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# Forward Indifference Pricing\*

Hillairet Caroline <sup>†</sup>      Jakani Manal <sup>‡</sup>      Mrad Mohamed <sup>§</sup>

## Abstract

This paper studies forward indifference prices for financial claims. Indifference prices measure the compensation required to offset the risk associated with providing or receiving a claim based on a utility criterion. Our approach relies on the framework of consistent dynamic utilities - also called forward utilities - which allows us to account for changes in the preferences as well as any uncertain evolution of the financial and economic environment. The indifference price is determined by equalizing the value function of a standard optimization problems of utility of terminal wealth, with the value function of another optimization problem that integrates the claim in the portfolio. We first determine the optimal strategy for the latter, in the context of consistent dynamic utilities. Then the indifference price is characterized as a solution of a backward stochastic differential equation of quadratic type. As an example, the case of consistent exponential utility is studied.

**2000 AMS subject classifications:** 49L20, 60H30, 91B16, 91G10, 91G20.

**Key words:** Indifference pricing, Utility maximization, Consistent Dynamic (Forward) utility, Optimal portfolio, Forward-Backward Stochastic Differential Equation (FBSDE).

## 1 Introduction

The valuation of contingent claims in financial markets involves multiple factors. In complete markets, the classical arbitrage-free pricing methodology can be applied. It relies upon the idea of risk replication, which induces a unique price for the contingent claim: by no arbitrage, the price of a replicable claim is the cost of the replicating portfolio hedging away the market risk. Hence, in a complete market, the price of the contingent claim is the expected future discounted cash-flows, calculated by the unique risk-neutral probability measure. Nevertheless, the replication technique is only possible for markets with high liquidity and for deeply

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traded assets. In incomplete markets, there is no universal pricing probability measure, and the range of no-arbitrage prices becomes significantly wider. A more appropriate strategy involves utility maximization. Following Hodges and Neuberger [HN89], the maximum price that an agent is willing to pay is relative to their indifference towards the transaction and according to their individual preferences. Utility indifference pricing offers a method for determining the price that an investor is willing to pay today in exchange for a future contingent claim, such that their expected utility remains at least as high as it would have been without the claim. That is, the investor decides whether to include the claim in their portfolio based on their risk preferences, which are accounted for in their decision criterion, but also on their current wealth. Consequently, the problem of optimization includes the portfolio's position and, as a result, provides the optimal investment strategy. One main challenge lies in the choice of the utility function  $U$  that best describes both the investor's attitude toward risk and their preferences. Utility indifference pricing has been studied in the literature, among others in [Hen02, HN89, MZ04b, MZ04a, HH09, PZ13, LH16, CS25], see also Carmona [Car08] which provides an overview of several of these contributions. However, these studies focus on deterministic utilities, which do not incorporate the instantaneous changes of the market, which leads to consistency issues.

Consistent dynamic utilities have proven successful in representing dynamic and consistent preferences, as they take into account both the evolution of the financial market and the portfolio's position. Musiela and Zariphopoulou [MZ09] were the first to suggest using consistent dynamic (also called forward) utilities instead of the classic criterion. Then, this theory has been extended by El Karoui and Mrad [EKM13] to a large class of dynamic utilities that are Itô semimartingales. It provides a characterization of consistent dynamic utilities as solutions of forward SPDEs and gives sufficient conditions for consistency. Explicit constructions of consistent dynamic utilities are developed in the literature. Homothetic forward utilities are studied by Liang and Zariphopoulou [LZ15], where the consistency property translates into ergodic BSDEs. Later on, using randomization of risk aversion, a new class of consistent utilities is provided in [Mra21]. To the best of our knowledge, very few papers in the literature are dedicated to indifference prices in the framework of a general probability space and stochastic utilities. One can cite the work on dynamic marginal utility prices (also called Davis prices) with application for long term discounting, see El Karoui et al. [EKHM22]. In a discrete time setting, Musiela et al. [MSZ10] study dynamic indifference pricing under exponential preferences in incomplete binomial market models. More recently, Freitas et al. [FHM25] give a representation of indifference prices for a class of (non-consistent) random utilities, in the setting of infinite probability spaces, while Mostovyi and Siorpaes [MS24] study large financial markets and the asymptotic behavior of such prices. Also, the certainty equivalent, a notion closely related to the indifference price, has been studied and developed by Frittelli and Maggis [FM11] in the framework of dynamic utilities

which are not necessarily consistent, and by Liang et al. [LSZ23] with consistency property. Finally, in the same framework of [LZ15], Chong et al. [CHLZ19] characterize forward entropic risk measures, which are maturity-independent dynamic risk measures.

The aim of this paper is to study indifference prices under a consistent dynamic preference criterion. Based on the findings of [EKM13], we first study the utility maximization problem for terminal wealth including a claim  $C_T$  at some maturity  $T$ , for general consistent dynamic utilities. Specifically, we identify the price  $p_t^q$  that ensures a balance for the agent, who becomes indifferent between investing with  $q$  quantity of the claim  $C_T$  in their portfolio or not. This results in satisfying an equilibrium of the form  $V^c(t, X_t - p_t^q, q) = V(t, X_t)$  where the value function of the primal optimization problem  $V(t, X_t)$  coincides with the value function  $V^c(t, X_t - p_t^q, q)$  associated with the investment with the claim  $C_T$ . The optimal policy for the optimization problem with claim is given in terms of the optimal one without claim, and identified using a forward-backward stochastic differential equation (FBSDE for short), whose coefficients depend on the characteristics of the consistent dynamic utility. The indifference price that achieves the desired equilibrium is provided explicitly and satisfies economically reasonable properties. Our analysis shows that the indifference price can be characterized as a solution of a BSDE of quadratic type.

The paper is organized as follows. In Section 2, we present the market model and we review the key results on consistent dynamic utilities, which are the main tools in our research. Section 3 provides a thorough study of indifference prices. First, we set the optimization problem with claim for an agent who is willing to invest in the claim  $C_T$  and we then determine several properties of the associated indifference price. In Section 4, we derive a FBSDE which characterizes both the optimal investment strategy and a key auxiliary process identified from an equation involving the marginal utility and the claim  $C_T$ . In the particular case where the claim  $C_T$  is replicable, the indifference price, which is the value of the replicating portfolio, turns out to be the auxiliary process itself. Finally, we show that the indifference price satisfies a quadratic BSDE whose coefficients depend on the latter FBSDE, that to our knowledge, has not been thoroughly addressed in the existing literature. This forms the main contribution of the paper. We conclude in Section 5 with an example based on consistent exponential utility, in which we study in detail the corresponding FBSDE and the quadratic BSDE for the indifference price.

## 2 The market model and consistent dynamic utilities

### 2.1 The investment universe

In this paper, we consider an incomplete Itô financial market, defined on a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . On this space, we define an  $n$ -dimensional Brownian motion  $W = (W^1, \dots, W^n)^\top$  (where the superscript  $\top$  indicates the transpose) and  $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$  denotes

the  $\mathbb{P}$ -augmented filtration generated by  $W$ . We adopt in the sequel the convention:  $xy$  denotes the scalar product of two vectors  $x$  and  $y$  having the same dimension. The financial market is characterized by a short rate  $(r_t)_{t \geq 0}$  and an  $n$ -dimensional risk premium vector  $(\eta_t)_{t \geq 0}$ , that are assumed to satisfy the following assumption:

$$\int_0^T (r_t + \|\eta_t\|^2) dt < +\infty, \forall T > 0, \text{ a.s.} \quad (2.1)$$

where  $T > 0$  represents a finite horizon. Throughout the paper, all equalities and inequalities involving processes (resp. random variables) are to be understood as holding  $dt \otimes d\mathbb{P}$  (resp.  $d\mathbb{P}$ ) almost surely.

## The market model

The strategy of investment in the risky assets should be chosen in an admissible set of self-financing strategies that is defined mathematically as follows:

### Definition 2.1. HEDGING PORTFOLIO

An Itô semimartingale  $X^\pi$  is called an admissible wealth process if

(i)  $X^\pi$  satisfies the self-financing dynamics :  $\forall t \geq 0$ ,

$$dX_t^\pi = X_t^\pi r_t dt + \pi_t (dW_t + \eta_t dt), \quad X_0^\pi \in \mathbb{R}, \quad (2.2)$$

with  $\int_0^T \|\pi_t\|^2 dt < +\infty, \forall T > 0$ , where  $\pi$  is an  $n$ -dimensional progressive vector which stands for the allocation.

(ii) For any  $t \geq 0$ , the allocation  $\pi_t \in \mathcal{R}_t$  where  $(\mathcal{R}_t)_{t \geq 0}$  is a family of linear spaces.

(iii) We introduce the following sets of admissible wealth processes (set of hedging strategies):

$$\begin{aligned} \mathcal{X} &\triangleq \{X^\pi : \pi_t \in \mathcal{R}_t \text{ for any } t, X^\pi \text{ satisfies (2.2)}\}, \\ \mathcal{X}(x) &\triangleq \{X^\pi \in \mathcal{X} \text{ such that } X_0^\pi = x\}. \end{aligned}$$

We denote by  $x^{\mathcal{R}}$  and  $x^\perp$  the projection of  $x$  on  $\mathcal{R}$  and its orthogonal respectively. Since from (2.2), the impact of the risk premium on the wealth dynamics only appears through the term  $\pi_t \eta_t$ , there is a "minimal" risk premium  $\eta^{\mathcal{R}}$ , the projection of  $\eta$  on the space  $\mathcal{R}$ , to which we refer in the sequel.

In this incomplete financial market, the class of discounted densities of martingale measures, also called state price density processes or discounted pricing kernels, are characterized as follows:

### Definition 2.2. STATE PRICE DENSITY

(i) An Itô semimartingale  $Y^\nu$  is called a state price density process if for any wealth process  $X^\pi \in \mathcal{X}$ , with  $\pi \in \mathcal{R}$ , the process  $(Y^\nu X^\pi)$  is a local martingale. Then,  $Y^\nu$  evolves according to the following dynamics:  $\forall t \geq 0$ ,

$$dY_t^\nu = Y_t^\nu [-r_t dt + (\nu_t - \eta_t^{\mathcal{R}}) dW_t], \quad \text{with } \nu_t \in \mathcal{R}_t^\perp \text{ and } Y_0^\nu \in \mathbb{R}_+^*. \quad (2.3)$$

(ii) Let  $\mathcal{Y}$  represent the family of all state price densities, i.e.

$$\mathcal{Y} \triangleq \{Y^\nu : \nu_t \in \mathcal{R}_t^\perp \text{ for any } t, Y^\nu \text{ satisfies (2.3)}\}.$$

In the case of a complete financial market, the set  $\mathcal{Y}$  is reduced to the minimal discounted martingale measure  $Y^0$  ( $\nu = 0$ ), and any contingent claim  $C_T$  with maturity  $T$  is replicable by a self-financing strategy, starting from the initial wealth  $\mathbb{E}[C_T \frac{Y_T^0}{Y_0^0}]$ . By no arbitrage, its price at any time  $0 \leq t \leq T$  is the value at time  $t$  of the replicating strategy and is given by  $\mathbb{E}[C_T \frac{Y_T^0}{Y_t^0} | \mathcal{F}_t]$ . In our framework of an incomplete financial market, any discounted martingale measure  $Y^\nu$  in the set  $\mathcal{Y}$  provides a no-arbitrage price  $\mathbb{E}[C_T \frac{Y_T^\nu}{Y_t^\nu} | \mathcal{F}_t]$  for  $C_T$  (which are all equal to  $\mathbb{E}[C_T \frac{Y_T^0}{Y_t^0} | \mathcal{F}_t]$  if  $C_T$  is replicable). To evaluate such non-replicable claims, we propose to investigate their indifference prices, in a dynamic setting. Indifference price, as defined e.g. in [EKR00, Car08], is the value at which the investor would be indifferent between investing or not in the claim, taking into account their preferences. In our dynamic setting, these preferences are modeled using dynamic utilities that are consistent with the market model introduced above.

## 2.2 Dynamic formulation of the convex set of admissible wealth processes

To characterize the dynamics of the indifference price, we need a dynamic formulation of the classes of admissible wealth processes. This requires defining the corresponding optimization problems at each intermediate date  $s$ . Accordingly, we consider wealth processes starting at time  $s$  from the same  $\mathcal{F}_s$ -attainable wealth  $\xi$ . The attainability is defined as follows:

**Definition 2.3.** *Let  $s \in [0, +\infty)$  and let  $\xi$  be a positive  $\mathcal{F}_s$ -measurable square integrable random variable. Then,  $\xi$  is said to be a  $\mathcal{F}_s$ -attainable wealth if there exists  $x \in \mathbb{R}$  and an admissible wealth  $X^\pi \in \mathcal{X}(x)$  such that  $X_s^\pi = \xi$ .*

In this framework, the notation of stochastic flows turns out to be particularly useful. We begin by introducing the concept of dynamic utility, while the property of consistency will be introduced afterwards.

**Definition 2.4** (Dynamic utility). *A dynamic utility is a continuous progressive random field  $U(t, x)$  on  $[0, +\infty) \times \mathbb{R} \times \Omega \rightarrow \mathbb{R} \cup \{-\infty\}$  such that the mapping  $x \mapsto U(\cdot, x)$  is strictly increasing and strictly concave on its domain  $\mathcal{D}_t := \{x, \text{ such that } U(t, x) > -\infty\}$ . Moreover,  $U$  is a  $\mathcal{C}^2$ -random field on the interior of the domain  $\overset{\circ}{\mathcal{D}}_t$ , with continuous first and second derivatives random fields  $U_z$  and  $U_{zz}$ , and satisfying the Inada Condition  $U_z(t, \overset{\circ}{\mathcal{D}}_t) = ]0, +\infty[$ .*

From now on, we fix a dynamic utility  $U$  and for any  $s \in [0, +\infty)$ , we define the random variable:

$$x_s^U := \inf\{x \in \mathbb{R} : U(s, x) > -\infty\}.$$

Consider a fixed time  $s \in [0, +\infty)$  and denote by  $\xi$  a  $\mathcal{F}_s$ -measurable random variable that is square integrable. Assuming  $\xi$  is attainable and  $\xi > x_s^U$ , we define the following set

$$\mathcal{X}(s, \xi) \triangleq \{(X_t^\pi)_{t \geq s} : X^\pi \in \mathcal{X}(x) \text{ such that } X_s^\pi = \xi \text{ and } X_t^\pi > x_t^U \text{ for all } t \geq s\}.$$

When needed, we will employ the flow notation  $(X_t(s, \xi))_{t \geq s}$  to refer to the elements of this class, indicating that each element  $X$  evolves from the wealth  $\xi$  at time  $s$ .

**Lemma 2.5.** *For any  $s \geq 0$  and any random variable  $\xi$  which is  $\mathcal{F}_s$ -attainable, the set  $\mathcal{X}(s, \xi)$  is convex.*

*Proof.* Considering two wealth processes  $X^1$  and  $X^2$  of  $\mathcal{X}(s, \xi)$  with dynamics (for  $i = 1, 2$ )

$$dX_t^i = X_t^i r_t dt + \pi_t^i (dW_t + \eta_t^{\mathcal{R}} dt), \quad \pi_t^i \in \mathcal{R}_t,$$

and  $\lambda \in [0, 1]$  then  $X^\lambda := \lambda X^1 + (1 - \lambda)X^2$  is characterized by the initial wealth  $\xi$  and the dynamics

$$dX_t^\lambda = X_t^\lambda r_t dt + \underbrace{(\lambda \pi_t^1 + (1 - \lambda) \pi_t^2)}_{\pi_t^\lambda} (dW_t + \eta_t^{\mathcal{R}} dt),$$

where  $\pi^\lambda \in \mathcal{R}$  since the family of strategies  $\mathcal{R}$  is convex. □

### 2.3 Consistent dynamic utilities

We now incorporate the concept of investor satisfaction within the established framework. The dynamic utility should capture the consistency property of the investors' preferences. This translates into the notion of dynamic utilities that are consistent with the financial market, a property that is defined and characterized below. In this paper, we consider dynamic utilities that are Itô semimartingales of the following form:

$$dU(t, z) = \beta(t, z)dt + \gamma(t, z)dW_t, \quad U(0, z) = u(z), \tag{2.4}$$

where  $(\beta, \gamma)$  are the local characteristics of  $U$  and  $u$  is a given deterministic utility function that represents the initial utility. The existence of a monotonic and concave solution to the dynamics described in (2.4) is not straightforward. We refer to [EKM13] for a detailed discussion of this issue. Essentially, the local characteristics  $\beta$  and  $\gamma$  must satisfy specific regularity requirements. More precisely, they are assumed to belong to the respective Kunita spaces<sup>1</sup>  $\mathcal{K}_{\text{loc}}^{3, \delta}$  and  $\overline{\mathcal{K}}_{\text{loc}}^{3, \delta}$  for some  $\delta \in (0, 1]$ , along with boundedness conditions on the ratios  $\frac{\gamma_z}{U_z}$  and  $\frac{\gamma_{zz}}{U_{zz}}$ . These assumptions are formally stated in Assumption 2.7, together with additional conditions related to the market-consistency property introduced below.

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<sup>1</sup>See the Annex for the definition of these spaces.

**Definition 2.6** ( $\mathcal{X}$ -consistent dynamic utility).

A  $\mathcal{X}$ -consistent dynamic utility (also called forward utility)  $U$  is a dynamic utility satisfying:

- *Consistency with the investment universe:* For any admissible wealth  $(X_t^\pi)_{t \geq s} \in \mathcal{X}(x)$  such that  $\mathbb{E}[|U(t, X_t^\pi)|] < \infty$ , we have for all  $s \leq t$

$$\mathbb{E}[U(t, X_t^\pi) | \mathcal{F}_s] \leq U(s, X_s^\pi).$$

- *Existence of an optimal wealth:* There exists a random field  $(X_t^*(s, \cdot))_{s \leq t}$  such that for any  $\mathcal{F}_s$ -attainable wealth  $\xi$ ,  $(X_t^*(s, \xi))_{t \geq s} \in \mathcal{X}(s, \xi)$  satisfying  $\mathbb{E}[|U(t, X_t^*(s, \xi))|] < \infty$  and

$$\mathbb{E}[U(t, X_t^*(s, \xi)) | \mathcal{F}_r] = U(r, X_r^*(s, \xi)), \quad 0 \leq s \leq r \leq t.$$

This means that for any  $\mathcal{X}$ -admissible wealth process  $X^\pi$ , the mapping  $t \mapsto U(t, X_t^\pi)$  is a supermartingale. In addition, for any time  $s$  and any  $\mathcal{F}_s$ -attainable wealth  $X_s$ ,  $(X_t^*(s, \xi))_{s \leq t}$  is the unique optimal wealth, for any maturity  $t \geq s$ : in particular,  $(U(t, X_t^*(s, \xi)))_{s \leq t}$  is a martingale.

Hereafter, the index  $\mathcal{X}$  is omitted since there is no ambiguity regarding the market.

Based on the results in [EKM13], the consistency of  $U$  translates into an Hamilton-Jacobi-Bellman (HJB) constraint on the drift component  $\beta$  given by

$$\beta(t, z) = -zr_t U_z(t, z) + \frac{1}{2U_{zz}(t, z)} \|U_z(t, z)\eta_t^{\mathcal{R}} + \gamma_z^{\mathcal{R}}(t, z)\|^2. \quad (2.5)$$

In this case, the optimal policy  $\pi^*$ , which provides the optimal wealth for the utility  $U$  is given by:

$$\pi_t^*(z) = -\frac{1}{U_{zz}(t, z)} (U_z(t, z)\eta_t^{\mathcal{R}} + \gamma_z^{\mathcal{R}}(t, z)). \quad (2.6)$$

In this paper, we consider consistent dynamic utilities of the Itô form (2.4), satisfying the following assumptions.

**Assumption 2.7.** *The consistent dynamic utility  $U$  and its local characteristics  $(\beta, \gamma)$  satisfy*

- (i)  $U$  is of class  $\mathcal{K}_{loc}^{3, \delta}$ ,  $\delta \in (0, 1]$ .
- (ii)  $\beta$  satisfies the HJB-SPDE (2.5).
- (iii) There exist two positive  $\mathcal{F}$ -adapted stochastic bounds  $K$  and  $\bar{K}$  that satisfy  $\int_0^T K_t^2 dt < +\infty$  and  $\int_0^T \bar{K}_t^2 dt < +\infty$ , for any  $T$  such that:

$$\|\gamma_z(t, z)\| \leq K_t |U_z(t, z)|, \quad \|\gamma_{zz}(t, z)\| \leq \bar{K}_t |U_{zz}(t, z)|.$$

**Remark 2.8.** *As stated in [EKM13], conditions (i) and (ii) imply that  $U(\cdot, X_\cdot)$  is a supermartingale for any admissible wealth process  $X$  and that  $\pi^*$  defined by (2.6) is a candidate strategy for the optimal control. Condition (iii) ensures that the SDE (2.2) associated with  $\pi^*$*

admits a unique strong solution, denoted by  $X^* = X^{\pi^*}$ , so that the consistency property holds with  $U(\cdot, X^*)$  being a martingale. Moreover, assertion (i) also implies that the marginal utility  $U_z$  is a positive, increasing random field which is a semimartingale satisfying an explicit HJB-type SPDE. This equation is obtained by differentiating term by term the SPDE satisfied by  $U$ , meaning that the local characteristics of  $U_z$  are given by  $(\beta_z, \gamma_z)$ . Under Assumption (iii), the first-order condition for optimality further implies that  $U_z(\cdot, X^*)$  coincides with the optimal state price density process, as detailed below.

## 2.4 Convex conjugate utility

As usual in convex analysis, to the dynamic utility function  $U$ , we associate its convex conjugate  $\tilde{U}$  (also called Fenchel transform) defined by:

$$\tilde{U}(t, y) \triangleq \inf_{z \in \mathbb{Q}} \{U(t, z) - zy\}. \quad (2.7)$$

Then  $\tilde{U}(t, y)$  is a decreasing convex dual utility with dynamics:

$$d\tilde{U}(t, y) = \tilde{\beta}(t, y)dt + \tilde{\gamma}(t, y)dW_t,$$

where the local characteristics  $(\tilde{\beta}, \tilde{\gamma})$  are given in terms of the first and second derivatives  $\tilde{U}_y$  and  $\tilde{U}_{yy}$ . More precisely, under Assumption 2.7-(i), it follows from Theorem 1.5 in [EKM13] that the volatility component satisfies

$$\tilde{\gamma}(t, y) = \gamma(t, -\tilde{U}_y(t, y)) \text{ and } \tilde{\gamma}_y(t, y) = -\gamma_z(t, -\tilde{U}_y(t, y))\tilde{U}_{yy}(y).$$

As for the drift  $\tilde{\beta}$ , it is mainly obtained from the consistency of  $U$  that results in the HJB constraint (2.5). Indeed, if  $U$  is an consistent utility then  $\tilde{U}$  is a  $\mathscr{Y}$ -consistent dual utility (see [EKM13]), namely, for any  $\mathscr{Y}$ -state price density process  $Y^\nu$ , the mapping  $t \mapsto \tilde{U}(t, Y_t^\nu)$  is a submartingale and a martingale for the optimal dual strategy that we denote by  $\nu^*$ . Then the drift  $\tilde{\beta}$  is given by the HJB constraint:

$$\tilde{\beta}(t, y) = y\tilde{U}_y(t, y)r_t + \frac{1}{2\tilde{U}_{yy}(t, y)} (\|\tilde{\gamma}_y(t, y)\|^2 - \|\tilde{\gamma}_y^{\mathcal{R}}(t, y) + y\tilde{U}_{yy}(t, y)\eta_t^{\mathcal{R}}\|^2).$$

In this case, the optimal policy  $\nu^*$  is given by:

$$y\nu_t^*(y) = \gamma_z^\perp(t, -\tilde{U}_y(t, y)). \quad (2.8)$$

Then, the existence of the optimal wealth processes  $Y^* := Y^{\nu^*}$  and  $X^* := X^{\pi^*}$  is ensured by Assumption 2.7-(iii). Indeed, using Theorem 4.2 in [EKM13], the optimal process  $Y^*$  exists and is unique, moreover it is strictly positive and strictly increasing with respect to the initial wealth  $y$ . As for the SDE (2.2), it has a priori a maximal solution  $X^*$ , but indistinguishable from  $\tilde{U}_y(t, Y_t^*(u_z(x)))$  (see Theorem 2.7 [EKM13]). As a by-product, we deduce the existence

and uniqueness of the optimal wealth process  $X^*$  which is strictly increasing with respect to the initial wealth. Finally, the random field  $U$  is an consistent dynamic utility with optimal wealth  $X^*$ . This also implies the dual identity

$$Y_t^*(y) = U_z(t, X_t^*(-\tilde{u}_y(y))), \quad (2.9)$$

where  $\tilde{u}$  denotes the convex conjugate utility associated with  $u$ . A dynamic version of this identity, using flow's notations, will be useful in the derivation of the indifference price:

$$Y_t^*(s, U_z(s, \xi)) = U_z(t, X_t^*(s, \xi)), \quad \forall s \leq t \text{ and where } \xi \text{ is a } \mathcal{F}_s\text{-attainable wealth.} \quad (2.10)$$

### 3 Indifference price: definition and properties

In this section, we first provide the definition of the indifference price in terms of consistent dynamic utilities and then establish its key properties. More precisely, let  $C_T$  be a positive claim which pays a predetermined random amount at time  $T$  and assume that  $C_T$  is a  $\mathcal{F}_T$ -measurable, square-integrable random variable. Once the maturity  $T$  of the claim is fixed, all processes will be defined on  $[0, T]$  and by a slight abuse of notation,  $\mathcal{F}$  now stands for  $(\mathcal{F}_t)_{t \in [0, T]}$  the filtration up to time  $T$ . The indifference price is determined by equalizing the stochastic value functions of two different stochastic optimization problems of expected utility over the set  $\mathcal{X}(t, \xi)$  and any random variable  $\xi$  which is  $\mathcal{F}_t$ -attainable.

- $(P_p)$  - The first stochastic optimization problem represents the primal problem without the claim and whose value function is given by:

$$V(t, \xi) \triangleq \operatorname{ess\,sup}_{X^\pi \in \mathcal{X}(t, \xi)} \mathbb{E}[U(T, X_T^\pi) | \mathcal{F}_t]. \quad (P_p)$$

- $(P_c)$  - In the second stochastic optimization problem, the investor's terminal utility  $U(T, \cdot)$  is perturbed by the payment  $qC_T$  of a  $q$ -quantity of the claim  $C_T$  if they are willing to buy (for  $q > 0$ )/sell (for  $q < 0$ ) the claim, which leads to the value function:

$$V^c(t, \xi, q) \triangleq \operatorname{ess\,sup}_{X^\pi \in \mathcal{X}(t, \xi)} \mathbb{E}[U(T, X_T^\pi + qC_T) | \mathcal{F}_t]. \quad (P_c)$$

As explained in Remark 2.8, the optimization problem without claim  $(P_p)$  has been fully solved in the literature. As for the optimization problem with claim  $(P_c)$ , to the best our knowledge, there is no result that guarantees whether the supremum is actually attained, in a general setting. We will show in Theorem 4.2 that the existence of an optimal strategy for  $(P_c)$  is related to the existence of a pair of solutions to a system of coupled forward-backward stochastic differential equations (FBSDEs).

The utility indifference price is then defined as follows. The definition has already been introduced in earlier studies (see e.g. [HN89],[MZ04b]).

**Definition 3.1** (Dynamic forward indifference price). *The indifference price of a  $q$ -quantity of an  $\mathcal{F}_T$ -measurable claim  $C_T$  for  $q \neq 0$  is an  $\mathcal{F}$ -adapted process  $(p_t^q)_{t \in [0, T]}$  which provides the initial wealth that compensates for the loss/gain incurred in providing/receiving  $qC_T$ . Mathematically, it is defined by the following representation:  $\forall t \in [0, T], \forall \xi$   $\mathcal{F}_t$ -attainable wealth,*

$$V^c(t, \xi - p_t^q, q) = V(t, \xi). \quad (3.1)$$

Observe that in the optimization problem  $(P_p)$ , the supremum is taken over the set  $\mathcal{X}(t, \xi)$ . Then, as per Definition 2.6, the primal problem can be represented as  $V(t, \xi) = U(t, \xi)$ . This implies that equality (3.1) can be written as

$$V^c(t, \xi - p_t^q, q) = U(t, \xi). \quad (3.2)$$

If the quantity  $q > 0$ , the process  $p^q$  is the positive cash amount at which the buyer is equally inclined to invest optimally either in the market without the claim, or with a  $q$ -quantity of the claim  $C_T$  in their portfolio but with an initial wealth reduced by the positive amount  $p^q$ . Conversely, if the quantity  $q < 0$ , the investor is indifferent between accepting to provide the  $q$ -quantity of  $C_T$  or not. Their utility maximization can be expressed as follows (with  $-q < 0$  and  $p^{(-q)} < 0$ ):

$$V^c(t, \xi - p_t^{(-q)}, -q) = V(t, \xi) = U(t, \xi). \quad (3.3)$$

This means that the seller is equally inclined to invest optimally either in the market without the claim, or by providing a  $q$ -quantity of the claim  $C_T$  in exchange for the positive additional amount  $(-p^{-q})$ . Also, relation (3.1) can alternatively be rewritten as:

$$V^c(t, \xi, q) = U(t, \xi + p_t^q). \quad (3.4)$$

Then, representation (3.4) can be simplified as follows, where  $U^{-1}$  denotes the inverse of  $U$ :

$$V^c(t, \xi, q) = U(t, \xi + p_t^q) \Leftrightarrow p_t^q = -\xi + U^{-1}(t, V^c(t, \xi, q)). \quad (3.5)$$

The following proposition provides standard properties of the indifference price, proved here in our dynamic framework.

**Proposition 3.2.** *Let  $q$  be fixed. If there exists an  $\mathcal{F}$ -adapted process  $(p_t^q)_{t \in [0, T]}$  that satisfies the equilibrium relation (3.1), then it is unique by strict concavity of  $U$ . This means that in case of existence, there exists a unique indifference price at which the investor is indifferent between opting to purchase or abstain (resp. sell or refrain from selling) the  $q$ -quantity of the claim  $C_T$  at time  $T$ . In addition, the following properties hold:*

- (i) *Relation buy/sell indifference price: Let  $p^{-q}$  be the indifference price for the quantity  $-q$  of the claim  $C_T$  defined by (3.2). Then*

$$p^{(-q)} = -p^q. \quad (3.6)$$

(ii) *Bounds of non-arbitrage:* Let  $h^{up}(qC_T)$  and  $h^{low}(qC_T)$  be respectively the super-replicating price and the sub-replicating price of  $qC_T$  defined by:

$$\begin{cases} h^{up}(qC_T) = \inf\{x \geq 0, \exists \pi \in \mathcal{R}, X^\pi \text{ satisfies (2.2) such that } X_t^\pi = x \text{ and } X_T^\pi - qC_T > 0\}, \\ h^{low}(qC_T) = \sup\{x \geq 0, \exists \pi \in \mathcal{R}, X^\pi \text{ satisfies (2.2) such that } X_t^\pi = -x \text{ and } X_T^\pi + qC_T > 0\}. \end{cases}$$

Then the indifference price lies inside the non-arbitrage bounds:

$$h^{low}(qC_T) \leq p^q \leq h^{up}(qC_T). \quad (3.7)$$

(iii) *Monotonicity:* Let  $q' \geq q$ , then  $p^{q'} \geq p^q$ .

(iv) *Convexity:* Let  $\lambda \in [0, 1]$  and denote by  $\bar{p}^q$  and  $\tilde{p}^q$  the respective indifference prices to buy a  $q$ -quantity of the claim  $\bar{C}_T$  and  $\tilde{C}_T := \lambda C_T + (1 - \lambda)\bar{C}_T$ . Then, we have:

$$\tilde{p}^q \geq \lambda p^q + (1 - \lambda)\bar{p}^q.$$

The proof of Proposition 3.2 follows the ideas provided in [EKR00]. However, for the sake of completeness, the details of the proof in this forward setting are postponed to the appendix.

**Remark 3.3.** *The property of monotonicity (iii) can be stated differently with two different claims  $C_T$  and  $\bar{C}_T$  such that  $C_T \geq \bar{C}_T$ , then the respective indifference prices  $p^q$  and  $\bar{p}^q$  are such that  $p^q \geq \bar{p}^q$  for the same quantity  $q$ . Note also that the indifference price for  $-qC_T$  that we have denoted  $p^{(-q)}$  is concave with respect to the claim since we have  $p^{(-q)} = -p^q$ .*

## 4 Characterization of the Indifference Price and its Dynamics

In this section, we derive a characterization of the indifference price for a  $q$ -quantity of the claim  $C_T$  as a solution of a BSDE with quadratic growth. The coefficients of this BSDE depend on the consistent dynamic utility  $U$  and its derivatives, as well as on the optimal wealth process of the optimization problem  $(P_c)$ , whenever the latter exists. The proof of our main result Theorem 4.2 follows two main steps. First we proceed by verification to find the policy  $\pi^{*,q}$ , candidate to be the optimal strategy. Second, we characterize the corresponding optimal wealth process  $X^{*,q} = X^{\pi^{*,q}}$  via a system of FBSDEs. Without loss of generality<sup>2</sup>, we assume zero interest rate (that is  $r = 0$ ). Let us start by introducing the following set of assumptions, which will hold for the rest of the analysis.

**Assumption 4.1.** *Throughout this paper, we assume:*

**a)** *There exists  $p > 1$  such that  $\mathbb{E}(|U_z(T, X_T^{*,q}(x) + qC_T)|^p) < \infty$ .*

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<sup>2</sup>One could consider discounted assets.

b) For any  $\pi \in \mathcal{R}$  and  $s \in [0, T]$ , we introduce the wealth process bifurcated at time  $s$  by

$$X_{s,t}^{*,\varepsilon\pi}(x) := X_t^{*,q}(x) + \varepsilon \mathbb{1}_{t \in [s, T]} \int_s^t \pi_u (dW_u + \eta_u^{\mathcal{R}} du),$$

and we assume that the random variable

$$\frac{1}{\varepsilon} \left( U(T, X_{s,T}^{*,\varepsilon\pi}(x) + qC_T) - U(T, X_T^{*,q}(x) + qC_T) \right)$$

is uniformly integrable in  $\varepsilon \in (0, 1)$ .

#### 4.1 Identification of the optimal strategy

The first step consists in determining the optimal strategy  $\pi^{*,q}$  in terms of the optimal strategy  $\pi^* = \pi^{*,0}$  of the primal problem ( $P_p$ ) (without claim) and with an appropriate shift process  $\zeta^q$  which is added to the current wealth process. This  $\mathcal{F}$ -adapted shift process  $\zeta^q$  equals  $qC_T$  at time  $T$  and for any  $t \in [0, T]$  it satisfies

$$U_z(t, X_t^{*,q}(x) + \zeta_t^q) = \mathbb{E}(U_z(T, X_T^{*,q}(x) + qC_T) | \mathcal{F}_t). \quad (4.1)$$

We emphasize here that  $(\zeta_t^q)_t$  can be interpreted as the certainty equivalent of  $qC_T$  associated with the marginal utility  $U_z$ . It is given by

$$\zeta_t^q = U_z^{-1} \left( t, \mathbb{E}(U_z(T, X_T^{*,q}(x) + qC_T) | \mathcal{F}_t) \right) - X_t^{*,q}(x), \quad t \in [0, T]. \quad (4.2)$$

The joint dynamics of  $(X^{*,q}, \zeta^q)$  is explicitly given in the following theorem, as it plays a fundamental role in determining the dynamics of the indifference price.

In what follows, for the sake of notational simplicity, we omit the dependence of the wealth processes on their initial condition  $x$  at time  $t = 0$ , except when this dependence is relevant. Moreover, to adopt consistent notations for the claims size  $q$ , the optimal processes  $\pi^*$  and  $X^*$  of the problem without claims ( $q = 0$ ) will be denoted by  $\pi^{*,0}$  and  $X^{*,0}$ .

**Theorem 4.2.** *Under Assumptions 2.7 and 4.1, the optimal strategy for the problem ( $P_c$ ) is given by*

$$\begin{aligned} \pi^{*,q}(t, X_t^{*,q}) &= - \frac{U_z(t, X_t^{*,q} + \zeta_t^q) \eta_t^{\mathcal{R}} + \gamma_z^{\mathcal{R}}(t, X_t^{*,q} + \zeta_t^q)}{U_{zz}(t, X_t^{*,q} + \zeta_t^q)} - \sigma_t^{\zeta, \mathcal{R}} \\ &= \pi^{*,0}(t, X_t^{*,q} + \zeta_t^q) - \sigma_t^{\zeta, \mathcal{R}}, \end{aligned} \quad (4.3)$$

whenever there exists a solution  $(X^{*,q}, \zeta^q, \sigma^\zeta)$  to the following FBSDE: for  $0 \leq t \leq T$ ,

$$\begin{cases} dX_t^{*,q} &= (\pi^{*,0}(t, X_t^{*,q} + \zeta_t^q) - \sigma_t^{\zeta, \mathcal{R}}) (dW_t + \eta_t^{\mathcal{R}} dt), \quad X_0^{*,q} = x \\ d\zeta_t^q &= \left\{ \left[ \frac{\gamma_{zz}^\perp}{U_{zz}} \sigma_t^{\zeta, \perp} - \frac{1}{2} U_{zz} \|\sigma_t^{\zeta, \perp}\|^2 \partial_z \left( \frac{1}{U_{zz}} \right) \right] (t, X_t^{*,q} + \zeta_t^q) + \sigma_t^{\zeta, \mathcal{R}} \eta_t^{\mathcal{R}} \right\} dt + \sigma_t^\zeta dW_t, \\ \zeta_T^q &= qC_T. \end{cases} \quad (4.4)$$

Conversely, the existence of a solution  $(X^{*,q}, \zeta^q, \sigma^\zeta)$  to the coupled system (4.4) ensures that  $X^{*,q}$  is the optimal wealth process for the optimization problem with claim ( $P_c$ ).

The system (4.4) is a coupled FBSDE which, to the best of our knowledge, has not been studied in the literature. The complexity arises from the fact that the diffusion term of  $X^{*,q}$  depends on the solution  $\zeta^q$  of the backward equation, as well as on its own diffusion term  $\sigma^\zeta, \mathcal{R}$ . Moreover, the driver of the backward component exhibits quadratic growth. As a consequence, existing results cannot be applied. The existence and uniqueness of solution of the FBSDE (4.4) remains an open question, which we will investigate in a separate work given the challenges it entails. We check that for replicable claims, the solution of the FBSDE leads to an indifference price which coincides with the value of the replicating portfolio, as we shall see in the next paragraph. We identify in Section 5 a tractable setting in which the FBSDE (4.4) can be solved explicitly, namely consistent dynamic exponential utility, allowing us to derive explicit dynamics for the indifference price.

*Proof.* The proof is carried out in three parts. The first part consists in characterizing the optimal strategy  $\pi^{*,q}$ , if it exists. It is characterized through the process  $\zeta^q$  defined by (4.2). The second part derives the FBSDE satisfied by  $(X^{*,q}, \zeta^q, \sigma^\zeta)$  in order to ensure the existence of the optimal strategy, where  $\sigma^\zeta$  is an  $\mathcal{F}$ -adapted process that denotes the volatility of  $\zeta^q$  (see for instance Horst et al. [HHI<sup>+</sup>14], whose approach is similar). The third part proves the converse implication.

#### Part 1: Optimal strategy

- We provide a first relation satisfied by the optimal strategy.

Let  $\varepsilon \in (0, 1)$ ,  $\pi \in \mathcal{R}$  and  $0 \leq s \leq T$ . The process bifurcated at time  $s$ ,

$$X_{s,t}^{*,\varepsilon\pi} := X_t^{*,q} + \varepsilon \mathbb{1}_{t \in [s, T]} \int_s^t \pi_u (dW_u + \eta_u^{\mathcal{R}} du),$$

is an admissible wealth starting from  $X_s^{*,q}$  at time  $s$  and such that

$$X_{s,t}^{*,\varepsilon\pi} = X_t^{*,q} + \int_s^t (\pi_u^{*,q} + \varepsilon \mathbb{1}_{t \in [s, T]} \pi_u) (dW_u + \eta_u^{\mathcal{R}} du).$$

By differentiability of  $U$ , there exists some  $\bar{\alpha} \in [0, 1]$  such that

$$\begin{aligned} U(T, X_{s,T}^{*,\varepsilon\pi} + qC_T) - U(T, X_T^{*,q} + qC_T) \\ &= \varepsilon \int_0^1 U_z(T, X_{s,T}^{*,\alpha\varepsilon\pi} + qC_T) d\alpha \int_s^T \pi_u (dW_u + \eta_u^{\mathcal{R}} du) \\ &= \varepsilon U_z(T, X_{s,T}^{*,\bar{\alpha}\varepsilon\pi} + qC_T) \int_s^T \pi_u (dW_u + \eta_u^{\mathcal{R}} du). \end{aligned}$$

Then taking the expectation, by the optimality of  $X^{*,q}$  we have

$$\mathbb{E} \left( U_z(T, X_{s,T}^{*,\bar{\alpha}\varepsilon\pi} + qC_T) \int_s^T \pi_u (dW_u + \eta_u^{\mathcal{R}} du) | \mathcal{F}_s \right) \leq 0,$$

which, when  $\varepsilon$  tends to zero and using the dominated convergence theorem combined with uniform integrability (Assumption 4.1-b), leads to

$$\mathbb{E} \left( U_z(T, X_T^{*,q} + qC_T) \int_s^T \pi_u (dW_u + \eta_u^{\mathcal{R}} du) | \mathcal{F}_s \right) \leq 0.$$

Same inequality holds also for  $(-\pi)$ . Therefore for any  $\pi \in \mathcal{R}$  and  $0 \leq s \leq T$

$$\mathbb{E}\left(U_z(T, X_T^{*,q} + qC_T) \int_s^T \pi_u(dW_u + \eta_u^{\mathcal{R}} du) | \mathcal{F}_s\right) = 0,$$

which can be written as

$$\mathbb{E}(M_{T,T} A_{s,T}^\pi | \mathcal{F}_s) = 0, \text{ for any } s \leq T \text{ and } \pi \in \mathcal{R}, \quad (4.5)$$

where for  $0 \leq s \leq t \leq T, \pi \in \mathcal{R}$ :

$$A_{s,t}^\pi := \int_s^t \pi_u(dW_u + \eta_u^{\mathcal{R}} du), \quad (4.6)$$

$$M_{s,T} := \mathbb{E}\left(U_z(T, X_T^{*,q} + qC_T) | \mathcal{F}_s\right) = U_z(s, X_s^{*,q} + \zeta_s^q) \quad (4.7)$$

by definition (4.1) of the process  $\zeta^q$ .

- Characterization of the dynamics of  $(M_{s,T})_s$ .

Thanks to the integrability Assumption 4.1-a),  $M$  is a martingale. This implies that there exists an  $\mathcal{F}$ -adapted process  $Z$ , that has to be determined, such that  $\mathbb{E}\left[\int_0^T |Z_s|^2 ds\right] < \infty$  and

$$dM_{t,T} = Z_t dW_t, \quad t \in [0, T]. \quad (4.8)$$

Furthermore, the process  $\zeta^q$  satisfies

$$\zeta_t^q = U_z^{-1}(t, M_{t,T}) - X_t^{*,q}, \quad t \in [0, T], \text{ with terminal condition } \zeta_T^q = qC_T. \quad (4.9)$$

Let us recall the dynamics of  $U_z^{-1} = -\tilde{U}_y$  (see [EKM13, Theorem 2.8])

$$\begin{cases} d\tilde{U}_y(t, y) &= \frac{1}{2} \partial_y \left[ \frac{1}{\tilde{U}_{yy}} (|\tilde{\gamma}_y|^2 - \|\tilde{\gamma}_y^{\mathcal{R}} - y\tilde{U}_{yy}\eta_t^{\mathcal{R}}\|^2) \right] (t, y) dt + \tilde{\gamma}_y(t, y) dW_t, \\ \tilde{\gamma}(t, y) &= \gamma(t, -\tilde{U}_y(t, y)). \end{cases}$$

Thus, applying Itô-Ventzel's formula (see [Ven65]),

$$\begin{aligned} d\tilde{U}_y(t, M_{t,T}) &= \frac{1}{2} \left\{ \partial_y \left[ \frac{1}{\tilde{U}_{yy}} (|\tilde{\gamma}_y|^2 - \|\tilde{\gamma}_y^{\mathcal{R}} - y\tilde{U}_{yy}\eta_t^{\mathcal{R}}\|^2) \right] + \tilde{U}_{yyy} \|Z_t\|^2 + 2\tilde{\gamma}_{yy} Z_t \right\} (t, M_{t,T}) dt \\ &+ \left( \tilde{\gamma}_y + \tilde{U}_{yy} Z_t \right) (t, M_{t,T}) dW_t. \end{aligned} \quad (4.10)$$

This, together with (4.9) implies that the volatility component of the process  $\zeta^q$  is given by:

$$\sigma_t^\zeta = -\pi^{*,q}(t, X_t^{*,q}) - \tilde{\gamma}_y(t, M_{t,T}) - \tilde{U}_{yy}(t, M_{t,T}) Z_t.$$

Therefore, the optimal strategy  $\pi^{*,q}$  is characterized as the projection onto  $\mathcal{R}$  of

$-\left(\tilde{\gamma}_y(t, M_{t,T}) + \tilde{U}_{yy}(t, M_{t,T}) Z_t + \sigma_t^\zeta\right)$ , and the diffusion coefficient of the martingale  $M$  is

$$Z_t = -\frac{\pi^{*,q}(t, X_t^{*,q}) + \tilde{\gamma}_y(t, M_{t,T}) + \sigma_t^\zeta}{\tilde{U}_{yy}(t, M_{t,T})}. \quad (4.11)$$

Besides, we have from the definition of the dual  $\tilde{U}$ ,

$$-\tilde{U}_y(t, M_{t,T}) = X_t^{*,q} + \zeta_t^q \quad \text{and} \quad \tilde{\gamma}(t, y) = \gamma(t, -\tilde{U}_y(t, y))$$

which yields

$$\tilde{U}_{yy}(t, M_{t,T}) = \frac{-1}{U_{zz}(t, X_t^{*,q} + \zeta_t^q)} \quad \text{and} \quad \tilde{\gamma}_y(t, M_{t,T}) = \frac{\gamma_z(t, X_t^{*,q} + \zeta_t^q)}{U_{zz}(t, X_t^{*,q} + \zeta_t^q)}.$$

Plugging this into (4.11) yields the following dynamics for  $M$

$$dM_{t,T} = \left( U_{zz}(t, X_t^{*,q} + \zeta_t^q) (\sigma_t^\zeta + \pi^{*,q}(t, X_t^{*,q})) + \gamma_z(t, X_t^{*,q} + \zeta_t^q) \right) dW_t. \quad (4.12)$$

• We now determine the optimal strategy  $\pi^{*,q}$ .

For  $0 \leq s \leq T$ , recall the processes  $(A_{s,t}^\pi)_{s \leq t}$  and  $(M_{t,T})_{t \leq T}$  defined by (4.6) and (4.7). Then, by applying Itô's formula to the process  $(M_{t,T} A_{s,t}^\pi)_{t \in [s, T]}$

$$\begin{aligned} d(M_{t,T} A_{s,t}^\pi) &= \left[ A_{s,t}^\pi \left( U_{zz}(t, X_t^* + \zeta_t^q) (\sigma_t^\zeta + \pi^{*,q}(t, X_t^{*,q})) + \gamma_z(t, X_t^{*,q} + \zeta_t^q) \right) + M_{t,T} \pi_t \right] dW_t \\ &\quad + \left( U_{zz}(t, X_t^{*,q} + \zeta_t^q) (\sigma_t^\zeta + \pi^{*,q}(t, X_t^{*,q})) + \gamma_z(t, X_t^{*,q} + \zeta_t^q) + \eta_t^{\mathcal{R}} M_{t,T} \right) \pi_t dt. \end{aligned}$$

Since  $A_{s,s}^\pi = 0$  and in view of (4.5), integrating between  $s$  and  $T$  and taking the conditional expectation yield, for any  $\pi \in \mathcal{R}$  and for any  $s \in [0, T]$

$$\begin{aligned} &\mathbb{E}(M_{T,T} A_{s,T}^\pi | \mathcal{F}_s) \\ &= \mathbb{E} \left( \int_s^T \left( U_{zz}(t, X_t^{*,q} + \zeta_t^q) (\sigma_t^\zeta + \pi^{*,q}(t, X_t^{*,q})) + \gamma_z(t, X_t^{*,q} + \zeta_t^q) + M_{t,T} \eta_t^{\mathcal{R}} \right) \pi_t dt | \mathcal{F}_s \right) \\ &= 0. \end{aligned}$$

This implies that for any  $t \in [s, T]$

$$\left[ U_{zz}(t, X_t^{*,q} + \zeta_t^q) (\sigma_t^\zeta + \pi^{*,q}(t, X_t^{*,q})) + \gamma_z(t, X_t^{*,q} + \zeta_t^q) + M_{t,T} \eta_t^{\mathcal{R}} \right]^{\mathcal{R}} = 0. \quad (4.13)$$

Using  $M_{t,T} = U_z(t, X_t^{*,q} + \zeta_t^q)$  and projecting onto  $\mathcal{R}$  yields

$$\begin{aligned} \pi^{*,q}(t, X_t^{*,q}) &= - \frac{U_z(t, X_t^{*,q} + \zeta_t^q) \eta_t^{\mathcal{R}} + \gamma_z^{\mathcal{R}}(t, X_t^{*,q} + \zeta_t^q)}{U_{zz}(t, X_t^{*,q} + \zeta_t^q)} - \sigma_t^{\zeta, \mathcal{R}} \\ &= \pi^{*,0}(t, X_t^{*,q} + \zeta_t^q) - \sigma_t^{\zeta, \mathcal{R}}, \end{aligned} \quad (4.14)$$

which characterizes the optimal strategy for the problem  $(P_c)$ .

### Part 2: Dynamics of $\zeta^q$

As  $\zeta_t^q = U_z^{-1}(t, M_{t,T}) - X_t^{*,q}$  (see (4.9)), it follows from the dynamics of  $X^{*,q}$  and  $\tilde{U}_y(t, M_{t,T})$  given by (4.10), that  $d\zeta_t^q = \mu_t^\zeta dt + \sigma_t^\zeta dW_t$  with

$$\begin{aligned} \mu_t^\zeta &:= -\pi^{*,q}(t, X_t^{*,q}) \eta_t^{\mathcal{R}} - \frac{1}{2} \left\{ \partial_y \left[ \frac{1}{\tilde{U}_{yy}} (\|\tilde{\gamma}_y\|^2 - \|\tilde{\gamma}_y^{\mathcal{R}} - y \tilde{U}_{yy} \eta_t^{\mathcal{R}}\|^2) \right] + \tilde{U}_{yyy} \|Z_t\|^2 + 2\tilde{\gamma}_{yy} Z_t \right\} (t, M_{t,T}) \\ \sigma_t^\zeta &:= (-\pi^{*,q}(t, X_t^{*,q}) - (\tilde{\gamma}_y + \tilde{U}_{yy} Z_t)) (t, M_{t,T}), \end{aligned} \quad (4.15)$$

where  $Z_t$  can be rewritten as, using (4.12) and (4.13),

$$\begin{aligned} Z_t &= -U_z(t, X_t^{*,q} + \zeta_t^q) \eta_t^{\mathcal{R}} + \gamma_z^\perp(t, X_t^{*,q} + \zeta_t^q) + U_{zz}(t, X_t^{*,q} + \zeta_t^q) \sigma_t^{\zeta, \perp} \\ &= \left[ U_z \left( \frac{\gamma_z^\perp}{U_z} + \frac{U_{zz}}{U_z} \sigma_t^{\zeta, \perp} - \eta_t^{\mathcal{R}} \right) \right] (t, X_t^{*,q} + \zeta_t^q). \end{aligned} \quad (4.16)$$

To simplify the expression of the drift  $\mu_t^\zeta$  given in (4.15), we compute each of its terms using the following identities for the dual utility  $\tilde{U}$

$$-\tilde{U}_y(t, U_z(t, z)) = z, \quad \tilde{U}_{yy}(t, U_z(t, z)) = \frac{-1}{U_{zz}(t, z)}, \quad \tilde{U}_{yyy}(t, U_z(t, z)) = \frac{U_{zzz}(t, z)}{U_{zz}^3(t, z)}, \quad (4.17)$$

$$\tilde{\gamma}_y(t, y) = -\tilde{U}_{yy}(t, y) \gamma_z(t, -\tilde{U}_y(t, y)) \text{ and } \forall f, g, f_y \circ g(z) = \frac{(f \circ g)_z(z)}{g_z(z)}. \quad (4.18)$$

So, on the one hand

$$\begin{aligned} & \partial_y \left[ \frac{1}{\tilde{U}_{yy}(t, y)} (\|\tilde{\gamma}_y(t, y)\|^2 - \|\tilde{\gamma}_y^{\mathcal{R}}(t, y) - y \tilde{U}_{yy}(t, y) \eta_t^{\mathcal{R}}\|^2) \right] \circ U_z(t, z) \\ &= \frac{1}{U_{zz}(t, z)} \partial_z \left[ -U_{zz} (\|\frac{\gamma_z^\perp}{U_{zz}}\|^2 - \|\frac{\gamma_z^{\mathcal{R}}}{U_{zz}} + \frac{U_z}{U_{zz}} \eta_t^{\mathcal{R}}\|^2) \right] (t, z) \\ &= \frac{1}{U_{zz}(t, z)} \left[ -\partial_z (U_{zz} \|\frac{\gamma_z^\perp}{U_{zz}}\|^2) + \partial_z [2U_{zz} \frac{\gamma_z^{\mathcal{R}}}{U_{zz}} \frac{U_z}{U_{zz}} \eta_t^{\mathcal{R}} + U_{zz} (\frac{U_z}{U_{zz}})^2 \|\eta_t^{\mathcal{R}}\|^2] \right] (t, z) \\ &= \left[ -\frac{1}{U_{zz}} \partial_z (\|\frac{\gamma_z^\perp}{U_{zz}}\|^2) + \frac{2}{U_{zz}} \partial_z [\frac{\gamma_z^{\mathcal{R}}}{U_{zz}} U_z] \eta_t^{\mathcal{R}} + \frac{1}{U_{zz}} \partial_z [\frac{U_z^2}{U_{zz}}] \|\eta_t^{\mathcal{R}}\|^2 \right] (t, z). \end{aligned} \quad (4.19)$$

On the other hand, we can easily observe that

$$\tilde{U}_{yyy} \circ U_z(t, z) = \left( \frac{U_{zzz}}{U_{zz}^3} \right) (t, z) = -\left( \frac{1}{U_{zz}} \partial \left( \frac{1}{U_{zz}} \right) \right) (t, z), \quad (4.20)$$

$$2(\tilde{\gamma}_{yy}) \circ U_z(t, z) = 2 \left( \frac{1}{U_{zz}} \partial_z \left( \frac{\gamma_z}{U_{zz}} \right) \right) (t, z). \quad (4.21)$$

Then, since  $U_z(t, X_t^*(x) + \zeta_t^q) = M_{t,T}$  and by recalling (4.16) along with (4.19), (4.20) and (4.21), we deduce that

$$\begin{aligned} & \left\{ \partial_y \left[ \frac{1}{\tilde{U}_{yy}} (\|\tilde{\gamma}_y\|^2 - \|\tilde{\gamma}_y^{\mathcal{R}} - y \tilde{U}_{yy} \eta_t^{\mathcal{R}}\|^2) + \tilde{U}_{yyy} \|Z_t\|^2 + 2\tilde{\gamma}_{yy} Z_t \right] (t, M_{t,T}) \right. \\ &= \left[ -\frac{1}{U_{zz}} \partial_z (\|\frac{\gamma_z^\perp}{U_{zz}}\|^2) + \frac{2}{U_{zz}} \partial_z [\frac{\gamma_z^{\mathcal{R}}}{U_{zz}} U_z] \eta_t^{\mathcal{R}} + \frac{1}{U_{zz}} \partial_z [\frac{U_z^2}{U_{zz}}] \|\eta_t^{\mathcal{R}}\|^2 \right] - \left( \frac{1}{U_{zz}} \partial \left( \frac{1}{U_{zz}} \right) \|Z_t\|^2 \right) \\ &+ \left. \left( 2 \frac{1}{U_{zz}} \partial_z \left( \frac{\gamma_z}{U_{zz}} \right) Z_t \right) \right\} (t, X_t^{*,q}(x) + \zeta_t^q) \\ &= \left[ -\frac{\|\gamma_z^\perp\|^2}{U_{zz}} \partial_z \left( \frac{1}{U_{zz}} \right) - 2 \frac{\gamma_z^\perp \gamma_{zz}^\perp}{U_{zz}^2} + 2 \frac{\gamma_z^{\mathcal{R}}}{U_{zz}} \eta_t^{\mathcal{R}} + 2 \frac{U_z}{U_{zz}^2} \gamma_{zz}^{\mathcal{R}} \eta_t^{\mathcal{R}} + 2 \frac{U_z}{U_{zz}} \gamma_z^{\mathcal{R}} \eta_t^{\mathcal{R}} \partial_z \left( \frac{1}{U_{zz}} \right) + 2 \frac{U_z}{U_{zz}} \|\eta_t^{\mathcal{R}}\|^2 \right. \\ &+ \frac{U_z^2}{U_{zz}} \|\eta_t^{\mathcal{R}}\|^2 \partial_z \left( \frac{1}{U_{zz}} \right) - \frac{1}{U_{zz}} \|\gamma_z^\perp\|^2 \partial_z \left( \frac{1}{U_{zz}} \right) - 2 \gamma_z^\perp \sigma_t^{\zeta, \perp} \partial_z \left( \frac{1}{U_{zz}} \right) - U_{zz} \|\sigma_t^{\zeta, \perp}\|^2 \partial_z \left( \frac{1}{U_{zz}} \right) \\ &- \frac{U_z^2}{U_{zz}} \|\eta_t^{\mathcal{R}}\|^2 \partial_z \left( \frac{1}{U_{zz}} \right) + 2 \frac{\gamma_z^\perp \gamma_{zz}^\perp}{U_{zz}^2} + 2 \frac{\gamma_{zz}^\perp \sigma_t^{\zeta, \perp}}{U_{zz}} - 2 \frac{U_z}{U_{zz}^2} \gamma_{zz}^{\mathcal{R}} \eta_t^{\mathcal{R}} + 2 \frac{\|\gamma_z^\perp\|^2}{U_{zz}} \partial_z \left( \frac{1}{U_{zz}} \right) \\ &\left. + 2 \gamma_z^\perp \sigma_t^{\zeta, \perp} \partial_z \left( \frac{1}{U_{zz}} \right) - 2 \frac{U_z}{U_{zz}} \gamma_z^{\mathcal{R}} \eta_t^{\mathcal{R}} \partial_z \left( \frac{1}{U_{zz}} \right) \right] (t, X_t^{*,q}(x) + \zeta_t^q) \end{aligned}$$

which simplifies to

$$\begin{aligned} & \left[ 2 \frac{\gamma_z^{\mathcal{R}} + U_z \eta_t^{\mathcal{R}}}{U_{zz}} \eta_t^{\mathcal{R}} + 2 \frac{\gamma_{zz}^\perp}{U_{zz}} \sigma^{\zeta, \perp} - U_{zz} \|\sigma_t^{\zeta, \perp}\|^2 \partial_z \left( \frac{1}{U_{zz}} \right) \right] (t, X_t^{*,q}(x) + \zeta_t^q) \\ & \stackrel{(4.14)}{=} -2(\pi^{*,q}(t, X_t^{*,q}) + \sigma_t^{\zeta^q, \mathcal{R}}) \eta_t^{\mathcal{R}} + \left[ 2 \frac{\gamma_{zz}^\perp}{U_{zz}} \sigma^{\zeta, \perp} - U_{zz} \|\sigma_t^{\zeta, \perp}\|^2 \partial_z \left( \frac{1}{U_{zz}} \right) \right] (t, X_t^{*,q}(x) + \zeta_t^q). \end{aligned}$$

Therefore, the dynamics of  $\zeta^q$  are given by

$$d\zeta_t^q = \left\{ \left[ \frac{\gamma_{zz}^\perp}{U_{zz}} \sigma^{\zeta, \perp} - \frac{1}{2} U_{zz} \|\sigma^{\zeta, \perp}\|^2 \partial_z \left( \frac{1}{U_{zz}} \right) \right] (t, X_t^{*,q}(x) + \zeta_t^q) + \sigma_t^{\zeta, \mathcal{R}} \eta_t^{\mathcal{R}} \right\} dt + \sigma_t^\zeta dW_t.$$

This, together with the terminal condition  $\zeta_T^q = qC_T$  and the self-financing dynamics of  $X_t^{*,q}$ , leads to the FSBDE (4.4).

**Part 3:** Conversely, assume that  $(X^{*,q}, \zeta^q, \sigma^\zeta)$  is a solution to the coupled system (4.4).  $X^{*,q}$  is an admissible wealth associated to the policy  $\pi^{*,q}$  given by (4.3). On the one hand,  $\pi^{*,q}$  is the unique admissible strategy that makes the expression (4.13) equal to zero. This implies that

$$\mathbb{E}(M_{T,T} A_{s,T}^\pi | \mathcal{F}_s) = 0, \quad \forall s \leq T \text{ and any } \pi \in \mathcal{R}, \quad (4.22)$$

where  $(A_{s,t}^\pi)_{s \leq t}$  and  $(M_{s,T})_{s \leq T}$  are defined in (4.6) and (4.7) respectively. On the other hand, consider an admissible strategy  $\pi \in \mathcal{R}$  and let  $X^\pi$  be the associated wealth process. Denote the  $\mathcal{F}_s$ -attainable wealth  $\xi := X_s^\pi$  and  $X_t^{*,q}(s, \xi)$  the solution of the forward equation of (4.4), which starts at time  $s$  from the  $\mathcal{F}_s$ -measurable random variable  $\xi$ . Then by concavity and differentiability of  $U$ , we have

$$\begin{aligned} U(T, X_T^\pi + qC_T) - U(T, X_T^{*,q}(s, \xi) + qC_T) & \leq U_z(T, X_T^{*,q}(s, \xi) + qC_T) \left( X_T^\pi(x) - X_T^{*,q}(s, \xi) \right) \\ & \leq U_z(T, X_T^{*,q}(s, \xi) + qC_T) \int_s^T (\pi_t - \pi_t^{*,q}) (dW_t + \eta_t^{\mathcal{R}} dt) \\ & = M_{T,T} A_{s,T}^{\pi - \pi^{*,q}}. \end{aligned}$$

Taking the conditional expectation with respect to  $\mathcal{F}_s$  and using (4.22), we get

$$\mathbb{E} \left( U(T, X_T^\pi + qC_T) - U(T, X_T^{*,q}(s, \xi) + qC_T) | \mathcal{F}_s \right) \leq \mathbb{E} \left( M_{T,T} A_{s,T}^{\pi - \pi^{*,q}} | \mathcal{F}_s \right) = 0,$$

that is

$$\mathbb{E} \left( U(T, X_T^\pi + qC_T) | \mathcal{F}_s \right) \leq \mathbb{E} \left( U(T, X_T^{*,q}(s, \xi) + qC_T) | \mathcal{F}_s \right), \text{ for any } \pi \in \mathcal{R}.$$

This proves the optimality of  $\pi^{*,q}$  and concludes the proof.  $\square$

This theorem allows us to obtain the following characterization of the indifference price of the claim  $qC_T$ .

**Corollary 4.3.** *Assume 2.7 and 4.1, and let  $X^{*,q}$  denote the optimal wealth process associated with the optimization problem  $(P_c)$  if it exists. Then, based on (3.5), the indifference price is given as follows:*

$$p_t^q := p(t, X_t^{*,q}, T, q) = -X_t^{*,q} + U^{-1}(t, \mathbb{E}(U(T, X_T^{*,q} + qC_T) | \mathcal{F}_t)).$$

As a by-product, we have the following characterization:

$$p_t^q = -X_t^{*,q} + U^{-1}\left(t, U(t, X_t^{*,q} + \zeta_t^q) + \mathbb{E}\left(\int_t^T \psi(s, X_s^{*,q}, \zeta_s^q) ds | \mathcal{F}_t\right)\right), \quad (4.23)$$

with

$$\psi(s, x, y) := \left[\frac{1}{2}U_{zz}\|\sigma_s^{\zeta, \perp}\|^2(1 - U_z\partial_z(\frac{1}{U_{zz}})) + \sigma_s^{\zeta, \perp}\left(\frac{U_z}{U_{zz}}\gamma_{zz}^\perp + \gamma_z^\perp\right)\right](s, x + y). \quad (4.24)$$

*Proof.* Applying Itô-Ventzel's formula, and using the (HJB) constraint (2.5) along with the dynamics of  $X^{*,q}$  and  $\zeta^q$ , we get

$$\begin{aligned} dU(t, X_t^{*,q} + \zeta_t^q) &= \left[\frac{1}{2}U_{zz}\|\sigma_t^{\zeta, \perp}\|^2(1 - U_z\partial_z(\frac{1}{U_{zz}})) + \sigma_t^{\zeta, \perp}\left(\frac{U_z}{U_{zz}}\gamma_{zz}^\perp + \gamma_z^\perp\right)\right](t, X_t^{*,q} + \zeta_t^q)dt \\ &\quad + \left[\gamma_z + U_z\pi^{*,0} + \gamma_z^\perp\sigma_t^{\zeta, \perp}\right](t, X_t^{*,q} + \zeta_t^q)dW_t. \end{aligned} \quad (4.25)$$

Under Assumptions 2.7 and 4.1, we integrate between  $t$  and the horizon  $T$  and take the conditional expectation, it then follows that the value function of  $(P_c)$  is

$$\mathbb{E}(U(T, X_T^{*,q} + qC_T) | \mathcal{F}_t) = U(t, X_t^{*,q} + \zeta_t^q) + \mathbb{E}\left[\int_t^T \psi(s, X_s^{*,q}, \zeta_s^q) ds | \mathcal{F}_t\right],$$

where  $\psi$  is defined in (4.24). Using representation (3.5) completes the proof.  $\square$

## 4.2 The case where $C_T$ is replicable

When the claim  $C_T$  is replicable, it is known that the indifference price corresponds to the value of the replicating portfolio. In this setting, by solving explicitly the FBSDE (4.4) we show that the associated indifference price coincides indeed with the cost of the hedging strategy.

**Proposition 4.4.** *Assume that the claim  $C_T$  is replicable, and denote  $\pi^C$  the replicating strategy and  $(C_t = X_t^{\pi^C})_{t \in [0, T]}$  the replicating portfolio value process. Then the solution of the FBSDE (4.4) is  $(X^{*,q}(x) = X^{*,0}(x + qC_0) - qC, \zeta^q = qC, \sigma^\zeta = q\pi^C)$  and the indifference price is  $(p_t^q = \zeta_t^q = qC_t)_{0 \leq t \leq T}$ .*

*Proof.* Assume that the claim  $C_T$  is replicable, i.e., there exists an admissible strategy  $\pi^C \in \mathcal{R}$  such that

$$X_s^{\pi^C} = C_s, \quad \text{for all } s \in [0, T].$$

Moreover, for any  $\pi \in \mathcal{R}$ , define  $\tilde{\pi}^C := \pi + \pi^C \in \mathcal{R}$  since  $\mathcal{R}$  is a linear space. Then  $X^{\tilde{\pi}^C}(s, X_s + qC_s)$  is an admissible wealth process starting at time  $s$  from the initial condition  $X_s + qC_s$ , where  $X_s$  is an  $\mathcal{F}_s$ -attainable wealth. Consequently,

$$\begin{aligned} V^c(s, X_s, q) &= \operatorname{ess\,sup}_{X^\pi \in \mathcal{X}(s, X_s)} \mathbb{E}[U(T, X_T^\pi + qC_T) \mid \mathcal{F}_s] \\ &= \operatorname{ess\,sup}_{X^\pi \in \mathcal{X}(s, X_s + qC_s)} \mathbb{E}[U(T, X_T^\pi) \mid \mathcal{F}_s] \\ &= V(s, X_s + qC_s). \end{aligned}$$

Therefore, the optimal wealth  $X^{*,q}$  for the problem with claim  $(P_c)$  satisfies

$$X_T^{*,q}(s, X_s) + qC_T = X_T^{*,0}(s, X_s + qC_s), \forall 0 \leq s \leq T, \quad (4.26)$$

and by absence of arbitrage, we get  $X_t^{*,q}(s, X_s) + qC_t = X_t^{*,0}(s, X_s + qC_s)$  for any  $0 \leq s \leq t \leq T$ . Besides, (2.10) implies

$$Y_t^*(s, U_z(s, X_s + qC_s)) = U_z\left(s, X_t^{*,0}(s, X_s + qC_s)\right), \quad t \in [s, T],$$

which defines a positive local martingale with integrable terminal value, and is thus a martingale. Then, from the definition (4.2) of  $\zeta^q$  and using the above observation, we obtain

$$\begin{aligned} \zeta_s^q &= U_z^{-1}\left(s, \mathbb{E}[U_z(T, X_T^{*,q}(s, X_s) + qC_T) \mid \mathcal{F}_s]\right) - X_s \\ &\stackrel{(4.26)}{=} U_z^{-1}\left(t, \mathbb{E}[U_z(T, X_T^{*,0}(s, X_s + qC_s)) \mid \mathcal{F}_s]\right) - X_s \\ &= U_z^{-1}\left(s, \mathbb{E}[Y_T^*(s, U_z(s, X_s + qC_s)) \mid \mathcal{F}_s]\right) - X_s \\ &= U_z^{-1}\left(s, Y_s^*(s, U_z(s, X_s + qC_s))\right) - X_s \\ &= U_z^{-1}\left(s, U_z(s, X_s + qC_s)\right) - X_s \\ &= qC_s, \quad s \in [0, T]. \end{aligned}$$

This implies in particular that  $\sigma^{\zeta, \mathcal{R}} = q\pi^C$  and  $\sigma^{\zeta, \perp} = 0$ . On the one hand, this shows that the FBSDE (4.4) reduces to the following decoupled system

$$\begin{cases} dX_t^{*,q} &= (\pi^{*,0}(t, X_t^{*,q} + qC_t) - q\pi_t^C)(dW_t + \eta_t^{\mathcal{R}} dt), \quad X_0^{*,q} = x \\ d(qC_t) &= q\pi_t^C(dW_t + \eta_t^{\mathcal{R}} dt), \quad t \leq T \end{cases} \quad (4.27)$$

whose terminal condition is  $qC_T$ . On the other hand, the function  $\psi$  given by (4.24) is then equal to zero since  $\sigma^{\zeta, \perp} = 0$ , hence the indifference price is given by

$$p_t^q = -X_t^{*,q} + U^{-1}\left(t, U(t, X_t^{*,q} + \zeta_t^q) + \mathbb{E}\left(\int_t^T \psi(s, X_s^{*,q}, \zeta_s^q) ds \mid \mathcal{F}_t\right)\right) = \zeta_t^q = qC_t. \quad (4.28)$$

□

### 4.3 Dynamics of the Indifference Price

In this subsection, we derive the dynamics of the indifference price process  $p_t^q = p(t, T, X_t^{*,q}, q)$ . Recalling its definition (3.4), we have

$$U(t, X_t^{*,q} + p_t^q) = V^c(t, X_t^{*,q}, q) = \mathbb{E}(U(T, X_T^{*,q} + qC_T) | \mathcal{F}_t). \quad (4.29)$$

**Theorem 4.5.** *We assume 2.7 and 4.1, and that a solution  $(X^{*,q}, \zeta^q, \sigma^\zeta)$  of the FBSDE (4.4) exists. Then the indifference price  $p^q = p(t, T, X_t^{*,q}, q)$  is well defined and it satisfies the following non-linear quadratic BSDE,*

$$\begin{cases} dp_t^q &= \left\{ \sigma_t^p \eta_t^{\mathcal{R}} - \frac{1}{2} \frac{U_{zz}(t, X_t^{*,q} + p_t^q)}{U_z(t, X_t^{*,q} + p_t^q)} \|\pi^{*,0}(t, X_t^{*,q} + p_t^q) - \pi^{*,0}(t, X_t^{*,q} + \zeta_t^q) + \sigma_t^{\zeta, \mathcal{R}} - \sigma_t^p\|^2 \right. \\ &\quad \left. - (\frac{\gamma^{\perp}}{U_z})(t, X_t^{*,q} + p_t^q) \sigma_t^p \right\} dt + \sigma_t^p dW_t, \\ p_T^q &= \zeta_T^q = qC_T. \end{cases}$$

Moreover, the diffusion coefficient  $\sigma^p$  is given by

$$\sigma_t^p = \frac{Z_t^U - \gamma(t, X_t^{*,q} + p_t^q)}{U_z(t, X_t^{*,q} + p_t^q)} - \pi^{*,0}(t, X_t^{*,q} + \zeta_t^q) + \sigma_t^{\zeta, \mathcal{R}} \quad (4.30)$$

where  $Z^U$  is the diffusion coefficient of the martingale  $(\mathbb{E}(U(T, X_T^{*,q} + qC_T) | \mathcal{F}_t))_{0 \leq t \leq T}$ .

*Proof.* The process  $(U(t, X_t^{*,q} + p_t^q) = \mathbb{E}(U(T, X_T^{*,q} + qC_T) | \mathcal{F}_t))_{0 \leq t \leq T}$  being a martingale, there exists an  $\mathcal{F}$ -adapted process  $Z^U$  with  $\mathbb{E} \left[ \int_0^T |Z_s^U|^2 ds \right] < \infty$  such that

$$dU(t, X_t^{*,q} + p_t^q) = Z_t^U dW_t. \quad (4.31)$$

We now apply Itô-Ventzel's formula to obtain an alternative expression for the dynamics of  $(U(t, X_t^{*,q} + p_t^q))_{0 \leq t \leq T}$ , that will be identified with (4.31). For this, we use the dynamics of  $X^{*,q}$  whose local characteristics are  $(\pi^{*,q}(t, X_t^{*,q}) \eta_t^{\mathcal{R}}, \pi^{*,q}(t, X_t^{*,q}))$  and that of the performance process  $U$  defined by (2.4) with drift  $\beta$  given in (2.5). In addition, the process  $p^q$  is an  $\mathcal{F}$ -adapted semimartingale. We denote by  $\mu^p$  and  $\sigma^p$  its drift and diffusion coefficients, respectively. We provide the detailed calculations as we have to be careful with the different arguments of the

functionals.

$$\begin{aligned}
dU(t, X_t^{*,q} + p_t^q) &= \left\{ \frac{1}{2} U_{zz}(t, X_t^{*,q} + p_t^q) \|\pi^{*,0}(t, X_t^{*,q} + p_t^q)\|^2 + \frac{1}{2} U_{zz}(t, X_t^{*,q} + p_t^q) \|\pi^{*,q}(t, X_t^{*,q}) + \sigma_t^p\|^2 \right. \\
&\quad \left. + U_z(t, X_t^{*,q} + p_t^q) (\mu_t^p + \pi^{*,q}(t, X_t^{*,q}) \eta_t^{\mathcal{R}}) + \gamma_z(t, X_t^{*,q} + p_t^q) (\pi^{*,q}(t, X_t^{*,q}) + \sigma_t^p) \right\} dt \\
&\quad + (\gamma(t, X_t^{*,q} + p_t^q) + U_z(t, X_t^{*,q} + p_t^q) (\pi^{*,q}(t, X_t^{*,q}) + \sigma_t^p)) dW_t \\
&\stackrel{(4.3)}{=} \left\{ \frac{1}{2} U_{zz}(t, X_t^{*,q} + p_t^q) \left[ \|\pi^{*,0}(t, X_t^{*,q} + p_t^q)\|^2 + \|\pi^{*,0}(t, X_t^{*,q} + \zeta_t^q) - \sigma_t^{\zeta, \mathcal{R}} + \sigma_t^p\|^2 \right] \right. \\
&\quad \left. + U_z(t, X_t^{*,q} + p_t^q) (\mu_t^p + (\pi^{*,0}(t, X_t^{*,q} + \zeta_t^q) - \sigma_t^{\zeta, \mathcal{R}}) \eta_t^{\mathcal{R}}) \right. \\
&\quad \left. + \gamma_z(t, X_t^{*,q} + p_t^q) (\pi^{*,0}(t, X_t^{*,q} + \zeta_t^q) - \sigma_t^{\zeta, \mathcal{R}} + \sigma_t^p) \right\} dt \\
&\quad + (\gamma(t, X_t^{*,q} + p_t^q) + U_z(t, X_t^{*,q} + p_t^q) (\pi^{*,q}(t, X_t^{*,q}) + \sigma_t^p)) dW_t \\
&= \left\{ \frac{1}{2} U_{zz}(t, X_t^{*,q} + p_t^q) \left[ \|\pi^{*,0}(t, X_t^{*,q} + p_t^q)\|^2 + \|\pi^{*,0}(t, X_t^{*,q} + \zeta_t^q) - \sigma_t^{\zeta, \mathcal{R}} + \sigma_t^p\|^2 \right] \right. \\
&\quad \left. + U_z(t, X_t^{*,q} + p_t^q) (\mu_t^p + (\pi^{*,0}(t, X_t^{*,q} + \zeta_t^q) - \sigma_t^{\zeta, \mathcal{R}} + \sigma_t^p) \eta_t^{\mathcal{R}}) - U_z(t, X_t^{*,q} + p_t^q) \sigma_t^p \eta_t^{\mathcal{R}} \right. \\
&\quad \left. + \gamma_z(t, X_t^{*,q} + p_t^q) (\pi^{*,0}(t, X_t^{*,q} + \zeta_t^q) - \sigma_t^{\zeta, \mathcal{R}} + \sigma_t^p) \right\} dt \\
&\quad + (\gamma(t, X_t^{*,q} + p_t^q) + U_z(t, X_t^{*,q} + p_t^q) (\pi^{*,q}(t, X_t^{*,q}) + \sigma_t^p)) dW_t \\
&= \left\{ \frac{1}{2} U_{zz}(t, X_t^{*,q} + p_t^q) \left[ \|\pi^{*,0}(t, X_t^{*,q} + p_t^q)\|^2 + \|\pi^{*,0}(t, X_t^{*,q} + \zeta_t^q) - \sigma_t^{\zeta, \mathcal{R}} + \sigma_t^p\|^2 \right] \right. \\
&\quad \left. + (\gamma_z^{\mathcal{R}}(t, X_t^{*,q} + p_t^q) + U_z(t, X_t^{*,q} + p_t^q) \eta_t^{\mathcal{R}}) (\pi^{*,0}(t, X_t^{*,q} + \zeta_t^q) - \sigma_t^{\zeta, \mathcal{R}} + \sigma_t^p) \right. \\
&\quad \left. + \gamma_z^{\perp}(t, X_t^{*,q} + p_t^q) \sigma_t^p + U_z(t, X_t^{*,q} + p_t^q) (\mu_t^p - \sigma_t^p \eta_t^{\mathcal{R}}) \right\} dt \\
&\quad + (\gamma(t, X_t^{*,q} + p_t^q) + U_z(t, X_t^{*,q} + p_t^q) (\pi^{*,q}(t, X_t^{*,q}) + \sigma_t^p)) dW_t.
\end{aligned}$$

Plugging  $\pi^{*,0}(t, z) = -\frac{\gamma_z^{\mathcal{R}}(t, z) + U_z(t, z) \eta_t^{\mathcal{R}}}{U_{zz}(t, z)}$  (see (2.6)) in the second line of the last equality, we get

$$\begin{aligned}
dU(t, X_t^{*,q} + p_t^q) &= \left\{ \frac{1}{2} U_{zz}(t, X_t^{*,q} + p_t^q) \left[ \|\pi^{*,0}(t, X_t^{*,q} + p_t^q)\|^2 + \|\pi^{*,0}(t, X_t^{*,q} + \zeta_t^q) - \sigma_t^{\zeta, \mathcal{R}} + \sigma_t^p\|^2 \right] \right. \\
&\quad - U_{zz}(t, X_t^{*,q} + p_t^q) \pi^{*,0}(t, X_t^{*,q} + p_t^q) (\pi^{*,0}(t, X_t^{*,q} + \zeta_t^q) - \sigma_t^{\zeta, \mathcal{R}} + \sigma_t^p) \\
&\quad \left. + \gamma_z^{\perp}(t, X_t^{*,q} + p_t^q) \sigma_t^p + U_z(t, X_t^{*,q} + p_t^q) (\mu_t^p - \sigma_t^p \eta_t^{\mathcal{R}}) \right\} dt \\
&\quad + (\gamma(t, X_t^{*,q} + p_t^q) + U_z(t, X_t^{*,q} + p_t^q) (\pi^{*,q}(t, X_t^{*,q}) + \sigma_t^p)) dW_t.
\end{aligned}$$

Thus, by rearranging the terms, we obtain:

$$\begin{aligned}
dU(t, X_t^{*,q} + p_t^q) &= \left\{ \frac{1}{2} U_{zz}(t, X_t^{*,q} + p_t^q) \left[ \|\pi^{*,0}(t, X_t^{*,q} + p_t^q) - \pi^{*,0}(t, X_t^{*,q} + \zeta_t^q) + \sigma_t^{\zeta, \mathcal{R}} - \sigma_t^p\|^2 \right] \right. \\
&\quad \left. + \gamma_z^{\perp}(t, X_t^{*,q} + p_t^q) \sigma_t^p + U_z(t, X_t^{*,q} + p_t^q) (\mu_t^p - \sigma_t^p \eta_t^{\mathcal{R}}) \right\} dt \\
&\quad + (\gamma(t, X_t^{*,q} + p_t^q) + U_z(t, X_t^{*,q} + p_t^q) (\pi^{*,q}(t, X_t^{*,q}) + \sigma_t^p)) dW_t. \tag{4.32}
\end{aligned}$$

Finally, identifying with the drift and the diffusion coefficient of the dynamics (4.31), we get

$$\begin{cases} \sigma_t^p = \frac{Z_t^U - \gamma(t, X_t^{*,q} + p_t^q)}{U_z(t, X_t^{*,q} + p_t^q)} - \pi^{*,q}(t, X_t^{*,q}) \\ \mu_t^p = \sigma_t^p \eta_t^{\mathcal{R}} - \frac{1}{2} \frac{U_{zz}(t, X_t^{*,q} + p_t^q)}{U_z(t, X_t^{*,q} + p_t^q)} \left[ \|\pi^{*,0}(t, X_t^{*,q} + p_t^q) - \pi^{*,0}(t, X_t^{*,q} + \zeta_t^q) + \sigma_t^{\zeta, \mathcal{R}} - \sigma_t^p\|^2 \right] \\ \quad - \left( \frac{\gamma_z^{\perp}}{U_z} \right)(t, X_t^{*,q} + p_t^q) \sigma_t^p. \end{cases}$$

The proof is now complete.  $\square$

It is remarkable that in the dynamics of the indifference price, the term  $\|\pi^{*,0}(t, X_t^{*,q} + p_t^q) - \pi^{*,0}(t, X_t^{*,q} + \zeta_t^q) + \sigma_t^{\zeta, \mathcal{R}} - \sigma_t^p\|$  takes the form of a deviation measure of the same function  $(S^\sigma, \sigma) \mapsto \pi^{*,0}(\cdot, X^{*,q} + S^\sigma) + \sigma$  evaluated at two different points, similar to an expansion around the reference process  $\zeta$ .

## 5 Application: Consistent dynamic exponential utility

This section is devoted to a particular example of time-separable consistent utility, namely CARA (exponential) case, for which the previous results are illustrated. More precisely, let  $\mu^Z$  and  $\delta^Z$  be two  $\mathcal{F}$ -adapted processes such that  $\int_0^T (\mu_t^Z + \|\delta_t^Z\|^2) dt < \infty$  and define the positive process  $Z$  satisfying the following dynamics:

$$\frac{dZ_t}{Z_t} = \mu_t^Z dt + \delta_t^Z dW_t, \quad Z_0 = 1.$$

Let us fix  $\alpha > 0$ , and define the random field

$$U^\alpha(t, z) := -Z_t e^{-\alpha z}.$$

Here  $\alpha$  denotes the constant absolute risk aversion parameter (for CARA utility). Under adequate conditions on  $\mu^Z$  and  $\delta^Z$  (as recalled below, see [Mra21] for more details), the above representation provides a class of consistent dynamic utilities. In fact, by recalling the HJB constraint (2.5) and using Itô's formula, the random field  $U^\alpha(t, x)$  is an consistent dynamic utility if and only if the drift process  $\mu_t^Z$  satisfies the following equation:

$$\mu_t^Z = \frac{1}{2} \|\eta_t^{\mathcal{R}} + \delta_t^{Z, \mathcal{R}}\|^2, \quad \forall t \geq 0. \quad (5.1)$$

The following result holds for consistent dynamic exponential utilities:

**Proposition 5.1.** *Consider a CARA consistent dynamic utility  $U^\alpha(t, z) = -Z_t e^{-\alpha z}$  characterized by its volatility  $\delta_t^Z$ . Assume that the processes  $\eta^{\mathcal{R}}$  and  $\delta^Z$  are uniformly bounded and that  $\mu^Z$  satisfies (5.1). Then, there exists a triple of processes  $(X^{*,q}, \zeta^q, \sigma^\zeta)$  such that*

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} (|X_t^{*,q}|^2 + |\zeta_t^q|^2) + \int_0^T \|\sigma_t^\zeta\|^2 dt \right] < \infty,$$

that satisfies the FBSDE (4.4) which in the setting of consistent dynamic exponential utility  $U^\alpha$  is given as below:

$$\begin{cases} dX_t^{*,q} &= [\frac{1}{\alpha}(\eta_t^{\mathcal{R}} + \delta_t^{Z, \mathcal{R}}) - \sigma_t^{\zeta, \mathcal{R}}](dW_t + \eta_t^{\mathcal{R}} dt), \\ d\zeta_t^q &= \{(\eta_t^{\mathcal{R}} + \delta_t^{Z, \perp})\sigma_t^\zeta - \frac{\alpha}{2}\|\sigma_t^{\zeta, \perp}\|^2\}dt + \sigma_t^\zeta dW_t, \\ \zeta_T^q &= qC_T. \end{cases}$$

Moreover, if the claim  $qC_T$  satisfies  $\mathbb{E}[e^{-\alpha qC_T}] < \infty$ , then the optimal investment strategy for the exponential utility maximization problem  $(P_c)$  is given explicitly by:

$$\pi^{*,q}(t, X_t^{*,q}) = \frac{1}{\alpha}(\eta_t^{\mathcal{R}} + \delta_t^{Z,\mathcal{R}}) - \sigma_t^{\zeta,\mathcal{R}}.$$

Therefore, the corresponding indifference price  $p^q$  is given by

$$p_t^q = p(t, T, x, q) = -x + \frac{1}{\alpha} \ln \left( \frac{\alpha}{Z_t} \left( e^{-\alpha(x+\zeta_t^q)} - \mathbb{E} \left( \int_t^T e^{-\alpha(X_s^{*,q} + \zeta_s^q)} (2\delta_s^{Z,\perp} \sigma_s^{\zeta,\perp} - \alpha \|\sigma_t^{\zeta,\perp}\|^2) ds \middle| \mathcal{F}_t \right) \right) \right) \quad (5.2)$$

and  $(p^q, \sigma^p)$  is a solution of the quadratic BSDE

$$\begin{cases} dp_t^q &= \left\{ (\eta_t^{\mathcal{R}} - \delta_t^{Z,\perp}) \sigma_t^p + \frac{\alpha}{2} \|\sigma_t^{\zeta,\mathcal{R}} - \sigma_t^p\|^2 \right\} dt + \sigma_t^p dW_t, \\ p_T^q &= qC_T. \end{cases} \quad (5.3)$$

*Proof.* First, let us recall the following derivatives of  $U^\alpha$

$$U^\alpha(t, z) = -Z_t e^{-\alpha z}, \quad U_z^\alpha(t, z) = \alpha Z_t e^{-\alpha z}, \quad U_{zz}^\alpha(t, z) = -\alpha^2 Z_t e^{-\alpha z} \quad \text{and} \quad \frac{U_z^\alpha}{U_{zz}^\alpha}(t, z) = -\frac{1}{\alpha},$$

as well as the derivatives of the volatility of  $U^\alpha$

$$\gamma(t, z) = U^\alpha(t, z) \delta_t^Z, \quad \text{and} \quad \gamma_z(t, z) = U_z^\alpha(t, z) \delta_t^Z.$$

Then, in the case of consistent dynamic exponential utilities, the optimal strategy (2.6) is given by  $\pi_t^{*,0}(x) = \frac{1}{\alpha}(\eta_t^{\mathcal{R}} + \delta_t^{Z,\mathcal{R}})$ , and the FBSDE (4.4) reads

$$\begin{cases} dX_t^{*,q} &= [\frac{1}{\alpha}(\eta_t^{\mathcal{R}} + \delta_t^{Z,\mathcal{R}}) - \sigma_t^{\zeta,\mathcal{R}}](dW_t + \eta_t^{\mathcal{R}} dt), \\ d\zeta_t^q &= \{(\eta_t^{\mathcal{R}} + \delta_t^{Z,\perp})\sigma_t^\zeta - \frac{\alpha}{2} \|\sigma_t^{\zeta,\perp}\|^2\} dt + \sigma_t^\zeta dW_t, \\ \zeta_T^q &= qC_T. \end{cases}$$

Note that the backward part of the above system does not depend on the forward part. Thus, the problem reduces to solving the following quadratic BSDE:

$$\begin{cases} d\zeta_t^q &= \{(\eta_t^{\mathcal{R}} + \delta_t^{Z,\perp})\sigma_t^\zeta - \frac{1}{2}\alpha \|\sigma_t^{\zeta,\perp}\|^2\} dt + \sigma_t^\zeta dW_t, \\ \zeta_T^q &= qC_T. \end{cases} \quad (5.4)$$

The coefficient of the above BSDE takes the form

$$f(t, z) = (\eta_t^{\mathcal{R}} + \delta_t^{Z,\perp})z - \frac{1}{2}\alpha \|z^\perp\|^2.$$

Let us define the process  $\widetilde{W}_t := W_t + \int_0^t (\eta_s^{\mathcal{R}} + \delta_s^{Z,\perp}) ds$ . Under the boundedness assumption on the parameters  $\eta^{\mathcal{R}}$  and  $\delta^{Z,\perp}$ ,  $\widetilde{W}$  is a Brownian motion under the probability measure  $\mathbb{Q}$  given by  $\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp \left( - \int_0^T (\eta_s^{\mathcal{R}} + \delta_s^{Z,\perp}) dW_s - \frac{1}{2} \int_0^T \|\eta_s^{\mathcal{R}} + \delta_s^{Z,\perp}\|^2 ds \right)$ .

Thus, BSDE (5.4) becomes:

$$\begin{cases} d\zeta_t^q &= -\frac{1}{2}\alpha \|\sigma_t^{\zeta,\perp}\|^2 dt + \sigma_t^\zeta d\widetilde{W}_t, \\ \zeta_T^q &= qC_T. \end{cases} \quad (5.5)$$

We use the result of Bahlali [Bah20] which establishes the existence of solutions for quadratic BSDEs with generator  $f(t, y, z) = h(y)|z|^2$ , where  $h$  is assumed to be only locally integrable on  $\mathbb{R}$ , with no integrability assumption required for the terminal condition. Moreover, under additional integrability assumptions on the terminal condition, uniqueness can also be ensured (see Propositions 2.1 and 2.2 [Bah20]). In our case, the generator is given by  $f(t, y, z) = -\frac{\alpha}{2}|z|^2$  which fits within this framework. Under the assumption  $\mathbb{E}[e^{-\alpha q C_T}] < \infty$ , BSDE (5.5) has a unique solution. Therefore, the initial BSDE (5.4) also has a unique solution  $(\zeta^q, \sigma^\zeta)$ , where  $\sigma^\zeta$  is square-integrable. By plugging the process  $\sigma^\zeta$  into the forward SDE

$$\begin{cases} dX_t^{*,q} &= [\frac{1}{\alpha}(\eta_t^{\mathcal{R}} + \delta_t^{Z,\mathcal{R}}) - \sigma_t^{\zeta,\mathcal{R}}](dW_t + \eta_t^{\mathcal{R}}dt), \\ X_0^{*,q} &= x, \end{cases}$$

we obtain a standard linear SDE whose solution exists and is unique.

The representation (5.2) and its characterization as a solution of the quadratic BSDE (5.3) follow directly from Corollary 4.3 and Theorem 4.5.  $\square$

## Conclusion

In summary, works on utility-based pricing have evolved from deterministic formulations to dynamic frameworks based on consistent utilities. In this context, our study introduces forward indifference prices, which extend classical indifference valuation to settings where preferences and market conditions evolve over time. We show that the indifference price can be characterized as a solution of a BSDE with quadratic growth, which is linked to a forward-backward structure arising from the optimization problem. This FBSDE (4.4) is generally difficult to analyze due to its nonlinearities, possible singularities, and strong coupling between the forward and backward components. Work in progress aims to further study the associated FBSDE and its applications to marginal indifference prices.

## 6 Annex

### 6.1 Definition of Kunita spaces

We recall here the definition of Kunita spaces, which are classes of regularity on the random fields. This spaces of Sobolev type random semi-norms allows to control (locally or globally) the growth of the random field and its derivatives.

Let  $\phi$  be a continuous  $\mathbb{R}^k$ -valued progressive random field,  $m$  a non-negative integer, and  $\delta$  a real number in  $(0, 1]$ . Let  $\phi$  be in the class  $\mathcal{C}^{m,\delta}([0, +\infty[)$ , i.e.  $m$ -times continuously differentiable with  $\phi^{(m)}$  is  $\delta$ -Hölder in  $x$  for any  $t$  a.s.

For any subset  $K \subset ]0, +\infty[$ , we define the family of random (Hölder)  $K$ -semi-norms

$$\begin{cases} \|\phi\|_{m:K}(t, \omega) = \sup_{x \in K} \frac{\|\phi(t, x, \omega)\|}{x} + \sum_{1 \leq j \leq m} \sup_{x \in K} \|\partial_x^j \phi(t, x, \omega)\| \\ \|\phi\|_{m, \delta:K}(t, \omega) = \|\phi\|_{m:K}(t, \omega) + \sup_{x, y \in K} \frac{\|\partial_x^m \phi(t, x, \omega) - \partial_x^m \phi(t, y, \omega)\|}{|x - y|^\delta}. \end{cases} \quad (6.1)$$

The previous semi-norms are related to the spatial parameter, we add the temporal dimension in assuming these semi-norms (or the square of the semi-norm) to be integrable in time with respect to the Lebesgue measure on  $[0, T]$  for all  $T$ . That is  $\mathcal{K}_{loc}^{m, \delta}$  (resp.  $\bar{\mathcal{K}}_{loc}^{m, \delta}$ ) denotes the set of all  $\mathcal{C}^{m, \delta}$ -random fields such that for any compact  $K \subset ]0, +\infty[$ , and any  $T$ ,  $\int_0^T \|\phi\|_{m, \delta:K}(t, \omega) < \infty$ , (resp.  $\int_0^T \|\phi\|_{m, \delta:K}^2(t, \omega) dt < \infty$ ). Then, as in Lebesgue's Theorem, we can differentiate, pass to the limit, commute limit and integral for random fields in such spaces. Calligraphic notation recalls that these semi-norms are random.

## 6.2 Proof of Proposition 3.2

*Proof.* Let  $t \in [0, T]$  and  $\xi$  an  $\mathcal{F}_t$ -attainable wealth. First, note that if there exists an optimal admissible wealth  $X^{*,q}$  for the optimization problem  $(P_c)$ , then uniqueness follows from the strict concavity of  $U$  and Lemma 2.5. In this case, we can write

$$V^c(t, \xi, q) = \mathbb{E}(U(T, X_T^{*,q} + qC_T) | \mathcal{F}_t), \quad (6.2)$$

and based on the representation (3.5), we retrieve the uniqueness of the indifference price  $p^q$ . Moreover, using the monotonicity of  $U(t, \cdot)$  and by relying once more on the representation (6.2), we observe that  $V^c(t, \cdot, q)$  is also increasing. We can now use these observations to show the above properties.

(i) The link between  $p^q$  and  $p^{(-q)}$  follows from Relation (3.3), as the indifference price for the seller of the quantity  $qC_T$  given by (3.3) is nothing else but the indifference price of the negative quantity  $-qC_T$  drawn by the main optimization problem (3.1). This reads

$$V^c(t, \xi - p_t^{(-q)}, -q) = V(t, \xi) = V^c(t, \xi + p_t^q, -q).$$

Then, by uniqueness of the indifference price, we can see that  $p^q = -p^{(-q)}$ .

(ii) For the upper bound, it follows from the definition of the super-replicating price that there exists a strategy  $\pi^{up} \in \mathcal{R}$  such that  $X^{up} := X^{\pi^{up}}$  is a wealth process of the form (2.2) with  $X_t^{up} = h^{up} := h^{up}(qC_T)$ . Besides, using Lemma 2.5 and its proof, we have that if  $X^\pi \in \mathcal{X}(t, \xi)$  then  $X^\pi + X^{up}$  is a wealth process that belongs to  $\mathcal{X}(t, \xi + h^{up})$  and

$$\begin{aligned} V^c(t, \xi + h^{up}, -q) &\geq \operatorname{ess\,sup}_{X^\pi \in \mathcal{X}(t, \xi)} \mathbb{E}[U(T, X_T^\pi + X_T^{up} - qC_T) | \mathcal{F}_t] \\ &\geq \operatorname{ess\,sup}_{X^\pi \in \mathcal{X}(t, \xi)} \mathbb{E}[U(T, X_T^\pi) | \mathcal{F}_t] \\ &= V(t, \xi). \end{aligned}$$

But  $V(t, \xi) = U(t, \xi) = V^c(t, \xi - p_t^{(-q)}, -q)$ . This means that  $h^{up} \geq -p^{(-q)}$  as  $V^c$  is increasing with respect to the spatial variable  $x$ . Then since  $p = -p^{(-q)}$ , we obtain  $h^{up} \geq p^q$ .

To derive the lower bound, we follow a similar procedure by considering a strategy  $\pi^{low} \in \mathcal{R}$  such that  $X^{low} := X^{\pi^{low}}$  satisfies (2.2) with  $X_t^{low} = -h^{low} := -h^{low}(qC_T)$ . More precisely, the process  $X^\pi + X^{low}$  is an admissible wealth that satisfies  $X_t^\pi + X_t^{low} = \xi - h^{low}$ , where  $\pi \in \mathcal{R}$ . This shows that:

$$\begin{aligned} V^c(t, \xi - h^{low}, q) &\geq \operatorname{ess\,sup}_{X^\pi \in \mathcal{X}(t, \xi)} \mathbb{E}[U(T, X_T^\pi + X_T^{low} + qC_T) | \mathcal{F}_t] \\ &\geq \operatorname{ess\,sup}_{X^\pi \in \mathcal{X}(t, \xi)} \mathbb{E}[U(T, X_T^\pi) | \mathcal{F}_t] \\ &= U(t, \xi) \\ &= V^c(t, \xi - p_t^q, q). \end{aligned}$$

Using the monotonicity of  $V^c(t, \cdot, q)$  once again, we conclude that  $p^q \geq h^{low}$ .

(iii) Recall that  $X^{*,q}$  denotes the optimum for the problem  $(P_c)$  with the claim  $qC_T$ . Then, to show the monotonicity of the indifference price with respect to the claim, it suffices to see that

$$\begin{aligned} V^c(t, \xi, q') &= \operatorname{ess\,sup}_{X^\pi \in \mathcal{X}(t, \xi)} \mathbb{E}[U(T, X_T^\pi + q'C_T) | \mathcal{F}_t] \\ &\geq \mathbb{E}[U(T, X_T^{*,q} + q'C_T) | \mathcal{F}_t] \\ &\geq \mathbb{E}[U(T, X_T^{*,q} + qC_T) | \mathcal{F}_t] \\ &= \operatorname{ess\,sup}_{X^\pi \in \mathcal{X}(t, \xi)} \mathbb{E}[U(T, X_T^\pi + qC_T) | \mathcal{F}_t] \\ &= V^c(t, \xi, q). \end{aligned}$$

Then, from the representation (3.5) and the monotonicity of the inverse  $U^{-1}(t, \cdot)$ , we deduce that  $p^{q'} = -\xi + U^{-1}(t, V^c(t, \xi, q')) \geq -\xi + U^{-1}(t, V^c(t, \xi, q)) = p^q$ .

(iv) Let  $\lambda \in [0, 1]$  and let  $X^{*,q}$  and  $\bar{X}^{*,q}$  denote the optimal processes associated with the problem  $(P_c)$  with the claim  $qC_T$  and  $q\bar{C}_T$  respectively. Then the process  $\tilde{X}^q := \lambda X^{*,q} + (1 - \lambda)\bar{X}^{*,q}$  is an admissible wealth process in the class  $\mathcal{X}(t, \xi - \lambda p^q - (1 - \lambda)\bar{p}^q)$ . Therefore, the following holds:

$$\begin{aligned} V^{\bar{c}}(t, \xi - \lambda p^q - (1 - \lambda)\bar{p}^q, q) &= \operatorname{ess\,sup}_{X^\pi \in \mathcal{X}(t, \xi - \lambda p^q - (1 - \lambda)\bar{p}^q)} \mathbb{E}[U(T, X_T^\pi + q\tilde{C}_T) | \mathcal{F}_t] \\ &\geq \mathbb{E}[U(T, \tilde{X}_T^q + q\tilde{C}_T) | \mathcal{F}_t] \\ &\geq \lambda \mathbb{E}[U(T, X_T^{*,q} + qC_T) | \mathcal{F}_t] + (1 - \lambda) \mathbb{E}[U(T, \bar{X}_T^{*,q} + q\bar{C}_T) | \mathcal{F}_t], \end{aligned}$$

where we have used the concavity of  $U$ . Then, from the definition of the optimal processes above, we deduce that

$$\begin{aligned} V^{\tilde{c}}(t, \xi - \lambda p^q - (1 - \lambda)\bar{p}^q, q) &\geq \lambda V^c(t, \xi - p^q, q) + (1 - \lambda)V^{\bar{c}}(t, \xi - \bar{p}^q, q) \\ &= U(t, \xi) \\ &= V^{\tilde{c}}(t, \xi - \tilde{p}^q, q). \end{aligned}$$

Then using the monotonicity of  $V^{\tilde{c}}(t, \cdot, q)$ , we conclude that  $\tilde{p} \geq \lambda p^q + (1 - \lambda)\bar{p}^q$ .  $\square$

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