_j^d(x))$ . Combining this with the last two inequalities gives, for large  $j$ s,

$$L_j^f(x) \leq C \max(\tilde{C}^2 2^{j\varepsilon}, e^{\phi_\mu(2^{-j})} 2^{Mj\varepsilon}) \mu(\lambda_j^d(x)) \leq C e^{\phi_\mu(2^{-j})} 2^{Mj\varepsilon} \mu(\lambda_j^d(x)).$$

Hence, for all  $j \in \mathbb{N}$  large enough

$$\log L_j^f(x) \leq \log C + \phi_\mu(2^{-j}) + \log \mu(\lambda_j^d(x)) + \log(2^{Mj\varepsilon})$$

and

$$\frac{\log L_j^f(x)}{\log(2^{-j})} \geq \frac{\log C}{\log(2^{-j})} + \frac{\phi_\mu(2^{-j})}{\log(2^{-j})} + \frac{\log \mu(\lambda_j^d(x))}{\log(2^{-j})} - M\varepsilon.$$

Taking the  $\liminf_{j \in \mathbb{N}}$  on both sides gives  $h_f(x) \geq \underline{h}_\mu(x) - M\varepsilon$ . Since this holds for all  $\varepsilon \in (0, \min(1, s_1))$ , one concludes that  $h_f(x) \geq \underline{h}_\mu(x)$ .  $\square$

**Remark 4.6.** In Lemma 4.5, one can replace  $\tilde{b}_{\infty,q}^{\mu,\Psi}([0, 1]^d)$  by  $b_{\infty,q}^{\mu,\Psi}([0, 1]^d)$ , by simply setting  $\varepsilon$  to zero in all the computations.

### 4.3 Proof of Theorem 1.14

The key result to obtain the prevalent value of the singularity spectrum of functions in  $\tilde{b}_{\infty,q}^{\mu,\Psi}([0, 1]^d)$  in Theorem 1.14 is the following:

**Proposition 4.7.** *The set*

$$\mathcal{F}^\mu = \left\{ f \in \tilde{b}_{\infty,q}^{\mu,\Psi}([0, 1]^d) : \forall x \in [0, 1]^d, h_f(x) \leq \bar{h}_\mu(x) \right\} \quad (4.4)$$

*is prevalent in  $\tilde{b}_{\infty,q}^{\mu,\Psi}([0, 1]^d)$ .*

The proof of Proposition 4.7 is postponed to Subsection 4.3.1. For the moment, we admit it and explain how this allows us to deduce Theorem 1.14.

*Proof of Theorem 1.14.* Using Lemma 4.5 and Proposition 4.7, for all  $f \in \mathcal{F}^\mu$ , for every  $x \in [0, 1]^d$ ,

$$\underline{h}_\mu(x) \leq h_f(x) \leq \bar{h}_\mu(x). \quad (4.5)$$

Fix  $h \in \mathbb{R}_+$ .

Step 1: let us start with *the upper bound*  $\sigma_f \leq \sigma_\mu$ . By (4.5), one directly gets

$$\begin{aligned} E_f(h) &\subset \{x \in \mathbb{R}^d : h_f(x) \leq h\} \subset \{x \in \mathbb{R}^d : \underline{h}_\mu(x) \leq h\} = \underline{E}_\mu^\leq(h), \\ E_f(h) &\subset \{x \in \mathbb{R}^d : h_f(x) \geq h\} \subset \{x \in \mathbb{R}^d : \bar{h}_\mu(x) \geq h\} = \bar{E}_\mu^\geq(h). \end{aligned}$$

We now apply Proposition 2.3 to  $\mu$  which satisfies the SMF, therefore

- if  $h \leq \tau'_\mu(0^+)$ , (ii) gives

$$\sigma_f(h) = \dim_H(E_f(h)) \leq \dim_H(\underline{E}_\mu^\leq(h)) = \tau_\mu^*(h) = \sigma_\mu(h),$$

- if  $\tau'_\mu(0^+) \leq h \leq \tau'_\mu(0^-)$ , with (iv), one has  $\sigma_f(h) \leq d = \sigma_\mu(h)$ ,

- if  $h \geq \tau'_\mu(0^-)$ , (iii) gives

$$\sigma_f(h) = \dim_H(E_f(h)) \leq \dim_H(\overline{E}_\mu^{\geq}(h)) = \tau_\mu^*(h) = \sigma_\mu(h).$$

Hence one gets the upper bound.

Step 2: let us move to *the lower bound*  $\sigma_f \geq \sigma_\mu$ . For all  $x \in E_\mu(h)$ , by Definition 2.1, one has  $\underline{h}_\mu(x) = \overline{h}_\mu(x) = h_\mu(x)$ . Thus, by (4.5), one deduces  $h_\mu(x) = h_f(x)$  and  $E_\mu(h) \subset E_f(h)$ . Finally, Proposition 2.3 gives

$$\sigma_f(h) = \dim_H(E_f(h)) \geq \dim_H(E_\mu(h)) = \sigma_\mu(h).$$

□

#### 4.3.1 Prevalence property of the auxiliary set $\mathcal{F}^\mu$

We want to show that  $(\mathcal{F}^\mu)^\complement = \widetilde{b}_{\infty,q}^{\mu,\Psi}([0,1]^d) \setminus \mathcal{F}^\mu$  is shy. The set  $(\mathcal{F}^\mu)^\complement$  can be written as

$$(\mathcal{F}^\mu)^\complement = \left\{ f \in \widetilde{b}_{\infty,q}^{\mu,\Psi}([0,1]^d) : \exists x \in [0,1]^d, h_f(x) > \overline{h}_\mu(x) \right\}.$$

We simplify the problem by including the complement of  $\mathcal{F}^\mu$  in a countable union of simpler ancillary sets. For  $p \in \mathbb{N}^*$ ,  $\gamma > 0$  and  $M = (M_1, M_2) \in (\mathbb{N}^*)^2$ , let

$$\mathcal{O}^{p,\gamma,M} = \left\{ f \in \widetilde{b}_{\infty,q}^{\mu,\Psi}([0,1]^d) : \begin{array}{l} \exists x \in [0,1]^d, \\ \forall \lambda_j = (j,k,l) \in \Lambda \times L^d, |c_{\lambda_j}^f| \leq M_1 2^{-j\gamma} (1 + |2^j x - k|)^\gamma \\ \forall j > M_2, \mu(\lambda_j(x)) \geq 2^{-j(\gamma-1/p)} \end{array} \right\}.$$

Remembering Proposition 2.6, the sets  $\mathcal{O}^{p,\gamma,M}$  shall be understood as the set of functions in  $\widetilde{b}_{\infty,q}^{\mu,\Psi}([0,1]^d)$  such that for some  $x \in [0,1]^d$ ,  $f \in C^\gamma(x)$  and simultaneously the  $\mu$ -mass of the dyadic cubes  $\lambda_j(x)$  is bounded below. This two properties will allow us to control the ratio  $\frac{c_{\lambda_j(x)}^f}{\mu(\lambda_j(x))}$ .

**Proposition 4.8.** *One has*

$$(\mathcal{F}^\mu)^\complement \subset \bigcup_{p \in \mathbb{N}^*} \bigcup_{\gamma \in \mathbb{Q}_+} \bigcup_{M \in (\mathbb{N}^*)^2} \mathcal{O}^{p,\gamma,M}.$$

*Proof.* We write  $(\mathcal{F}^\mu)^\complement = \bigcup_{p \in \mathbb{N}^*} \mathcal{F}_p$ , where

$$\mathcal{F}_p = \left\{ f \in \widetilde{b}_{\infty,q}^{\mu,\Psi}([0,1]^d) : \exists x \in [0,1]^d, h_f(x) - \overline{h}_\mu(x) > \frac{2}{p} \right\}.$$

Supposing that  $f \in \mathcal{F}_p$ , there exists  $\gamma \in \mathbb{Q}_+$  and  $x \in [0,1]^d$  such that  $h_f(x) > \gamma$  and  $\overline{h}_\mu(x) \leq \gamma - \frac{2}{p}$ . Using (2.6) of Proposition 2.6, one has

$$\begin{aligned} h_f(x) > \gamma &\implies f \in C^\gamma(x) \\ &\implies \exists M_1 \in \mathbb{N}, \forall \lambda_j = (j,k,l) \in \Lambda \times L^d, |c_{\lambda_j}^f| \leq M_1 2^{-j\gamma} (1 + |2^j x - k|)^\gamma. \end{aligned}$$

In addition, by (2.1), one has

$$\overline{h}_\mu(x) \leq \gamma - \frac{2}{p} \implies \exists M_2 \in \mathbb{N}, \sup_{j > M_2} \frac{\log \mu(\lambda_j(x))}{\log 2^{-j}} \leq \gamma - \frac{1}{p}$$

$$\implies \exists M_2 \in \mathbb{N}, \forall j > M_2, \mu(\lambda_j(x)) \geq 2^{-j(\gamma-1/p)}.$$

Hence,  $f \in \mathcal{O}^{p,\gamma,M}$  with  $M = (M_1, M_2)$ . One gets that if  $f \notin \mathcal{F}^\mu$ , then there exists  $p \in \mathbb{N}^*$ ,  $f \in \mathcal{F}_p^\mu$ . Then, for  $p \in \mathbb{N}^*$ , there exists  $\gamma \in \mathbb{Q}_+$  and an  $M = (M_1, M_2) \in (\mathbb{N}^*)^2$  such that  $f \in \mathcal{O}^{p,\gamma,M}$ .  $\square$

To use the prevalence theory, it remains to prove the universal measurability of  $\mathcal{O}^{p,\gamma,M}$ . Analytic sets in Polish spaces are defined as continuous images of Borel sets. Unfortunately, as we work with  $\tilde{b}_{\infty,q}^{\mu,\Psi}([0,1]^d)$  which is not separable, we cannot use the Polish spaces characterisation and have to consider a more general definition, from Choquet [14], valid for any Hausdorff topological space  $X$  endowed with a Borel  $\sigma$ -algebra  $\mathcal{B}(X)$ .

Choquet's capacitability theorem [14, Theorem 30.1] with the following definition of analyticity [14, Theorem 5.1] give the right setting, as was observed in [4].

**Definition 4.9.** *For a compact topological space  $K$ , the collection of its closed subsets is written  $\mathcal{K}$ . Let  $(\mathcal{B}(X) \times \mathcal{K})_{\sigma\delta}$  be the collection of countable intersections of countable unions of sets that are Cartesian products of a Borel set in  $X$  and a closed set in  $K$ .*

*A set  $A \subset X$  is said to be analytic if there exists a compact space  $K$  and  $\mathcal{T} \in (\mathcal{B}(X) \times \mathcal{K})_{\sigma\delta}$  such that*

$$A = \pi(\mathcal{T}) \tag{4.6}$$

*with  $\pi : X \times K \rightarrow X$  the canonical projection map  $\pi(x, y) = x$ .*

Then Choquet's capacitability theorem yields the following theorem.

**Theorem 4.10.** *Every analytic set of a Hausdorff topological space is universally measurable.*

By Theorem 4.10, the universal measurability of  $\mathcal{O}^{p,\gamma,M}$  is equivalent to its analyticity.

**Proposition 4.11.** *Let  $p \in \mathbb{N}^*$ ,  $M \in (\mathbb{N}^*)^2$  and  $\gamma \in \mathbb{Q}_+$ . The set  $\mathcal{O}^{p,\gamma,M}$  is universally measurable in  $\tilde{b}_{\infty,q}^{\mu,\Psi}([0,1]^d)$ .*

*Proof.* We are going to prove the analyticity of  $\mathcal{O}^{p,\gamma,M}$  by taking the right sets  $X$ ,  $K$  and  $\mathcal{T}$  in Definition 4.9, then writing it as  $\pi(\mathcal{T})$ . Denote  $X := \left(\tilde{b}_{\infty,q}^{\mu,\Psi}([0,1]^d), d\right)$  with  $d$  defined in (1.20) and  $K := ([0,1]^d, \|\cdot\|_\infty)$ .

Step 1: For  $\lambda_j = (j, k, l) \in \Lambda \times L^d$ , let us introduce the maps  $\phi_{\lambda_j}, \phi : X \times K \rightarrow \mathbb{R}$  defined as

$$\begin{aligned} \phi_{\lambda_j} : (f, x) &\mapsto M_1 2^{-j\gamma} (1 + |2^j x - k|)^\gamma - \left| c_{\lambda_j}^f \right| \\ \phi : (f, x) &\mapsto \inf_{\lambda \in \Lambda \times L^d} \phi_\lambda(f, x). \end{aligned}$$

**Lemma 4.12.** *Each  $\phi_{\lambda_j}$  is continuous on  $X \times K$ .*

*Proof.* Let  $\varepsilon > 0$ . Recalling (1.20), for any  $f, g \in X$  such that  $d(f, g) < \varepsilon$ , one gets that, for any  $n := \lfloor \max(1, s_1^{-1}) \rfloor + 1$ ,

$$2^{-n} \frac{\|f - g\|_{b_{\infty,q}^{\mu(-1/n),\Psi}}}{1 + \|f - g\|_{b_{\infty,q}^{\mu(-1/n),\Psi}}} < \varepsilon.$$

This induces that  $\|f - g\|_{b_{\infty,q}^{\mu(-1/n),\Psi}} < 2^n \varepsilon$ . By definition of  $\|\cdot\|_{b_{\infty,q}^{\mu(-\varepsilon),\Psi}}$  in Remark 1.10,  $\|f - g\|_\infty < 2^n \varepsilon$ .

The application  $f \in X \mapsto c_{\lambda_j}^f = \langle f, \Psi_{\lambda_j} \rangle \in \mathbb{R}$  is a scalar product between  $f$  and a fixed wavelet  $\Psi_{\lambda_j}$ . With the classical remark that  $\langle f, \Psi_{\lambda_j} \rangle \leq \|f\|_\infty \|\Psi_{\lambda_j}\|_1$ ,  $f \mapsto c_{\lambda_j}^f$  is continuous. So the mapping  $(f, x) \mapsto c_{\lambda_j}^f$  is continuous and so is  $\phi_{\lambda_j}$  as a sum of continuous functions.  $\square$

Step 2: Denote

$$\mathcal{D}_\mu(j, \gamma - 1/p) := \left\{ \lambda_j \in \Lambda_j : \frac{\log \mu(\lambda_j)}{\log(2^{-j})} \leq \gamma - \frac{1}{p} \right\}. \quad (4.7)$$

Let us introduce

$$\mathcal{T} := \phi^{-1}([0, +\infty)) \cap \left( X \times \bigcap_{j \geq M_2} \bigcup_{\lambda \in \mathcal{D}_\mu(j, \gamma - 1/p)} \lambda \right).$$

**Lemma 4.13.** *The set  $\mathcal{T}$  belongs to  $(\mathcal{B}(X) \times \mathcal{K})_{\sigma\delta}$ .*

*Proof.* First, it is straightforward from the definition that, for all  $\lambda \in \Lambda$ ,  $X \times \lambda \in \mathcal{B}(X) \times \mathcal{K}$ , so taking a countable intersection of compact sets,

$$X \times \bigcap_{j \geq M_2} \bigcup_{\lambda \in \mathcal{D}_\mu(j, \gamma - 1/p)} \lambda \in (\mathcal{B}(X) \times \mathcal{K})_{\sigma\delta}.$$

Let  $n \in \mathbb{N}$  and  $m \in \mathbb{Z}$  and consider

$$F_{\lambda_j}(n, m) := \{f \in \tilde{b}_{\infty, q}^{\mu, \Psi}([0, 1]^d) : c_{\lambda_j}^f \in 2^{-n}[m, m+1]\}.$$

Since  $2^{-n}[m, m+1]$  is a closed interval,  $F_{\lambda_j}(n, m)$  is closed as the preimage of a closed set by the continuous function  $f \mapsto c_{\lambda_j}^f$ . Next, let us define

$$X_{\lambda_j}(n, m) := \left\{ x \in [0, 1]^d : \sup_{f \in F_{\lambda_j}(n, m)} \phi_{\lambda_j}(f, x) \geq 0 \right\}.$$

Observe that when  $f$  ranges in  $\tilde{b}_{\infty, q}^{\mu, \Psi}([0, 1]^d)$ , all possible values of  $c_{\lambda_j}^f \in 2^{-n}[m, m+1]$  are reached. Hence, one can rewrite  $X_{\lambda_j}(n, m)$  as

$$X_{\lambda_j}(n, m) = \left\{ x \in [0, 1]^d : \max_{c \in 2^{-n}[m, m+1]} M_1 2^{-j\gamma} (1 + |2^j x - k|)^\gamma - |c| \geq 0 \right\}.$$

The max can be in turn rewritten

$$X_{\lambda_j}(n, m) = \left\{ x \in [0, 1]^d : \exists c \in 2^{-n}[m, m+1], M_1 2^{-j\gamma} (1 + |2^j x - k|)^\gamma - |c| \geq 0 \right\}.$$

Denoting the mapping  $v : (x, c) \in \mathbb{R}^d \times \mathbb{R} \mapsto M_1 2^{-j\gamma} (1 + |2^j x - k|)^\gamma - |c| \in \mathbb{R}$ , one has, for  $\tilde{\pi}(x, c) = x$  the projection on the first coordinate,

$$X_{\lambda_j}(n, m) = \tilde{\pi} \left( v^{-1}(\mathbb{R}_+) \cap ([0, 1]^d \times 2^{-n}[m, m+1]) \right).$$

The mapping  $v$  is continuous from  $\mathbb{R}^d \times \mathbb{R}$  to  $\mathbb{R}$ , so  $v^{-1}(\mathbb{R}_+) \cap ([0, 1]^d \times 2^{-n}[m, m+1])$  is compact. As the projection is continuous,  $X_{\lambda_j}(n, m)$  is also compact.

**Lemma 4.14.** *One has*

$$\phi_{\lambda_j}^{-1}([0, +\infty)) = \bigcap_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{Z}} F_{\lambda_j}(n, m) \times X_{\lambda_j}(n, m).$$

*Proof.* First, let us prove that  $\phi_{\lambda_j}^{-1}([0, +\infty)) \subset \bigcap_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{Z}} F_{\lambda_j}(n, m) \times X_{\lambda_j}(n, m)$ .

Let  $(f, x) \in \phi_{\lambda_j}^{-1}([0, +\infty))$ . Then every  $n \in \mathbb{N}$ , there is an integer  $m \in \mathbb{Z}$  such that  $c_{\lambda_j}^f \in 2^{-n}[m, m+1]$  and so  $f \in F_{\lambda_j}(n, m)$ . In addition,  $\phi_{\lambda_j}(f, x) \geq 0$ , so  $\sup_{g \in F_{\lambda_j}(n, m)} \phi_{\lambda_j}(g, x) \geq 0$  and  $x \in X_{\lambda_j}(n, m)$ . Hence  $(f, x) \in F_{\lambda_j}(n, m) \times X_{\lambda_j}(n, m)$ .

Next, let us prove the converse inclusion  $\phi_{\lambda_j}^{-1}([0, +\infty)) \supset \bigcap_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{Z}} F_{\lambda_j}(n, m) \times X_{\lambda_j}(n, m)$ .

Let  $(f, x) \in \bigcap_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{Z}} F_{\lambda_j}(n, m) \times X_{\lambda_j}(n, m)$ . Then for all  $n \in \mathbb{N}$ , there exist an  $m \in \mathbb{Z}$  such that  $f \in F_{\lambda_j}(n, m)$  and  $x \in X_{\lambda_j}(n, m)$ .

One has  $c_{\lambda_j}^f \in 2^{-n}[m, m+1]$  and  $\sup_{g \in F_{\lambda_j}(n, m)} \phi_{\lambda_j}(g, x) \geq 0$ . One deduces

$$\begin{aligned} \left| \sup_{g \in F_{\lambda_j}(n, m)} \phi_{\lambda_j}(g, x) - \phi_{\lambda_j}(f, x) \right| &\leq \sup_{g \in F_{\lambda_j}(n, m)} |\phi_{\lambda_j}(g, x) - \phi_{\lambda_j}(f, x)| \\ &\leq \sup_{g \in F_{\lambda_j}(n, m)} |c_{\lambda_j}^g - c_{\lambda_j}^f| \leq 2^{-n}. \end{aligned}$$

As, this result holds for all  $n \in \mathbb{N}$ , one deduces that  $\phi_{\lambda_j}(f, x) \geq 0$  □

Hence,  $\phi_{\lambda_j}^{-1}([0, +\infty)) \in (\mathcal{B}(X) \times \mathcal{K})_{\sigma\delta}$ . We deduce that  $\phi^{-1}([0, +\infty)) = \bigcap_{\lambda \in \Lambda \times L^d} \phi_{\lambda}^{-1}([0, +\infty))$  also belongs to  $(\mathcal{B}(X) \times \mathcal{K})_{\sigma\delta}$ . This proves that the set  $\mathcal{T}$  indeed belongs to  $(\mathcal{B}(X) \times \mathcal{K})_{\sigma\delta}$ . □

Step 3: By Definition 4.9, the canonical projection map  $\pi : (x, y) \in X \times K \rightarrow x \in X$  gives that

$$\pi(\mathcal{T}) := \left\{ f \in X : \exists x \in K, (f, x) \in \phi^{-1}([0, +\infty)) \text{ and } \forall j > M_2, \mu(\lambda_j(x)) \geq 2^{-j(\gamma-1/p)} \right\}$$

is analytic in the space  $(X, \mathcal{B}(X))$ .

Step 4: To conclude, notice that one can rewrite

$$\mathcal{O}^{p, \gamma, M} = \pi(\mathcal{T}).$$

This yields that  $\mathcal{O}^{p, \gamma, M}$  is analytic and thus universally measurable. □

As the sets  $\mathcal{O}^{p, \gamma, M}$  are universally measurable, prevalence theory is now used to show the shyness of  $(\mathcal{F}^\mu)^\mathbb{G}$  through the shyness of  $\mathcal{O}^{p, \gamma, M}$ . To do so, the classical method is to use a finite dimensional subspace  $\mathcal{P}$  and the Lebesgue measure restricted to a compact subset of  $\mathcal{P}$  as a transverse measure.

Let  $d_1$  be an integer such that

$$d < d_1/2p. \tag{4.8}$$

**Definition 4.15.** Consider the saturating function  $\mathcal{G}_q$  whose wavelet coefficients are  $g_\lambda$  (see (4.2)).

Let  $\mathcal{P} = \text{Vect}_{i=1, \dots, d_1}(\mathcal{G}_q^{(i)}) \subset \tilde{b}_{\infty, q}^{\mu, \Psi}([0, 1]^d)$  where  $\mathcal{G}_q^{(i)}$  is the wavelet series whose wavelet coefficients  $g_\lambda^{(i)}$  are defined in the following way: for every  $\lambda = (j, k, l) \in \Lambda \times \{0, 1\}^d$ ,

$$g_\lambda^{(i)} = \begin{cases} g_\lambda & \text{if } j \pmod{d_1} = i, \\ 0 & \text{else.} \end{cases} \tag{4.9}$$

Take an arbitrary  $f \in \tilde{b}_{\infty,q}^{\mu,\Psi}([0,1]^d)$ , define the function  $f^\beta$  by

$$\begin{aligned} \mathbb{R}^{d_1} &\rightarrow f + \mathcal{P} \\ \beta &\mapsto f^\beta = f + \sum_{i=1}^{d_1} \beta_i \cdot \mathcal{G}_q^{(i)}. \end{aligned} \quad (4.10)$$

Denote  $c_\lambda^f$  (resp.  $c_\lambda^\beta$ ) the wavelet coefficient of  $f$  (resp.  $f^\beta$ ) and  $L_\lambda^f$  (resp.  $L_\lambda^\beta$ ) the associated wavelet leader.

**Proposition 4.16.** *For any function  $f \in \tilde{b}_{\infty,q}^{\mu,\Psi}([0,1]^d)$ , the set  $\{\beta \in \mathbb{R}^{d_1} : f^\beta \in \mathcal{O}^{p,\gamma,M}\}$  has  $d_1$ -dimensional Lebesgue measure  $\mathcal{L}_{d_1}$  zero.*

*Proof.* Let us rewrite  $\{\beta \in \mathbb{R}^{d_1} : f^\beta \in \mathcal{O}^{p,\gamma,M}\}$  as

$$\mathcal{B}_f := \left\{ \beta \in \mathbb{R}^{d_1} : \begin{array}{l} \exists x \in [0,1]^d, \\ \forall \lambda_j = (j,k,l) \in \Lambda \times L^d, \left| c_{\lambda_j}^\beta \right| \leq M_1 2^{-j\gamma} (1 + |2^j x - k|)^\gamma \\ \text{and } \forall j > M_2, \mu(\lambda_j(x)) \geq 2^{-j(\gamma-1/p)} \end{array} \right\}.$$

We start with a technical lemma.

**Lemma 4.17.** *The mapping  $\begin{cases} \mathbb{R}^{d_1} &\longrightarrow \mathbb{R} \\ \beta &\longmapsto \mathbf{1}_{\mathcal{B}}(\beta) \end{cases}$  is measurable.*

*Proof.* We will use the notation and function from the proof of Lemma 4.13.

Let  $\Phi : (\beta, x) \mapsto \inf_{\lambda_j \in \Lambda \times L^d} \phi_{\lambda_j}(f^\beta, x)$ . The map  $\Phi$  is Borel on  $\mathbb{R}^{d_1} \times [0,1]^d$  as an infimum of countably many continuous functions. One sees that

$$\mathcal{B}_f = \left\{ \beta \in \mathbb{R}^{d_1} : \exists x \in [0,1]^d, \Phi(\beta, x) \geq 0 \text{ and } \forall j > M_2, \mu(\lambda_j(x)) \geq 2^{-j(\gamma-1/p)} \right\}.$$

This set can be rewritten as

$$\mathcal{B}_f = \tilde{\pi} \left( \Phi^{-1}([0, +\infty)) \cap \left( \mathbb{R}^{d_1} \times \bigcap_{j \geq M_2} \bigcup_{\lambda \in \mathcal{D}_\mu(j, \gamma-1/p)} \lambda \right) \right), \quad (4.11)$$

where  $\tilde{\pi}(\beta, x) = \beta$  is the canonical projection on the first coordinate and  $\mathcal{D}_\mu(j, \gamma-1/p)$  is defined by (4.7). Since the set in brackets in (4.11) is a Borel set,  $\mathcal{B}_f$  is analytic by Definition 4.9. By Theorem 4.10, it is also Lebesgue-measurable.  $\square$

We are ready to prove that the set  $\mathcal{B}_f$  has  $d_1$ -dimensional Lebesgue measure  $\mathcal{L}_{d_1}$  zero. For any  $\lambda_0 := (j_0, k_0, l_0) \in \Lambda$ , denote

$$\mathcal{B}_{f,\lambda_0} = \left\{ \beta \in \mathbb{R}^{d_1} : \begin{array}{l} \exists x \in \lambda_0, \\ \forall \lambda_j = (j,k,l) \in \Lambda \times L^d, \left| c_{\lambda_j}^\beta \right| \leq M_1 2^{-j\gamma} (1 + |2^j x - k|)^\gamma \\ \text{and } \forall j > M_2, \mu(\lambda_j(x)) \geq 2^{-j(\gamma-1/p)} \end{array} \right\}. \quad (4.12)$$

so that

$$\mathcal{B}_f = \limsup_{j_0 \rightarrow \infty} \bigcup_{(j_0, k_0, l_0) \in \Lambda_{j_0}^d} \mathcal{B}_{f,\lambda_0} = \bigcap_{J \in \mathbb{N}^*} \bigcup_{j_0 \geq J} \bigcup_{(j_0, k_0, l_0) \in \Lambda_{j_0}^d} \mathcal{B}_{f,\lambda_0}. \quad (4.13)$$

Let us show that  $\mathcal{L}_{d_1}(\mathcal{B}_f) = 0$  by establishing an upper bound for each  $\mathcal{L}_{d_1}(\mathcal{B}_{f,\lambda_0})$  and using the Borel-Cantelli lemma. One can ensure that  $j_0$  is large enough so that  $j_0 \geq M_2 + d_1$ . Suppose that  $\beta$  and  $\tilde{\beta}$

belong to a same  $\mathcal{B}_{f,\lambda_0}$ . There exist  $x_\beta, x_{\tilde{\beta}} \in \lambda_0 \subset \mathbb{R}^d$  such that for every  $j \in \mathbb{N}^*$ ,

$$\left| c_{\lambda_j(x_\beta)}^\beta \right| \leq M_1 2^{-j\gamma} (1 + |2^j x_\beta - k_{j,\beta}|)^\gamma \quad \text{and} \quad \left| c_{\lambda_j(x_{\tilde{\beta}})}^{\tilde{\beta}} \right| \leq M_1 2^{-j\gamma} (1 + |2^j x_{\tilde{\beta}} - k_{j,\tilde{\beta}}|)^\gamma \quad (4.14)$$

for  $k_{j,\beta}$  and  $k_{j,\tilde{\beta}}$  the multi-integers associated respectively to  $\lambda_j(x_\beta)$  and  $\lambda_j(x_{\tilde{\beta}})$ .

For every  $j \in \{j_0 - d_1, \dots, j_0 - 1\}$ , one has  $\lambda_j(x_\beta) = \lambda_j(x_{\tilde{\beta}})$ . Let us denote  $\tilde{\lambda}_j = \lambda_j(x_\beta)$ .

Using  $x_\beta, x_{\tilde{\beta}} \in \lambda_0 \subset \tilde{\lambda}_j$  and (4.14), one deduces that there is a constant  $C > 0$  depending on  $M, j_0, d_1$  and  $\gamma$  such that, for every  $j \in \{j_0 - d_1, \dots, j_0 - 1\}$ ,  $\left| c_{\tilde{\lambda}_j}^\beta \right| \leq C 2^{-j\gamma}$  and  $\left| c_{\tilde{\lambda}_j}^{\tilde{\beta}} \right| \leq C 2^{-j\gamma}$ .

From all this, we deduce that, for  $j \in \{j_0 - d_1, \dots, j_0 - 1\}$  and  $i$  such that  $j \pmod{d_1} \equiv i$ ,

$$\begin{aligned} \left| (\beta_i - \tilde{\beta}_i) g_{\tilde{\lambda}_j}^{(i)} \right| &= \left| \left( c_{\tilde{\lambda}_j}^\beta + \beta_i g_{\tilde{\lambda}_j}^{(i)} \right) - \left( c_{\tilde{\lambda}_j}^{\tilde{\beta}} + \tilde{\beta}_i g_{\tilde{\lambda}_j}^{(i)} \right) \right| \\ &\leq \left| c_{\tilde{\lambda}_j}^\beta \right| + \left| c_{\tilde{\lambda}_j}^{\tilde{\beta}} \right| \leq 2C \cdot 2^{-(j_0-d_1)\gamma}. \end{aligned}$$

So, by (4.2) and (4.9),

$$\left| \beta_i - \tilde{\beta}_i \right| \leq 2C \frac{2^{-(j_0-d_1)\gamma}}{j^{-2/q} \mu(\tilde{\lambda}_j)}.$$

As for every  $j \in \{j_0 - d_1, \dots, j_0 - 1\}$ , one has  $j > M_2$ , then  $\mu(\tilde{\lambda}_j) \geq 2^{-j(\gamma-1/p)}$ , and we deduce that

$$\mu(\tilde{\lambda}_j) \geq 2^{-j_0(\gamma-1/p)}.$$

Going through all  $j \in \{j_0 - d_1, \dots, j_0 - 1\}$  induce that for every  $i \in \{1, \dots, d_1\}$ , one has

$$\left| \beta_i - \tilde{\beta}_i \right| \leq 2C \frac{2^{-(j_0-d_1)\gamma}}{j^{-2/q} 2^{-j_0(\gamma-1/p)}} \leq 2C j_0^{2/q} \frac{2^{-(j_0-d_1)\gamma}}{2^{-j_0(\gamma-1/p)}} \leq 2C 2^{d_1\gamma} 2^{-j_0 \left( \frac{1}{p} - \frac{2}{q} \frac{\log_2(j_0)}{j_0} \right)}.$$

Considering  $j_0$  large enough, one gets that for every  $i \in \{1, \dots, d_1\}$ ,

$$\left| \beta_i - \tilde{\beta}_i \right| \leq C 2^{-j_0/2p}.$$

This means that the distance between two elements of  $\mathcal{B}_{\lambda_0}$  is at most  $2^{-j_0/2p}$  up to a universal constant. Hence, the volume of  $\mathcal{B}_{f,\lambda_0}$  is given by

$$\mathcal{L}_{d_1}(\mathcal{B}_{\lambda_0}) \leq C 2^{-j_0 \cdot d_1/2p}.$$

Summing over all the  $2^{dj_0}$  dyadic cubes for  $k_0 \in \{0, \dots, 2^{j_0}\}^d$  at scale  $j_0$  gives

$$\mathcal{L}_{d_1} \left( \bigcup_{k_0 \in \mathbb{Z}_{j_0}^d} \mathcal{B}_{\lambda_0} \right) \leq C 2^{j_0(d-d_1/2p)}.$$

Because of our choice (4.8),  $d < d_1/2p$  so the Borel-Cantelli lemma implies that

$$\mathcal{L}_{d_1} \left( \limsup_{j_0 \rightarrow \infty} \bigcup_{k_0 \in \mathbb{Z}_{j_0}^d} \mathcal{B}_{\lambda_0} \right) = 0.$$

Finally by (4.13), one concludes that  $\mathcal{L}_{d_1}(\mathcal{B}) = 0$ , which is the claim of Proposition 4.16.  $\square$

We can now conclude regarding the shyness of the sets  $\mathcal{O}^{p,\gamma,M}$ .

**Proposition 4.18.** *For every  $p \in \mathbb{N}^*$ ,  $M \in (\mathbb{N}^*)^2$  and positive rational  $\gamma$ , the set  $\mathcal{O}^{p,\gamma,M}$  is shy.*

*Proof.* Let  $\vartheta = G_{\#}\mathcal{L}_{d_1}$  be the push-forward measure of the  $d_1$ -dimensional Lebesgue measure by

$$G : \begin{cases} [0, 1]^{d_1} & \rightarrow & \mathcal{P} \\ \beta & \mapsto & \sum_{i=1}^{d_1} \beta_i \cdot \mathcal{G}_q^{(i)}. \end{cases} \quad (4.15)$$

This measure is supported by a compact set of  $\mathcal{P} \subset \tilde{b}_{\infty,q}^{\mu,\Psi}([0, 1]^d)$ .

Fix any  $f \in \tilde{b}_{\infty,q}^{\mu,\Psi}([0, 1]^d)$ . Using Proposition 4.16, for  $\vartheta$ -almost all  $F \in \mathcal{P}$ ,  $f + F \notin \mathcal{O}^{p,\gamma,M}$ . Hence for any  $f \in \tilde{b}_{\infty,q}^{\mu,\Psi}([0, 1]^d)$ , the set  $\{f + \mathcal{O}^{p,\gamma,M}\}$  has  $\vartheta$ -measure equal to 0, i.e.,

$$\vartheta(\{f + \mathcal{O}^{p,\gamma,M}\}) = 0.$$

Since it is true for any  $f \in \tilde{b}_{\infty,q}^{\mu,\Psi}([0, 1]^d)$ , the set  $\mathcal{O}^{p,\gamma,M}$  is shy.  $\square$

Using Proposition 4.8, the set  $(\mathcal{F}^\mu)^\complement$  is shy as it is a subset of countable unions and intersections of shy sets. This yields that  $\mathcal{F}^\mu$  is prevalent, proving Proposition 4.7.

## 5 Extensions

We have identified the singularity spectrum for a prevalent set of functions in the case  $p = \infty$ . This leaves the finite case  $p < \infty$  for which the conjectured result is the analog of the result by Barral and Seuret for Baire typical function in  $\tilde{b}_{\infty,q}^{\mu,\Psi}([0, 1]^d)$  [9].

**Conjecture 5.1.** *Let  $\mu$  be an almost doubling capacity on  $[0, 1]^d$  satisfying (P) and verifying the SMF, let  $p, q \in [1, +\infty]$ . Consider the mappings*

$$\tau_{\mu,p}(t) = \begin{cases} \frac{p-t}{p} \tau_\mu\left(\frac{p}{p-t} t\right) & \text{if } t \in (-\infty, p) \\ d & \text{if } t \in [p, +\infty) \end{cases} \quad \text{and} \quad \theta_p : \alpha \in [\tau'_\mu(+\infty), \tau'_\mu(-\infty)] \mapsto \alpha + \frac{\tau_\mu^*(\alpha)}{p}. \quad (5.1)$$

For all  $f \in \tilde{b}_{p,q}^{\mu,\Psi}([0, 1]^d)$ ,

$$\sigma_f(h) \leq \begin{cases} \sigma_\mu(\theta^{-1}(h)) = \tau_{\mu,p}^*(h) & \text{if } h \leq \tau'_{\mu,p}(0) \\ d & \text{if } h > \tau'_{\mu,p}(0). \end{cases} \quad (5.2)$$

In addition, there exists a prevalent set of functions  $f \in \tilde{b}_{p,q}^{\mu,\Psi}([0, 1]^d)$  satisfying  $\sigma_f = \sigma_\mu(\theta^{-1}(\cdot)) = \tau_{\mu,p}^*$ .

A second possible extension is to relax the hypothesis of Theorem 1.14 to include capacities that are not almost doubling, such as the Bernoulli multinomial measures and multiplicative cascade.

## References

- [1] P. Abry, P. Ciuciu, M. Dumeur, S. Jaffard, and G. Saës, “Multifractal analysis based on the weak scaling exponent and applications to MEG recordings in neuroscience,” working paper or preprint, Nov. 2024.
- [2] J. L. Ansorena and O. Blasco, “Characterization of Weighted Besov Spaces,” *Mathematische Nachrichten*, vol. 171, no. 1, pp. 5–17, 1995.

- [3] J.-M. Aubry, F. Bastin, and S. Dispa, “Prevalence of multifractal functions in  $S^V$  spaces,” *Journal of Fourier analysis and applications*, vol. 13, pp. 175–185, 2007.
- [4] J.-M. Aubry, D. Maman, and S. Seuret, “Local behavior of traces of Besov functions: Prevalent results,” *Journal of Functional Analysis*, vol. 264, no. 3, pp. 631–660, 2013.
- [5] R. Balka, U. Darji, and M. Elekes, “Hausdorff and packing dimension of fibers and graphs of prevalent continuous maps,” *Advances in Mathematics*, vol. 293, pp. 221–274, 2016.
- [6] J. Barral, “Inverse problems in multifractal analysis of measures,” *Annales Scientifiques de l’École Normale Supérieure*, 6th ser., vol. 48, pp. 1457–1510, 2015.
- [7] J. Barral, F. Ben Nasr, and J. Peyriere, “Comparing multifractal formalisms: The neighboring boxes condition,” *Asian Journal of Mathematics*, vol. 7, no. 2, pp. 149–166, 2003.
- [8] J. Barral and S. Seuret, “The Frisch-Parisi conjecture I: Prescribed multifractal behavior, and a partial solution,” *Journal de Mathématiques Pures et Appliquées*, vol. 175, pp. 76–108, 2023.
- [9] J. Barral and S. Seuret, “The Frisch-Parisi conjecture II: Besov spaces in multifractal environment, and a full solution,” *Journal de Mathématiques Pures et Appliquées*, vol. 175, pp. 281–329, 2023.
- [10] F. Bayart and Y. Heurteaux, “On the Hausdorff dimension of graphs of prevalent continuous functions on compact sets,” in *Further Developments in Fractals and Related Fields: Mathematical Foundations and Connections*, J. Barral and S. Seuret, Eds. Birkhäuser Boston, 2013, pp. 25–34.
- [11] R. Bowen, J.-R. Chazottes, and D. Ruelle, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms* (Lecture Notes in Mathematics), 2nd ed. Springer, 2008, vol. 470.
- [12] G. Brown, G. Michon, and J. Peyrière, “On the multifractal analysis of measures,” *Journal of Statistical Physics*, vol. 66, no. 3–4, pp. 775–790, 1992.
- [13] F. Chamizo and A. Ubis, “Multifractal behavior of polynomial Fourier series,” *Advances in Mathematics*, vol. 250, pp. 1–34, 2014.
- [14] G. Choquet, “Theory of capacities,” *Annales de l’institut Fourier*, vol. 5, pp. 131–295, 1954.
- [15] J. P. R. Christensen, “On sets of Haar measure zero in abelian polish groups,” *Israel Journal of Mathematics*, vol. 13, no. 3, pp. 255–260, 1972.
- [16] A. Cohen, *Numerical Analysis of Wavelet Methods*, Elsevier, Ed. JAI Press, 2003, vol. Studies in Mathematics and its Applications.
- [17] P. Collet, J. L. Lebowitz, and A. Porzio, “The dimension spectrum of some dynamical systems,” *Journal of Statistical Physics*, vol. 47, no. 5, pp. 609–644, 1987.
- [18] L. Diening, P. Harjulehto, P. Hästö, and M. Ruzicka, *Lebesgue and Sobolev Spaces with Variable Exponents* (Lecture Notes in Mathematics), J.-M. Morel and B. Teissier, Eds. Springer Berlin, Heidelberg, 2011, vol. 2017.
- [19] W. Farkas and H.-G. Leopold, “Characterisations of function spaces of generalised smoothness,” *Annali di Matematica Pura ed Applicata*, vol. 185, pp. 1–62, 2006.
- [20] D.-J. Feng, “Gibbs properties of self-conformal measures and the multifractal formalism,” *Ergodic Theory and Dynamical Systems*, vol. 27, no. 3, pp. 787–812, 2007.
- [21] A. Fraysse, “Generic Validity of the Multifractal Formalism,” *SIAM Journal on Mathematical Analysis*, vol. 39, no. 2, pp. 593–607, 2007.
- [22] A. Fraysse and S. Jaffard, “How smooth is almost every function in a Sobolev space?” *Revista Matemática Iberoamericana*, vol. 22, no. 2, pp. 663–682, 2006.
- [23] A. Fraysse, S. Jaffard, and J.-P. Kahane, “Quelques propriétés génériques en analyse,” *Comptes Rendus Mathématique*, vol. 340, no. 9, pp. 645–651, 2005.
- [24] U. Frisch and G. Parisi, *On the singularity structure of fully developed turbulence*, Proceedings of the International School of Physic Enrico Fermi. North-Holland, 1985, vol. 88, pp. 84–88.
- [25] M. Gaczkowski, P. Górka, and D. J. Pons, “Sobolev spaces with variable exponents on complete manifolds,” *Journal of Functional Analysis*, vol. 270, no. 4, pp. 1379–1415, 2016.
- [26] B. R. Hunt, “The prevalence of continuous nowhere differentiable functions,” *Proceedings of the American Mathematical Society*, vol. 122, no. 3, pp. 711–717, 1994.
- [27] B. R. Hunt, T. Sauer, and J. A. Yorke, “Prevalence: a translation-invariant “almost every” on infinite-dimensional spaces,” *Bulletin of the American Mathematical Society*, vol. 27, no. 2, pp. 217–238, 1992.

- [28] S. Jaffard, “Exposants de Hölder en des points donnés et coefficients d’ondelettes,” *Comptes Rendus de l’Académie des sciences*, I, vol. 308, pp. 79–81, 1989.
- [29] S. Jaffard, “The spectrum of singularities of Riemann’s function,” *Revista Matemática Iberoamericana*, vol. 12, no. 2, pp. 441–460, 1996.
- [30] S. Jaffard, “Multifractal formalism for functions part I: Results valid for all functions,” *SIAM Journal on Mathematical Analysis*, vol. 28, no. 4, pp. 944–970, 1997.
- [31] S. Jaffard, “On the Frisch–Parisi conjecture,” *Journal de Mathématiques Pures et Appliquées*, vol. 79, no. 6, pp. 525–552, 2000.
- [32] S. Jaffard, “Wavelet techniques in multifractal analysis,” *Proc. Symp. Pure Math.*, vol. 72, Jan. 2004.
- [33] S. Jaffard, B. Lashermes, and P. Abry, “Wavelet leaders in multifractal analysis,” in *Wavelet Analysis and Applications*, T. Qian, M. I. Vai, and Y. Xu, Eds., Basel: Birkhäuser Basel, 2007, pp. 201–246.
- [34] S. Jaffard and Y. Meyer, “On the pointwise regularity of functions in critical Besov spaces,” *Journal of Functional Analysis*, vol. 175, no. 2, pp. 415–434, 2000.
- [35] S. Jaffard, G. Saës, W. Ben Nasr, F. Palacin, and V. Billat, “A Review of Univariate and Multivariate Multifractal Analysis Illustrated by the Analysis of Marathon Runners Physiological Data,” in *Analysis, Applications, and Computations*, Cham: Springer International Publishing, 2023, pp. 3–60.
- [36] T. Kleytssens and S. Nicolay, “Generalized  $S^\nu$  spaces,” *Mathematische Nachrichten*, vol. 297, no. 1, pp. 266–283, 2024. eprint: <https://onlinelibrary.wiley.com/doi/pdf/10.1002/mana.202200133>.
- [37] D. Kreit and S. Nicolay, “Some characterizations of generalized Hölder spaces,” *Mathematische Nachrichten*, vol. 285, no. 17-18, pp. 2157–2172, 2012.
- [38] D. Kreit and S. Nicolay, “Characterizations of the elements of generalized Hölder–Zygmund spaces by means of their representation,” *Journal of Approximation Theory*, vol. 172, pp. 23–36, 2013.
- [39] D. Kreit and S. Nicolay, “Generalized Pointwise Hölder Spaces Defined via Admissible Sequences,” *Journal of Function Spaces*, vol. 2018, no. 1, p. 11, 2018.
- [40] J. Lévy Véhel and R. Vojak, “Multifractal Analysis of Choquet Capacities,” *Advances in Applied Mathematics*, vol. 20, no. 1, pp. 1–43, 1998.
- [41] L. Loosveldt and S. Nicolay, “Some equivalent definitions of Besov spaces of generalized smoothness,” *Mathematische Nachrichten*, vol. 292, no. 10, pp. 2262–2282, 2019.
- [42] L. Loosveldt and S. Nicolay, “Generalized spaces of pointwise regularity: toward a general framework for the WLM,” *Nonlinearity*, vol. 34, no. 9, p. 6561, 2021.
- [43] L. Loosveldt and S. Nicolay, “Some Prevalent Sets in Multifractal Analysis: How Smooth is Almost Every Function in  $T_p^\alpha(x)$ ,” *Journal of Fourier Analysis and Applications*, vol. 28, no. 4, p. 58, 2022.
- [44] Y. Meyer, *Ondelettes et opérateurs I: Ondelettes*. Hermann, 1990.
- [45] S. D. Moura, “On some characterizations of Besov spaces of generalized smoothness,” *Mathematische Nachrichten*, vol. 280, no. 9-10, pp. 1190–1199, 2007.
- [46] L. Olsen, “A multifractal formalism,” *Advances in Mathematics*, vol. 116, no. 1, pp. 82–196, 1995.
- [47] L. Olsen, “Fractal and multifractal dimensions of prevalent measures,” *Indiana Univ. Math. J.*, vol. 59, pp. 661–690, 2 2010.
- [48] L. Olsen, “Prevalent  $L^q$ -dimensions of measures,” *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 149, no. 3, pp. 553–571, 2010.
- [49] Y. Pesin and H. Weiss, “The multifractal analysis of Gibbs measures: Motivation, mathematical Foundation, and examples,” *Chaos*, pp. 89–106, 1997.
- [50] S. Seuret and A. Ubis, “Local  $L^2$ -regularity of Riemann’s Fourier series,” *Annales de l’Institut Fourier*, vol. 67, no. 5, pp. 2237–2264, 2017.
- [51] P. Shmerkin, “A Modified Multifractal Formalism for a Class of Self-similar Measures with Overlap,” *Asian Journal of Mathematics*, vol. 9, no. 3, pp. 323–348, 2005.
- [52] P. Shmerkin and B. Solomyak, “Absolute continuity of self-similar measures, their projections and convolutions,” *Transactions of the American Mathematical Society*, vol. 9368, no. 7, pp. 5125–5151, 2016.
- [53] K. Simon, B. Solomyak, and M. Urbański, *Hausdorff dimension of limit sets for parabolic IFS with overlaps*. Mathematical Sciences Publishers, 2001, vol. 201, pp. 441–478.

- [54] J. Soria and P. Tradacete, “The least doubling constant of a metric measure space,” *Annales Fennici Mathematici*, vol. 44, no. 2, pp. 1015–1030, 2019.
- [55] H. Triebel, *Theory of Function Spaces* (Modern Birkhäuser Classics). Birkhäuser Basel, 1983.
- [56] H. Triebel, *Theory of Function Spaces III* (Monographs in Mathematics 1). Birkhäuser Basel, 2006.
- [57] H. Wendt, P. Abry, S. Jaffard, H. Ji, and Z. Shen, “Wavelet Leader multifractal analysis for texture classification,” in *2009 16th IEEE International Conference on Image Processing (ICIP)*, 2009, pp. 3829–3832.
- [58] H. Wendt, S. G. Roux, P. Abry, and S. Jaffard, “Wavelet leaders and bootstrap for multifractal analysis of images,” *Signal Processing*, vol. 89, no. 6, pp. 1100–1114, 2009.