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Frequency synchronization and phase ordering of bounded confidence oscillators

Trisha Srivastava[‡], Carmela Bernardo[‡], Paolo Frasca[†], Giacomo Casadei[†], Francesco Vasca[‡]

Abstract—Limited range of phase measurements motivates the analysis of coupled oscillators that interact if their geodesic distances are within a prescribed bound. Steady state behaviors of this system depend on the natural frequencies of the oscillators and on the underlying graph structure. A necessary and sufficient condition is provided for the graph to remain complete. If the graph is connected over time, it is proved that phase ordering is preserved among the oscillators according to their natural frequencies. The asymptotic convergence to frequency synchronization is proved if the graph is assumed to stay connected and all natural frequencies are the same. A comprehensive analysis of the three-oscillator case shows that phase ordering is not necessary for frequency synchronization, and that graph connectivity, together with an appropriate bound on the range of natural frequencies, ensures frequency synchronization.

I. INTRODUCTION

One of the most celebrated models for analyzing frequency synchronization in networks of coupled oscillators is the Kuramoto model [1]–[3]. The type of interactions among the oscillators shapes their time evolution [4]. In global interaction models, like the mean-field model [5], all oscillators influence each other’s time evolution. However, this globally connected interaction structure does not reflect how coupling occurs in practice. For instance, in microgrids operating under droop control, frequency adjustments are obtained through the computation of phase shifts based on local reference frames, for which measurements could be more accurate and practically available [6]. This motivates the use of limited range interaction models, where an oscillator interacts only with a particular subset of oscillators based on some predefined criteria [7].

Taking inspiration from the applications above, we consider a network of coupled oscillators where each one determines its instantaneous frequency by adding to its natural frequency an action given by the average of phase differences with its neighbors, which are the oscillators within a bounded confidence of phase differences. The concept of bounded confidence in the Kuramoto oscillator model was first introduced in [8], which draws its similarity with the sociophysical models of opinions regarding different types of synchronized states, such as consensus, polarization, and fragmentation. Coupling between oscillators with a proximity rule is also analyzed in [9]. A system of oscillators in a ring structure where a pair of oscillators interact only if they are sufficiently close to each other is analyzed in [10]. In [11], the bounded confidence Kuramoto model is considered to study cluster formation at steady state, based on the amplitude of the confidence bound in the case of

bimodal distribution of the natural frequencies.

The bounded confidence oscillators model considered in [12] investigates the situation where a pair of oscillators interacts only if their geodesic phase difference is sufficiently small. Under this bounded confidence assumption, the sine of the phase differences that appears in the classical Kuramoto model is substituted by the phase differences themselves. The model can also be seen as a relevant addition to the study of opinion dynamics, since exogenous inputs are typically not incorporated into standard models of opinion dynamics [13]. This class of bounded confidence oscillators model is considered herein by proposing new results on frequency synchronization and phase ordering.

In [12], convergence to frequency synchronization is proved under the assumption of a constant graph, which allows one to exploit convergence properties of row-stochastic matrices. However, the hypothesis of the graph being constant over time is quite restrictive and intuition leads to a conjecture that frequency synchronization is achieved even under the weaker condition of the graph staying connected throughout the evolution, which is an open problem. In this paper it is shown that, by adding the condition of all natural frequencies being the same to the assumption of graph staying connected, frequency synchronization is reached.

The natural question arising from the considerations above is whether it is possible to obtain formal results by relaxing the assumption of equal natural frequencies. First, we investigate for some structure in the time evolution. It is shown that if the graph stays connected over time, then the phase order of oscillators will be preserved if, at a time instant during the evolution, they get ordered according to their non-decreasing natural frequencies — a result that had been proved in the literature only for complete graphs [12]. Moreover, the comprehensive analysis of the case of three oscillators proposed in this paper shows that frequency synchronization is obtained if a suitable bound on the range of natural frequencies is imposed, a condition which is conjectured to hold also for any number of oscillators.

Another research question investigated in this paper is whether it is possible to relax the assumption of the connectivity of the graph over time. Our analysis shows that connectivity can be lost even if the bound on the range of natural frequencies is satisfied, and the oscillators are only initially connected. Instead, we show that the same bound is an iff condition for the graph to stay complete once it becomes complete, which then leads to frequency synchronization.

The rest of the paper is organized as follows. In Section II

the geodesic bounded confidence model is presented. Section III examines the ordering of oscillators during evolution, while Section IV explores frequency synchronization under various connectivity conditions. Section V focuses on a comprehensive analysis of the specific case of three oscillators. Section VI concludes with a summary of the findings.

II. BOUNDED CONFIDENCE OSCILLATORS MODEL

Consider a set of oscillators, each having a constant and possibly distinct natural frequency $\omega_i \in \mathbb{R}_+$, $i \in \mathcal{N} = \{1, 2, \dots, N\}$, $N \in \mathbb{N}$, where \mathbb{R}_+ and \mathbb{N} are the set of nonnegative real and natural numbers, respectively. Phase dynamics of the i -th oscillator can be represented by the following scalar difference equation

$$\theta_i^+ - \theta_i = h\omega_i + \frac{h\eta}{|\mathcal{N}_i^+(\theta)|} \sum_{j \in \mathcal{N}_i^+(\theta)} \sigma(\theta_i, \theta_j) \|\theta_j - \theta_i\|, \quad (1)$$

where $\theta_i \in \mathbb{R}_+$ stands for $\theta_i(k)$, θ_i^+ for $\theta_i(k+1)$, $k \in \mathbb{N}$ being a time step (in the sequel for the sake of notation the time-dependence of θ will be omitted), with $\theta = \text{col}((\theta_i)_{i \in \mathcal{N}})$, the symbol ‘col’ meaning the arguments stacked in a column vector, $\|\theta_i - \theta_j\|$ being the geodesic distance between the two oscillators,

$$\sigma(\theta_i, \theta_j) = \begin{cases} +1, & \text{if } \text{mod}_{[-\pi, \pi]}(\theta_j - \theta_i) > 0, \\ -1, & \text{otherwise,} \end{cases} \quad (2)$$

and $\mathcal{N}_i(\theta)$ denoting the set of indices of the neighbors of the i -th oscillator, $i \in \mathcal{N}$, defined by

$$j \in \mathcal{N}_i(\theta) \iff \|\theta_j - \theta_i\| \leq \Delta, \quad (3)$$

where $\Delta \in (0, \pi)$ is the confidence threshold. Note that the symmetry of the model, i.e., $i \in \mathcal{N}_j$ if and only if $j \in \mathcal{N}_i$, implies that $\sigma(\theta_i, \theta_j) = -\sigma(\theta_j, \theta_i)$ for all $i, j \in \mathcal{N}$. In (1), $\eta \in \mathbb{R}_+$ is the coupling gain and $h \in \mathbb{R}_+$ is the sampling period which is used to keep correspondence of (1) with the continuous-time bounded confidence oscillators model, see [12]. The term $(\theta_i^+ - \theta_i)/h$ is called instantaneous frequency of the i -th oscillator, $i \in \mathcal{N}$. The set $\mathcal{N}_i^+(\theta)$ ($\mathcal{N}_i^-(\theta)$) of successors (predecessors) of the i -th oscillator are defined as those $j \in \mathcal{N}_i(\theta)$ such that $\sigma(\theta_i, \theta_j) > 0$ ($\sigma(\theta_i, \theta_j) < 0$), with $i \in \mathcal{N}$, where $\sigma(\theta_i, \theta_j)$ is given by (2). At each time step, the values of $\mathcal{N}_i(\theta)$, $i \in \mathcal{N}$, induce a graph whose nodes are the oscillators. The main assumption in our analysis is that this graph remains connected over time and there exists a phase interval lower bounded by the confidence threshold Δ in which there are no oscillators. This condition can be formally expressed as follows.

Assumption 1. *The graph induced by the oscillators is connected for all $k \in \mathbb{N}$. Moreover, for any $k \in \mathbb{N}$, there exists at least one oscillator $i \in \mathcal{N}$ such that $\mathcal{N}_i^+(\theta)$ is empty and $|\theta_j - \theta_i| < 2\pi - \Delta$ for all $i, j \in \mathcal{N}$.*

Since (1) is invariant for any constant shift applied to all oscillators’ phases, without loss of generality one can assume $\theta_i(0) \in [0, 2\pi - \Delta)$ for all $i \in \mathcal{N}$. Then, under Assumption 1 it follows that $\sigma(\theta_i, \theta_j) \|\theta_j - \theta_i\| = \theta_j - \theta_i$ for all $i, j \in \mathcal{N}$.

Then (1) can be rewritten as

$$\theta_i^+ = \theta_i + h\omega_i + \frac{h\eta}{|\mathcal{N}_i^+(\theta)|} \sum_{j \in \mathcal{N}_i^+(\theta)} (\theta_j - \theta_i), \quad (4)$$

together with (3), $i \in \mathcal{N}$, $k \in \mathbb{N}$. The average of the neighbors’ phases of the i -th oscillator neighbors is given by

$$\theta_i^{\text{av}} = \frac{1}{|\mathcal{N}_i^+(\theta)|} \sum_{\ell \in \mathcal{N}_i^+(\theta)} \theta_\ell \quad (5)$$

for all $i \in \mathcal{N}$. By using (5), the expression (4) can be rewritten as

$$\theta_i^+ = (1 - h\eta)\theta_i + h\omega_i + h\eta\theta_i^{\text{av}} \quad (6)$$

for all $i \in \mathcal{N}$.

Intuitively one could expect that Assumption 1 implies the convergence to frequency synchronization, i.e., at steady state all oscillators rotate with the same instantaneous frequency. Clearly, if the oscillators remain separated in groups, this synchronization frequency must be the same for all groups. Although extensive numerical simulations suggest that the conjecture holds, a formal proof remains an open problem. On the other hand, later in the discussion it is shown that by adding some constraints on the oscillators’ natural frequencies, it is possible to prove frequency synchronization.

III. PHASE ORDER PRESERVATION

The connectivity assumption implies some structure in the solution of (4). This section discusses phase ordering, which reflects the emergence of collective behavior of oscillators also during transient. The following definition formalizes the concept of order preservation.

Definition 1. *Consider (4) under Assumption 1. The oscillators are said to satisfy the order preservation property if the implications $\theta_j > \theta_i \implies \theta_j^+ > \theta_i^+$, with $i, j \in \mathcal{N}$, hold.*

Additionally, if two phases are equal at the initial time step, then their order will be defined at the first time step when they become different, if it exists.

In the case of complete graph, a phase order is established among the oscillators in accordance with their non-decreasing natural frequencies and preserved thereafter [12]. In the following it is shown that, if this phases ordering is obtained at some time step, it will persist under the weaker assumption that the graph stays connected over time. The following lemma is a preliminary step we need.

Lemma 2. *Consider (4) under Assumption 1. For any $i, j \in \mathcal{N}$ such that $\theta_j \geq \theta_i$ at some $k \in \mathbb{N}$, the following condition*

$$\theta_j^{\text{av}} \geq \theta_i^{\text{av}} \quad (7)$$

holds. In particular, the equality condition in (7) holds if and only if $\mathcal{N}_i(\theta) = \mathcal{N}_j(\theta)$.

Proof. Let $\mathcal{N}_{ij}(\theta) = \mathcal{N}_i(\theta) \cap \mathcal{N}_j(\theta)$ the set of common neighbors of the oscillators i and j . Consider the case $\mathcal{N}_{ij}(\theta) = \emptyset$. From (4) and the hypothesis $\theta_j > \theta_i$, it follows that for any $\ell_1 \in \mathcal{N}_j(\theta)$ and $\ell_2 \in \mathcal{N}_i(\theta)$, it is $\theta_{\ell_1} > \theta_{\ell_2}$, thus implying (7) with the strict inequality sign. Consider

the case $\mathcal{N}_{ij}(\theta) \neq \emptyset$. Say $\widehat{\mathcal{N}}_i(\theta) = \mathcal{N}_i(\theta) \setminus \mathcal{N}_{ij}(\theta)$ and $\widehat{\mathcal{N}}_j(\theta) = \mathcal{N}_j(\theta) \setminus \mathcal{N}_{ij}(\theta)$. Assume that $\widehat{\mathcal{N}}_i(\theta) \neq \emptyset$ and $\widehat{\mathcal{N}}_j(\theta) \neq \emptyset$. The average phases in the three sets $\mathcal{N}_{ij}(\theta)$, $\widehat{\mathcal{N}}_i(\theta)$, and $\widehat{\mathcal{N}}_j(\theta)$ are given by

$$\bar{\theta}_{ij} = \frac{\sum_{\ell \in \mathcal{N}_{ij}(\theta)} \theta_\ell}{|\mathcal{N}_{ij}(\theta)|}, \quad \bar{\theta}_i = \frac{\sum_{\ell \in \widehat{\mathcal{N}}_i(\theta)} \theta_\ell}{|\widehat{\mathcal{N}}_i(\theta)|}, \quad \bar{\theta}_j = \frac{\sum_{\ell \in \widehat{\mathcal{N}}_j(\theta)} \theta_\ell}{|\widehat{\mathcal{N}}_j(\theta)|},$$

respectively. Clearly, it is $\bar{\theta}_j > \bar{\theta}_{ij}$ ($\bar{\theta}_i < \bar{\theta}_{ij}$) because $\theta_j > \theta_i$ by hypothesis. Then, it follows

$$\theta_j^{\text{av}} = \frac{1}{|\mathcal{N}_j(\theta)|} \sum_{\ell \in \mathcal{N}_j(\theta)} \theta_\ell = \frac{|\widehat{\mathcal{N}}_j(\theta)|\bar{\theta}_j + |\mathcal{N}_{ij}(\theta)|\bar{\theta}_{ij}}{|\mathcal{N}_j(\theta)|} > \bar{\theta}_{ij} \quad (8)$$

and similarly it can be shown that $\theta_i^{\text{av}} < \bar{\theta}_{ij}$. By combining the inequalities above, (7) holds with strict inequality sign. Consider the case $\mathcal{N}_{ij}(\theta) \neq \emptyset$, $\widehat{\mathcal{N}}_i(\theta) = \emptyset$ and $\widehat{\mathcal{N}}_j(\theta) \neq \emptyset$ ($\widehat{\mathcal{N}}_i(\theta) \neq \emptyset$ and $\widehat{\mathcal{N}}_j(\theta) = \emptyset$). By following the arguments above, it is $\theta_j^{\text{av}} > \bar{\theta}_{ij}$ ($\theta_i^{\text{av}} < \bar{\theta}_{ij}$); moreover, since $\mathcal{N}_i(\theta) = \mathcal{N}_{ij}(\theta)$ ($\mathcal{N}_j(\theta) = \mathcal{N}_{ij}(\theta)$) it is $\theta_i^{\text{av}} = \bar{\theta}_{ij}$ ($\theta_j^{\text{av}} = \bar{\theta}_{ij}$) and (7) holds with strict inequality sign. If $\mathcal{N}_{ij}(\theta) \neq \emptyset$ and $\widehat{\mathcal{N}}_i(\theta) = \widehat{\mathcal{N}}_j(\theta) = \emptyset$, it is $\mathcal{N}_i(\theta) = \mathcal{N}_j(\theta)$ and the equality in (7) directly follows because the two oscillators have the same neighbors. \square

By exploiting the lemma above, it is possible to prove our main result regarding phase order preservation.

Theorem 3. *Consider (4) under Assumption 1 and assume $h\eta \in [0, 1]$. For any $i, j \in \mathcal{N}$ such that $\theta_j \geq \theta_i$ with $\omega_j \geq \omega_i$ holds at some $k \in \mathbb{N}$ it will be $\theta_j^+ \geq \theta_i^+$. Moreover, if $\theta_j = \theta_i$ and $\omega_j > \omega_i$ ($\omega_j = \omega_i$) then $\theta_j^+ > \theta_i^+$ ($\theta_j^+ = \theta_i^+$).*

Proof. By considering (6) for two oscillators i and j , one can write

$$\theta_j^+ - \theta_i^+ = (1 - h\eta)(\theta_j - \theta_i) + h(\omega_j - \omega_i) + h\eta(\theta_j^{\text{av}} - \theta_i^{\text{av}}) \quad (9)$$

for all $i, j \in \mathcal{N}$. Consider (9) for the case $\theta_j > \theta_i$. Given the condition $h\eta \in [0, 1]$, the hypothesis on the natural frequencies, and by using Lemma 2 it directly follows from (9) that $\theta_j^+ > \theta_i^+$. For the case $\theta_j = \theta_i$, it is $\theta_j^{\text{av}} = \theta_i^{\text{av}}$. If $\omega_j = \omega_i$, then from (9) it is $\theta_j^+ = \theta_i^+$, i.e., the two oscillators will have the same phase for all $k \in \mathbb{N}$. If $\omega_j > \omega_i$, from (9) it follows that $\theta_j^+ > \theta_i^+$, i.e., the order of the two oscillators will persist for all $k \in \mathbb{N}$, which completes the proof. \square

An interpretation of Theorem 3 is that if the graph stays connected over time, a necessary condition to have ‘‘crossing’’ between the j -th and i -th oscillators starting from $\theta_j > \theta_i$ is that $\omega_j < \omega_i$. As a consequence, one can prove the following result.

Corollary 4. *Consider (4) under Assumption 1 and assume $h\eta \in [0, 1]$. If there exists a time step \bar{k} such that the phases of the oscillators are ordered according to non-decreasing natural frequencies, i.e., $\theta_j(\bar{k}) > \theta_i(\bar{k})$ for all $i, j \in \mathcal{N}$ such that $\omega_j \geq \omega_i$, then the order preservation property in Definition 1 holds for any $k \geq \bar{k}$.*

Proof. The proof follows straightforwardly by a recursive application of Theorem 3. \square

Note that phase ordering is not a necessary condition for frequency synchronization. In other words, the system can converge to frequency synchronization (with connected but not complete graph) even without being ordered in phase. A simple numerical test of (4) in the case $N = 3$, $\omega = \text{col}(1.00, 1.33, 1.34)$, $\eta = 1.00$, $\Delta = 0.50$, $h = 0.06$ and initial conditions close to the phases’ steady state situation $\bar{\theta}_2 - \bar{\theta}_1 = 0.66$, and $\bar{\theta}_3 - \bar{\theta}_1 = 0.48$, makes this evident.

IV. FREQUENCY SYNCHRONIZATION

This section deals with frequency synchronization, which can be formalized as follows.

Definition 5. *The system (4) is said to converge to frequency synchronization if there exists $\omega_s \in \mathbb{R}$ such that*

$$\lim_{k \rightarrow \infty} \frac{\theta^+ - \theta}{h} = \omega_s \mathbf{1}, \quad (10)$$

where $\mathbf{1}$ represents a column vector of all ones of suitable dimension.

An intuitive reason for expecting frequency synchronization is the fact that the oscillator with the maximum (minimum) phase slows down (speeds up) with respect to its natural frequency because of the influence of its predecessors (successors). Indeed, by using (6) for $\theta_M = \max_{i \in \mathcal{N}} \theta_i$ one can write $\frac{\theta_M^+ - \theta_M}{h} - \omega_M = \eta(\theta_M^{\text{av}} - \theta_M)$, where ω_M is the natural frequency corresponding to θ_M (not necessarily the maximum natural frequency), which is negative because θ_M is the maximum phase and there exists at least one $j \in \mathcal{N}_M$ such that $\theta_j < \theta_M$. Analogously, one can show that the instantaneous frequency of the oscillator with the minimum phase, say θ_m , is larger than its natural frequency ω_m . Although the range of phases $\theta_M - \theta_m$ may increase, Assumption 1 implies that the range of phases is upper bounded by $2\pi - \Delta$, which motivates the conjecture that if the graph remains connected, then frequency synchronization is reached. Proving frequency synchronization using just Assumption 1, however, is an open issue. In this section, we show that by adding the condition of a complete graph or the assumption that all natural frequencies are the same, frequency synchronization can be proved.

A. Complete graph

In the particular case that the graph is complete over time, it is possible to prove that the system converges to frequency synchronization. First, let us show a condition for the graph to preserve completeness over time.

Lemma 6. *Consider (4) with $h\eta \in [0, 1]$. Assume that the initial phases are such that the graph is complete. The graph remains complete over time if and only if*

$$\omega^{\max} - \omega^{\min} \leq \eta\Delta, \quad (11)$$

where $\omega^{\max} = \max_{i \in \mathcal{N}} \omega_i$ and $\omega^{\min} = \min_{i \in \mathcal{N}} \omega_i$.

Proof. Assume that (11) holds. Say $\theta_{M_1}(\theta_{m_1})$ the maximum (minimum) phase at step 1. By using (9) at the first step, since $\theta_i^{\text{av}}(0) = \theta_j^{\text{av}}(0)$ for all $i, j \in \mathcal{N}$ because the initial graph is complete by hypothesis, then one can write

$$\begin{aligned}\theta_{M_1} - \theta_{m_1} &= (1 - h\eta)(\theta_{M_0} - \theta_{m_0}) + h(\omega_{M_1} - \omega_{m_1}) \\ &\leq (1 - h\eta)(\theta_{M_0} - \theta_{m_0}) + h(\omega^{\max} - \omega^{\min}) \\ &\leq (1 - h\eta)\Delta + h\eta\Delta \leq \Delta,\end{aligned}\quad (12)$$

where (11) has been used. The inequality (12) means that at $k = 1$ the graph is complete. By applying recursively (12), the sufficient part of the proof directly follows.

The necessary part is proved by contradiction. Assume that $\omega^{\max} - \omega^{\min} > \eta\Delta$ and that the graph stays complete. Since in this case $\theta_j^{\text{av}} = \theta_i^{\text{av}}$ for all $i, j \in \mathcal{N}$, from (9) it follows that there exists a finite time step \bar{k} such that the oscillators are ordered by non-decreasing values of their natural frequencies and from Corollary 4 this condition will hold also for any $k > \bar{k}$. Then ω^{\max} (ω^{\min}) is the natural frequency of $\theta_M(k)$ ($\theta_m(k)$) for any $k \geq \bar{k}$. By using (9), one can write

$$\theta_M^+ - \theta_m^+ = (1 - h\eta)(\theta_M - \theta_m) + h(\omega^{\max} - \omega^{\min}) \quad (13)$$

and then from $\omega^{\max} - \omega^{\min} > \eta\Delta$ it follows that there exists a finite time step $\hat{k} \geq \bar{k}$ such that $\theta_M(\hat{k}) - \theta_m(\hat{k}) > \Delta$ which contradicts the hypothesis that the graph remains complete over time. \square

The following result proves that if the graph remains complete over time, then the instantaneous frequencies of the oscillators will synchronize to the average of their natural frequencies.

Lemma 7. *Consider (4). If the graph is complete over time, then the following conditions*

$$\lim_{k \rightarrow \infty} (\theta_j - \theta_i) = \frac{\omega_j - \omega_i}{\eta} \quad (14)$$

hold for all $i, j \in \mathcal{N}$ and the system converges asymptotically to frequency synchronization according to (10) with

$$\omega_s = \frac{1}{N} \sum_{i \in \mathcal{N}} \omega_i. \quad (15)$$

Proof. Assume that the graph is complete. Then it is $\theta_j^{\text{av}} = \theta_i^{\text{av}}$ for all $i, j \in \mathcal{N}$ and by applying (9) the asymptotic condition (14) directly follows.

In order to prove (15), from (6) one obtains

$$\lim_{k \rightarrow \infty} \frac{\theta_i^+ - \theta_i}{h} = \omega_i + \eta \lim_{k \rightarrow \infty} (\bar{\theta}^{\text{av}} - \theta_i), \quad (16)$$

where $\bar{\theta}^{\text{av}} = \frac{1}{N} \sum_{i \in \mathcal{N}} \theta_i$ for all $i \in \mathcal{N}$ because the graph is complete. By adding (16) for all $i \in \mathcal{N}$, one can write

$$\sum_{i \in \mathcal{N}} \lim_{k \rightarrow \infty} \frac{\theta_i^+ - \theta_i}{h} = \sum_{i \in \mathcal{N}} \omega_i. \quad (17)$$

The sum of all instantaneous frequencies converges to a constant. On the other hand, condition (14) implies that for any pair of oscillators their phase differences converge to a constant and then all instantaneous frequencies will converge to a constant, say ω_s , and (15) directly follows from (17). \square

By combining the results above, we provide a less restrictive condition that if the graph becomes complete at any time instant during the evolution, and (11) holds, then the system will converge to frequency synchronization.

Theorem 8. *Consider (4) and $h\eta \in [0, 1]$. If condition (11) holds and at some time step the graph becomes complete, then the system converges asymptotically to frequency synchronization with (14) and (15).*

Proof. By hypothesis, there exists a time step such that the graph is complete. Then, since (11) holds by hypothesis, by applying Lemma 6, it follows that the graph remains complete, and by using Lemma 7, the proof is complete. \square

Theorem 8 means that the synchronization frequency will be the mean value of the natural frequencies and at steady state the oscillators will be ordered by increasing values of their natural frequencies, i.e., from (14) if $\omega_j \geq \omega_i$ then $\lim_{k \rightarrow \infty} (\theta_j - \theta_i) \geq 0$ for all $i, j \in \mathcal{N}$.

B. Connected graph with equal natural frequencies

Another interesting situation is when all natural frequencies are the same. Note that Assumption 1 of the subgraph being connected must be present, as condition (11) is not sufficient to ensure the connectivity of the graph once it becomes connected. In order to analyze this scenario, let us reformulate the phase dynamics (4) of all oscillators in matrix form. By defining $\omega = \text{col}((\omega_i)_{i \in \mathcal{N}})$, (4) can be rewritten as

$$\theta^+ = R_k \theta + h\omega, \quad (18)$$

where, with some abuse of notation, $R_k = I + M_k$, $M_k = h\eta(A_k - I)$, $k \in \mathbb{N}_0$, $A_k = \text{diag}\left(\text{col}\left(\frac{1}{|\mathcal{N}_i(\theta)|}\right)\right) A_k$, and A_k is the adjacency matrix of the graph corresponding to the phase vector θ , $k \in \mathbb{N}_0$. Applying (18) recursively, starting from the initial condition $\theta(0)$, one obtains

$$\theta = \prod_{n=k-1}^0 R_n \theta(0) + Q_{k-1} h\omega \quad (19)$$

for all $k \in \mathbb{N}$, where

$$Q_{k-1} = I + R_{k-1} + R_{k-1}R_{k-2} + \dots + R_{k-1} \cdot \dots \cdot R_1 \quad (20)$$

for all $k \in \mathbb{N}$ with $Q_0 = I$. It is now possible to prove that if the natural frequencies are zero, Assumption 1 ensures that the phases will converge to a common value.

Lemma 9. *Consider (4) under Assumption 1 and assume $h\eta \in [0, 1]$. If $\omega_i = 0$ for all $i \in \mathcal{N}$, then there exists a constant $\alpha \in \mathbb{R}_+$ such that $\lim_{k \rightarrow \infty} \theta = \alpha \mathbf{1}$.*

Proof. By assuming $\omega = 0$, from (19) one obtains

$$\lim_{k \rightarrow \infty} \theta = \lim_{k \rightarrow \infty} \prod_{n=k-1}^0 R_n \theta(0). \quad (21)$$

From (4) it follows that all matrices A_k , $k \in \mathbb{N}_0$, are row stochastic, i.e., $A_k \mathbf{1} = \mathbf{1}$ for all $k \in \mathbb{N}_0$, and then M_k are zero row sum and R_k are row stochastic for all $k \in \mathbb{N}_0$. Moreover, all matrices A_k , $k \in \mathbb{N}_0$, have positive diagonal entries less than 1. Then, all entries of the main diagonal of M_k belong to the interval $(-h\eta, 0)$ for all $k \in \mathbb{N}_0$. As a

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consequence, the elements of the main diagonal of the row stochastic matrix R_k belong to the interval $(1 - h\eta, 1)$ and, for any $h\eta \in [0, 1]$, are strictly positive and less than 1 for all $k \in \mathbb{N}_0$. Then, by using Corollary 7 in [13] which shows that the product of row stochastic matrices with strictly positive diagonal elements converges to a constant matrix, one can write $\lim_{k \rightarrow \infty} \prod_{n=k-1}^0 R_n = \mathcal{R}^\infty$, where \mathcal{R}^∞ is a constant matrix. By substituting in (21), it follows that when k goes to infinity, the phases converge to the constant vector $\bar{\theta} = \mathcal{R}^\infty \theta(0)$. By using (6) with $\bar{\theta}_i^+ = \bar{\theta}_i$ and $\omega_i = 0$, $i \in \mathcal{N}$, it follows that $\bar{\theta}_i = \bar{\theta}_i^{\text{av}}$, $i \in \mathcal{N}$. By considering $i = M$, since $\mathcal{N}_M^+ = \emptyset$, there exists a constant α such that $\bar{\theta}_i = \bar{\theta}_M = \alpha$ for all $i \in \mathcal{N}_M(\bar{\theta})$ otherwise it would be $\bar{\theta}_M \neq \bar{\theta}_M^{\text{av}}$, which is not possible. By using Assumption 1, it follows that $\mathcal{N}_M(\bar{\theta}) = \mathcal{N}$ and then $\bar{\theta}_i = \alpha$ for all $i \in \mathcal{N}$, which completes the proof. \square

It should be noticed that Assumption 1 is weaker than the assumption in [12], in the sense that the analysis therein requires the graph connections to be constant after some time step in order to achieve frequency synchronization, i.e., $\mathcal{N}_i(\theta) = \bar{\mathcal{N}}_i$, $i \in \mathcal{N}$, for all $k \geq \bar{k}$. Under this assumption, frequency synchronization is reached with the synchronization frequency given by

$$\omega_s = \frac{1}{\sum_{i \in \mathcal{N}} |\bar{\mathcal{N}}_i|} \sum_{i \in \mathcal{N}} |\bar{\mathcal{N}}_i| \omega_i. \quad (22)$$

The following result shows that if all oscillators have the same natural frequencies, the graph does not require the assumption to be constant, but only to be connected over time to converge to frequency synchronization.

Theorem 10. *Consider (4) under Assumption 1 and assume $h\eta \in [0, 1]$. If all natural frequencies are equal, i.e., $\omega_i = \omega_s$ for all $i \in \mathcal{N}$ with some ω_s , then the system converges asymptotically to frequency synchronization with (10).*

Proof. By using (19) and Lemma 9, one can write

$$\lim_{k \rightarrow \infty} \theta = \alpha \mathbb{1} + \lim_{k \rightarrow \infty} Q_{k-1} h \omega. \quad (23)$$

From (20) one can write

$$\begin{aligned} Q_k - Q_{k-1} &= R_k Q_{k-1} + I - Q_{k-1} \\ &= (R_k - I) Q_{k-1} + I = M_k Q_{k-1} + I. \end{aligned} \quad (24)$$

Taking the difference of (23) at two consecutive steps we get

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{\theta^+ - \theta}{h} &= \lim_{k \rightarrow \infty} (Q_k - Q_{k-1}) \omega \\ &= \lim_{k \rightarrow \infty} (M_k Q_{k-1} + I) \omega. \end{aligned} \quad (25)$$

Since by hypothesis all oscillators have the same natural frequency, one can write $\omega = \omega_s \mathbb{1}$. By substituting this expression in (25), by noticing from (20) that Q_{k-1} is the summation of symmetric row stochastic matrices and by using the condition that M_k is symmetric zero row sum, it follows that $M_k Q_{k-1} \mathbb{1}$ is zero for all $k \in \mathbb{N}$ and then from (25) one obtains that (10) holds. \square

The analysis in the previous section shows two scenarios that ensure the convergence of the system to frequency synchronization: first, the graph being complete together with condition (11) (Theorem 8), and second, oscillators under Assumption 1 and having the same natural frequencies (Theorem 10). In this section, we investigate the case of three oscillators with distinct natural frequencies and the role of Assumption 1 and the bound (11) on the natural frequencies. Proving convergence to frequency synchronization in more general situations, like connected oscillators having distinct natural frequencies, is a direction for future research.

Without loss of generality assume that it is $\omega_3 \geq \omega_2 \geq \omega_1$ (without assuming any prior order of the phases) and say α , β , and γ the indices of the oscillators corresponding to the phases at the time step k such that the order $\theta_\gamma \geq \theta_\beta \geq \theta_\alpha$ holds, with α , β , and γ being distinct values belonging to the set $\{1, 2, 3\}$. Note that since at any time step the position of oscillators may change with respect to that at the previous time step, it is possible that the value of α , β , and γ may change over time.

Under Assumption 1 with $N = 3$, there are only two possible configurations: a) the graph is complete, i.e., $\mathcal{N}_\alpha = \mathcal{N}_\beta = \mathcal{N}_\gamma = \mathcal{N} = \{1, 2, 3\}$; b) the graph is not complete (though connected), with the oscillator in the middle directly connected with those at the extremes and the oscillators at the extremes not directly connected, i.e., $\mathcal{N}_\alpha = \{\alpha, \beta\}$, $\mathcal{N}_\beta = \{\alpha, \beta, \gamma\}$, and $\mathcal{N}_\gamma = \{\beta, \gamma\}$. In the following, these two scenarios will be analyzed under the assumption that (11) holds, i.e., $\omega_3 - \omega_1 \leq \eta \Delta$.

Consider the case that the initial graph is complete. Let us define $\Theta_{\beta\alpha} = \theta_\beta - \theta_\alpha$, $\Theta_{\gamma\alpha} = \theta_\gamma - \theta_\alpha$, and $\Theta_{\gamma\beta} = \theta_\gamma - \theta_\beta$. Then, from (4) the dynamics can be written as

$$\Theta_{\beta\alpha}^+ = (1 - h\eta) \Theta_{\beta\alpha} + h(\omega_\beta - \omega_\alpha), \quad (26a)$$

$$\Theta_{\gamma\alpha}^+ = (1 - h\eta) \Theta_{\gamma\alpha} + h(\omega_\gamma - \omega_\alpha), \quad (26b)$$

and $\Theta_{\gamma\beta}^+ = \Theta_{\gamma\alpha}^+ - \Theta_{\beta\alpha}^+$. By using Lemma 6, Lemma 7, and Theorem 8, it follows that the graph will maintain its completeness over time and the system will asymptotically converge to frequency synchronization with order creation, i.e., at steady state $\alpha = 1$, $\beta = 2$, and $\gamma = 3$ together with $\bar{\Theta}_{21} = (\omega_2 - \omega_1)/\eta$, $\bar{\Theta}_{31} = (\omega_3 - \omega_1)/\eta$, $\bar{\Theta}_{32} = (\omega_3 - \omega_2)/\eta$, and $\omega_s = (\omega_1 + \omega_2 + \omega_3)/3$.

Now consider that the graph is connected but not complete. By using (4), the system dynamics can be written as

$$\Theta_{\beta\alpha}^+ = \left(1 - \frac{7}{6} h\eta\right) \Theta_{\beta\alpha} + \frac{1}{3} h\eta \Theta_{\gamma\alpha} + h(\omega_\beta - \omega_\alpha) \quad (27a)$$

$$\Theta_{\gamma\alpha}^+ = \left(1 - \frac{1}{2} h\eta\right) \Theta_{\gamma\alpha} + h(\omega_\gamma - \omega_\alpha) \quad (27b)$$

$$\Theta_{\gamma\beta}^+ = \left(1 - \frac{7}{6} h\eta\right) \Theta_{\gamma\beta} + \frac{1}{3} h\eta \Theta_{\gamma\alpha} + h(\omega_\gamma - \omega_\beta). \quad (27c)$$

Evaluating (27) at steady state, and using (22), one can write $\bar{\Theta}_{\beta\alpha} = \frac{2(\omega_s - \omega_\alpha)}{\eta}$, $\bar{\Theta}_{\gamma\alpha} = \frac{2(\omega_\gamma - \omega_\alpha)}{\eta}$, $\bar{\Theta}_{\gamma\beta} = \frac{2(\omega_\gamma - \omega_s)}{\eta}$

with $\omega_s = \frac{1}{7}(2\omega_\alpha + 3\omega_\beta + 2\omega_\gamma)$. The graph connections imply that it must be $\Theta_{\gamma\beta} \leq \Delta$, $\Theta_{\beta\alpha} \leq \Delta$, and $\Delta < \Theta_{\gamma\alpha} \leq 2\Delta$, and then it follows that the natural frequencies must satisfy the following constraints $\omega_\gamma - \omega_\alpha \in (0.5\eta\Delta, \eta\Delta]$, $\omega_s - \omega_\alpha \leq 0.5\eta\Delta$, and $\omega_\gamma - \omega_s \leq 0.5\eta\Delta$. From these inequalities it follows that at steady state it must be $\alpha \neq 3$ and $\gamma \neq 1$, and then the only three possible combinations for the distinct values of $\{\alpha, \beta, \gamma\}$ at steady state are $\{1, 2, 3\}$, $\{1, 3, 2\}$, and $\{2, 1, 3\}$. It is tedious though straightforward to explore the evolution of all possible combinations of three oscillators, and it follows that they synchronize to one of the above-mentioned steady states. Therefore, the analysis above shows that not all configurations are allowed at steady state.

The following analysis shows that without Assumption 1, condition (11) is not sufficient for the system to reach frequency synchronization. A simple numerical test of the model (4) with parameters $\omega = 10^{-2} \cdot \text{col}(1.0, 1.1, 5.2)$, $\Delta = 0.5$, $\eta = 0.1$, and $h = 0.12$ shows that for the initial phases $\theta(0) = \text{col}(3, 2, 1)$, frequency synchronization is achieved; while solely by changing the initial phases to $\theta(0) = \text{col}(1, 2, 3)$, permanent oscillations are observed. Moreover, (11) is not sufficient to ensure the connectivity over time even if the graph is initially connected. Indeed, consider $\{\alpha, \beta, \gamma\} = \{1, 2, 3\}$, $\Theta_{21} = \epsilon_0\Delta$, $\Theta_{32} = \epsilon_1\Delta$, $\omega_2 - \omega_1 = \epsilon_2\eta\Delta$, with some $\epsilon_i \in [0, 1]$, $i = 0, 1, 2$, and $\epsilon_0 + \epsilon_1 > 1$. Then from (27a) it follows that, by taking for instance $\epsilon_0 = 1$, any pair ϵ_1, ϵ_2 such that $\frac{1}{3}\epsilon_1 + \epsilon_2 > \frac{5}{6}$ leads to $\Theta_{21}^\dagger > \Delta$, i.e., the graph becomes disconnected.

Frequency synchronization is achieved for $N = 3$ with $\omega_i = \omega_s$, $i = 1, 2, 3$, and the less restrictive assumption of the graph being only initially connected, so as shown in the following. Clearly, (11) holds in this case. From Corollary 4 it is clear that the three oscillators will not cross each other. If the subgraph is initially connected then $\Theta_{\gamma\beta} \leq \Delta$, $\Theta_{\beta\alpha} \leq \Delta$, and $\Theta_{\gamma\alpha} \leq 2\Delta$. Applying these conditions to (27) and by separately considering $(1 - \frac{7}{6}h\eta)$ to be positive or not, it is easy to show that the graph remains connected although not necessarily complete. On the other hand, numerical simulations show that this is not always true for a larger number of oscillators, e.g., $N = 6$, with $\theta(0) = \text{col}(0.1, 0.6, 1.1, 1.6, 2.1, 2.6)$, $\eta = 0.1$, $\Delta = 0.5$, $h = 0.12$, and $\omega_i = 0.1$ for all $i \in \mathcal{N}$.

In synthesis, the analysis of three oscillators clearly brings up challenges and insights for the case of larger number of oscillators. In particular, the discussion above clarifies the following: first, frequency synchronization is possible without phase ordering; second, frequency synchronization depends on the initial topology of the oscillators; third, Assumption 1 is crucial for analyzing frequency synchronization among oscillators with distinct natural frequencies, as there is no known condition that guarantees their connectivity over time; and fourth, if the graph of oscillators with the same natural frequency is initially connected, it stays connected for $N = 3$, but it is not always true for $N > 3$.

VI. CONCLUSION

Convergence of oscillators with bounded interaction is analyzed. Two kinds of graph connectivity are considered: complete graphs and connected graphs. A necessary and sufficient condition on the oscillator's natural frequencies is given for the graph to remain complete throughout the evolution, if it becomes complete at any time. It is also shown that the same condition is not sufficient for the graph to remain connected without reaching completeness. In case of complete graph, phase order in the oscillators is established and maintained in accordance with the ordering of their natural frequencies. On the other hand, under the assumption of connected only graph, order is preserved, but frequency synchronization is possible without the oscillators being ordered. If all oscillators have the same natural frequency, the system achieves frequency synchronization. Future work will focus on finding conditions for frequency synchronization and phase ordering in the case of oscillators with different natural frequencies, and formally proving the conjecture that with the assumption that the range of natural frequencies is less or equal to the product of the coupling gain and the confidence threshold, a connected graph leads to frequency synchronization.

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