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| Pierre Gaillard. Solutions to the equation of order 6 of the NLS hierarchy. 2025. <hal-05391463>

HAL Id: hal-05391463

<https://hal.science/hal-05391463v1>

Preprint submitted on 1 Dec 2025

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Solutions to the equation of order 6 of the NLS hierarchy

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Abstract

The equation of order 6 of the NLS hierarchy is considered and particular solutions are given. Explicit expressions of these solutions are constructed for the first orders depending on multi-parameters. We study patterns of these solutions in the (x, t) plane according to the different values of the parameters.

Key Words : NLS hierarchy, rational solutions.

PACS numbers :
33Q55, 37K10, 47.10A-, 47.35.Fg, 47.54.Bd

1 Introduction

We are interested here in the hierarchy of the NLS equation, in particular its order 6 equation which can be written in the form

$$\begin{aligned} u_t + u_{7x} + 14|u|^2 u_{5x} + 42\bar{u}u_x u_{4x} + 28u\bar{u}_x u_{4x} + 14uu_x \bar{u}_{4x} + 70|u|^4 u_{3x} \\ + 98|u_x|^2 u_{3x} + 70\bar{u}u_{2x} u_{3x} + 42u\bar{u}_{2x} u_{3x} + 28u_x^2 \bar{u}_{3x} + 28uu_{2x} \bar{u}_{3x} + 280\bar{u}|u|^2 u_x u_{2x} \\ + 140|u|^2 u\bar{u}_x u_{2x} + 140|u|^2 uu_x \bar{u}_{2x} + 70\bar{u}u_{2x}^2 + 112u_x |u_{2x}|^2 + 70\bar{u}^2 u_x^3 + 280|u|^2 |u_x|^2 u_x \\ + 70u^2 |u_x|^2 \bar{u}_x + 140|u|^6 u_x \end{aligned} \quad (1)$$

the subscripts meaning partial derivatives and \bar{u} the complex conjugate of u . The first equation of this NLS hierarchy is the well known NLS equation [1]. The second one is the mKdV equation [2]. The third one is the LPD equation [3].

A lot of works has been done on these three equations. One can mention, for the first equation [4], for the second one [5] and the third one [3, 6, 7]. But, for the following orders of this hierarchy, very few studies have been carried

out. We construct here explicit solutions of the equation of order 6 of this hierarchy for the first orders and study the patterns of the modulus of the solutions in the (x, t) plane.

2 Solution of order 1 to the sixth equation of the NLS hierarchy

Theorem 2.1 *The function $v(x, t)$ defined by*

$$v(x, t) = \frac{-3 + 4x^2 - 1120xt + 78400t^2}{1 + 4x^2 - 1120xt + 78400t^2} \quad (2)$$

is a solution to the (NLS6) equation (1).

Proof: It is sufficient to replace the expression of the solution given by (2) and check that (1) is satisfied.

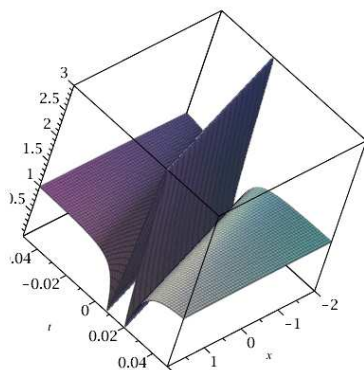


Figure 1. Solution of order 1 to (NLS6).

It is important to emphasize that we get a non smooth solution of the equation (1) as in the case of the mKdV equation [?] or the equation of order 4 of this NLS hierarchy [12].

3 Solutions of order 2 to the sixth equation of the NLS hierarchy

Theorem 3.1 *The function $v(x, t)$ defined by*

$$v(x, t) = \frac{n(x, t)}{d(x, t)} \quad (3)$$

with

$$\begin{aligned}
n(x, t) = & -(-64x^6 + 53760tx^5 + 2304b_1x^5 + 144x^4 - 34560b_1^2x^4 + 768ia_1x^4 - \\
& 18816000t^2x^4 - 1612800b_1tx^4 - 768a_1^2x^4 + 430080ta_1^2x^3 + 19353600b_1^2tx^3 + \\
& 18432b_1a_1^2x^3 + 276480b_1^3x^3 - 188160tx^3 - 430080ia_1tx^3 - 18432ia_1x^3b_1 - \\
& 4992b_1x^3 + 3512320000t^3x^3 + 451584000b_1t^2x^3 - 368793600000t^4x^2 - 165888b_1^2a_1^2x^2 + \\
& 6144ia_1^3x^2 + 90316800ia_1t^2x^2 + 4032000b_1tx^2 - 7741440b_1ta_1^2x^2 + 5760a_1^2x^2 + \\
& 58752b_1^2x^2 - 4064256000b_1^2t^2x^2 + 7741440ia_1b_1tx^2 - 90316800t^2a_1^2x^2 + 62092800t^2x^2 - \\
& 1244160b_1^4x^2 + 180x^2 - 63221760000b_1t^3x^2 - 3072a_1^4x^2 - 116121600b_1^3tx^2 - \\
& 1152ia_1x^2 + 165888ia_1b_1^2x^2 - 322560ta_1^2x + 4425523200000b_1t^4x + 1083801600b_1t^2a_1^2x + \\
& 379330560000b_1^2t^3x - 835430400b_1t^2x + 348364800b_1^4tx + 16257024000b_1^3t^2x + \\
& 46448640b_1^2ta_1^2x - 4608ia_1xb_1 - 1083801600ia_1xb_1t^2 - 7902720000t^3x + 2985984b_1^5x - \\
& 28062720b_1^2tx + 663552b_1^3a_1^2x - 50688b_1a_1^2x - 1720320ia_1^3tx + 8429568000t^3a_1^2x - \\
& 46448640ia_1xb_1^2t - 73728ia_1^3xb_1 + 36864b_1a_1^4x - 663552ia_1xb_1^3 - 5616b_1x - \\
& 967680ia_1tx - 290304b_1^3x + 20652441600000t^5x + 860160ta_1^4x - 8429568000ia_1t^3x - \\
& 292320tx - 720ia_1 + 1536ia_1^3 - 50577408000b_1t^3a_1^2 - 5160960b_1ta_1^4 + 995328ia_1b_1^4 + \\
& 10321920ia_1^3b_1t - 645120b_1ta_1^2 - 45 + 8386560ia_1b_1t + 350353920000t^4 + 63866880b_1^3t + \\
& 12288ia_1^5 - 92897280b_1^3ta_1^2 + 295034880000ia_1t^4 + 96768b_1^2a_1^2 - 67737600t^2a_1^2 + \\
& 51631104000b_1t^3 + 120422400ia_1^3t^2 - 7761600t^2 + 947520b_1t + 221184ia_1^3b_1^2 - \\
& 481890304000000t^6 - 3251404800b_1^2t^2a_1^2 - 132765696000000b_1^2t^4 + 2777241600b_1t^2 - \\
& 24385536000b_1^4t^2 - 758661120000b_1^3t^3 - 995328b_1^4a_1^2 - 110592b_1^2a_1^4 - 123914649600000b_1t^5 - \\
& 295034880000t^4a_1^2 - 60211200t^2a_1^4 + 69120ia_1b_1^2 + 158054400ia_1t^2 + 92897280ia_1b_1^3t + \\
& 3251404800ia_1b_1^2t^2 + 50577408000ia_1b_1t^3 - 418037760b_1^5t + 8448a_1^4 + 518400b_1^4 - \\
& 2985984b_1^6 - 4096a_1^6 + 1872a_1^2 + 18000b_1^2)e^{2ia_1}
\end{aligned}$$

and

$$\begin{aligned}
d(x, t) = & 64x^6 - 2304b_1x^5 - 53760tx^5 + 34560b_1^2x^4 + 1612800b_1tx^4 + 48x^4 + \\
& 18816000t^2x^4 + 768a_1^2x^4 - 430080ta_1^2x^3 + 384b_1x^3 - 19353600b_1^2tx^3 - 18432b_1a_1^2x^3 - \\
& 3512320000t^3x^3 + 80640tx^3 - 276480b_1^3x^3 - 451584000b_1t^2x^3 - 2096640b_1tx^2 + \\
& 368793600000t^4x^2 - 1152a_1^2x^2 + 108x^2 + 165888b_1^2a_1^2x^2 + 7741440b_1ta_1^2x^2 + \\
& 4064256000b_1^2t^2x^2 + 116121600b_1^3tx^2 + 63221760000b_1t^3x^2 + 90316800t^2a_1^2x^2 - \\
& 39513600t^2x^2 + 1244160b_1^4x^2 - 17280b_1^2x^2 + 3072a_1^4x^2 - 1083801600b_1t^2a_1^2x + \\
& 16450560b_1^2tx - 967680ta_1^2x - 110880tx - 379330560000b_1^2t^3x + 5795328000t^3x + \\
& 124416b_1^3x - 348364800b_1^4tx - 16257024000b_1^3t^2x - 4608b_1a_1^2x - 4425523200000b_1t^4x - \\
& 2448b_1x - 46448640b_1^2ta_1^2x - 663552b_1^3a_1^2x - 2985984b_1^5x - 36864b_1a_1^4x - \\
& 860160ta_1^4x + 564480000b_1t^2x - 20652441600000t^5x - 8429568000t^3a_1^2x + \\
& 5160960b_1ta_1^4 - 40642560b_1^3t + 69120b_1^2a_1^2 - 1964390400b_1^2t^2 - 38986752000b_1t^3 + \\
& 58564800t^2 + 158054400t^2a_1^2 + 8386560b_1ta_1^2 - 276595200000t^4 + 13276569600000b_1^2t^4 + \\
& 418037760b_1^5t + 9 + 758661120000b_1^3t^3 + 995328b_1^4a_1^2 + 110592b_1^2a_1^4 + 24385536000b_1^4t^2 + \\
& 295034880000t^4a_1^2 + 60211200t^2a_1^4 + 2116800b_1t + 123914649600000b_1t^5 + \\
& 481890304000000t^6 + 92897280b_1^3ta_1^2 + 3251404800b_1^2t^2a_1^2 + 50577408000b_1t^3a_1^2 + \\
& 6912a_1^4 - 269568b_1^4 + 2985984b_1^6 + 4096a_1^6 + 1584a_1^2 + 20016b_1^2
\end{aligned}$$

is a solution to the (NLS6) equation (1).

Proof: We have to replace the expression of the solution given by (3), and verify that the relation (1) is verified.

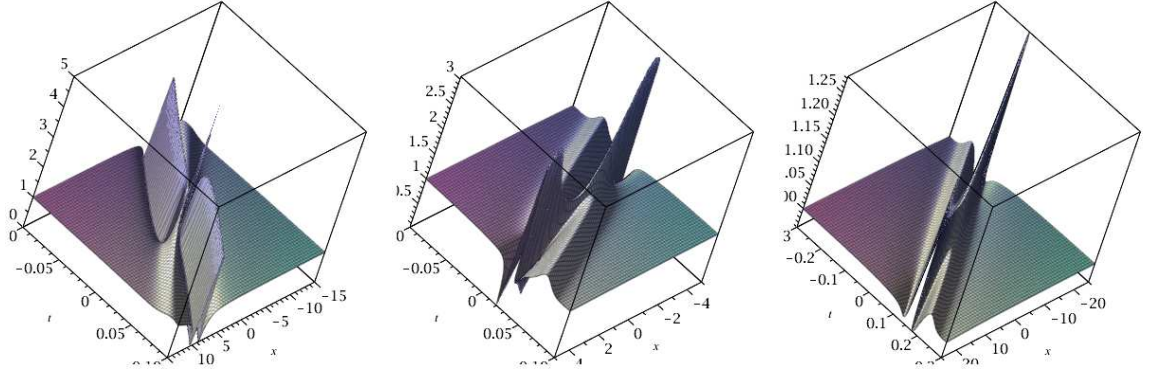


Figure 2. Solution of order 2 to the equation (1); to the left $a_1 = 0, b_1 = 0$; in the center $a_1 = 1, b_1 = 0$; to the right $a_1 = 3, b_1 = 0$.

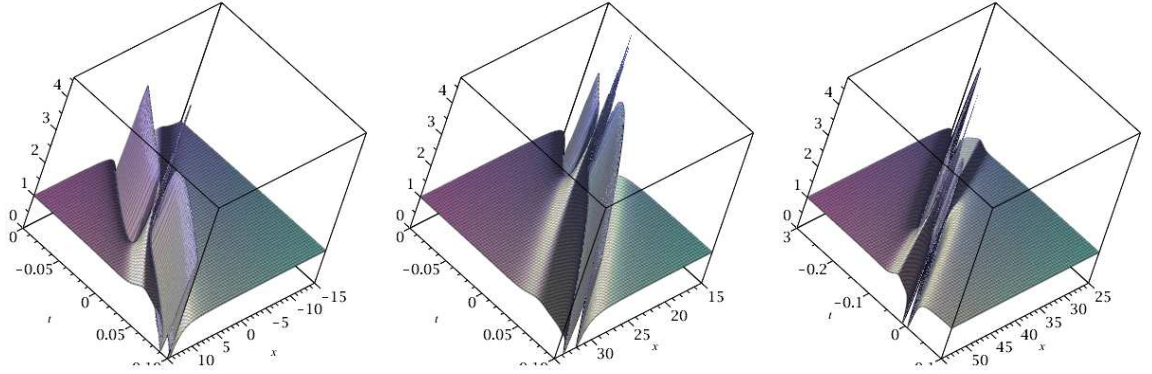


Figure 3. Solution of order 2 to the equation (1); to the left $a_1 = 0, b_1 = 1$; in the center $a_1 = 0, b_1 = 4$; to the right $a_1 = 0, b_1 = 10$.

We get also a non smooth solution of the equation (1). It is the same as in the case of the mKdV equation [?] or the equation of order 4 of this NLS hierarchy [12].

4 Solutions of order 3 to the sixth equation of the NLS hierarchy

The solution order 3 to the sixth equation of the NLS hierarchy is given by the following expression.

Theorem 4.1 *The function $v(x, t)$ defined by*

$$v(x, t) = \left(1 - 24 \frac{n(x, t)}{d(x, t)} \right) e^{i(2a_1 - 6a_2 + 20t)} \quad (4)$$

with

$$\begin{aligned}
n(x, t) = & 675 - 86400(8b_1 - 80b_2 + 560t)(32b_2 - 448t) + 15(1 + (4a_1 - 24a_2)^2)(2x - \\
& 12b_1 + 60b_2 - 280t)^8 + (210 - 60(4a_1 - 24a_2)^2 + 50(4a_1 - 24a_2)^4 + 7680(4a_1 - \\
& 24a_2)a_2)(2x - 12b_1 + 60b_2 - 280t)^6 + (-720(4a_1 - 24a_2)^2(8b_1 - 80b_2 + 560t) - \\
& 5760b_1 - 11520b_2 + 564480t)(2x - 12b_1 + 60b_2 - 280t)^5 - 172800(4a_1 - 24a_2)^3a_2 + \\
& (450(4a_1 - 24a_2)^2 - 150(4a_1 - 24a_2)^4 + 70(4a_1 - 24a_2)^6 + 19200(4a_1 - \\
& 24a_2)^3a_2 - 450 + 5400(8b_1 - 80b_2 + 560t)^2 - 460800a_2^2 + 57600(4a_1 - 24a_2)a_2)(2x - \\
& 12b_1 + 60b_2 - 280t)^4 + (-2400(4a_1 - 24a_2)^4(8b_1 - 80b_2 + 560t) + 460800(4a_1 - \\
& 24a_2)(8b_1 - 80b_2 + 560t)a_2 + 57600b_1 - 806400b_2 + 7257600t - 7200(4a_1 - \\
& 24a_2)^2(16b_1 - 128b_2 + 672t))(2x - 12b_1 + 60b_2 - 280t)^3 + (6750(4a_1 - 24a_2)^4 + \\
& 420(4a_1 - 24a_2)^6 + 45(4a_1 - 24a_2)^8 - 2700(4a_1 - 24a_2)^2(5 + 4(8b_1 - 80b_2 + \\
& 560t)^2 - 3072a_2^2) - 675 - 10800(8b_1 - 80b_2 + 560t)^2 - 2764800a_2^2 + 691200(4a_1 - \\
& 24a_2)a_2 - 230400(4a_1 - 24a_2)^3a_2)(2x - 12b_1 + 60b_2 - 280t)^2 + (-1680(4a_1 - \\
& 24a_2)^6(8b_1 - 80b_2 + 560t) - 460800(4a_1 - 24a_2)^3(8b_1 - 80b_2 + 560t)a_2 - \\
& 10800(4a_1 - 24a_2)^2(8b_1 - 272b_2 + 3248t) + 86400b_1 - 1209600b_2 + 10886400t + \\
& 43200(8b_1 - 80b_2 + 560t)^3 + 11059200(8b_1 - 80b_2 + 560t)a_2^2 + 3600(4a_1 - \\
& 24a_2)^4(8b_1 + 80b_2 - 1680t) - 86400(4a_1 - 24a_2)(16(8b_1 - 80b_2 + 560t)a_2 + \\
& 16a_2(32b_2 - 448t)))(2x - 12b_1 + 60b_2 - 280t) + 450(4a_1 - 24a_2)^4(-17 + \\
& 28(8b_1 - 80b_2 + 560t)^2 + 3072a_2^2) + 10800(4a_1 - 24a_2)(-16a_2 + 64(8b_1 - 80b_2 + \\
& 560t)^2a_2 + 16384a_2^3) + 675(4a_1 - 24a_2)^2(-3 + 16(8b_1 - 80b_2 + 560t)^2 + 4096a_2^2 - \\
& 128(8b_1 - 80b_2 + 560t)(32b_2 - 448t)) + i(-345600(8b_1 - 80b_2 + 560t)^2a_2 + (4a_1 - \\
& 24a_2)(2x - 12b_1 + 60b_2 - 280t)^{10} + (-60a_1 + 840a_2 + 5(4a_1 - 24a_2)^3)(2x - \\
& 12b_1 + 60b_2 - 280t)^8 + (-600a_1 - 240a_2 - 140(4a_1 - 24a_2)^3 + 10(4a_1 - 24a_2)^5 + \\
& 3840(4a_1 - 24a_2)^2a_2)(2x - 12b_1 + 60b_2 - 280t)^6 + (-240(4a_1 - 24a_2)^3(8b_1 - \\
& 80b_2 + 560t) - 23040(8b_1 - 80b_2 + 560t)a_2 + 720(4a_1 - 24a_2)(8b_1 - 176b_2 + \\
& 1904t))(2x - 12b_1 + 60b_2 - 280t)^5 + 417600(4a_1 - 24a_2)^4a_2 + (-450(4a_1 - \\
& 24a_2)^3 - 210(4a_1 - 24a_2)^5 + 10(4a_1 - 24a_2)^7 + 4800(4a_1 - 24a_2)^4a_2 + 450(4a_1 - \\
& 24a_2)(-3 + 12(8b_1 - 80b_2 + 560t)^2 - 1024a_2^2) - 14400a_2 + 28800(4a_1 - \\
& 24a_2)^2a_2)(2x - 12b_1 + 60b_2 - 280t)^4 + (-480(4a_1 - 24a_2)^5(8b_1 - 80b_2 + 560t) + \\
& 230400(4a_1 - 24a_2)^2(8b_1 - 80b_2 + 560t)a_2 + 7200(4a_1 - 24a_2)(8b_1 - 48b_2 + \\
& 112t) - 2400(4a_1 - 24a_2)^3(16b_1 - 128b_2 + 672t) - 230400(8b_1 - 80b_2 + 560t)a_2 - \\
& 230400a_2(32b_2 - 448t))(2x - 12b_1 + 60b_2 - 280t)^3 + (1710(4a_1 - 24a_2)^5 - \\
& 60(4a_1 - 24a_2)^7 + 5(4a_1 - 24a_2)^9 - 900(4a_1 - 24a_2)^3(7 + 4(8b_1 - 80b_2 + \\
& 560t)^2 - 3072a_2^2) + 675(4a_1 - 24a_2)(7 + 16(8b_1 - 80b_2 + 560t)^2 + 4096a_2^2) - \\
& 345600a_2 - 345600(8b_1 - 80b_2 + 560t)^2a_2 - 88473600a_2^3 - 115200(4a_1 - \\
& 24a_2)^4a_2)(2x - 12b_1 + 60b_2 - 280t)^2 + (-240(4a_1 - 24a_2)^7(8b_1 - 80b_2 + \\
& 560t) - 115200(4a_1 - 24a_2)^4(8b_1 - 80b_2 + 560t)a_2 + 10800(4a_1 - 24a_2)(24b_1 - \\
& 400b_2 + 3920t + 4(8b_1 - 80b_2 + 560t)^3 + 1024(8b_1 - 80b_2 + 560t)a_2^2) + \\
& 3600(4a_1 - 24a_2)^3(24b_1 - 176b_2 + 784t) + 720(4a_1 - 24a_2)^5(56b_1 - 400b_2 + \\
& 1680t) + 345600(8b_1 - 80b_2 + 560t)a_2 + 691200a_2(32b_2 - 448t) - 43200(4a_1 - \\
& 24a_2)^2(16(8b_1 - 80b_2 + 560t)a_2 + 16a_2(32b_2 - 448t))(2x - 12b_1 + 60b_2 - \\
& 280t) + 90(4a_1 - 24a_2)^5(-107 + 28(8b_1 - 80b_2 + 560t)^2 + 3072a_2^2) + 5400(4a_1 - \\
& 24a_2)^2(176a_2 + 64(8b_1 - 80b_2 + 560t)^2a_2 + 16384a_2^3) - 225(4a_1 - 24a_2)^3(11 + \\
& 80(8b_1 - 80b_2 + 560t)^2 + 20480a_2^2 + 128(8b_1 - 80b_2 + 560t)(32b_2 - 448t)) - \\
& 675(4a_1 - 24a_2)(-7 + 56(8b_1 - 80b_2 + 560t)^2 + 22528a_2^2 - 128(8b_1 - 80b_2 + \\
& 560t)(32b_2 - 448t) - 128(32b_2 - 448t)^2) - 1440(4a_1 - 24a_2)^8a_2 - 9600(4a_1 -
\end{aligned}$$

$$24 a_2)^6 a_2 - 2764800 (8 b_1 - 80 b_2 + 560 t) a_2 (32 b_2 - 448 t) - 870 (4 a_1 - 24 a_2)^7 + 25 (4 a_1 - 24 a_2)^9 + (4 a_1 - 24 a_2)^{11} - 151200 a_2 + 265420800 a_2^3) - 11520 (4 a_1 - 24 a_2)^7 a_2 - 195840 (4 a_1 - 24 a_2)^5 a_2 + 86400 (32 b_2 - 448 t)^2 + (2 x - 12 b_1 + 60 b_2 - 280 t)^{10} + 27000 (8 b_1 - 80 b_2 + 560 t)^2 + 2190 (4 a_1 - 24 a_2)^6 + 495 (4 a_1 - 24 a_2)^8 + 11 (4 a_1 - 24 a_2)^{10} + 23500800 a_2^2$$

and

$$d(x, t) = 2024 - 777600 (8 b_1 - 80 b_2 + 560 t) (32 b_2 - 448 t) + 120 (8 b_1 - 80 b_2 + 560 t) (2 x - 12 b_1 + 60 b_2 - 280 t)^9 + 1440 (32 b_2 - 448 t) (2 x - 12 b_1 + 60 b_2 - 280 t)^7 - 115200 (4 a_1 - 24 a_2)^7 a_2 - 518400 (4 a_1 - 24 a_2)^5 a_2 + 265420800 (8 b_1 - 80 b_2 + 560 t)^2 a_2^2 + (-120 (4 a_1 - 24 a_2)^2 + 5760 (4 a_1 - 24 a_2) a_2 + 120) (2 x - 12 b_1 + 60 b_2 - 280 t)^8 + (480 (4 a_1 - 24 a_2)^2 - 240 (4 a_1 - 24 a_2)^4 + 15360 (4 a_1 - 24 a_2)^3 a_2 + 2320 + 2160 (8 b_1 - 80 b_2 + 560 t)^2 + 1290240 a_2^2 - 92160 (4 a_1 - 24 a_2) a_2) (2 x - 12 b_1 + 60 b_2 - 280 t)^6 + (-720 (4 a_1 - 24 a_2)^4 (8 b_1 - 80 b_2 + 560 t) - 276480 (4 a_1 - 24 a_2) (8 b_1 - 80 b_2 + 560 t) a_2 + 4320 (4 a_1 - 24 a_2)^2 (8 b_1 - 176 b_2 + 1904 t) - 51840 b_1 + 103680 b_2 + 2177280 t) (2 x - 12 b_1 + 60 b_2 - 280 t)^5 + (-1440 (4 a_1 - 24 a_2)^4 + 11520 (4 a_1 - 24 a_2)^5 a_2 + 240 (4 a_1 - 24 a_2)^2 (56 + 135 (8 b_1 - 80 b_2 + 560 t)^2 - 11520 a_2^2) + 518400 (4 a_1 - 24 a_2) a_2 + 345600 (4 a_1 - 24 a_2)^3 a_2 + 3360 + 32400 (8 b_1 - 80 b_2 + 560 t)^2 - 13824000 a_2^2 + 86400 (8 b_1 - 80 b_2 + 560 t) (32 b_2 - 448 t)) (2 x - 12 b_1 + 60 b_2 - 280 t)^4 + (-960 (4 a_1 - 24 a_2)^6 (8 b_1 - 80 b_2 + 560 t) + 921600 (4 a_1 - 24 a_2)^3 (8 b_1 - 80 b_2 + 560 t) a_2 - 43200 (4 a_1 - 24 a_2)^2 (24 b_1 - 272 b_2 + 2128 t) - 7200 (4 a_1 - 24 a_2)^4 (48 b_1 - 448 b_2 + 2912 t) + 345600 b_1 - 5529600 b_2 + 53222400 t - 86400 (8 b_1 - 80 b_2 + 560 t)^3 - 22118400 (8 b_1 - 80 b_2 + 560 t) a_2^2 + 172800 (4 a_1 - 24 a_2) (16 (8 b_1 - 80 b_2 + 560 t) a_2 - 16 a_2 (32 b_2 - 448 t))) (2 x - 12 b_1 + 60 b_2 - 280 t)^3 + (13440 (4 a_1 - 24 a_2)^6 + 240 (4 a_1 - 24 a_2)^8 - 240 (4 a_1 - 24 a_2)^4 (-326 + 45 (8 b_1 - 80 b_2 + 560 t)^2 - 34560 a_2^2) + 480 (4 a_1 - 24 a_2)^2 (-76 + 135 (8 b_1 - 80 b_2 + 560 t)^2 + 311040 a_2^2) - 4147200 (4 a_1 - 24 a_2)^3 a_2 - 414720 (4 a_1 - 24 a_2)^5 a_2 - 64800 (4 a_1 - 24 a_2) (-96 a_2 + 64 (8 b_1 - 80 b_2 + 560 t)^2 a_2 + 16384 a_2^3) + 12144 - 97200 (8 b_1 - 80 b_2 + 560 t)^2 + 8294400 a_2^2 + 518400 (32 b_2 - 448 t)^2) (2 x - 12 b_1 + 60 b_2 - 280 t)^2 + (-360 (4 a_1 - 24 a_2)^8 (8 b_1 - 80 b_2 + 560 t) - 276480 (4 a_1 - 24 a_2)^5 (8 b_1 - 80 b_2 + 560 t) a_2 - 1440 (4 a_1 - 24 a_2)^6 (8 b_1 - 240 b_2 + 2800 t) + 32400 (4 a_1 - 24 a_2)^4 (8 b_1 + 112 b_2 - 2128 t) + 64800 (4 a_1 - 24 a_2)^2 (-40 b_1 + 752 b_2 - 7728 t + 4 (8 b_1 - 80 b_2 + 560 t)^3 + 1024 (8 b_1 - 80 b_2 + 560 t) a_2^2) - 777600 (4 a_1 - 24 a_2) (16 (8 b_1 - 80 b_2 + 560 t) a_2 + 32 a_2 (32 b_2 - 448 t)) - 172800 (4 a_1 - 24 a_2)^3 (48 (8 b_1 - 80 b_2 + 560 t) a_2 + 16 a_2 (32 b_2 - 448 t)) - 648000 b_1 + 8553600 b_2 - 74390400 t + 259200 (8 b_1 - 80 b_2 + 560 t)^3 + 331776000 (8 b_1 - 80 b_2 + 560 t) a_2^2 - 1036800 (8 b_1 - 80 b_2 + 560 t)^2 (32 b_2 - 448 t) + 265420800 a_2^2 (32 b_2 - 448 t)) (2 x - 12 b_1 + 60 b_2 - 280 t) + 80 (4 a_1 - 24 a_2)^6 (191 + 63 (8 b_1 - 80 b_2 + 560 t)^2 + 6912 a_2^2) + 21600 (4 a_1 - 24 a_2)^3 (-368 a_2 + 64 (8 b_1 - 80 b_2 + 560 t)^2 a_2 + 16384 a_2^3) + 240 (4 a_1 - 24 a_2)^4 (599 + 135 (8 b_1 - 80 b_2 + 560 t)^2 - 57600 a_2^2 - 360 (8 b_1 - 80 b_2 + 560 t) (32 b_2 - 448 t)) - 16200 (4 a_1 - 24 a_2) (496 a_2 + 1280 (8 b_1 - 80 b_2 + 560 t)^2 a_2 + 65536 a_2^3 + 2048 (8 b_1 - 80 b_2 + 560 t) a_2 (32 b_2 - 448 t)) + 24 (4 a_1 - 24 a_2)^2 (3881 + 12150 (8 b_1 - 80 b_2 + 560 t)^2 + 7257600 a_2^2 + 21600 (8 b_1 - 80 b_2 + 560 t) (32 b_2 - 448 t) + 21600 (32 b_2 - 448 t)^2) - 1920 (4 a_1 - 24 a_2)^9 a_2 + 518400 (32 b_2 - 448 t)^2 + 356400 (8 b_1 - 80 b_2 + 560 t)^2 + 518400 (8 b_1 - 80 b_2 + 560 t)^4 + 3720 (4 a_1 - 24 a_2)^8 + 120 (4 a_1 - 24 a_2)^{10} + (1 + (2 x - 12 b_1 + 60 b_2 - 280 t)^2 + (4 a_1 - 24 a_2)^2)^6 + 33973862400 a_2^4 + 223948800 a_2^2$$

is a solution to the (NLS6) equation (1).

Proof: It is sufficient to check that the relation (1) is verified if we replace the expression of the solution given by (4).

We present the patterns of the modules of the solutions according to different values of the parameters.

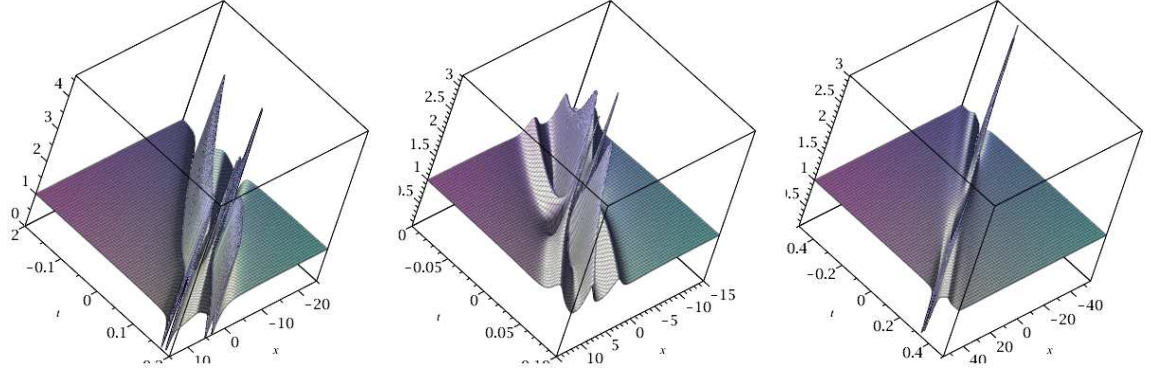


Figure 4. Solution of order 3 to (1); to the left $a_1 = 0, b_1 = 0, a_2 = 0, b_2 = 0$; in the center $a_1 = 1, b_1 = 0, a_2 = 0, b_2 = 0$; to the right $a_1 = 5, b_1 = 0, a_2 = 0, b_2 = 0$.

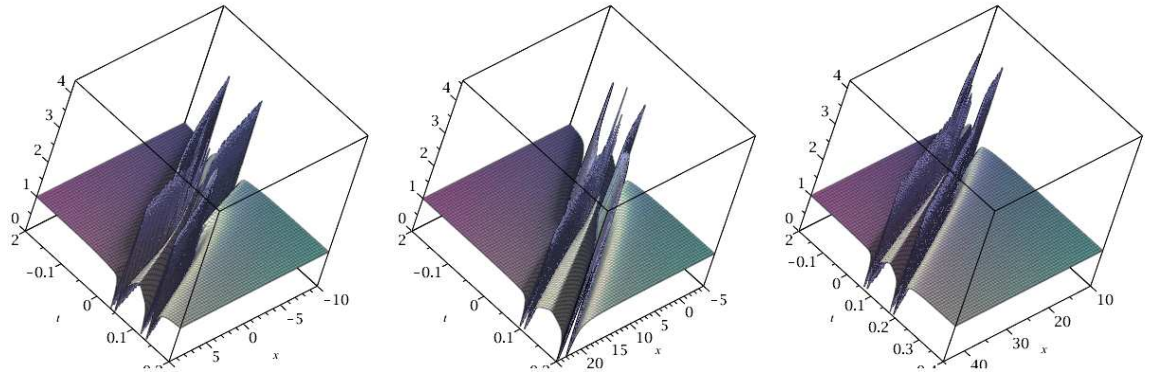


Figure 5. Solution of order 3 to (1); to the left $a_1 = 0, b_1 = 0, 1, a_2 = 0, b_2 = 0$; in the center $a_1 = 0, b_1 = 1, a_2 = 0, b_2 = 0$; to the right $a_1 = 0, b_1 = 5, a_2 = 0, b_2 = 0$.

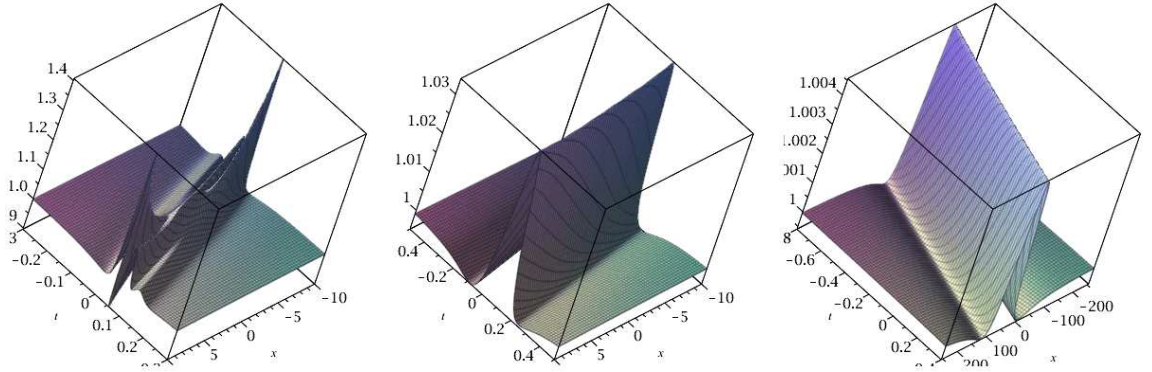


Figure 6. Solution of order 3 to (1); to the left $a_1 = 0, b_1 = 0, a_2 = 0, 5, b_2 = 0$; in the center $a_1 = 0, b_1 = 0, a_2 = 1, b_2 = 0$; to the right $a_1 = 0, b_1 = 5, a_2 = 3, b_2 = 0$.

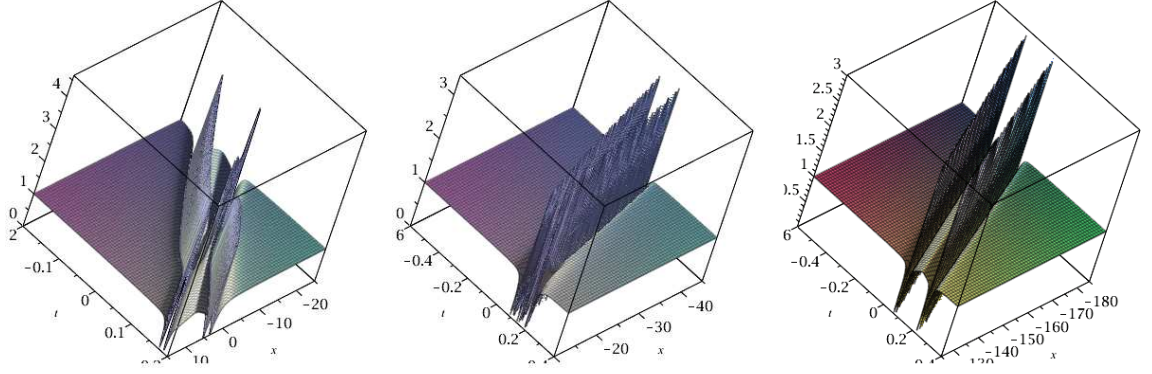


Figure 7. Solution of order 3 to (1); to the left $a_1 = 0, b_1 = 0, a_2 = 0, b_2 = 0, 5$; in the center $a_1 = 0, b_1 = 0, a_2 = 0, b_2 = 1$; to the right $a_1 = 0, b_1 = 5, a_2 = 0, b_2 = 5$.

We get also a non smooth solution of the equation (1).

5 Conclusion

For the first orders, rational solutions to the $(NLS6)$ equation have been constructed.

It is relevant to note that, we get a non smooth solution of the equation (1) as in the case of the mKdV equation [?] or the equation of order 4 of this NLS hierarchy [11].

The situation is contrary to odd-order equations, where we obtain rogue waves. This is the case for the equation of order 1 (NLS), for the equation of order 3 (LPD) and for the equation of order 5. In these cases, according to the order

N of the determinant giving the solutions of these equations, we obtain the following results.

Generically, for $N = 1$, there is a formation of a single peak, for $N = 2$, the formation of three peaks, and for $N = 3$, the formation of six peaks.

In all these N -order solutions we get quotient of a polynomial of degree $N(N+1)$ in x and t for the numerator by a polynomial of degree $N(N+1)$ in x and t for the denominator.

It is important to emphasize that unlike other equations belonging to this NLS hierarchy, as the NLS equation [8], the Lakshmanan Porsezian Daniel equation [10] or the equation of order 5 of this hierarchy [12], we do not recover the structure of the triangles with the peaks that appear depending on the different values of the parameters.

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