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On ω -categorical groups of finite dimension

Moreno Invitti

Abstract

We prove that a finite-dimensional ω -categorical group is finite-by-abelian-by-finite and that a finite-dimensional ω -categorical ring is virtually finite-by-null.

Introduction

This article analyzes the structure of ω -categorical groups and rings of finite dimension.

Definition. A structure \mathfrak{M} is said to be ω -categorical if its theory $T = T(\mathfrak{M})$ has a unique countable model up to isomorphism.

Some examples of ω -categorical structures are: dense linear ordered set without endpoints (this is the theory of $(\mathbb{Q}, <)$ that is ω -categorical by Cantor's Theorem), random graphs, infinite vector spaces over finite fields, and boolean atomless algebras. The fundamental result that holds for these structures, proved independently by Ryll-Nardzewski [8], Svenonius [12], and Engeler [13], states that an ω -categorical structure has only finitely many parameter-free definable sets, up to equivalence.

?(RyllNard)? **Theorem 0.1.** Let \mathfrak{M} be a first-order structure. Then, TFAE:

1. \mathfrak{M} is ω -categorical;
2. In any arity, there exist finitely many parameter-free definable sets;
3. For any $n < \omega$, there exists only finitely many orbits in M^n for the action of $\text{Aut}_\emptyset(\mathfrak{M})$.

From Theorem 0.1(3), we can derive the following easy corollary.

Corollary 0.2. Let X be a definable subset of an ω -categorical structure, $\{c_i\}_{i < n_C}$ a finite set of elements in X , $\{R_i\}_{i \in I}$ a family of relations definable over a finite set R and $\{f_j\}_{j \in J}$ a family of functions definable over a finite set F . Then, the structure $(X, \{c_i\}_{i < n_C}, \{R_i\}_{i \in I}, \{f_j\}_{j \in J})$ is ω -categorical.

Proof. Let D the set of definition of X and $n_D = |D|$. Let $n_R = |R|$ and $n_F = |F|$ and $N = n_D + n_C + n_F + n_R$.

It is sufficient to observe that, for any $n < \omega$, the action of $\text{Aut}(\mathfrak{M})$ on the set M^{n+N} has finitely many orbits. In particular, there are finitely many orbits for the subset $(d_1, \dots, d_{n_D}, c_1, \dots, c_{n_C}, f_1, \dots, f_{n_F}, r_1, \dots, r_{n_R}) \times X^n$. Since all the automorphisms of \mathfrak{M} fixing $(d_1, \dots, d_{n_D}, c_1, \dots, c_{n_C}, f_1, \dots, f_{n_F}, r_1, \dots, r_{n_R})$ are automorphisms of X , the proof is completed. \square

The study of ω -categorical groups has a long history. The following results are a straightforward application of Theorem 0.1.

Lemma 0.3. *An ω -categorical group is locally finite and of finite exponent.*

?(ChaSimSer)? **Lemma 0.4.** *Let G be an ω -categorical group. Then, there exists a chain of subgroups $G = G_0 \geq G_1 \geq \dots \geq G_n = \{0\}$ such that*

- G_i is characteristic and definable in G for any $i \leq n$;
- G_i/G_{i+1} is characteristically simple for any $i \leq n - 1$.

The main theorem about ω -categorical groups is the classification of ω -categorical characteristically simple groups obtained by Wilson in [3].

?(CharSimGro)? **Fact 0.5.** *For each infinite, countable, ω -categorical, characteristically simple group H , one of the following holds:*

- For some prime number p , H is an elementary abelian p -group;
- $H \simeq B(F)$ or $H \simeq B^-(F)$ for some non-abelian finite simple group F . $B(F)$ denotes the group of all the continuous functions from the Cantor space C to F (with the discrete topology) while $B^-(F)$ is the subgroup consisting of the functions $f \in B(F)$ such that $f(x_0) = e$ for a fixed element $x_0 \in C$.
- H is a perfect p -group for some prime p .

The important question is whether, under an assumption of "tameness", we can classify ω -categorical groups and ω -categorical rings. In particular, the two big meta-conjectures are the following.

Theorem 0.6. *1. A tame ω -categorical group is virtually nilpotent. A tame ω -categorical ring is virtually nilpotent.*

2. A supertame ω -categorical group is finite-by-abelian-by-finite. A supertame ω -categorical ring is finite-by-null-by-finite.

These have been proven under several conditions of tameness. If tame is substituted by stable, (1) has been proved by Felgner [10] for groups and by Baldwin and Rose [14] for rings, while (2) by Bauer, Cherlin, and Macintyre in [11]. In the simple case, (2) has been verified by Evans and Wagner in [7]. (1) holds also for NSOP and NIP theories as proved by Macpherson in [9] and Krupinski in [4] (under the assumption of f.s.g. for groups) respectively. Finally, (2) has been proved under the hypothesis of finite burden by Wagner and Dobrowolski in [1].

This article proves (2) under the assumption of finite-dimensionality, an important model-theoretic notion introduced by Wagner in [5]. In the first section, we introduce finite-dimensional theories and describe the properties of finite-dimensional groups. Moreover, we introduce the notion of dimensional-generic of a group. In the second section, we define bilinear quasi-forms. These, as in [1], are fundamental in the analysis of ω -categorical groups. In the third section, we define principal indiscernible dimensional-generic sequences. These are used in section 4 to prove the finite-dimensional version of [1, Theorem 7.5].

^{?(VirtTriv)?} **Theorem 0.7.** *A bilinear quasi-form definable in a finite-dimensional ω -categorical structure is virtually almost trivial.*

In the fifth section, we apply Theorem 0.7 to prove the following result.

Theorem 0.8. *An ω -categorical group of finite dimension is finite-by-abelian-by-finite, and an ω -categorical ring of finite dimension is virtually null-by-finite.*

1 Finite-dimensional groups

We define finite-dimensional theories.

Definition. *A theory T is finite-dimensional if there exists a function \dim from the class of all the interpretable subsets in any model \mathcal{M} of T into $\omega \cup \{-\infty\}$ such that for any $\phi(x, y)$ formula, X, Y interpretable sets in T and f interpretable function from X to Y , then*

- *If a, a' have the same type over \emptyset , $\dim(\phi(x, a)) = \dim(\phi(x, a'))$;*
- *$\dim(\emptyset) = -\infty$ and $\dim(X) = 0$ if and only if X is finite;*
- *$\dim(X \cup Y) = \max\{\dim(X), \dim(Y)\}$;*
- *If $\dim(f^{-1}(y)) \geq k$ for any $y \in Y$, then $\dim(X) \geq \dim(Y) + k$;*
- *If $\dim(f^{-1}(y)) \leq k$ for any $y \in Y$, then $\dim(X) \leq \dim(Y) + k$.*

Examples of finite-dimensional theories are superstable theories of finite Lascar rank, supersimple theories of finite Lascar rank, and o -minimal theories.

We now focus our attention on finite-dimensional groups. An important property of finite-dimensional groups is the ω -DCC.

Definition. *Let G be a definable group. Then, G has the ω -DCC property if there exists no infinite strictly descending chain of definable subgroups $\{G_i\}_{i < \omega}$ such that $|G_i : G_{i+1}| > \omega$.*

Groups definable in a finite-dimensional theory respect the ω -DCC (Corollary 2.4 of [5]). Moreover, finite-dimensional groups have the ucc, as a consequence of the following fundamental Lemma [15, Lemma 1.1].

^{?(boundedind)?} **Lemma 1.1.** *Let G be a definable group of finite dimension and $\{H_i\}_{i \in I}$ a family of uniformly definable subgroups. Then, there exist $n, d < \omega$ such that there is no $J = \{j_1, \dots, j_n\} \subseteq I$ of cardinality n with $|\bigcap_{i=1}^k H_{j_i} : \bigcap_{i=1}^{k+1} H_{j_i}| \geq d$ for any $k \leq n-1$.*

If we apply these results to the centralisers of elements in G , we obtain that a finite-dimensional definable group has the hereditarily $\widehat{\mathfrak{M}}_c$ -property.

Definition. A group G is hereditarily- $\widetilde{\mathfrak{M}}_c$ if, for any definable subgroups H, N such that N is normalised by H , there exist natural numbers n_{HN} and d_{HN} such that any sequence of centralisers

$$C_H(a_0/N) \geq C_H(a_0, a_1/N) \geq \dots \geq C_H(a_0, a_1, \dots, a_n/N) \geq \dots$$

with $|C_H(a_0, \dots, a_n/N) : C_H(a_0, \dots, a_{n+1}/N)| \geq d_{HN}$ has length at most n_{HN} .

The notions of almost containment and commensurability are fundamental in the study of groups definable in finite-dimensional theories.

Definition. • Let G be a group and $A, B \leq G$ subgroups. A almost contains B , denoted $A \gtrsim B$, if $|B : B \cap A|$ is finite.

- If either $A \lesssim B$ or $B \lesssim A$, we say that A and B are comparable;
- If $A \lesssim B$ and $B \lesssim A$, A and B are commensurable, denoted $A \sim B$. \sim is clearly an equivalent relation on the class of subgroups of G .
- Given a family of subgroups $\{G_i\}_{i \in I}$ in G , the family $\{G_i\}_{i \in I}$ is uniformly commensurable if there exists $n < \omega$ such that $|G_i/G_i \cap G_j| \leq n$ for every $i, j \in I$.

We also define the notion of almost direct sum.

Definition. Let G be a group and H, K subgroups of G that normalize each other. The sum $H + K$ is almost direct, denoted by $H \oplus K$, if $H \cap K$ is finite.

A consequence of the hereditarily $\widetilde{\mathfrak{M}}_c$ -condition is that the almost centralisers are definable.

Definition. Let G be a group, $K, H \leq G$ and N a subgroup of G normalized by both H and K . The almost centraliser in K of H/N , denoted $\widetilde{C}_K(H/N)$ is the subgroup

$$\{k \in K : C_H(k/N) \gtrsim H\}.$$

If we assume $N = 0$ and $K = H = G$, we have the notion of almost center.

Definition. Let G be a group. We define iteratively the n -almost center of G , denoted by $\widetilde{Z}^n(G)$, as follows:

- $\widetilde{Z}^1(G) = \widetilde{C}_G(G)$;
- $\widetilde{Z}^{n+1}(G)$ is such that $\widetilde{Z}(G/\widetilde{Z}^n(G)) = \widetilde{Z}^{n+1}(G)/\widetilde{Z}^n(G)$.

The following Lemma is proven in [2, Proposition 3.3].

Lemma 1.2. Let G be a hereditarily $\widetilde{\mathfrak{M}}_c$ -group and K, H, N as before. If K, H, N are definable, $\widetilde{C}_K(H/N)$ is definable.

An easy result for the almost center is the following.

^{?(Zfin)?} **Lemma 1.3.** Let G be a group such that $\widetilde{Z}(G)$ is finite. Then, $\widetilde{Z}(G/\widetilde{Z}(G)) = \widetilde{Z}(G)$.

Proof. Let $g \in \tilde{Z}(G/\tilde{Z}(G))$. Then, $[g, G/\tilde{Z}(G)] + \tilde{Z}(G)$ is finite. Since $\tilde{Z}(G)$ is finite, $[g, G]$ is finite i.e. $g \in \tilde{Z}(G)$. \square

Another important property of almost centralisers for definable groups is symmetry [2, Theorem 2.10].

^{?(sym)?} **Lemma 1.4.** *Let G be a definable group and H, K, N definable subgroups such that both H and K normalize N . Then, $H \lesssim \tilde{C}_G(K/N)$ iff $K \lesssim \tilde{C}_G(H/N)$.*

In the last section of this article, we will use a fundamental theorem of linearization for the action of an almost abelian group definable in a finite-dimensional theory.

Definition. *Let G be a group. G is almost abelian if $\tilde{Z}(G) = G$.*

We introduce the notion of absolutely minimal G -group.

Definition. *Let G be a definable group acting on the definable group A . A is absolutely G -minimal if, for any definable subgroup H of finite index in G , A is H -minimal.*

We introduce the notion of almost centralisers of an action.

Definition. *Let G be a group acting on a group A . The almost centraliser of the action in G , denoted by $\tilde{C}_G(A)$, is the subgroup $\{g \in G : C_A(g) \sim A\}$. The almost centraliser of the action in A , denoted by $\tilde{C}_A(G)$, is the subgroup $\{a \in A : C_G(a) \sim G\}$.*

It follows from Lemma 1.1 that the two almost centralisers are definable. We introduce almost trivial actions.

Definition. *Let G be a group acting on a group A . The action of G on A is almost trivial if either $\tilde{C}_G(A)$ is of finite index in G or $\tilde{C}_A(G)$ is of finite index in A .*

In the finite-dimensional case, these two conditions are equivalent by [15, Lemma 12.4]. The following result is fundamental in the analysis of almost trivial actions.

^{?(lemma 4)?} **Lemma 1.5.** *Let G be a definable almost abelian group acting on the definable almost abelian group A . Then,*

$$[\tilde{C}_G(A), \tilde{C}_A(G)] = \langle [g, a] = ga - a : g \in \tilde{C}_G(A), a \in \tilde{C}_A(G) \rangle$$

is finite.

Proof. By Lemma 1.1, the almost centralisers $H = \tilde{C}_G(A), B = \tilde{C}_A(G)$ are definable subgroups such that B is G -invariant and H is G -normal. Since A is almost abelian, B' is finite and characteristic, and H acts on the abelian group B/B' . Since B is definable and every element has finite H -orbit, there exists $b \in B$ such that Hb is of maximal cardinality. Let h_1, \dots, h_n be a left transversal of $C_H(b)$ in H . Since every element $h \in H$ has centraliser in A of finite index, $C = C_B(h_1, \dots, h_n)$ is of finite index in B . Let b_1, \dots, b_m be a transversal of C in B . Denote by F the H -invariant finitely generated subgroup $\langle Hb, Hb_1, \dots, Hb_m \rangle$. Clearly, F has countable cardinality. Moreover, by definition, $B = F + C$. On the other hand, let $D = \{[h, b] : h \in H, b \in B\}$. D is clearly a definable subset of B and therefore either finite or unbounded. We verify that

$D \leq F$. Given $b \in B$, there exist $c \in C$ and $f \in F$ such that $b = c + f$. Therefore, $[h, b] = h(c + f) - c - f = h(c) - c + h(f) - f$. Since $h(f) \in F$, $[h, b]$ is contained in F iff $h(c) - c \in F$. Define $w = c + a$, then $h_i(w) = h_i(c) + h_i(b)$ but $c \in C$ and so this is equal to $c + h_i(b)$. Since $h_i(b) \neq h_j(b)$ when $i \neq j$ and by maximality of the cardinality of the orbit of b , the orbit of w is $c + g_i(b)$ for $i = 1, \dots, n$. Therefore, for any $h \in H$, $h(w) = c + h_i(b)$ for a certain i . Consequently,

$$h(c) - c = h(w) - h(b) - c = c + h_i(b) - h_j(b) - c = h_i(b) - h_j(b) \leq F.$$

Being D definable and not unbounded, it is finite. Therefore, the subgroup $[H, B]$ is finitely generated and, being B abelian, it is finite iff the order of any generator is finite. But $n(g - 1)(b) = g - 1(nb) \leq D$ that is finite so also $o(g - 1(b))$ is finite for every $g \in H$ and $b \in B$. Since H' is finite, this proves the theorem. \square

Finally, if G and A are two almost abelian groups such that G acts on A not almost trivially and A is absolutely G -minimal, then the action can be linearized [15, Theorem 12.6].

$?(LinInv)?$ **Theorem 1.6.** *Let G be a definable almost abelian group acting on a definable infinite almost abelian group A . Assume that:*

- *A is absolutely G -minimal;*
- *the action is not almost trivial.*

Then, $G/\tilde{C}_G(A)$ is abelian and there exists a definable field K such that $G/\tilde{C}_G(A)$ definably embeds in K^\times and $A/\tilde{C}_A(G)$ is isomorphic to K^+ .

1.1 Dimensional-generics

We introduce the notion of dimensional-generics of a definable group of finite dimension. We also recall the notion of subgroup-generic used in [1].

Definition. *Given G a group definable in a theory T , a definable subset X of G is:*

- *subgroup-generic if it is not contained in the union of finitely many translates of interpretable subgroups of infinite index;*
- *dimensional-generic if $\dim(G) = \dim(X)$ with T a finite-dimensional theory.*

A (possibly partial) type p is $$ -generic if every $\phi \in p$ is $*$ -generic (with $*$ equal to either subgroup or dimensional).*

We prove the existence of dimensional-generics.

$?(ExistGen)?$ **Lemma 1.7.** *Let G be a definable finite-dimensional group and A a set of parameters. Then, there always exists a dimensional-generic type $p \in S(A)$. Moreover, we can construct infinite indiscernible sequences $\{a_i \in G\}_{i \in I}$ over A such that $tp(a_i/\{a_j\}_{j < i} \cup A)$ is dimensional-generic.*

Proof. Assume that G is \emptyset -definable.

For the first part, let q be the partial type over A given by $\{X \subseteq G : \dim(G - X) < \dim(G)\}$. We verify that it is finitely consistent. Assume not, then there exists a finite subfamily $\{X_i\}_{i \leq n}$ of q such that $\bigcap_{i \leq n} X_i$ is empty. This implies that $G - \bigcap_{i \leq n} X_i = \bigcup_{i \leq n} G - X_i = G$. Therefore, $\dim(G) = \dim(\bigcup_{i \leq n} G - X_i) = \max_{i \leq n} \dim(G - X_i)$ by union. By hypothesis, $\dim(G) = \max_{i \leq n} \dim(G - X_i) < \dim(G)$, a contradiction. Therefore, let p be a completion of q in $S_A(G)$. Then, this type is dimensional-generic. Assume not, then there exists an A -definable set X in p such that $\dim(X) < \dim(G)$. By definition of q , $G - X \in q \subseteq p$ and so $X, G - X \in p$, contradicting the finite satisfiability of p .

For the second part, apply iteratively the first part to obtain an infinite sequence $\{a_i\}_{i \in I}$ of elements in G such that $tp(a_i/A \cup \{a_j\}_{j < i})$ is dimensional-generic. This sequence is also A -dimensional independent since $\dim(tp(a_i/A \cup \{a_j\}_{j < i})) = \dim(G) \geq \dim(tp(a_i/A))$ since $a_i \in G$. The conclusion follows by Ramsey's Theorem. \square

We verify that dimensional-generics are subgroup-generics.

^{?(dimgen)?} **Lemma 1.8.** *A dimensional-generic is subgroup-generic.*

Proof. Let p be a dimensional-generic for a definable group G of finite dimension. Assume, for a contradiction, that p is not subgroup-generic. Then, by definition, there exists $\phi(x) \in p$ such that $\phi(G)$ is contained in a translate of an interpretable subgroup H of infinite index in G . By fibration, $\dim(H) < \dim(G)$. Therefore, by invariance, $\dim(\phi(G)) \leq \dim(H) < \dim(G)$, contradicting the dimensional-genericity. \square

2 Bilinear quasi-forms

We introduce bilinear quasi-forms, which are a fundamental instrument in the analysis of ω -categorical groups and rings of finite dimension.

Definition. *A bilinear quasi-form λ from G, H to K abelian groups is a partial function from $G \times H$ to K such that, for every $g \in G$ and $h \in H$, the partial functions*

$$\lambda_g := \lambda(g, -) : h \in \text{dom}(\lambda(g, -)) \rightarrow \lambda(g, h) \in K$$

and

$$\lambda'_h := \lambda(-, h) : g \in \text{dom}(\lambda(-, h)) \rightarrow \lambda(g, h) \in K$$

are partial endomorphisms. We denote $\text{ann}_H(g) = \{h \in H : \lambda(g, h) = 0_K\}$ and similarly $\text{ann}_G(h)$ for $h \in H$.

We define the almost annihilator of a bilinear quasi-form.

Definition. *Let λ be a bilinear quasi-form from $G \times H$ to K and let $A \leq G$ and $B \leq H$. We denote by $\widetilde{\text{ann}}_H(A)$ the almost annihilator of A over H with respect to λ defined as*

$$\widetilde{\text{ann}}_H(A) := \{h \in H : \lambda(A, h) \text{ is finite}\} = \{h \in H : \text{ann}_G(h) \gtrsim A\}.$$

Similarly, we define the almost annihilator of B over G with respect to λ the subgroup

$$\widetilde{\text{ann}}_G(B) := \{g \in G : \lambda(g, B) \text{ is finite}\} = \{g \in G : \text{ann}_H(g) \succeq B\}.$$

If the quasi-form is clear, we call them almost annihilator of A over H and almost annihilator of B over G , respectively.

It follows from Lemma 1.1 that, if G, H, A, B and ϕ are definable in a finite-dimensional structure, the almost annihilators are definable.

?(definability)? **Lemma 2.1.** *Assume we are working in a finite-dimensional theory T . Let ϕ be a definable bilinear quasi-form from $G \times H$ to K and A, B definable subgroups of G, H respectively. Then, $\widetilde{\text{ann}}_H(A)$ and $\widetilde{\text{ann}}_G(B)$ are definable.*

Proof. From Lemma 1.1 applied the uniformly definable family of subgroups $\{\text{ann}_G(h)\}_{h \in \widetilde{\text{ann}}_H(A)}$, there exists $N \in \mathbb{N}$ such that $N > |G : \text{ann}_G(h)|$. This implies that $\widetilde{\text{ann}}_H(A) = \{h \in H : |G : \text{ann}_G(h)| \leq n\}$ is definable. The conclusion for $\widetilde{\text{ann}}_G(B)$ follows similarly. \square

We introduce the notions of triviality, almost triviality, and virtual almost triviality.

Definition. *A bilinear quasi-form λ is trivial if $\lambda(g, h) = 0$ for all $g \in G, h \in H$. It is almost trivial if $\lambda(G, H) \leq S$ with S a finite subgroup of K . λ is virtually almost trivial if there exists $G_0 \leq G$ and $H_0 \leq H$ subgroups of finite index such that $\lambda|_{G_0 \times H_0}$ is almost trivial.*

We have a characterization of virtually almost trivial bilinear quasi-form given by [1, Corollary 5.11].

?(almosttrivial)? **Lemma 2.2.** *Let $\lambda : G \times H \rightarrow K$ be a definable bilinear quasi-form. Then, TFAE:*

- λ is virtually almost trivial;
- $\widetilde{\text{ann}}_G(H) \succeq G$;
- $\widetilde{\text{ann}}_H(G) \succeq H$.

From now on, we assume that the bilinear quasi-forms are definable in a finite-dimensional theory. We introduce the notion of reduced dimension of a bilinear quasi-form.

Definition. *Let $\lambda : G \times H \rightarrow K$ be a definable bilinear quasi-form of finite dimension. Then, we define*

- *the reduced dimension of G , denoted as $\text{rdim}_\lambda(G)$, as $\max\{\dim(G/\text{ann}_G(\bar{h}))\}_{\bar{h} \text{ finite tuple in } H}$;*
- *the reduced dimension of H denoted as $\text{rdim}_\lambda(H)$ as $\max\{\dim(H/\text{ann}_H(\bar{g}))\}_{\bar{g} \text{ finite tuple in } G}$;*
- *the reduced dimension of λ denoted as $\text{rdim}(\lambda)$ as $\text{rdim}_\lambda(G) + \text{rdim}_\lambda(H) + \dim(K)$.*

We also introduce the notion of induced bilinear quasi-form.

Definition. A bilinear quasi-form $\bar{\lambda} : A \times B \rightarrow D/C$ is induced from $\lambda : G \times H \rightarrow K$ if $A \leq G, B \leq H, C, D \leq K$ and $\pi \circ \lambda : \lambda^{-1}(D) \cap A \times B \rightarrow D/C$ coincides with $\bar{\lambda}$ where π is the projection from D to D/C .

Observe that, in order to be $\bar{\lambda}$ well-defined, $A \times B$ must be almost contained in $\lambda^{-1}(D)$.

We verify that, given λ a bilinear quasi-form, the reduced dimension of any induced bilinear quasi-form is less than $\text{rdim}(\lambda)$.

^{?(induced)?} **Lemma 2.3.** *if $\bar{\lambda} : A \times B \rightarrow D/C$ is induced from λ , then $\text{rdim}_{\bar{\lambda}}(B) \leq \text{rdim}_{\lambda}(H)$ and $\dim(D/C) \leq \dim(K)$. In particular, $\text{rdim}(\bar{\lambda}) \leq \text{rdim}(\lambda)$.*

Proof. The latter conclusion follows easily from the previous one. For the first point, it is sufficient to prove that

$$\max\{\dim(H/\text{Ker}(\lambda_g))\}_{g \in G} \geq \max\{\dim(B/\text{Ker}(\bar{\lambda}_a))\}_{a \in A}.$$

For every $\bar{a} \in A$, denote λ_a^C the function that sends $h \in H$ to $\lambda_a(h) + C$. Then,

$$\dim(H/\text{ker}(\lambda_a)) \geq \dim(H/\text{ker}(\lambda_a^C)) \geq \dim(A/\text{ker}(\lambda_a^C \cap A)) = \dim(A/\text{ker}(\bar{\lambda}_a)).$$

This completes the proof. \square

Finally, we recall [1, Lemma 4.6].

^{?(Lemm 4.6)?} **Lemma 2.4.** *Let G be an ω -categorical abelian group and f a definable quasi-endomorphism of G . Then, there is $n < \omega$ such that $\text{im}(f^{\circ n}) \oplus \text{ker}(f^{\circ n}) \sim G$.*

3 Principal indiscernible sequences

We introduce principal dimensional-generic indiscernible sequences. As proven in Lemma 1.8, any dimensional-generic is subgroup-generic. Therefore, these sequences are particular examples of principal subgroup-generic sequences introduced in [1]. On the other hand, the existence of principal indiscernibles finite-dimensional sequences will substantially simplify the proof of Theorem 0.7.

Definition. Let G be an infinite definable group and A a set of parameters. An A -indiscernible sequence $\langle (g_i, \bar{a}_i); i \in I \rangle$ with $g_i \in G$ for each $i \in I$ is principal indiscernible if, for any $i \in I$ and $A_i = A \cup \{g_j, \bar{a}_j : j \neq i\}$, whenever C is an A_i -definable coset of some subgroup H and $g_i \in C$, then $g_i \in H_{A_i}^0$. A sequence $\{g_i\}_{i \in I}$ is principal indiscernible dimensional-generic if it is principal, indiscernible, and dimensional-generic.

We modify slightly [1, Proposition 6.4].

^{?(Prinseq)?} **Proposition 3.1.** *Let $\langle (g_i, \bar{a}_i) : i \in \mathbb{Q} \cup (\mathbb{Q} + \varepsilon) \rangle$, with ε an infinitesimal, be an A -indiscernible sequence. Then, the sequence $\langle (h_i, \bar{a}_i \bar{a}_{i+\varepsilon}) : i \in \mathbb{Q} \rangle$ with $h_i = g_{i+\varepsilon}^{-1} g_i$ is principal indiscernible. If, moreover, $\langle (g_i, \bar{a}_i) : i \in \mathbb{Q} \cup (\mathbb{Q} + \varepsilon) \rangle$ is dimensional-generic, then also $\langle (h_i, \bar{a}_i \bar{a}_{i+\varepsilon}) : i \in \mathbb{Q} \rangle$ is dimensional-generic. In particular, principal indiscernible dimensional-generic sequences exist.*

Proof. The first part follows from [1, Proposition 6.4]. The second part by the fact that if $\text{tp}(g_{i+\varepsilon}/A, \{g_j, \bar{a}_j : j \leq i\})$ is dimensional-generic, also $\text{tp}(g_{i+\varepsilon}^{-1}/A, \{g_j, \bar{a}_j : j \leq i\})$ and $\text{tp}(g_{i+\varepsilon}^{-1}g_i/A, \{g_j, \bar{a}_j : j \leq i\})$ are dimensional-generic. From Lemma 1.7, there exists an indiscernible dimensional independent sequence. Therefore, applying the construction of the first part, we may obtain a principal indiscernible dimensional-generic sequence. This completes the proof of the Lemma. \square

4 Virtual almost triviality

We now prove Theorem 0.7. By Lemma 2.2, it is sufficient to show that one of the two almost annihilators is of finite index in G or H , respectively.

We prove Theorem 0.7 by induction on the reduced dimension of the bilinear quasi-form λ . Since for dimension 0, the conclusion is clear, we may assume that $\text{rdim}(\lambda) = N$ and that any bilinear quasi-form of reduced dimension strictly less than N is virtually almost trivial.

We need some preliminary lemmas.

^{?(Comparability)?} **Lemma 4.1.** *Let $\lambda : G \times H \rightarrow K$ be a definable bilinear quasi-form, and that G, H, K are ω -categorical groups of finite dimension.*

Assume that any induced quasi-form of strictly smaller reduced dimension is virtually almost trivial. Let $\langle y_i \rangle_{i \in \mathbb{Q}}$ be a principal indiscernible sequence in G over \emptyset . Then, $\text{im}(\lambda_{y_i})$ and $\text{im}(\lambda_{y_j})$ are comparable. The same holds for the annihilators.

Proof. Let $C = \sum_{i=0}^n \text{im}(\lambda_{y_i})$ be a sum of maximal dimension among the finite sums of images of λ_y for $y \in \langle y_i \rangle_{i \in \mathbb{Q}}$. Moreover, we may assume that, for any $j \leq n$, $\dim(\sum_{i=0, i \neq j}^n \text{im}(\lambda_{y_i})) < \dim(\sum_{i=0}^n \text{im}(\lambda_{y_i}))$. By maximality of the dimension, for every $y \in \langle y_i \rangle_{i \in \mathbb{Q}}$, $\text{im}(\lambda_y)$ is almost contained in the previous sum.

We consider the induced bilinear quasi-form $\bar{\lambda} : G \times H \rightarrow K/C$ and denote by A the almost annihilator of $\bar{\lambda}$ in G and by B the subgroup $\{h \in H : \lambda_h(A) \text{ is finite}\}$. A, B are definable by Lemma 2.1, and moreover, H is almost contained in B by Lemma 1.4. By definition, $\lambda_a[B] \leq \lambda_a[H] \lesssim C$ for any $a \in A$ and similarly for $h \in B$. The restriction

$$\lambda : A \times B \rightarrow C$$

is an induced $\{y_i\}_{i \in \mathbb{Q}}$ form as is

$$\lambda_i : A \times B \rightarrow C/\text{im}(\lambda_i) \cap C.$$

Each of the λ_i has dimension strictly less than $\dim(\lambda)$ (by minimality of n) and so it must be virtually almost trivial. Therefore, the almost annihilator is a $\{y_i\}_{i \leq n}$ -definable subgroup of finite index in A i.e. it contains $A_{\{y_i : i \leq n\}}^0$. Since y is principal, it must be contained in the connected component and so $\text{im}(\lambda_y) \lesssim \text{im}(\lambda_{y_i})$. By principal indiscernibility, the conclusion follows.

For the annihilators, take an intersection $B := \bigcap_{i \leq n} \text{ann}(\lambda_{y_i})$ of minimal dimension possible and with n minimal among the intersections of minimal dimension.

Let $A := \{g \in G : B \lesssim \text{ann}_H(g)\}$ and consider the following quasi bilinear form

$$\lambda_i : A \times \text{ann}_H(y_i) \rightarrow K.$$

We show that the reduced dimension of λ_i is strictly less than $\text{rdim}(\lambda)$. It is sufficient to prove that $\text{rdim}_{\lambda_i}(\text{ann}_H(y_{l_i})) < \text{rdim}_{\lambda}(H)$. Taken $\bar{g} \in G$, then $\text{ann}_H(\bar{g}) \supseteq B$ and so the dimension of $\text{ann}_H(y_{l_i})/\text{ann}_{\text{ann}_H(y_{l_i})}(\bar{g})$ is less than $\dim(\text{ann}_H(y_{l_i})/B)$. By minimality, $\dim(\bigcap_{j=1, j \neq i} \text{ann}_H(y_j)/B) > 0$ and so

$$\dim(\text{ann}_H(y_{l_i})/B) < \dim(\text{ann}_H(y_{l_i})/B) + \dim\left(\bigcap_{j=1, j \neq i} \text{ann}_H(y_j)/B\right) \leq \dim(H/B)$$

that is less than the reduced dimension of H . Therefore, λ_i is virtually almost trivial. Consequently, the almost annihilator is a subgroup of finite index in A and so it contains the connected component. This implies that $\text{ann}_H(y_{l_i}) \lesssim \text{ann}_H(y_{l_j})$. \square

^{?(NilpoIsom)?} **Lemma 4.2.** *Suppose that λ is definable in a finite-dimensional theory and that any induced bilinear quasi-form of strictly smaller dimension is virtually almost trivial. Let $\langle y_i \rangle_{i \in \mathbb{Q}}$ be a principal dimensional-generic sequence. Then, for any $j \in \mathbb{Q}$, any definable quasi-endomorphism f of $H/\text{ann}_H(y_j)$ is either invertible or nilpotent.*

Proof. By Lemma 2.4, there exists $n < \omega$ such that $G \lesssim \text{im}(f^n) \oplus \ker(f^n)$. The group

$$A := \{g \in G : \text{ann}_H(y_j) \lesssim \text{ann}_H(g)\}$$

is a definable subgroup by Lemma 1.1. Denote B_1, B_2 the preimages in H of $\text{im}(f^n)$ and $\ker(f^n)$ respectively. Then $B_1 \cap B_2$ is a finite extension of $\text{ann}_H(y_j)$. If f is neither invertible nor nilpotent, then both the B_i are infinite and the restriction of λ to $A \times B_i$ has strictly smaller dimension. Therefore, it is virtually almost trivial and the same for the bilinear quasi-form

$$\lambda' : A \times H \rightarrow K$$

given by the sum of the two. By Lemma 4.1, $y_i \in A$ for either $i < j$ or $j > i$. Therefore, $y_i \in \text{ann}_G(H)$ and the latter must be of finite index since y_i is dimensional-generic. Therefore, by Lemma 2.2, λ is virtually almost trivial. \square

We start the proof of Theorem 0.7.

Proof. Let λ be a counterexample of Theorem 0.7 of minimal dimension. Up to add finitely many parameters, we can assume that everything is \emptyset -definable. Let $\langle (x_i, x'_i) : i \in \mathbb{Q} \cup \mathbb{Q} + \varepsilon \rangle$ be a dimensional-generic sequence, that exists by Lemma 1.7. Define $y_i = x_i - x_{i+\varepsilon}$ and $y'_i = x'_i - x'_{i+\varepsilon}$. Then, by Lemma 3.1, $\langle (y_i, y'_i) : i \in \mathbb{Q} \rangle$ is a principal indiscernible dimensional-generic sequence.

Claim. *For $i < j$, $\text{ann}_H(y_j) \lesssim \text{ann}_H(y_i)$ and $\text{im}(\lambda_i) \lesssim \text{im}(\lambda_j)$.*

Proof. The second is clearly a consequence of the first and of Lemma 4.1. $\widetilde{\text{ann}}_H(G)$ is definable and the indexes $|G : \text{ann}_G(h)|$ takes only finitely many values. Let n be the maximal one and let h realize a generic type for $\widetilde{\text{ann}}_H(G)$. Then, taken x_0, x_1, \dots, x_n , there exist $i \neq j$ such that $x_i - x_j \in \text{ann}_G(h)$ and so $h \in \text{ann}_H(x_i - x_j)$. By dimensional-genericity of h , the subgroup $\text{ann}_H(x_i - x_j) \cap \widetilde{\text{ann}}_H(G)$ is of finite index in $\widetilde{\text{ann}}_H(G)$ i.e. $x_i - x_j \in \widetilde{\text{ann}}_G(\widetilde{\text{ann}}_H(G))$. By indiscernibility, $\text{ann}_H(y_0) \supseteq \widetilde{\text{ann}}_H(G)$. Suppose $\text{ann}_H(y_0) \lesssim \text{ann}_H(y_1)$. Then $y_1 \in \widetilde{\text{ann}}_G(\text{ann}_G(y_0))$. By dimensional-genericity,

$\widehat{\text{ann}}_G(\text{ann}_G(y_0))$ should be of finite index. Consequently, $\widehat{\text{ann}}_H(G) \succeq \widehat{\text{ann}}_H(y_0)$ and the same must hold for every y_i for $i \in \mathbb{Q}$ by indiscernibility. This implies that $\text{ann}_H(y_0) \sim \widehat{\text{ann}}_H(G) \sim \text{ann}_H(y_1)$. If $\text{ann}_H(y_1) \lesssim \text{ann}_H(y_0)$, the conclusion follows. \square

We define $\lambda_{y_1, y_j} = \lambda_{y_j}^{-1} \circ \lambda_{y_1}$ to be the definable additive relation in $H \times H$ given by

$$\{(h, h') : \lambda(y_1, h) = \lambda(y_j, h')\}.$$

Clearly, the kernel of this relation is $\text{ann}_H(y_1)$ and $\text{kat}(\lambda_{y_1, y_j}) = \text{ann}_H(y_j)$. By Claim 1, this defines a quasi-homomorphism $\overline{\lambda}_{y_1, y_j}$ of $H/\text{ann}_H(y_1)$. For the definition of additive relation and quasi-homomorphism see section 3 of [1].

Claim. For $i \neq j$, $\text{ann}_H(y_i) \not\sim \text{ann}_H(y_j)$.

Proof. Suppose otherwise and put $\overline{H} = H/\text{ann}_H(y_0)$. If $\text{ann}_H(y_0)$ is of finite index, $y_0 \in \widehat{\text{ann}}_G(H)$ that, being y_0 dimensional-generic, should imply that $\widehat{\text{ann}}_G(H)$ is of finite index in G . Consequently, λ is virtually almost trivial.

Assume, for a contradiction, that $\text{ann}_H(y_0) \sim \text{ann}_H(y_i)$. Then $\text{ann}_H(y_0) \sim \text{ann}_H(y_i)$ for every i by indiscernibility. This implies that $\text{im}(\lambda_{y_i}) \sim \text{im}(\lambda_{y_j})$ since $\text{im}(\lambda_0) \lesssim \text{im}(\lambda_i)$ and

$$\dim(\text{im}(\lambda_0)) = \dim(H) - \dim(\text{ann}_H(y_0)) = \dim(H) - \dim(\text{ann}_H(y_i)) = \dim(\text{im}(\lambda_i)).$$

Let R be the ring of quasi-endomorphisms of \overline{H} modulo equivalence. Then, every element of R is nilpotent or invertible by Lemma 4.2. The subset $I = \{r \in R : r \text{ is nilpotent}\}$ is an ideal in R . Indeed, it is clearly invariant for left and right multiplication. On the other hand, if $f, g \in I$, then $f + g$ is nilpotent. If not, there is $h \in R$ such that $h(f + g) = 1$. But $(id - hg)(id + hg + (hg)^2 + \dots) = id$ (this sum is finite since hg is nilpotent) implies that $id - hg = hf$ is invertible, a contradiction. Therefore, R/I is a division ring. Let \overline{r} be a finite tuple in R/I definable over a finite set A . Then, by ω -categoricity, \overline{r} generates a finite subring of R . Therefore, by little Wedderburn's theorem, this is a locally finite field. Any quasi-endomorphism λ_{y_i, y_j} is a quasi-automorphism on \overline{H} . Indeed, $\ker(\lambda_y) = \ker(\lambda_{y'})$ and therefore λ_y is a quasi-isomorphism from \overline{H} to $\text{im}(\lambda_0)$. This implies that the composition of λ_y and $\lambda_{y'}^{-1}$ is a quasi-automorphism. In particular, it is not nilpotent. By local finiteness and indiscernibility, $\lambda_{y, y'}$ in R/I has finite order equal to N . Since there are only finitely many N -roots of the unity, there are finitely many couples $\{(y_i, y_j)\}_{(i, j) \in \mathbb{Q}^2}$ such that, for any $y, y' \in \{y_i\}_{i \in \mathbb{Q}}$, there exists a couple (y_i, y_j) such that $\lambda_{y_i, y_j} \sim \lambda_{y, y'}$. Therefore, by indiscernibility, we may assume that all λ_{y_i, y_j} are equal to the same $\eta + I$. Then, $\eta^2 + I = \lambda_{y_j, y_k} \lambda_{y_i, y_j} + I = \eta + I$ and so $\eta = 1$ (since $\eta \neq 0$).

Since $\lambda_{x_1 - x_3, x_2 - x_3} - id_H \in I$ then $B := \text{im}(\lambda_{x_1 - x_3, x_2 - x_3} - id_H)$ is of infinite index in H almost containing $\text{ann}_H(y_0)$.

Consequently, for any $h \in H_{x_1, x_2, x_3}^\circ \leq \text{dom}(\lambda_{x_1 - x_3, x_2 - x_3} - id_H)$, there exists $b \in B$ such that $h + b \in \lambda_{x_1 - x_3, x_2 - x_3}(h)$. Hence,

$$\lambda(x_1 - x_3, h) = \lambda(x_2 - x_3, h + b) = \lambda(x_2 - x_3, h) + \lambda(x_2 - x_3, b)$$

whence

$$\lambda(x_1 - x_2, h) = \lambda(x_1 - x_3, h) - \lambda(x_2 - x_3, h) = \lambda(x_2 - x_3, b).$$

This means that $\text{im}(\lambda_{x_2-x_3}) \simeq \text{im}(\lambda_{x_1-x_2})$ is almost contained in $\lambda_{x_2-x_3}[B]$. But $\ker(\lambda_{x_2-x_3}) \lesssim B$ so

$$\dim(\lambda_{x_2-x_3}[B]) = \dim(B) + \dim(\ker(\lambda_{x_2-x_3})) < \dim(\lambda_{x_2-x_3}).$$

This is clearly a contradiction. \square

We now complete the proof.

Observe that $\text{im}(\lambda_{y_i})$ is of infinite index in K for $i \in \mathbb{Q}$. If not, there exists $n < \omega$ such that the following formula is satisfied by $(x_i, x_{i+\varepsilon})$:

$$\exists k_1, \dots, k_n \in K \forall k \in K \exists h \in H \bigvee_{i=1}^n k_i \lambda(x_i - x_{i+\varepsilon}, h) = k.$$

By indiscernibility, every y_j realizes this formula. Therefore, the images of λ_{y_j} are all of finite index in K , implying that $\text{ann}_H(y_j) \sim \text{ann}_H(y_i)$ for all $i, j \in \mathbb{Q}$, a contradiction by the previous claim.

Take $0 < k \in \mathbb{Q}$ and put

$$A := \{g \in G : \text{im}(\lambda_g) \lesssim \text{im}(\lambda_{y_k})\}$$

and

$$H' := \{h \in H : \lambda'_h[A] \lesssim \text{im}(\lambda_{y_k})\}.$$

We verify that $\dim(H') = \dim(H)$ and so $H' \sim H$. Let

$$X := \{(a, h) \in A \times H : \lambda(a, h) \in \text{im}(\lambda_{y_k})\}.$$

X is clearly definable. We prove that $\dim(X) = \dim(A \times H)$. Taken the first projection π_1 , this is a surjective map from X to A with fibers $\pi_1^{-1}(a) = \{h \in H : \lambda(a, h) \in \text{im}(\lambda_{y_k})\}$ that is of dimension equal to $\dim(H)$ by the assumption on A . By fibration, $\dim(X) = \dim(H) + \dim(A)$. Taken the second projection π_2 , the image is H and the fibers are $\pi_2^{-1}(h) = \{a \in A : \lambda(a, h) \in \text{im}(\lambda_{y_k})\}$. H' is clearly the set (definable by Lemma 1.1) of $h \in H$ with fibers of maximal dimension. By union, $\dim(X) \leq \max\{\dim(H') + \dim(A), \dim(H - H') + \dim(A) - 1\}$. $\dim(X) = \dim(A) + \dim(H)$ and necessarily $\dim(H') = \dim(H)$.

We now study the bilinear quasi-form

$$\tilde{\lambda} : A \times H' \rightarrow \text{im}(\lambda_{y_k}).$$

This has reduced dimension strictly less than $\text{rdim}(\lambda)$. Indeed, $\text{rdim}_{\tilde{\lambda}}(A) \leq \text{rdim}_{\lambda}(G)$ by Lemma 2.3. The same holds for $\text{rdim}_{\tilde{\lambda}}(H')$. Finally, $\dim(\lambda_{y_i}) < \dim(K)$ by previous proof. By assumption on the minimality of $\dim(\lambda)$, $\tilde{\lambda}$ is virtually almost trivial and so $\widetilde{\text{ann}}_A(H')$ is a $(x_k, x_{k+\varepsilon})$ -definable subgroup of finite index in A such that, for all $g \in \widetilde{\text{ann}}_A(H')$, $\lambda(g, \text{ann}_{H'}(A))$ is finite. Since $\widetilde{\text{ann}}_{H'}(A)$ is of finite index in H' and so also in H , $\widetilde{\text{ann}}_A(H') \leq \widetilde{\text{ann}}_G(H)$. Taken y_l for $l < k$, $\text{im}(y_l) \lesssim \text{im}(y_k)$ by Claim 1 and so $y_l \in A$. By principality, $y_l \in A_{y_k}^\circ$. This implies that $y_l \in \widetilde{\text{ann}}_G(H)$. By indiscernibility, this holds for every y_i . Since y_0 is dimensional-generic, this implies that $\widetilde{\text{ann}}_G(H)$ is of finite index in G . Finally, the proof follows from Lemma 2.2. \square

5 Groups and rings

We show that finite-dimensional ω -categorical groups are finite-by-abelian-by-finite and that finite-dimensional ω -categorical rings are virtually finite-by-null.

We start by verifying that the case (2) in Theorem 0.5 is impossible for finite-dimensional groups. In other words, we need to prove that a definable characteristically simple group G of finite dimension cannot be isomorphic to $B(F)$ or $B^-(F)$. This is a consequence of the following Lemma.

^{?(B(F))?} **Lemma 5.1.** *The groups $B(F)$ and $B^-(F)$ are not of finite dimension i.e. there is no finite-dimensional theory T in which $B(F)$ and $B^-(F)$ can be defined.*

Proof. It is sufficient to observe that there exists an infinite chain of definable subgroups $\{C_i\}_{i < \omega}$ such that $|C_i/C_{i+1}|$ is infinite, contradicting ω -DCC.

Let $\{A_i\}_{i < \omega}$ be a strictly ascending chain of clopen subsets in the Cantor set (this clearly exists). We take, for f a non central element in F , the function g_1 such that $g_1(A_1) = \{f\}$ and $g_1(C - A_1) = \{0\}$. Let C_1 be the centraliser in $B(F)$ of g_1 . This contains all the functions g such that $g(A_1) \leq C_F(f)$ and g assumes arbitrary values outside A_1 . Let g_2 be a function with $g_2(A_2) = f$ and let C_2 be the centraliser of g_2 in $B(F)$. We verify that $|C_2 : C_1|$ is infinite. Indeed, there exist pairwise disjoint clopen subsets $\{D_j\}_{j < \omega}$ of C such that $D_j \subseteq A_2$ and $D_j \cap A_1 = \emptyset$. Take the functions $s_j \in B(F)$ such that $s_j(A_1) = f$ and $s_j(D_j) = d_j \in F - C_F(f)$. Then, $s_j^{-1}s_i \in C_1$ but not in C_2 . This verifies that the index of C_2 in C_1 is infinite. Iterating the proof for any i , we obtain the contradiction. The proof for $B^-(F)$ is similar. \square

The second step is to show that a ω -categorical finite-dimensional definable group is virtually nilpotent. To do this, we can proceed in two ways: either as in [1] or using the two following results.

Lemma 5.2. *A locally nilpotent definable group of finite dimension is virtually nilpotent.*

Proof. It is sufficient to prove that, given G a locally nilpotent definable group of finite dimension, $\tilde{Z}(G)$ is infinite. Iterating the proof, we conclude that G is almost nilpotent. Consequently, G is virtually nilpotent from [2, Proposition 3.29].

Assume, by contradiction, that the almost center is finite. Then $G/\tilde{Z}(G)$ is a definable locally nilpotent group of finite dimension with trivial almost center by Lemma 1.3. We verify that this is impossible. Let C be a centraliser of a tuple of minimal dimension between all the centralisers. If $\dim(C) = 0$, take C_1 a minimal finite centraliser of a finite tuple. Then $C_1 \cap C_G(g)$ is not empty by locally nilpotency. By minimality, $C_1 \leq C_G(g)$ for all $g \in G$ i.e. $C_1 \leq Z(G) = \{0\}$, a contradiction. Assume that C is infinite. Then, $C \cap C_G(g)$ has same dimension as C and so $C \leq C_G(g)$ for all $g \in G$ i.e. $G \leq \tilde{C}_G(C)$. By Lemma 1.4, $C \leq \tilde{Z}(G)$. This is again contradictory. \square

Since an ω -categorical group of exponent p is locally nilpotent, the third case of Lemma 0.5 is impossible. Since the second case is excluded by Lemma 5.1, we have the following conclusion.

^{?⟨AbeChaSim⟩?} **Lemma 5.3.** *Let G be an ω -categorical finite-dimensional characteristically simple group. Then, G is an elementary abelian p -group.*

By Lemma 0.4 and Lemma 5.3, for any ω -categorical group G of finite dimension, there exists a series of definable characteristic subgroups $\{G_i\}_{i \leq n}$ such that:

- G_i/G_{i+1} is characteristically simple;
- G_i/G_{i+1} is an elementary abelian p -group.

Therefore, we have the following conclusion.

Lemma 5.4. *Let G be a soluble finite-dimensional group. Then, G is soluble and of finite exponent.*

Finally, the virtual nilpotency follows from the next Lemma.

Lemma 5.5. *Let G be a finite-dimensional ω -categorical group. Then, G is virtually nilpotent.*

Proof. By [2, Proposition 3.29], it is sufficient to prove that G is almost nilpotent. We prove the Lemma by induction on the dimension. Clearly, if G has infinite almost center, the conclusion follows by induction hypothesis. Let $\{G_i\}_{i \leq n}$ be a series of definable subgroups of G such that G_i/G_{i+1} is infinite and almost abelian. This exists by [2, Proposition 3.21]. Clearly, the almost abelian group G_{n-2}/G_{n-1} acts on the abelian group G_{n-1}/G'_{n-1} . Let A be a definable subgroup of G_{n-1}/G'_{n-1} of minimal dimension between all the subgroup that are minimal for a definable subgroup of finite index in G_{n-2}/G_{n-1} . Let $H \leq G_{n-2}/G_{n-1}$ be a definable subgroup of finite index such that A is H -invariant. Then, by construction, A is absolutely H -minimal. By Linearization Theorem 1.6, if the action is not almost trivial, there exists a finite subgroup F_1 in A and a definable subgroup F_2 not of finite index in H such that $A/F_1 \simeq R^+$ and H/F_2 embeds in R^\times . The latter contradicts the fact that H is of finite exponent. Therefore, the action is almost trivial. This implies that $\tilde{C}_{G_{n-1}}(G_{n-2}) =: C$ is a definable infinite G -normal subgroup in G_{n-1} . By Lemma 1.4, $\tilde{C}_{G_{n-2}}(C)$ is of finite index in G_{n-2} . Moreover, by Lemma 1.5, $[C, \tilde{C}_{G_{n-2}}(G)] =: C_1$ is a finite G -normal subgroup. $G_{n-3}/\tilde{C}_{G_{n-2}}$ is almost abelian and acts on C/C_1 . Iterating the proof for any G_i , we obtain that that $\tilde{Z}(G)$ is infinite. \square

We can also verify that G is virtually nilpotent-by-finite with a proof similar to [1]. We need the following lemma.

^{?⟨BooAlg⟩?} **Lemma 5.6.** *An atomless Boolean algebra cannot have finite dimension (i.e. it cannot be defined in a finite-dimensional theory).*

Proof. Assume, for a contradiction, that there exists a finite-dimensional theory T and B an atomless Boolean algebra definable in T . Assume B of minimal dimension n among all the infinite atomless Boolean algebras definable in T .

Let $a \in B$ be an element not equal to 0 or 1 and define the function

$$f_a : (B, \vee) \rightarrow (B_a, \vee)$$

that sends b in $f_a(b) = a \wedge b$ where $B_a = \{b \in B : b \leq a\} = \{b \in B : \exists c \ a \wedge c = b\}$. This map is clearly surjective. We verify that B_a is an atomless Boolean algebra. Since B_a is clearly definable, it is either finite or its dimension is equal to n . It is closed for \wedge and \vee since $a \wedge (b \vee c) = a \wedge b \vee a \wedge c$ and $a \wedge (a \wedge b) = a \wedge b$ (since $a \wedge 1 = a$, this also proves that f_a is an homomorphism of monoids). Therefore, B_a is closed for \vee . It is closed also for \wedge since $a \wedge b \wedge a \wedge c = a \wedge (b \wedge c)$. The other properties of Boolean algebra follow easily (assuming $1_{B_a} = a$). Finally, we prove that it is atomless. It is sufficient to show that, for any $a \wedge c \in B_a$, there exists $d \in B_a$ such that $d \neq 0, a \wedge c$ and $d < a \wedge c$. Since B is an atomless Boolean algebra and $a \wedge c \in B$, there exists $d \in B$ such that $0 < d < a \wedge c \leq a$ and so $d \in B_a$. Therefore, $n = \dim(B) \geq \dim(B_a) \geq n$ so $\dim(B_a) = n$. If we verify that $f_a^{-1}(c)$ is infinite for every $c \in B_a$, it implies that $n = \dim(B) \geq 1 + \dim(B_a) = 1 + n$ by fibration, a contradiction. We prove that $f_a^{-1}(0) = \{b \in B : b \leq \neg a\}$. The latter set is infinite since B is atomless and $\neg a = 0$ iff $a = 1$. Assume $b \leq \neg a$. Then, $a \wedge b = a \wedge (\neg a \wedge b) = (a \wedge \neg a) \wedge b = 0 \wedge b = 0$. On the other hand, if $a \wedge b = 0$, then $b = 1 \wedge b = (a \vee \neg a) \wedge b = (a \wedge b) \vee (\neg a \wedge b) = \neg a \wedge b$ and so $b \leq \neg a$.

Given $b \in f_a^{-1}(0)$ and $d \in f_a^{-1}(c)$, then $a \wedge (b \vee d) = (a \wedge b) \vee (a \wedge d) = a \wedge d = c$. Clearly, $f_a(d) = f_a(a \wedge d)$. Therefore, any element $(a \wedge d) \vee b$ with $b \in f_a^{-1}(0)$ is in $f_a^{-1}(c)$. Assume that, for $b \neq b_1 \leq \neg a$, $(a \wedge d) \vee b = (a \wedge d) \vee b_1$. Then, $\neg a \wedge ((a \wedge d) \vee b) = \neg a \wedge ((a \wedge d) \vee b_1)$. But the first is equal to $\neg a \wedge b$ and the second to $\neg a \wedge b_1$. Since $b, b_1 \leq \neg a$, these are equal to b and b_1 , contradicting the assumption that $b \neq b_1$. Therefore, each fiber is infinite, and we reach a contradiction. \square

Since any ω -categorical group can be written as a chain of characteristic subgroups with characteristically simple factors, we may assume that every ω -categorical finite-dimensional subgroup is soluble. If it is not nilpotent-by-finite, an atomless Boolean algebra can be defined on it by [6], contradicting Lemma 5.6.

Therefore, we may assume that every ω -categorical group of finite dimension is nilpotent-by-finite. From [2, Proposition 3.29], any ω -categorical finite-dimensional subgroup has a characteristic nilpotent subgroup of finite index. Finally, we prove that any nilpotent ω -categorical subgroup of finite dimension is finite-by-abelian-by-finite. This verifies the meta-conjecture for the group case.

Lemma 5.7. *A nilpotent ω -categorical group of finite dimension is finite-by-abelian-by-finite.*

Proof. We may take a counterexample of minimal nilpotent class possible.

By minimality, $G/Z(G)$ is finite-by-abelian-by-finite. Since G is not finite-by-abelian-by-finite, $Z(G)$ cannot be finite. The pre-image of $\tilde{Z}(G/Z(G))$ is a characteristic subgroup of finite index in G , and so we may replace G with it. By [2, Proposition 3.27], $F := G'Z(G)$ is a finite extension of $Z(G)$. Again by [2, Proposition 3.27], F' is finite and characteristic. Therefore, the quotient G/F' is again ω -categorical and we can assume $F' = \{0\}$. As $F/Z(G)$ is characteristic, $C_G(F/Z(G))$ is a characteristic definable subgroup of finite index. Up to take $G = C_G(F/Z(G))$, we may assume $[G, F] \leq Z(G)$ i.e. $F \leq Z^2(G)$. Consequently, the map

$$[-, g] : G \rightarrow F/Z(G)$$

that sends g' in $[g, g'] + Z(G)$ is a definable homomorphism. The kernel of H_g has finite index in G for any g . Consequently, the map

$$[-, g] : h \in H_g \rightarrow [h, g] \in Z(G)$$

is a homomorphism with kernel containing $H'_g Z(G)$, since the image is abelian. Let G_0 be a definable subgroup in G of finite index such that $|G'_0 : G'_0 \cap Z(G)|$ is minimal. Then, for any definable subgroup H of finite index, $G'_0 Z(G)$ and $(H \cap G_0)' Z(G)$ are finite extensions of $Z(G)$ with $G'_0 Z(G) \geq (G_0 \cap H)' Z(G)$. Moreover, $|G'_0 Z(G) / Z(G)| = |G'_0 / Z(G) \cap G'_0|$ and $|(G_0 \cap H)' Z(G) : Z(G)| = |(G_0 \cap H)' / Z(G) \cap (G_0 \cap H)'|$. By minimality, the latter must be equal to the first. Therefore, $G'_0 Z(G) = (H_g \cap G_0)' Z(G) \leq H'_g Z(G) \leq C_G(g)$. This implies that $G'_0 \leq Z(G)$ and so $Z_2(G)$ is of finite index in G . Since $Z_2(G)$ is a characteristic subgroup, we may assume that G is nilpotent of class 2. Then the function

$$[-, -] : (g_1, g_2) \in G \times G \rightarrow [g_1, g_2] \in Z(G)$$

is a bilinear quasi-form. By Theorem 0.7, it should be virtually almost trivial, and this is equivalent to saying that G is finite-by-abelian-by-finite. \square

In conclusion, we have proven the following theorem.

Theorem 5.8. *Let G be an ω -categorical group of finite dimension. Then G is finite-by-abelian-by-finite.*

For rings, the proof is a straightforward corollary of Theorem 0.7.

Lemma 5.9. *Let R be an ω -categorical ring of finite dimension. Then, R is virtually finite-by-null.*

Proof. It is sufficient to observe that the function

$$-\cdot- : R \times R \rightarrow R$$

that sends (r, r') in $r \cdot r'$ is a bilinear form. By Theorem 0.7, it must be virtually almost trivial. Therefore, R is virtually finite-by-null. \square

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