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# An abstract setting for the Fredholm backstepping transformation: self-adjoint case

Ludovick Gagnon\*, Amaury Hayat†, Swann Marx‡,  
Shengquan Xiang§ and Christophe Zhang¶

## Abstract

In this paper we address the existence of a Fredholm backstepping transformation for self-adjoint operators  $A$ . We prove, under assumptions on the control operator  $B$  and  $A$ , that there exists a Fredholm backstepping transformation for operators  $A$  of order strictly greater than 1. One of the main contribution of the paper is to exhibit the underlying isomorphism used to construct  $T$ . This isomorphism allows to deduce sharp estimates on  $\|T\|_{\mathcal{L}(H;H)}$  and  $\|T^{-1}\|_{\mathcal{L}(H;H)}$  with respect to the damping parameter  $\lambda > 0$ . This allow us to prove the finite-time stabilization for a large class of self-adjoint operators.

## Contents

<b>1</b>	<b>Introduction</b>	<b>3</b>
1.1	Main results . . . . .	4
1.2	The Fredholm backstepping method . . . . .	6
1.2.1	Abstract approach . . . . .	7
1.2.2	Riesz basis . . . . .	7
1.2.3	The quadratically close criterion . . . . .	8
1.3	Previous results . . . . .	9
1.4	Outline of the paper . . . . .	9

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<b>2</b>	<b>Finite-dimensional example</b>	<b>10</b>
2.1	Solving the main equation . . . . .	10
2.2	The uniqueness condition and expression of the feedback. . . . .	11
2.3	The explicit isomorphism . . . . .	11
2.4	Invertibility of the transformation $T$ . . . . .	13
<b>3</b>	<b>Construction and invertibility of <math>T</math></b>	<b>14</b>
3.1	Functional framework . . . . .	14
3.2	Definition of the transformation $T$ and the feedback $K$ . . . . .	17
3.2.1	Some technical estimates . . . . .	18
3.2.2	Study of the auxiliary transformations $\check{T}$ and $\check{T}^{-1}$ . . . . .	19
3.2.3	Uniqueness condition and existence of the feedback. . . . .	25
3.3	Properties of $K$ . . . . .	27
3.4	Invertibility of $T$ . . . . .	29
<b>4</b>	<b>Quantitative estimates for the transformation</b>	<b>29</b>
4.1	Exhibiting the underlying isomorphism of $T$ . . . . .	29
4.2	Cost of the backstepping transformation with respect to the damping parameter $\lambda$ . . . . .	32
4.3	Rearrangement of $k_n b_n$ . . . . .	33
4.4	Behavior of $J_n(\lambda)$ . . . . .	35
4.5	Behavior of $F_n$ . . . . .	39
4.6	Estimation of $\text{Dist}_\alpha(\lambda)$ . . . . .	46
<b>5</b>	<b>Well-posedness and exponential stability of the closed-loop system</b>	<b>48</b>
5.1	Operator equality and domain of the closed-loop operator . . . . .	48
5.2	Well-posedness and stability of the closed-loop system . . . . .	53
<b>6</b>	<b>Finite-time stabilization</b>	<b>57</b>
6.1	Small-time null-controllability with a feedback form control . . . . .	57
6.2	Small-time stabilization with a bounded time periodic feedback law . . . . .	60
6.2.1	Well-posedness in $D(A^{1/2})$ . . . . .	60
6.2.2	Definition of the feedback law . . . . .	63
<b>A</b>	<b>Technical lemmas on the spectrum of <math>A</math></b>	<b>69</b>

# 1 Introduction

Consider the abstract system,

$$\begin{cases} \frac{d}{dt}y(t) = Ay(t) + Bu(t), & t \geq 0 \\ y(0) = y_0, \end{cases} \quad (1)$$

with  $y_0 \in H$ . Here,  $H$  denotes a complex Hilbert space endowed with the norm  $\|\cdot\|_H$ , the operator  $A : D(A) \subset H \rightarrow H$  is densely defined and the control operator  $B \in \mathcal{L}(U; D(A^*))$  is such that  $\dim U < +\infty$ <sup>1</sup>. We furthermore assume that  $A$  generates a strongly continuous semigroup that we denote by  $(e^{tA})_{t \geq 0}$ .

A classical problem in control theory is to establish the rapid stabilization of equations under the form (1), that is to say, for every  $\nu > 0$ , to seek a feedback operator  $K : D(K) \subseteq H \rightarrow \mathbb{R}$  such that the control in feedback form  $u(t) = Ky(t)$  and the solution  $y(t)$  of the closed-loop system,

$$\begin{cases} \frac{d}{dt}y(t) = (A + BK)y(t), & t \geq 0, \\ y(0) = y_0, \end{cases} \quad (2)$$

satisfies,

$$\|y(t)\|_H \leq Ce^{-\nu t} \|y_0\|_H,$$

for  $C > 0$ . Many techniques exist in the literature to establish the rapid stabilization of (2) from controllability properties stemming from (1) (for instance the Riccati theory [2, 4, 27], Gramian method [36], equivalence between observability and stabilizability [34], quantitative finite time stabilization [39, 38], various Lyapunov approaches [13, 23], among others). The backstepping method for PDEs, initially introduced by Balogh and Krstić for boundary control systems [1, 3], is a powerful stabilization method which consists in finding explicit feedbacks using an appropriate Volterra transformation of the control system. It has since then been generalized to internal control systems, and has spawned several variants. These generalizations show that the backstepping method is also a way to relate controllability properties to the rapid stabilization of (2).

The use of the Fredholm alternative for the backstepping method has seen recent progress. Although more intricate, the Fredholm transformation possesses the advantage of having a systematic approach, based on controllability and spectral assumptions on (1). This allows for instance to treat PDEs where the existence of a Volterra transformation is difficult to establish, due in particular to the non-trivial boundary condition on the diagonal (see for instance [26]). An important question for the backstepping method is therefore: are there any sufficient or necessary conditions for the method to work? This question has been addressed in a recent paper for skew-adjoint operators with superlinear growth of

---

<sup>1</sup>Remark that if  $\dim U = m$ , with  $1 < m < +\infty$ , and if  $A$  admits an orthonormal basis of eigenfunctions, we can write  $y = \sum_{i=1}^m y_i(t)$  with  $y_i$  satisfying  $\dot{y}_i = Ay_i + B_i u_i$  (see Assumption 1). Hence, we consider the case  $m = 1$  for the rest of the paper.

the eigenvalues [20]. An important step toward necessary or sufficient conditions was the introduction of the duality/compactness method in [20] to remove the critical threshold  $3/2$  imposed by the quadratically close criterion for the existence of a Riesz basis (see Section 1.2.2). This is a critical step to solve the so-called uniqueness condition  $TB = B$ , paramount to establish many properties such as the continuity and invertibility of the Fredholm transformation as well as the well-posedness of the closed-loop system (2).

## 1.1 Main results

In this sequence of articles, we prove necessary and sufficient conditions on the existence of Fredholm backstepping transformations for self-adjoint operators and skew-adjoint operators under classical assumptions of controllability and superlinear growth of the eigenvalues, achieving rapid stabilization for these systems. The principal breakthrough in these papers is to circumvent the use of a Riesz basis altogether by exhibiting the explicit isomorphism behind the Fredholm backstepping transformation. We state here the existence result of the Fredholm backstepping transformation for self-adjoint operators.

**Theorem 1** *Assume  $A : D(A) \subset H \rightarrow H$  and  $B \in \mathcal{L}(U; D(A)')$ , such that  $\dim U < +\infty$ , satisfy Assumption 1. Then, there exists  $T \in \mathcal{L}(H; H)$  invertible from  $H$  to itself and a feedback  $K \in D(A^s)'$  for  $s > 1/2\alpha$  such that*

$$\begin{cases} TA + BK = (A - \lambda I)T, \\ TB = B, \text{ in } D(A)', \end{cases} \quad (3)$$

*holds for  $\lambda > 0$  non-resonant<sup>2</sup>. Moreover, for any  $\lambda > 0$ , there exists a feedback  $K \in D(A^s)'$  such that the solution of the closed-loop system (2) satisfies*

$$\|e^{(A+BK)t}y_0\|_H \leq Ce^{-\lambda t}\|y_0\|_H.$$

This result holds for operators  $A$  of order  $\alpha$  strictly greater than 1, that is the eigenvalues of  $A$  satisfies  $|\lambda_n| \simeq n^\alpha, n \rightarrow \infty$ . This allows us to cross the critical threshold  $3/2$  for self-adjoint operators. Note that Theorem 1 holds in a wider range of spaces. We refer to Proposition 11 for a precise statement on the regularity of the domain and image of  $T$ . where the precise sense in which the first equation of (3) make sense is also precised. This is done in Section 5.

The explicit isomorphism underlying  $T$  allows us to solve directly the so-called uniqueness condition  $TB = B$ . The existence of a Riesz basis is now a consequence rather than a necessity. A further analysis of this explicit isomorphism allows us to characterize the cost of the backstepping transformation  $T = T_\lambda$  with respect to the damping parameter  $\lambda > 0$ .

**Theorem 2** *For any  $N \in \mathbb{N}$ , there exists  $\lambda \in [N, N + 1]$  such that the transformation  $T_\lambda$  of Theorem 1 satisfies*

$$\|T_\lambda\|_{\mathcal{L}(H;H)} + \|T_\lambda^{-1}\|_{\mathcal{L}(H;H)} \leq Ce^{c\lambda^{1/\alpha}},$$

---

<sup>2</sup>The non-resonant condition is stated in (24)

where  $c, C > 0$  do not depend on  $N$  nor  $\lambda$ , and where  $\alpha > 1$  is the order of the operator  $A$ .

The cost of the backstepping transformation, together with well-posedness results of the feedback system and estimates on the feedback  $K_\lambda$  allow us to derive the second breakthrough of the paper, that is a general finite-time stabilization result for self-adjoint operators (see for instance Proposition 15).

**Theorem 3** *For any  $s \in [0, 1 - 1/2\alpha)$ , there exists a piecewise constant feedback  $K(t), t \in (0, T)$  such that the closed-loop system*

$$\begin{cases} \frac{d}{dt}y(t) = (A + BK(t))y(t), & t \in (0, T) \\ y(0) = y_0, \end{cases}$$

with  $y_0 \in D(A^s)$ , is well-posed in  $D(A^s)$  and satisfies

$$y(T) = 0.$$

Although reaching the equilibrium in finite time, the design of the feedback in Theorem 3 may diverge in norm close to the final time  $T$ , which is not desirable in practice. To circumvent this, one is able to design a time-varying feedback law  $F$  ensuring semi-global finite-time stabilization and uniform stability, under additional regularity hypothesis.

**Theorem 4** *Let  $T > 0$  and  $\Gamma > 0$ . Then there exists a time-varying feedback law  $F : \mathbb{R}^+ \times D(A^{1/2}) \rightarrow \mathbb{C}$  such that the unique solution to*

$$\begin{cases} \frac{d}{dt}y(t) = Ay(t) + BF(t, y(t)) \\ y(0) = y_0 \end{cases} \quad (4)$$

with  $y_0 \in D(A^{1/2})$  satisfies:

1. *Finite-time stabilization:*

$$\forall y_0 \in D(A^{1/2}) \text{ such that } \|y_0\|_{D(A^{1/2})} \leq \Gamma \implies y(2T) = 0.$$

2. *Uniform stability: for all  $\epsilon > 0$ , there exists  $\eta > 0$  such that*

$$\forall y_0 \in D(A^{1/2}) \text{ such that } \|y_0\|_{D(A^{1/2})} \leq \eta \implies \|y(t)\|_{D(A^{1/2})} \leq \epsilon.$$

We refer to Section 6.2 for a precise meaning of solutions to (4), and the design (179) and properties  $(\mathcal{P}_1) - (\mathcal{P}_4)$  of the time-varying feedback.

## 1.2 The Fredholm backstepping method

The Fredholm backstepping method has been successful lately to deduce the rapid stabilization of PDEs from their controllability properties. The general idea is to seek a feedback operator  $K$  and a transformation  $T$  mapping the resulting closed-loop system onto a stable target system. The target system often has the form,

$$\begin{cases} \frac{d}{dt}w(t) = (A - \lambda I)w(t), & t \geq 0, \\ w(0) = w_0. \end{cases} \quad (5)$$

More general exponentially stable target systems can be considered, but (5) has the advantage of using the same domain of definition  $D(A)$  for both the equation to stabilize and the target system. We refer for instance to [14] for a case where the exponential stability comes from a change of boundary conditions, changing the domain of the target system and resulting in a more involved analysis, but leading successfully to the rapid stabilization with the backstepping method.

Denote,

$$\omega_0(e^{tA}) := \inf_{t \in (0, \infty)} \log \|e^{tA}\|_{\mathcal{L}(H, H)},$$

then the solution  $w$  of (5) satisfies the estimate:

$$\|w(t)\|_H \leq C e^{(\omega_0(e^{tA}) - \lambda)t} \|w_0\|_H,$$

hence, taking  $\lambda > 0$  sufficiently large, one can prescribe any exponential decay rate for solution of (5). Then, assuming that  $T \in \mathcal{L}(H, H)$  is invertible from  $H$  to itself, choosing  $w_0 = Ty_0$  implies the rapid stabilization of (2),

$$\|y(t)\| \leq \|T^{-1}\| \|w(t)\|_H \leq e^{-\nu t} \|T^{-1}\| \|w_0\|_H \leq e^{-\nu t} \|T^{-1}\| \|T\| \|y_0\|_H.$$

Operators  $T$  and  $K$  thus play a fundamental role to link the system to stabilize (2) to the rapidly stable target system (5). Let us deduce the equation  $T$  and  $K$  have to satisfy formally. The rigorous justification of the exponential stability and the functional framework will be specified in Section 3.1. First, differentiating  $Ty(t) = w(t)$  in time leads to,

$$T(A + BK)y(t) = (A - \lambda I)Ty(t).$$

The previous equation holds for every solution  $y(t)$ , implying that the operators  $T$  and  $K$  satisfy:

$$T(A + BK) = (A - \lambda I)T. \quad (6)$$

In the context of the Fredholm backstepping method, it is classical to impose the so-called *uniqueness condition*,

$$TB = B,$$

which reduces (6) to,

$$\begin{cases} TA + BK = (A - \lambda I)T, \\ TB = B. \end{cases} \quad (7)$$

### 1.2.1 Abstract approach

To illustrate how to solve (7) by the generalized backstepping method, we consider a diagonalizable operator  $A$  with eigenfunctions  $\varphi_n$  and eigenvalues  $\lambda_n$ . First, apply  $(A - \lambda I)T - TA = BK$  on the eigenbasis  $\{\varphi_n\}_{n \in \mathbb{N}^*}$ ,

$$(A - (\lambda_n + \lambda)I)T\varphi_n = BK\varphi_n.$$

Since  $k_n := K\varphi_n$  is a scalar, we can define, assuming  $\lambda \notin \sigma(A - \lambda_n I), \forall n \in \mathbb{N}^*$ ,

$$g_n := (A - (\lambda_n + \lambda)I)^{-1}B. \quad (8)$$

With this definition, we have  $T\varphi_n = k_n g_n$ . It then remains to identify the coefficients  $k_n$  and to establish the continuity of the transformation  $T$  and its invertibility. The coefficients of  $k_n$  are obtained through the uniqueness condition. Indeed,

$$TB = T \left( \sum_{n \in \mathbb{N}^*} b_n \varphi_n \right) = \sum_{n \in \mathbb{N}^*} b_n k_n g_n,$$

and therefore  $TB = B$  amounts to,

$$\sum_{n \in \mathbb{N}^*} b_n k_n g_n = \sum_{n \in \mathbb{N}^*} b_n \varphi_n. \quad (9)$$

The controllability property of (1) allows to ensure that  $b_n$  are bounded from above and below in a norm specific to the space of controllability (see for instance the framework of [8]). Together with spectral properties of  $A$ , the usual approach is to prove that the  $\{g_n\}_{n \in \mathbb{N}^*}$  are a Riesz basis, allowing to solve (9) and therefore the uniqueness condition  $TB = B$ .

### 1.2.2 Riesz basis

We briefly recall the definition and some basic properties of Riesz basis, and refer to [6] for more details.

**Definition 1** *Let  $H$  be a Hilbert space. A family of vectors  $\{\xi_n\}_{n \in \mathcal{I}}$ , where  $\mathcal{I} = \mathbb{Z}, \mathbb{N}$ , or  $\mathbb{N}^*$  is said to be*

- (1) **Minimal** in  $H$ , if for every  $k \in \mathcal{I}$ ,  $\xi_k \notin \overline{\text{span}\{\xi_i; i \in \mathcal{I} - \{k\}\}}$ .
- (2) **Complete** in  $H$ , if  $\overline{\text{span}\{\xi_i; i \in \mathcal{I}\}} = H$ .
- (3)  **$\omega$ -independent** in  $H$ , if

$$\sum_{k \in \mathcal{I}} c_k \xi_k = 0 \text{ in } H \text{ with } \{c_n\}_{n \in \mathcal{I}} \in l^2(\mathcal{I}) \implies c_n = 0, \forall n \in \mathcal{I}. \quad (10)$$

(4) **Quadratically close** to a family of vectors  $\{e_n\}_{n \in \mathcal{I}}$ , if

$$\sum_{k \in \mathcal{I}} \|\xi_k - e_k\|_H^2 < +\infty. \quad (11)$$

(5) **Riesz basis** of  $H$ , if it is the image of some orthonormal basis by an isomorphism on  $H$ .

(5)' **Riesz basis** of  $H$  (an equivalent definition of (5)), if it is complete in  $H$  and if there exist  $C_1, C_2 > 0$  such that for any  $\{a_n\}_{n \in \mathcal{I}} \in l^2(\mathcal{I}; \mathbb{C})$  we have

$$C_1 \sum_{k \in \mathcal{I}} |a_k|^2 \leq \left\| \sum_{k \in \mathcal{I}} a_k \xi_k \right\|_H^2 \leq C_2 \sum_{k \in \mathcal{I}} |a_k|^2. \quad (12)$$

Riesz bases have the following useful characterization:

**Proposition 1** *Let  $H$  be a Hilbert space. Let  $\{\xi_n\}_{n \in \mathcal{I}}$  (where  $\mathcal{I} = \mathbb{Z}, \mathbb{N}$ , or  $\mathbb{N}^*$ ) be a family of vectors, and  $\{e_n\}_{n \in \mathcal{I}}$  be an orthonormal basis of  $H$ . Assume that  $\{\xi_n\}$  and  $\{e_n\}$  are quadratically close. Then, if  $\{\xi_n\}$  is complete, or  $\omega$ -independent, it is a Riesz basis of  $H$ .*

### 1.2.3 The quadratically close criterion

When  $|\lambda_n| \simeq n^\alpha$  with  $\alpha > 3/2$ , it can be shown that the family  $(g_n)$  is a Riesz basis through the quadratically close criterion of Proposition 1 under classical assumptions, whether  $A$  is self-adjoint or skew-adjoint. Indeed, assuming  $|\lambda_n - \lambda_k| > n^{\alpha-1}|n - k|$ ,  $\forall n, k \in \mathbb{N}$  to simplify the computations, the quadratically close criterion amounts roughly to,

$$\sum_{n \in \mathbb{N}} \|\varphi_n - q_n\|^2 \leq C \sum_{n \in \mathbb{N}} \sum_{k \neq n} \left( \frac{1}{|n|^{\alpha-1}|k - n|} \right)^2$$

which converges if  $\alpha > 3/2$ . We refer to [8, 18] for the application of this method for a skew-adjoint case (Schrödinger equation) and an adjoint case (degenerate heat equation) respectively.

When  $|\lambda_n| \simeq n^\alpha$  with  $1 < \alpha \leq 3/2$ , then the growth of the eigenvalues at infinity is not sufficient to prove the existence of a Riesz basis through the quadratically close criterion, a threshold that has been longstanding since the introduction of the Fredholm backstepping method. In the recent paper [20], we proved for skew-adjoint operators  $A$  with the spectral gap and other under classical assumptions that the family  $\{g_n\}_{n \in \mathbb{N}^*}$  defined by (8) is still a Riesz basis. The proof relies on a duality/compactness method, together with the Fredholm alternative. The duality is used to prove that the family  $\{g_n\}_{n \in \mathbb{N}^*}$  is complete and  $\omega$ -independent, whereas the compactness method consists in proving a sharp regularizing property for the operator  $\varphi_n \mapsto g_n - \frac{1}{\lambda} \varphi_n$ , which implies its compactness. In turn, we prove that the family  $\{g_n\}_{n \in \mathbb{N}^*}$  is obtained as an invertible Fredholm transformation of the orthonormal basis  $\varphi_n$ , which yields the Riesz basis property from Definition 1 item (5).

**Remark 1** *We underline that the Fredholm backstepping method in the case  $\alpha = 1$  was not discussed above, as this case is very specific and often requires techniques of its own (see [14] and the references therein).*

### 1.3 Previous results

We find in the literature two different approaches for the rapid stabilization with the backstepping method with a Fredholm transformation : either the operator  $A$  is of first order ( $\alpha = 1$ ) or of second order strictly greater than 1 ( $\alpha > 1$ ). The case  $\alpha = 1$  was essentially not treated in the discussion above as it seems to be a very specific case with techniques of its own. We refer for instance to cases where the rapid stabilization for hyperbolic systems was established in [9, 10] through direct methods or by identifying the isomorphism applied to the eigenbasis leading to the Riesz basis [14, 42, 43]. Most of the results found in the literature for operator of order strictly greater than 1 ( $\alpha > 1$ ) were established through the quadratically close criterion, thanks to the sufficient growth of the eigenvalues. Following the steps described in the previous section, the rapid stabilization was obtained for the linearized bilinear Schrödinger equation [8], the KdV equation [11], the Kuramoto-Sivashinski equation [12], a degenerate parabolic operator [19, 29] and finally the heat equation for which the backstepping is proved in sharp spaces [22]. More recently, we have established necessary and sufficient conditions for the Fredholm backstepping method for skew-adjoint operators of order strictly greater than 1 with the introduction of the duality/compactness method [21], removing the restriction on the growth of the eigenvalues.

The finite-time stabilization was previously known only in specific cases. Among others : heat equations with variable coefficients in [15, 39], the KdV equation in [37], the viscous Burgers equation in [16], the linear transport equation in [44], and Navier-Stokes equations [38].

We highlight as well the link between the backstepping method and the pole shifting methods ([25], [30], [32], [40]). Some of these techniques have inspired the computations developed in Section 5.

### 1.4 Outline of the paper

This first paper focuses on the self-adjoint case. First, we present the strategy of the paper in the finite-dimensional case. We highlight the key role of the Cauchy matrix in solving the  $TB = B$  condition. This is done in Section 2. In Section 3, we establish the existence of a Fredholm backstepping transformation for (2) by establishing the link between an underlying isomorphism and the finite-dimensional Cauchy matrix. Preliminary estimates with respect to  $\lambda$  and its distance to the non-resonant set is also begun in this section. These estimates are completed in Section 4, where most of the technical estimates on  $\|T\|$  and  $\|T^{-1}\|$  are performed. These sharp estimates allows us to conclude on finite-time stabilization results in Section 6. Prior to this last section, we prove well-posedness results of the closed-loop system in Section 5.

## 2 Finite-dimensional example

We start this article by illustrating the methodology used in this article using a finite-dimensional example, highlighting the differences with the existing literature.

We use the eigenvectors  $\{\varphi_n\}_{1 \leq n \leq d} \in \mathbb{R}^d$  of  $A$  to construct a set of vectors  $\{g_n\}_{1 \leq n \leq d}$  solving the first equation  $TA + BK = (A - \lambda I)T$ . Usual methods for the Fredholm backstepping here is to prove that the  $\{g_n\}_{1 \leq n \leq d}$  forms a basis of  $\mathbb{R}^d$ . This basis is then used to solve the equation  $TB = B$ , thereby retrieving the coefficients  $\{k_n\}_{1 \leq n \leq d}$  of the feedback  $K$ .

Instead, we inject the expression of  $\{g_n\}_{1 \leq n \leq d}$  into the  $TB = B$  equation, and exhibit an invertible mapping, allowing us to recover the feedback coefficients  $\{k_n\}_{1 \leq n \leq d}$ . This gives another proof of,

**Theorem 5** ([5, 7]) *Let the pair  $(A, B) \in M(\mathbb{R}^d) \times \mathbb{R}^d$  be controllable. Then, for any  $\lambda > 0$ , there exists a unique transformation  $T \in GL(\mathbb{R}^d)$  and a unique feedback  $K^* \in \mathbb{R}^d$  such that*

$$\begin{cases} TA + BK = (A - \lambda I)T, \\ TB = B. \end{cases}$$

The advantage of the Cauchy is the explicit expression of its inverse, with the damping parameter  $\lambda > 0$  being tractable. The generalization to infinite dimensional systems of the Cauchy matrix at the same time to overcome the quadratically close criterion and to have sharp estimates on the cost of the backstepping method.

### 2.1 Solving the main equation

Consider a matrix  $A \in M_d(\mathbb{R})$ ,  $B \in \mathbb{R}^d$ , and suppose that  $A$  is diagonalizable with simple eigenvalues  $\{\lambda_n\}_{1 \leq n \leq d}$  and eigenvectors  $\{\varphi_n\}_{1 \leq n \leq d}$ . Assume that  $(A, B)$  is controllable and recall that we aim to solve

$$\begin{cases} TA + BK = (A - \lambda I)T, \\ TB = B. \end{cases} \quad (13)$$

Applying  $(A - \lambda I)T - TA = BK$  to the  $\varphi_n$ , we get:

$$(A - (\lambda_n + \lambda)I)T\varphi_n = BK\varphi_n.$$

Since  $k_n := K\varphi_n$  is a scalar, assuming  $\lambda \notin \sigma(A - \lambda_n I), \forall n \in \{1, \dots, d\}$ , we deduce,

$$T\varphi_n = k_n(A - (\lambda_n + \lambda)I)^{-1}B,$$

and therefore

$$T_{n,k} := \langle T\varphi_n, \varphi_k \rangle = \frac{k_n b_k}{\lambda_k - \lambda_n - \lambda}, \quad \forall n, k \in \{1, \dots, d\}, \quad (14)$$

$$T\varphi_n = \sum_{k=1}^d T_{n,k} \varphi_k, \quad \forall n \in \{1, \dots, d\}. \quad (15)$$

## 2.2 The uniqueness condition and expression of the feedback.

The uniqueness condition (9) then writes:

$$TB = \sum_{n=1}^d b_n T \varphi_n = \sum_{n=1}^d b_n k_n (A - (\lambda_n + \lambda)I)^{-1} B,$$

where  $B = \sum_{n=1}^d b_n \varphi_n$ . At this point, the standard procedure calls to prove that the family  $\{g_n\}_{1 \leq n \leq d}$  is a Riesz basis, where  $g_n := (A - (\lambda_n + \lambda)I)^{-1} B$ . Here, we proceed differently.

Using again the decomposition of  $B$ , one obtains:

$$\begin{aligned} \sum_{n=1}^d b_n k_n (A - (\lambda_n + \lambda)I)^{-1} B &= \sum_{n=1}^d b_n k_n \left( \sum_{k=1}^d b_k (A - (\lambda_n + \lambda)I)^{-1} \varphi_k \right) \\ &= \sum_{n=1}^d b_n k_n \left( \sum_{k=1}^d \frac{b_k}{\lambda_k - \lambda_n - \lambda} \varphi_k \right) \\ &= \sum_{k=1}^d \sum_{n=1}^d \frac{b_n k_n b_k}{\lambda_k - \lambda_n - \lambda} \varphi_k \end{aligned}$$

Hence  $TB = B$  amounts to,

$$\sum_{k=1}^d \sum_{n=1}^d \frac{k_n b_n b_k}{\lambda_k - \lambda_n - \lambda} \varphi_k = \sum_{k=1}^d b_k \varphi_k,$$

and since the controllability assumption requires (using Fattorini-Hautus, for example),  $b_k \neq 0$ , for any  $k = 1, \dots, d$  (otherwise the controllability of this mode fails), this is equivalent to,

$$\sum_{n=1}^d \frac{k_n b_n}{\lambda_k - \lambda_n - \lambda} = 1, \quad k = 1, \dots, d \quad (16)$$

## 2.3 The explicit isomorphism

This last equality can be viewed as solving the matrix system,

$$\tilde{T} \begin{pmatrix} k_1 b_1 \\ k_2 b_2 \\ \vdots \\ k_d b_d \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \quad (17)$$

where the entries of the matrix  $\tilde{T} \in M_d(\mathbb{C})$  are given by,

$$\tilde{T}_{i,j} = \frac{1}{\lambda_i - \lambda_j - \lambda}.$$

This induces a new transformation  $\tilde{T}$ :

$$\langle \tilde{T}\varphi_j, \varphi_i \rangle := \tilde{T}_{i,j} \text{ thus } {}^3 \tilde{T}\varphi_j = \sum_{i=1}^d \tilde{T}_{i,j}\varphi_i. \quad (18)$$

This is a particular case of a Cauchy matrix,

$$C_{ij} = \frac{1}{x_i - y_j},$$

with  $x_i \neq y_j$  for  $1 \leq i, j \leq d$ . Here we define, for  $\tilde{T}$ , the entries  $x_i = \lambda_i$ ,  $y_j = \lambda_j + \lambda$ . Notice that  $x_i \neq y_j$  is ensured by imposing the classical non-resonance condition  $\lambda \notin \sigma(A - \lambda_j I)$ ,  $j = 1, \dots, d$ . Moreover, since the eigenvalues of  $A$  are simple, we also have  $x_i \neq x_j$  and  $y_i \neq y_j$  for  $i \neq j$ .

Under these hypothesis, the invertibility of the Cauchy matrix  $C$  is ensured (see for instance [31]) and has the form,

$$C_{ij}^{-1} = (x_j - y_i)A_j(y_i)B_i(x_j),$$

with  $A_i, B_i$  being the Lagrange polynomials for  $x_i$  and  $y_i$  respectively:

$$A_i(x) = \frac{A(x)}{A'(x_i)(x - x_i)} = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}, \quad B_i(x) = \frac{B(x)}{B'(y_i)(x - y_i)} = \prod_{j \neq i} \frac{x - y_j}{y_i - y_j},$$

with

$$A(x) = \prod_{i=1}^d (x - x_i), \quad B(x) = \prod_{i=1}^d (x - y_i).$$

Hence, we deduce the explicit expression of the matrix  $\tilde{T}_{i,j}^{-1}$ ,

$$\begin{aligned} (\tilde{T}^{-1})_{ij} &= (x_j - y_i) \left( \frac{\prod_{n \neq j} (y_i - x_n)}{\prod_{n \neq j} (x_j - x_n)} \right) \left( \frac{\prod_{m \neq i} (x_j - y_m)}{\prod_{m \neq i} (y_i - y_m)} \right) \\ &= -\frac{1}{x_j - y_i} \left( \frac{\prod_n (y_i - x_n)}{\prod_{n \neq j} (x_j - x_n)} \right) \left( \frac{\prod_m (x_j - y_m)}{\prod_{m \neq i} (y_i - y_m)} \right) \\ &= -\frac{1}{x_j - y_i} \left( \frac{\prod_m (y_i - x_m)}{\prod_{m \neq i} (y_i - y_m)} \right) \left( \frac{\prod_n (x_j - y_n)}{\prod_{n \neq j} (x_j - x_n)} \right) \\ &= \frac{\lambda^2}{(\lambda_j - \lambda_i - \lambda)} \prod_{\substack{m=1 \\ m \neq i}}^d \left( 1 + \frac{\lambda}{\lambda_i - \lambda_m} \right) \prod_{\substack{n=1 \\ n \neq j}}^d \left( 1 - \frac{\lambda}{\lambda_j - \lambda_n} \right). \end{aligned} \quad (19)$$

Then, the coefficients  $k_j$  are obtained through (17), using again the fact that  $b_j \neq 0$ .

We define a new transformation  $\tilde{T}^{-1}$  by

$$\langle \tilde{T}^{-1}\varphi_j, \varphi_i \rangle := \tilde{T}_{i,j}^{-1} \text{ thus } \tilde{T}^{-1}\varphi_j = \sum_{i=1}^d \tilde{T}_{i,j}^{-1}\varphi_i. \quad (20)$$

---

<sup>3</sup>Note that this expression is slightly different from the one in (15) concerning the transformation  $T$ : in (15) the sum is over the second index of  $T$ , while in this expression we sum over the first index of  $\tilde{T}$ .

## 2.4 Invertibility of the transformation $T$ .

The coefficient of the feedback  $\{k_n\}_{1 \leq n \leq d}$  having been identified, we are able to go back to the transformation  $T$  and establish its invertibility,

$$\begin{aligned}
Tf &= \sum_{n=1}^d f_n T \varphi_n \\
&= \sum_{n=1}^d f_n k_n (A - (\lambda_n + \lambda)I)^{-1} B \\
&= \sum_{n=1}^d f_n k_n \sum_{k=1}^d \frac{b_k}{\lambda_k - \lambda_n - \lambda} \varphi_k \\
&= \sum_{k=1}^d \sum_{n=1}^d f_n k_n \frac{b_k}{\lambda_k - \lambda_n - \lambda} \varphi_k \\
&= \tau_B \check{T} \tau_K f
\end{aligned} \tag{21}$$

where,

$$\tau_K : \varphi_n \mapsto k_n \varphi_n, \quad \tau_B : \varphi_n \mapsto b_n \varphi_n, \quad \check{T} : \varphi_n \mapsto \sum_{k=1}^d \frac{1}{\lambda_k - \lambda_n - \lambda} \varphi_k \tag{22}$$

$\check{T}$  is given by (18), and  $\tau_B$  is invertible by hypothesis on the  $b_n$ . Hence, the invertibility of  $T$  hinges on the invertibility of  $\tau_K$  and therefore on  $k_j$  being non-zero. This can be proved directly from the explicit representation of  $k_n$  through  $\check{T}^{-1}$ , but it is quite technical. In the infinite-dimensional setting, this will be established through some technical estimates on the infinite sums and products involved.

Formally we have,

$$T^{-1} = \tau_K^{-1} \check{T}^{-1} \tau_B^{-1} \tag{23}$$

Another way to obtain the invertibility of  $T$  in finite-dimension is to prove that  $\ker T = 0$ . To that end, note that if  $x \in \ker(T^*)$ , then by  $TB = B$ , we have

$$B^* x = B^* T^* x = 0.$$

On the other hand, from the operator equation (6), we have

$$T^*(A^* - \lambda I)x = (A + BK)^* T^* x = 0,$$

which implies that  $\ker(T^*)$  is stable by  $A^*$ , hence invariant under the action of the semi-group  $e^{tA^*}$ . Combining the above two assertions yields

$$B^* e^{tA^*} x = 0,$$

which, by weak observability, implies that  $x = 0$ . Hence,  $T$  is invertible.

### 3 Construction and invertibility of $T$

We now move on to the construction of  $T$  and  $K$  in the infinite dimensional case. We begin by introducing the functional framework and the assumptions on  $A : D(A) \subset H \rightarrow H$  and  $B \in \mathcal{L}(U; D(A)')$ .

#### 3.1 Functional framework

Let us specify the functional framework of our main results. We start by some assumptions on the operators of the control system and the target system. First, let us define the following countable set of numbers

$$\mathcal{N} := \{\lambda_i - \lambda_j : i, j \in \mathbb{N}\}, \quad (24)$$

We can now state the assumptions of this paper

**Assumption 1** *1. The operator  $A$  is real, negative, self-adjoint, and has compact resolvent. Hence the eigenvalues of the operator are real, countable, and*

$$0 < -\lambda_1 \leq -\lambda_2 \leq \dots \leq -\lambda_n \rightarrow +\infty$$

*2. The pair  $(A, B)$  is controllable in  $H$  in the sense that,*

$$0 < c \leq |b_n| \leq C, \quad \forall n \in \mathbb{N} \quad (25)$$

*3. The Hilbert space  $H$  can be decomposed as a direct sum of **finitely** many subspaces:*

$$H = \bigoplus_{i=1}^h H_i, \quad (26)$$

*such that for any  $i \in \{1, 2, \dots, h\}$ , the operator  $A|_{H_i}$  is diagonalizable, with simple eigenvalues. We denote its eigenpairs by*

$$\{(\lambda_{i,n}, \varphi_{i,n}) : \varphi_{i,n} \in H_i, A\varphi_{i,n} = \lambda_{i,n}\varphi_{i,n}, \forall n \in \mathbb{N}^*\}. \quad (27)$$

*4. On each subspace  $H_i$ ,  $i \in \{1, \dots, h\}$ , the eigenvalues satisfy*

$$n^{\alpha-1} \lesssim |\lambda_{i,n+1} - \lambda_{i,n}| \lesssim n^{\alpha-1}, \quad \forall n \in \mathbb{N}^*, \quad \forall i \in \{1, \dots, h\}. \quad (28)$$

*and the gap estimate*

$$ck^{\alpha-1}|k - n| \leq |\lambda_{i,k} - \lambda_{i,n}|, \quad \forall k, n \in \mathbb{N}^*. \quad (29)$$

*5. We assume  $\lambda \notin \mathcal{N}$ .*

**Remark 2** *In Assumption 1, item 1, we assume that  $A$  is negative to simplify the computations. In particular, we assume that no eigenvalues are equal to zero. Note that this is without loss of generality : the more general framework with a finite number of unstable eigenvalues (that is, with positive real part of the eigenvalues) can be treated by the change of variables  $x(t) := e^{\mu t}y(t)$ , leading to the operator  $A - \mu I$  being negative for  $\mu > 0$  large enough.*

*Item 4 implies that one can handle the case where the all the eigenvalues are simple, without loss of generality: the general case reduces to a finite number of spaces on which one considers an operator with simple eigenvalues.*

**Remark 3** *It is easy to see in our context that Assumption 1, item 2, implies that the operator  $B$  is admissible in  $H$  (see for instance [35] for a definition).*

**Remark 4** *Assumption 1, item 5, is classical in the Fredholm backstepping method. Recall that the decay at rate  $\tilde{\lambda} \in \mathcal{N}$  is obtained by choosing any  $\lambda \notin \mathcal{N}$  such that  $\tilde{\lambda} < \lambda$ . In the first part of this article, we derive explicit bounds on  $\|T\|$  and  $\|T^{-1}\|$  with respect to  $\lambda$  and its distance to  $\mathcal{N}$ . In the second part of this article, we prove that there exist infinitely many  $\lambda > 0$  such that this bound is indeed positive, and in fact lower bounded by  $e^{-c\lambda^{\frac{1}{\alpha}}}$  where  $c$  is a constant independent of  $\lambda$  (see Proposition 8). This, in turn, allow us to derive a quantitative estimation of  $\|T\|$  and  $\|T^{-1}\|$  (see Theorem 6).*

**Remark 5** *Let us underline that items 4 and 5 of Assumption (1) are satisfied in many cases, in particular if the operator is self-adjoint with simple eigenvalues given by  $\lambda_n \sim -n^\alpha$  with  $\alpha > 1$ .*

Finally, we point out that estimate (29) in item 5 implies a third gap estimate:

**Lemma 1** *Assuming (29), then there exists  $c > 0$  such that,*

$$|\lambda_n - \lambda_k| \geq c|n - k|^\alpha, \quad \alpha > 1, \quad \forall p, n \in \mathbb{N}^*, \quad (30)$$

**Proof :** Without loss of generality, assume  $k > n$ , the case  $n = k$  being trivial. Then, from (29),

$$|\lambda_n - \lambda_k| \geq ck^{\alpha-1}|k - n| \text{ and } |\lambda_n - \lambda_k| \geq cn^{\alpha-1}|k - n|$$

and thus, the conclusion comes if we are able to prove,

$$\max\{k^{\alpha-1}, n^{\alpha-1}\} \geq |k - n|^{\alpha-1},$$

but this comes from the monotonicity of the function  $x \in \mathbb{R}^+ \mapsto x^{\alpha-1}$  with  $\alpha > 1$ .  $\square$

For the remainder of the article, since our method is essentially designed for operators with simple eigenvalues, we will present our results in this framework. We now consider a real self-adjoint unbounded operator  $A$  on  $H$ , with simple eigenvalues  $\{\lambda_n\}_{n \in \mathbb{N}^*}$  and an orthogonal family of corresponding eigenvectors  $\{\varphi_n\}_{n \in \mathbb{N}^*}$ . As usual, we define the domain of  $A$  by

$$D(A) := \{f \in H, Af \in H\},$$

and we further note that the growth of the  $\{\lambda_n\}_{n \in \mathbb{N}^*}$  assumed in Assumption 1 implies that  $A$  has compact resolvent.

Moreover, we introduce a family of rigged Banach spaces (and refer to [33] for the detailed construction and further properties), which can be defined spectrally since  $A$  is diagonalizable. For  $s \geq 0$ , we define

$$\begin{aligned}\mathcal{H}^s &:= \{f \in H, \sum_{n \in \mathbb{N}^*} n^{2s} |\langle f, \varphi_n \rangle|^2 < +\infty\}, \\ D(A^s) &:= \{f \in H, \sum_{n \in \mathbb{N}^*} |\lambda_n|^{2s} |\langle f, \varphi_n \rangle|^2 < +\infty\},\end{aligned}$$

Notice that if  $cn^\alpha \leq |\lambda_n| \leq Cn^\alpha$ , then  $D(A^s) = \mathcal{H}^{\alpha s}$ .

The spaces  $\mathcal{H}^s$  are naturally endowed with a Hilbert structure given by the scalar product

$$\langle f, g \rangle_s := \sum_{n \in \mathbb{N}^*} (n^{2s}) \langle f, \varphi_n \rangle \langle g, \varphi_n \rangle, \quad \forall f, g \in \mathcal{H}^s,$$

as well as a natural Hilbert basis given by  $\left( \frac{1}{\sqrt{1 + |n|^{2s}}} \varphi_n \right)_{n \in \mathbb{N}^*}$ .

The spaces  $D(A^s)$ ,  $s \geq 0$  can in turn be endowed with the closely related Hilbert structure (recall that  $\lambda_n \neq 0$ ):

$$\langle f, g \rangle_{D(A^s)} = \sum_{n \in \mathbb{N}^*} |\lambda_n|^{2s} \langle f, \varphi_n \rangle \langle g, \varphi_n \rangle, \quad \forall f, g \in D(A^s).$$

For  $s < 0$ ,  $\mathcal{H}^s$  (resp.  $D(A^s)$ ) can be defined by identifying it to the dual space  $(\mathcal{H}^s)'$  (resp.  $D(A^s)'$ ), with pivot space  $H$ .

For the remainder of this article, we will denote by:

- $\|\cdot\|$  the norm of the Hilbert space  $H$ ;
- $\|\cdot\|_s$  the norm associated to the scalar product  $\langle \cdot, \cdot \rangle_s$  on  $\mathcal{H}^s$ , for  $s \neq 0$ ;
- $\|\cdot\|_{D(A^s)}$  the norm associated to the scalar product  $\langle \cdot, \cdot \rangle_{D(A^s)}$  on  $D(A^s)$ , for  $s \neq 0$ .
- For linear mappings on  $H$  (resp.  $D(A^s)$  or  $D_s(A + BK)$  in Sections 5 and 6), the usual operator norm will be denoted  $\|\cdot\|$  (resp.  $\|\cdot\|_{D(A^s)}$  and  $\|\cdot\|_{D_s(A+BK)}$ ).

Importantly, for  $s < 0$ ,  $D(A^s)$  can also be defined as the completion of  $H$  with respect to the norm  $\|\cdot\|_{\rho, s} = \|(A - \rho I)^s \cdot\|$  for well-chosen  $\rho$  in the resolvent set of  $A$  (the fractional negative power of  $A - \rho I$  can be defined using the Laplace transform, see for example [41, Chapter IX, Section 11], or [28, Proposition 4.20]).

Finally, note that if  $s_1 < s_2$ , the injections  $\mathcal{H}^{s_2} \rightarrow \mathcal{H}^{s_1}$  and  $D(A^{s_1}) \rightarrow D(A^{s_2})$  are continuous and compact, due to the fact that  $A$  has compact resolvent.

### 3.2 Definition of the transformation $T$ and the feedback $K$

Recall that we work on a given invariant subspace  $H_i$  where  $A$  has simple eigenvalues. In the following, we denote  $H_i$  and  $(\lambda_{i,n}, \varphi_{i,n})$  by  $H$  and  $(\lambda_n, \varphi_n)$  to alleviate notations.

Let us define the operators, at least formally for now, used throughout this article. For  $\lambda > 0$  such that  $\lambda_j + \lambda - \lambda_i \neq 0, \forall i, j \in \mathbb{N}^*$ , we denote

$$K : \varphi_n \mapsto k_n \in \mathbb{R}, \quad \forall n \in \mathbb{N}^*, \quad (31)$$

$$T : \varphi_n \mapsto k_n(A - (\lambda_n + \lambda)I)^{-1}B \in H, \quad \forall n \in \mathbb{N}^*. \quad (32)$$

Thus

$$T_{n,k} := \langle T\varphi_n, \varphi_k \rangle = \frac{k_n b_k}{\lambda_k - \lambda_n - \lambda}, \quad \forall n, k \in \mathbb{N}^*, \quad (33)$$

$$T\varphi_n = \sum_{k \in \mathbb{N}^*} T_{n,k} \varphi_k, \quad \forall n \in \mathbb{N}^*. \quad (34)$$

Motivated by the finite dimensional example, one key element of this article is to introduce and prove several quantitative estimates, with respect to  $\lambda$ , on the auxiliary operator  $\tilde{T}$  associated to the infinite dimensional extension of the Cauchy matrix  $(\tilde{T}_{i,j})$ . Similar to (18), we define

$$\tilde{T}_{i,j} = \frac{1}{\lambda_i - \lambda_j - \lambda}, \quad (35)$$

and a transformation related to the matrix  $(\tilde{T}_{i,j})$ :

$$\langle \tilde{T}\varphi_j, \varphi_i \rangle := \tilde{T}_{i,j} \text{ thus } \tilde{T}\varphi_j = \sum_{i \in \mathbb{N}^*} \tilde{T}_{i,j} \varphi_i. \quad (36)$$

We also define another auxiliary transformation  $\check{T}^{-1}$  associated to the inverse of the Cauchy matrix  $(\tilde{T}_{i,j})$ , which we shall prove to be the left inverse of  $\tilde{T}$ , by generalizing identity (19): define

$$\check{T}_{i,j}^{-1} = \frac{\lambda^2}{(\lambda_j - \lambda_i - \lambda)} \prod_{\substack{m \in \mathbb{N}^* \\ m \neq i}} \left(1 + \frac{\lambda}{\lambda_i - \lambda_m}\right) \prod_{\substack{n \in \mathbb{N}^* \\ n \neq j}} \left(1 - \frac{\lambda}{\lambda_j - \lambda_n}\right), \quad \forall i, j \in \mathbb{N}^*, \quad (37)$$

and

$$\langle \check{T}^{-1}\varphi_j, \varphi_i \rangle := \check{T}_{i,j}^{-1} \text{ thus } \check{T}^{-1}\varphi_j = \sum_{i \in \mathbb{N}^*} \check{T}_{i,j}^{-1} \varphi_i. \quad (38)$$

Again we emphasize that (34) concerning the transformation  $T$  is slightly different from (38) on the transformation  $\check{T}$ .

Much of this article will revolve around the analysis of  $\tilde{T}$ , and  $\check{T}^{-1}$ . Its explicit representation will allow us in particular to deduce quantitative bounds on the operator norms  $\|T\|$ ,  $\|T^{-1}\|$ ,  $\|T\|_{D(A^s)}$ ,  $\|T^{-1}\|_{D(A^s)}$  with respect to  $\lambda$ . This allows to obtain stability estimates that are quantitative with  $\lambda$ , a key difference with the previous analysis, for instance [21]. This is also what allows us to derive a finite-time stabilisation (see Theorem 6).

In the remainder of this section, we proceed as follows: in Section 3.2.1 we provide some technical estimates that will be useful in the analysis; in Section 3.2.2, we show that  $\check{T}$  and  $\check{T}^{-1}$  defined similarly as in the finite-dimensional case by (18)–(19) are still well-defined operators in the infinite-dimensional setting and that  $\check{T}^{-1}$  is indeed the inverse of  $\check{T}$ ; in Section 3.2.3 we show that we may choose  $K$  appropriately such that the condition  $TB = B$  holds and we give an explicit expression of  $K$ . A major benefit of this approach, in contrast with previous approaches such as [18, 17, 20].

### 3.2.1 Some technical estimates

We begin by introducing a quantity that will play a fundamental role in the quantitative estimates,

$$\text{Dist}_\alpha(\lambda) := \inf_{i,j \in \mathbb{N}^*} |\lambda_j - \lambda_i + \lambda|. \quad (39)$$

Clearly,  $\text{Dist}_\alpha(\lambda) = 0$  if and only if  $\lambda \in \mathcal{N}$ . Assumption 1, item 6 asserts that  $\text{Dist}_\alpha(\lambda) > 0$ . We list here some properties related to  $\text{Dist}_\alpha(\lambda)$  that will be used throughout this article, as well as technical estimates on the  $\lambda_j$ . The proofs are in Appendix A.

**Lemma 2** *Assume that  $\lambda \notin \mathcal{N}$ . For every  $i \in \mathbb{N}^*$ , define*

$$D_i(\lambda) = \min_{j \in \mathbb{N}^*} |\lambda_j - \lambda_i + \lambda|.$$

Then,

(i) *There exists at most two  $j_1, j_2 \in \mathbb{N}^*$  such that*

$$|\lambda_{j_1} - \lambda_i + \lambda| = |\lambda_{j_2} - \lambda_i + \lambda| = D_i(\lambda),$$

*and  $|j_1 - j_2| = 1$ .*

(ii)

$$\text{Dist}_\alpha(\lambda) \leq D_i(\lambda), \quad \forall i \in \mathbb{N}^*.$$

The following lemma ensures that the infinite product (19) is well-defined.

**Lemma 3** *There exists some effectively computable constant  $C > 0$ , such that for any given  $\lambda > 0$  and any  $i, j \in \mathbb{N}^*$  there is*

$$(i) \left| \prod_{\substack{m \in \mathbb{N}^* \\ m \neq i}} \left( 1 + \frac{\lambda}{\lambda_i - \lambda_m} \right) \right| \leq C e^{C\lambda^{\frac{1}{\alpha}}}, \quad \forall \lambda \in (0, +\infty), \forall i \in \mathbb{N};$$

$$(ii) \left| \prod_{\substack{n \in \mathbb{N}^* \\ n \neq j}} \left( 1 - \frac{\lambda}{\lambda_j - \lambda_n} \right) \right| \leq C e^{C\lambda^{\frac{1}{\alpha}}}, \quad \forall \lambda \in (0, +\infty), \forall j \in \mathbb{N};$$

Moreover, the power  $\frac{1}{\alpha}$  in (i)-(ii) is sharp.

Finally, the following estimate will be useful in Section 4 concerning the upper and lower bound of  $\|T^{-1}\|$ .

**Lemma 4** *There exists  $C > 0$  such that, for any  $\lambda > 0$  satisfying  $\lambda \notin \mathcal{N}$ , and any  $i, j \in \mathbb{N}^*$  there is*

$$(i) \sum_{j \in \mathbb{N}^*} \left| \frac{\lambda^2}{(\lambda_j - \lambda_i - \lambda)} \right| \leq C \left( \lambda^2 + \frac{\lambda^2}{\text{Dist}_\alpha(\lambda)} \right), \quad \forall i \in \mathbb{N};$$

$$(ii) \sum_{i \in \mathbb{N}^*} \left| \frac{\lambda^2}{(\lambda_j - \lambda_i - \lambda)} \right| \leq C \left( \lambda^2 + \frac{\lambda^2}{\text{Dist}_\alpha(\lambda)} \right), \quad \forall j \in \mathbb{N}.$$

### 3.2.2 Study of the auxiliary transformations $\check{T}$ and $\check{T}^{-1}$

Recall the definition of the critical set  $\mathcal{N}$  in (24). From now on we fix the value of  $\lambda \notin \mathcal{N}$ , and investigate the transformations  $\check{T} = \check{T}(\lambda)$  and  $\check{T}^{-1} = \check{T}^{-1}(\lambda)$  given by (36) and (38). For the ease of notations, we omit the index  $\lambda$ .

First we prove that the operator  $\check{T}^{-1}$  defined above is well-defined. Recall that  $\check{T}$  is related to the matrix  $(\check{T}_{i,j})$ , and  $\check{T}^{-1}$  is related to the matrix  $(\check{T}_{i,j}^{-1})$  given by (35) and (37).

**Proposition 2** *Let  $\lambda \notin \mathcal{N}$ . The entries  $(\check{T}^{-1})_{ij}, i, j \in \mathbb{N}^*$  are well-defined. Moreover,*

- (i) *for every  $j \in \mathbb{N}^*$ ,  $\{(\check{T}^{-1})_{ij}\}_{i \in \mathbb{N}^*} \in \ell^1(\mathbb{N}^*; \mathbb{C})$  and  $\sup_{j \in \mathbb{N}^*} \|\{(\check{T}^{-1})_{ij}\}_{i \in \mathbb{N}^*}\|_{\ell^1} \leq C < \infty$ .*
- (ii) *for every  $i \in \mathbb{N}^*$ ,  $\{(\check{T}^{-1})_{ij}\}_{j \in \mathbb{N}^*} \in \ell^1(\mathbb{N}^*; \mathbb{C})$  and  $\sup_{i \in \mathbb{N}^*} \|\{(\check{T}^{-1})_{ij}\}_{j \in \mathbb{N}^*}\|_{\ell^1} \leq C < \infty$ .*

**Proof of Proposition 2:** From assertion (i) and (ii) of Lemma 3, we deduce that,

$$|(\check{T}^{-1})_{ij}| \leq C e^{C\lambda^{\frac{1}{\alpha}}} \left| \frac{\lambda^2}{(\lambda_j - \lambda_i - \lambda)} \right|, \quad (40)$$

where the constant  $C$  is independent of  $\lambda$  and  $i, j \in \mathbb{N}^*$ . Moreover, from assertion (ii) of Lemma 4,

$$\sum_{i \in \mathbb{N}^*} |(\check{T}^{-1})_{ij}| \leq C e^{C\lambda^{\frac{1}{\alpha}}} \sum_{i \in \mathbb{N}^*} \left| \frac{\lambda^2}{\lambda_j - \lambda_i - \lambda} \right| \leq C e^{C\lambda^{\frac{1}{\alpha}}} \left( \lambda^2 + \frac{\lambda^2}{\text{Dist}_\alpha(\lambda)} \right),$$

where the constant  $C$  is independent of  $\lambda$  and  $j \in \mathbb{N}^*$ . The second property can be proved using the same argument.  $\square$

We now show that the entries  $(\check{T}^{-1})_{(i,j) \in \mathbb{N}^*}$  indeed define a continuous operator  $\check{T}^{-1}$  on  $H$ .

**Proposition 3** Let  $\lambda \notin \mathcal{N}$ . The operators  $\check{T}$  defined by (36), and  $\check{T}^{-1}$  defined by (38), i.e.

$$\check{T} : \varphi_n \rightarrow \sum_{i \in \mathbb{N}^*} \check{T}_{i,n} \varphi_i, \quad (41)$$

$$\check{T}^{-1} : \varphi_n \rightarrow \sum_{i \in \mathbb{N}^*} \check{T}_{i,n}^{-1} \varphi_i \quad (42)$$

are bounded linear operators from  $H$  to  $H$ . Moreover,

$$\|\check{T}\| \leq C \left( 1 + \frac{1}{\text{Dist}_\alpha(\lambda)} \right), \quad (43)$$

$$\|\check{T}^{-1}\| \leq C \left( 1 + \frac{1}{\text{Dist}_\alpha(\lambda)} \right) e^{C\lambda^{\frac{1}{\alpha}}}, \quad (44)$$

where  $C$  is independent of  $\lambda$ .

In fact, this bounded operator is indeed the inverse of  $\check{T}$ , as expected:

**Proposition 4** Let  $\lambda \notin \mathcal{N}$ . The operator  $\check{T}^{-1}$  defined by (38) is the inverse of  $\check{T}$  in  $H$ .

**Proof of Proposition 3:** In the proof of this proposition, the constants  $C$  may differ from line to line, but all of them are independent of the choice of  $\lambda \notin \mathcal{N}$ . Using the expressions of  $\check{T}$  and  $\check{T}^{-1}$ , we are able to prove that it is a bounded operator on  $H$  from Lemma 3 : for every  $f \in H$ , there is

$$\begin{aligned} \|\check{T}^{-1}f\|^2 &= \left\| \sum_{n \in \mathbb{N}^*} f_n \check{T}^{-1} \varphi_n \right\|^2 \\ &= \left\| \sum_{n \in \mathbb{N}^*} f_n \left( \sum_{k \in \mathbb{N}^*} (\check{T}^{-1})_{k,n} \varphi_k \right) \right\|^2 \\ &= \left\| \sum_{k \in \mathbb{N}^*} f_k (\check{T}^{-1})_{kk} \varphi_k + \sum_{k \in \mathbb{N}^*} \left( \sum_{n \in \mathbb{N}^* \setminus \{k\}} f_n (\check{T}^{-1})_{k,n} \right) \varphi_k \right\|^2 \\ &\leq C e^{C\lambda^{\frac{1}{\alpha}}} \|f\|^2 + 2 \sum_{k \in \mathbb{N}^*} \left| \sum_{n \in \mathbb{N}^* \setminus \{k\}} f_n (\check{T}^{-1})_{k,n} \right|^2 \\ &\leq C e^{C\lambda^{\frac{1}{\alpha}}} \|f\|^2 + 2C e^{C\lambda^{\frac{1}{\alpha}}} \sum_{k \in \mathbb{N}^*} \left| \sum_{n \in \mathbb{N}^* \setminus \{k\}} \frac{|f_n|}{|\lambda_n - \lambda_k - \lambda|} \right|^2, \end{aligned}$$

where in the last line we have used the inequality (40). Thanks to the Cauchy-Schwarz

inequality and Lemma 4,

$$\begin{aligned}
& \sum_{k \in \mathbb{N}^*} \left| \sum_{n \in \mathbb{N}^* \setminus \{k\}} \frac{|f_n|}{|\lambda_n - \lambda_k - \lambda|} \right|^2 \\
& \leq \sum_{k \in \mathbb{N}^*} \left( \sum_{n \in \mathbb{N}^* \setminus \{k\}} \frac{|f_n|^2}{|\lambda_n - \lambda_k - \lambda|} \right) \left( \sum_{n \in \mathbb{N}^* \setminus \{k\}} \frac{1}{|\lambda_n - \lambda_k - \lambda|} \right) \\
& \leq C \left( 1 + \frac{1}{\text{Dist}_\alpha(\lambda)} \right) \sum_{k \in \mathbb{N}^*} \sum_{n \in \mathbb{N}^* \setminus \{k\}} \frac{|f_n|^2}{|\lambda_n - \lambda_k - \lambda|} \\
& \leq C \left( 1 + \frac{1}{\text{Dist}_\alpha(\lambda)} \right) \sum_{k \in \mathbb{N}^*} \sum_{n \in \mathbb{N}^*} \frac{|f_n|^2}{|\lambda_n - \lambda_k - \lambda|} \\
& \leq C \left( 1 + \frac{1}{\text{Dist}_\alpha(\lambda)} \right) \sum_{n \in \mathbb{N}^*} \sum_{k \in \mathbb{N}^*} \frac{|f_n|^2}{|\lambda_n - \lambda_k - \lambda|} \\
& \leq C \left( 1 + \frac{1}{\text{Dist}_\alpha(\lambda)} \right)^2 \sum_{n \in \mathbb{N}^*} |f_n|^2 \\
& \leq C \left( 1 + \frac{1}{\text{Dist}_\alpha(\lambda)} \right)^2 \|f\|^2.
\end{aligned}$$

Thus

$$\|\check{T}^{-1}f\| \leq C \left( 1 + \frac{1}{\text{Dist}_\alpha(\lambda)} \right) e^{C\lambda^{\frac{1}{\alpha}}} \|f\|. \quad (45)$$

This ends the proof of boundedness of  $\check{T}^{-1}$  and gives directly (44). The proof of (43) is similar and even simpler, due to the simpler expression of  $\check{T}_{i,j}$ .  $\square$

We now turn to the proof of Proposition 4

**Proof of Proposition 4:** Let us fix the value of  $\lambda \notin \mathcal{N}$ . Recall the definition of  $\check{T}$ :

$$\check{T} : \varphi_n \mapsto \sum_p \check{T}_{p,n} \varphi_p = \sum_p \frac{\varphi_p}{\lambda_p - \lambda_n - \lambda} \quad \text{with} \quad \check{T}_{p,n} := \frac{1}{\lambda_p - \lambda_n - \lambda}. \quad (46)$$

Recall the definition of  $\check{T}^{-1}$

$$\check{T}^{-1} : \varphi_n \mapsto \sum_i \check{T}_{i,n}^{-1} \varphi_i, \quad (47)$$

where

$$\check{T}_{i,j}^{-1} := \frac{\lambda^2}{\lambda_j - \lambda_i - \lambda} \prod_{m \neq i} \left( 1 + \frac{\lambda}{\lambda_i - \lambda_m} \right) \prod_{n \neq j} \left( 1 - \frac{\lambda}{\lambda_j - \lambda_n} \right). \quad (48)$$

We will show that  $\check{T}^{-1}$  is in fact the inverse of  $\check{T}$ , i.e.

$$\check{T} \circ \check{T}^{-1} = \check{T}^{-1} \circ \check{T} = \text{Id} : H \rightarrow H.$$

We focus on the case of  $\check{T}^{-1} \circ \check{T}$ , while the other case can be treated similarly. Thanks to the boundedness of  $\check{T}$  and  $\check{T}^{-1}$  (see Proposition 3), this is a bounded operator from  $H$  to

itself. Since  $\{\varphi_n\}_{n \in \mathbb{N}^*}$  is an orthogonal basis of  $H$ , it suffices to show that

$$(\tilde{T}^{-1} \circ \tilde{T})\varphi_n = \varphi_n, \quad \forall n \in \mathbb{N}^*.$$

Since

$$\begin{aligned} (\tilde{T}^{-1} \circ \tilde{T})\varphi_n &= \tilde{T}^{-1}(\tilde{T})\varphi_n \\ &= \tilde{T}^{-1} \left( \sum_p \tilde{T}_{p,n} \varphi_p \right) \\ &= \sum_p \tilde{T}_{p,n} \sum_k \tilde{T}_{k,p}^{-1} \varphi_k \\ &= \sum_k \varphi_k \left( \sum_p \tilde{T}_{k,p}^{-1} \tilde{T}_{p,n} \right) \end{aligned}$$

which is equivalent to

$$\sum_{i \in \mathbb{N}^*} \tilde{T}_{ji}^{-1} \tilde{T}_{in} = \delta_{j,n}, \quad \forall j, n \in \mathbb{N}^*, \quad (49)$$

where  $\delta_{j,n}$  denotes the Kronecker symbol. The proof is inspired from the finite dimensional Cauchy matrix ([31]). For any  $N \in \mathbb{N}^*$  we introduce the finite (truncated) matrices

$$\tilde{T}^N := (\tilde{T}_{ij}^N)_{1 \leq i, j \leq N}, \quad (\tilde{T}^{-1})^N := ((\tilde{T}_{ij}^{-1})^N)_{1 \leq i, j \leq N} \in \mathbb{R}^{N \times N} \quad (50)$$

with,

$$\tilde{T}_{ij}^N = \tilde{T}_{ij} = \frac{1}{\lambda_i - \lambda_j - \lambda}, \quad 1 \leq i, j \leq N, \quad (51)$$

$$(\tilde{T}_{ij}^{-1})^N = \frac{\lambda^2}{\lambda_j - \lambda_i - \lambda} \prod_{\substack{m \neq i \\ 1 \leq m \leq N}} \left( 1 + \frac{\lambda}{\lambda_i - \lambda_m} \right) \prod_{\substack{n \neq j \\ 1 \leq n \leq N}} \left( 1 - \frac{\lambda}{\lambda_j - \lambda_n} \right), \quad 1 \leq i, j \leq N. \quad (52)$$

By denoting  $x_i := \lambda_i$  and  $y_j := \lambda_j + \lambda$ , we notice that  $(\tilde{T}_{i,j}^N)$  is a Cauchy matrix as  $\tilde{T}_{ij}^N = 1/(x_i - y_j)$ . This implies that the  $N \times N$  matrix  $(\tilde{T}_{i,j}^N)$  is invertible with its inverse being  $(\tilde{T}^{-1})^N = ((\tilde{T}_{ij}^{-1})^N)$ , namely

$$\sum_{1 \leq i \leq N} (\tilde{T}^{-1})_{ji}^N \tilde{T}_{in}^N = \delta_{j,n}, \quad \forall 1 \leq j, n \leq N. \quad (53)$$

For any fixed  $j, n \in \mathbb{N}^*$ , we consider  $N \geq |j + n|$ , the previous equation yields

$$\sum_{1 \leq i \leq N} (\tilde{T}^{-1})_{ji}^N \tilde{T}_{in}^N = \sum_{1 \leq i \leq N} (\tilde{T}^{-1})_{ji}^N \tilde{T}_{in}^N = \delta_{j,n}, \quad \forall N \geq |j + n|. \quad (54)$$

We know that for any  $N \geq |j + n|$  and any  $i$  satisfying  $i \leq N$  there is

$$\begin{aligned} \tilde{T}_{ji}^{-1} - (\tilde{T}_{ji}^{-1})^N &= \tilde{T}_{ji}^{-1} \left( 1 - \prod_{m \geq N+1} \left( 1 + \frac{\lambda}{\lambda_j - \lambda_m} \right)^{-1} \prod_{n \geq N+1} \left( 1 - \frac{\lambda}{\lambda_i - \lambda_n} \right)^{-1} \right) \\ &= \tilde{T}_{ji}^{-1} \left( 1 - \exp \left( - \sum_{m \geq N+1} \log \left( 1 + \frac{\lambda}{\lambda_j - \lambda_m} \right) - \sum_{n \geq N+1} \log \left( 1 - \frac{\lambda}{\lambda_i - \lambda_n} \right) \right) \right). \end{aligned}$$

For any  $N \geq |j + n| + (10\lambda/c)^{1/(\alpha-1)}$ , where the constant  $c$  is given in (29), and any  $i$  satisfying  $i \leq N$  there is,

$$\left| \frac{\lambda}{\lambda_j - \lambda_m} \right| \leq \frac{|\lambda|}{c|j - m|m^{\alpha-1}} \leq \frac{|\lambda|}{cN^{\alpha-1}} \leq \frac{1}{10}, \quad \forall m \geq N + 1,$$

and thus,

$$\log \left( 1 - \frac{\lambda}{|\lambda_j - \lambda_m|} \right) \leq \log \left( 1 + \frac{\lambda}{\lambda_j - \lambda_m} \right) \leq \log \left( 1 + \frac{\lambda}{|\lambda_j - \lambda_m|} \right), \quad \forall m \geq N + 1.$$

One has similar estimates for the term

$$\log \left( 1 - \frac{\lambda}{|\lambda_i - \lambda_n|} \right) \leq \log \left( 1 - \frac{\lambda}{\lambda_i - \lambda_n} \right) \leq \log \left( 1 + \frac{\lambda}{|\lambda_i - \lambda_n|} \right), \quad \forall n \geq N + 1.$$

Since the value of  $\lambda \notin \mathcal{N}$  is fixed, we know from Assumption 1 the existence of  $c_0 = c_0(\lambda) > 0$  such that

$$|\lambda_i - \lambda_j - \lambda| \geq c_0 |\lambda_i - \lambda_j|, \quad \forall i, j \in \mathbb{N}^*. \quad (55)$$

In conclusion, we fix some  $j, n \in \mathbb{N}^*$ , then for any  $N \geq 2|j + n| + (10\lambda/c)^{1/(\alpha-1)}$  and any  $i \leq N$ , there is,

$$\begin{aligned} \tilde{T}_{ji}^{-1} - (\tilde{T}_{ji}^{-1})^N &\leq |\tilde{T}_{ji}^{-1}| \left( 1 - \exp \left( - \sum_{m \geq N+1} \log \left( 1 + \frac{\lambda}{|\lambda_j - \lambda_m|} \right) \right. \right. \\ &\quad \left. \left. - \sum_{n \geq N+1} \log \left( 1 + \frac{\lambda}{|\lambda_i - \lambda_n|} \right) \right) \right) \\ &\leq |\tilde{T}_{ji}^{-1}| \left( 1 - \exp \left( - \sum_{m \geq N+1} \frac{\lambda}{|\lambda_j - \lambda_m|} - \sum_{n \geq N+1} \frac{\lambda}{|\lambda_i - \lambda_n|} \right) \right) \\ &\leq |\tilde{T}_{ji}^{-1}| \left( 1 - \exp \left( - \sum_{m \geq N+1} \frac{\lambda}{c|j - m|m^{\alpha-1}} - \sum_{n \geq N+1} \frac{\lambda}{c|i - n|n^{\alpha-1}} \right) \right) \\ &\leq |\tilde{T}_{ji}^{-1}| \left( 1 - \exp \left( - \sum_{m \geq N+1} \frac{2\lambda}{cm^\alpha} - \sum_{n \geq N+1} \frac{2\lambda}{cn^\alpha} \right) \right) \\ &\leq |\tilde{T}_{ji}^{-1}| \left( 1 - \exp \left( - \frac{4\lambda}{c(\alpha-1)} \cdot \frac{1}{N^{\alpha-1}} - \frac{C \log N}{N^{\alpha-1}} \right) \right) \\ &\leq Ce^{C\lambda^{\frac{1}{\alpha}}} \left| \frac{\lambda^2}{(\lambda_i - \lambda_j - \lambda)} \right| \left( 1 - \exp \left( - \frac{4\lambda}{c(\alpha-1)} \cdot \frac{1}{N^{\alpha-1}} - \frac{C \log N}{N^{\alpha-1}} \right) \right) \\ &\leq Ce^{C\lambda^{\frac{1}{\alpha}}} \left| \frac{\lambda^2}{(\lambda_i - \lambda_j - \lambda)} \right| \frac{\lambda + \log N}{N^{\alpha-1}} \\ &\leq \frac{C}{|\lambda_i - \lambda_j|} \frac{\log N}{N^{\alpha-1}}, \end{aligned}$$

where the constant  $C = C(\lambda)$  is independent of  $j, n$  and  $N$  larger than  $2|j + n| + (10\lambda/c)^{1/(\alpha-1)}$  and  $i \leq N$ . And similarly, together with the simple estimates,

$$\log(1 - |x|) \geq -2|x| \text{ and } 1 - e^x \geq -2x, \quad \forall x \in [0, 1/10],$$

we obtain

$$\begin{aligned}
\tilde{T}_{ji}^{-1} - (\tilde{T}_{ji}^{-1})^N &\geq |\tilde{T}_{ji}^{-1}| \left( 1 - \exp \left( - \sum_{m \geq N+1} \log \left( 1 - \frac{\lambda}{|\lambda_j - \lambda_m|} \right) \right. \right. \\
&\quad \left. \left. - \sum_{n \geq N+1} \log \left( 1 - \frac{\lambda}{|\lambda_i - \lambda_n|} \right) \right) \right) \\
&\geq |\tilde{T}_{ji}^{-1}| \left( 1 - \exp \left( - \sum_{m \geq N+1} \left( -\frac{2\lambda}{|\lambda_j - \lambda_m|} \right) - \sum_{n \geq N+1} \left( -\frac{2\lambda}{|\lambda_i - \lambda_n|} \right) \right) \right) \\
&= |\tilde{T}_{ji}^{-1}| \left( 1 - \exp \left( \sum_{m \geq N+1} \left( \frac{2\lambda}{|\lambda_j - \lambda_m|} \right) + \sum_{n \geq N+1} \left( \frac{2\lambda}{|\lambda_i - \lambda_n|} \right) \right) \right) \\
&\geq |\tilde{T}_{ji}^{-1}| \left( 1 - \exp \left( \frac{C(\lambda + \log N)}{N^{\alpha-1}} \right) \right) \\
&\geq -\frac{C}{|\lambda_i - \lambda_j|} \frac{\log N}{N^{\alpha-1}}.
\end{aligned}$$

Therefore, there exists some  $C = C(\lambda)$  such that for any  $j, n \in \mathbb{N}^*$ , for any  $N \geq 2|j+n| + (10\lambda/c)^{1/(\alpha-1)}$  and for any  $i \leq N$ , there is

$$\left| \tilde{T}_{ji}^{-1} - (\tilde{T}_{ji}^{-1})^N \right| \leq \frac{C \log N}{N^{\alpha-1} |\lambda_i - \lambda_j|}. \quad (56)$$

Using the estimate (56) we can prove the required equality (49). Fix  $j, n \in \mathbb{N}^*$ . By comparing (49) and (54) we obtain that as  $N \in (2|j+n| + (10\lambda/c)^{1/(\alpha-1)}, +\infty)$  tends to  $\infty$  there is

$$\begin{aligned}
&\left| \sum_{i \in \mathbb{N}^*} \tilde{T}_{ji}^{-1} \tilde{T}_{in} - \sum_{1 \leq i \leq N} (\tilde{T}^{-1})_{ji}^N \tilde{T}_{in} \right| \\
&= \left| \sum_{i \geq N+1} \tilde{T}_{ji}^{-1} \tilde{T}_{in} + \sum_{i=1}^N (\tilde{T}_{ji}^{-1} - (\tilde{T}_{ji}^{-1})^N) \tilde{T}_{in} \right| \\
&\leq \sum_{i \geq N+1} C e^{C\lambda^{\frac{1}{\alpha}}} \left| \frac{\lambda^2}{(\lambda_i - \lambda_j - \lambda)} \right| \frac{1}{|\lambda_i - \lambda_n - \lambda|} + \sum_{i=1}^N \frac{C}{|\lambda_i - \lambda_j|} \frac{\log N}{N^{\alpha-1}} \frac{1}{|\lambda_i - \lambda_n - \lambda|} \\
&\leq C \sum_{i \geq N+1} \frac{1}{|\lambda_i - \lambda_j|} \frac{1}{|\lambda_i - \lambda_n|} + C \sum_{i=1}^N \frac{1}{|\lambda_i - \lambda_j|} \frac{1}{|\lambda_i - \lambda_n|} \frac{\log N}{N^{\alpha-1}} \\
&\leq C \sum_{i \geq N+1} \frac{1}{\left(\frac{i}{2}\right)^\alpha} \frac{1}{\left(\frac{i}{2}\right)^\alpha} + C \sum_{i=1}^N \frac{1}{|i-j|^\alpha} \frac{1}{|i-n|^\alpha} \frac{\log N}{N^{\alpha-1}} \\
&\leq \frac{C}{N^{2\alpha-1}} + \frac{C \log N}{N^{\alpha-1}},
\end{aligned}$$

where we have used the estimate (55). Thus we conclude that

$$\sum_{i \in \mathbb{N}^*} \tilde{T}_{ji}^{-1} \tilde{T}_{in} = \lim_{N \rightarrow +\infty} \sum_{1 \leq i \leq N} (\tilde{T}^{-1})_{ji}^N \tilde{T}_{in} = \delta_{j,n}. \quad (57)$$

Therefore, the equality (49) holds implying that  $\check{T}^{-1} \circ \check{T} = Id$ . Using the same method we can also deduce that

$$\sum_{i \in \mathbb{N}^*} \check{T}_{ji} \check{T}_{in}^{-1} = \delta_{j,n}. \quad (58)$$

implying that  $\check{T} \circ \check{T}^{-1} = Id$ . This ends the proof of Proposition 4

□

### 3.2.3 Uniqueness condition and existence of the feedback.

We are now in position to solve the uniqueness condition  $TB = B$ , as given by (13). Recall that this condition may not be formulated as is, since  $B \notin H$  which may prevent  $TB$  from being well-defined *a priori*. To circumvent this technical point, we may exploit the infinite matrices representation of the operators  $\check{T}$  and  $\check{T}^{-1}$ , introduced in the previous subsection, to work on a weaker formulation, analogous to (59):

$$\sum_{n \in \mathbb{N}^*} \frac{k_n b_n}{\lambda_j - \lambda_n - \lambda} = 1, \quad \forall j \in \mathbb{N}^*. \quad (59)$$

Solving this uniqueness condition has a double impact: it defines the feedback coefficients  $k_n$  uniquely, and thereby defines a unique candidate transformation  $T$ . As we will show in Section 5, it is also crucial to prove that  $T$  and  $K$  do satisfy the operator equation (6), which ensures the equivalence between the closed-loop system and the target system.

**Corollary 1** *Let  $\lambda \notin \mathcal{N}$ . Then there exists  $\{k_n\}_{n \in \mathbb{N}^*} \in \ell^\infty(\mathbb{N}; \mathbb{R})$  such that*

$$\sum_{n \in \mathbb{N}} \frac{k_n b_n}{\lambda_j - \lambda_n - \lambda} = 1,$$

*is satisfied for every  $j \in \mathbb{N}^*$ .*

**Proof :** We have proven that

$$\sum_{n \in \mathbb{N}} \frac{k_n b_n}{\lambda_j - \lambda_n - \lambda} = 1, \quad \forall j \in \mathbb{N}^*,$$

is equivalent to

$$k_n b_n = \sum_{j \in \mathbb{N}^*} (\check{T}^{-1})_{nj}, \quad \forall n \in \mathbb{N}^* \quad (60)$$

Indeed, this is a consequence of Proposition 4. First, thanks to Proposition 2, we know that  $\{(\check{T}^{-1})_{nj}\}_{j \in \mathbb{N}^*} \in \ell^1(\mathbb{N}; \mathbb{R})$ . This in particular implies that the above series on the right-hand side is convergent. Next, by implementing the above defined  $\{k_n b_n\}_{n \in \mathbb{N}^*}$  into

the summation and using the identity (58) we obtain:

$$\begin{aligned} \sum_{n \in \mathbb{N}} \frac{k_n b_n}{\lambda_j - \lambda_n - \lambda} &= \sum_{n \in \mathbb{N}} \tilde{T}_{j,n} \sum_{k \in \mathbb{N}^*} (\tilde{T}^{-1})_{n,k} \\ &= \sum_{k \in \mathbb{N}^*} \sum_{n \in \mathbb{N}} \tilde{T}_{j,n} (\tilde{T}^{-1})_{n,k} \\ &= \sum_{k \in \mathbb{N}} \delta_{j,k} = 1. \end{aligned}$$

Moreover, by the controllability assumption, there exists  $0 < c \leq |b_n| \leq C, \forall n \in \mathbb{N}$ . With the fact that  $\sup_{n \in \mathbb{N}^*} \|\{(\tilde{T}^{-1})_{nj}\}_{j \in \mathbb{N}^*}\|_{\ell^1} < C$ , we deduce that  $\{k_n\}_{n \in \mathbb{N}^*} \in \ell^\infty(\mathbb{N}; \mathbb{R})$ .  $\square$

From Corollary 1 we deduce that there exists  $\{k_n\}_{n \in \mathbb{N}^*} \in \ell^\infty(\mathbb{N}; \mathbb{R})$  such that (59) holds. Note that, at this point, we have not yet proved the invertibility of  $T$  (in contrast with  $\tilde{T}$  which we know is invertible from Proposition 4): the properties established on the auxiliary transformation  $\tilde{T}$  suffice to study the uniqueness condition and define the feedback  $K$ . From (32) we know that with this choice of  $\{k_n\}_{n \in \mathbb{N}^*} \in \ell^\infty(\mathbb{N}; \mathbb{R})$  there exists a unique  $T \in \mathcal{L}(H)$ , and it satisfies a weak equivalent to the  $TB = B$  condition, namely (59). What we have shown is summarized in:

**Proposition 5** *Let  $\{k_n\}_{n \in \mathbb{N}^*}$  be given by Corollary 1, then  $T \in \mathcal{L}(H)$  given by (32), that is,*

$$T\varphi_n = k_n \sum_{p \in \mathbb{N}} \frac{b_p}{\lambda_p - \lambda - \lambda_n} \varphi_p, \quad \forall n \in \mathbb{N}$$

*is well defined and satisfies (59).*

In the following section, we show that this transformation  $T$  is invertible, and consequently that the initial system is exponentially stable with decay rate  $\lambda$  with the choice of feedback operator  $\{k_n\}_{n \in \mathbb{N}^*}$ . Moreover we will give additional estimates with respect to the dependency in  $\lambda$  of  $T$ .

In this section we turn to the operator  $T$ . Recall from (21)–(22) that  $T$  is linked to  $\tilde{T}$  by

$$T = \tau_B \tilde{T} \tau_K \tag{61}$$

and

$$T\varphi_n = k_n \sum_{j \in \mathbb{N}} b_j \tilde{T}_{j,n} \varphi_j. \tag{62}$$

This section is organized as follows: first we show some properties on  $K$  (Section 3.3). Then, we show that  $T$  is invertible thanks to our choice of  $K$  using a method similar to [8, 18, 17] (Section 3.4). Finally, we show precise quantitative estimates on the norm of  $T$  and  $T^{-1}$ , which is the main challenge of this section (Section 4). These sharp estimates will in turn be used to prove the finite-time stabilization of Section 6.

**Remark 6** *In fact, one can show that  $T$  can be written as follows:*

$$T = \frac{1}{\lambda} Id + T_c, \tag{63}$$

*where  $T_c$  is a compact operator (see [21, 24]).*

### 3.3 Properties of $K$

In this section, we discuss the properties of the feedback law  $K$ . First, we prove that in the present context  $K$  cannot be a bounded functional on  $H$  (see Lemma 5). This might be surprising, given that the solution of the system belongs to  $H$ . In fact, this is rendered possible by the fact that the construction of  $K$  depends on  $A$  and consequently  $K$  still make sense when applied to solutions of the systems. More precisely, we will prove that  $K$  is admissible for the strongly continuous semigroup  $(e^{At})_{t \geq 0}$  (see Lemma 6). First, we have the following result, classical for the Fredholm backstepping method.

**Lemma 5** *Let  $\{k_n\}_{n \in \mathbb{N}^*} \in \ell^\infty$ . If  $K : H \rightarrow \mathbb{R}$  were to be a bounded functional, then  $T : H \rightarrow H$  would be a compact operator and therefore not invertible over  $H$ .*

**Proof :** Assume  $K : H \rightarrow \mathbb{R}$  is a bounded functional. Then, for every  $f \in H$ ,

$$|Kf| = \left| \sum_{n \in \mathbb{N}^*} k_n f_n \right| < \infty. \quad (64)$$

We assume for the moment that  $\{f_n\}, \{k_n\} \in \ell^2(\mathbb{N}^*; \mathbb{R})$ . We deal with the general case  $\{f_n\}, \{k_n\} \in \ell^2(\mathbb{N}^*; \mathbb{C})$  later on. For every sequence  $\{f_n\} \in \ell^2(\mathbb{N}^*; \mathbb{R})$ , one can define  $g_n = \text{sign}(k_n)|f_n|$ , a sequence in  $\ell^2(\mathbb{N}^*; \mathbb{R})$ . Hence, defining  $g \in H$  such that

$$g = \sum_{n \in \mathbb{N}^*} g_n \varphi_n,$$

we deduce that

$$|Kg| = \left| \sum_{n \in \mathbb{N}^*} |k_n| |f_n| \right| < \infty,$$

and therefore  $\{|k_n| |f_n|\}_{n \in \mathbb{N}^*} \in \ell^1(\mathbb{N}^*; \mathbb{R})$ . But since  $\{f_n\} \in \ell^2(\mathbb{N}^*; \mathbb{R})$ , this amounts to  $\{k_n\} \in \ell^2(\mathbb{N}^*; \mathbb{R})$ . Indeed, consider the family of linear applications  $l_n : \ell^2(\mathbb{N}^*; \mathbb{R}) \rightarrow \mathbb{R}$ ,

$$l_n(\{f_j\}) = \sum_{j=1}^n |k_j| f_j.$$

The application  $l_n$  is a bounded functional of norm  $\|l_n\| = \left( \sum_{j=1}^n |k_j|^2 \right)^{1/2}$  by the Riesz's

representation Theorem. But from the Banach-Steinhaus Theorem, we deduce that  $\|l_n\|$  is bounded and converge to  $\|\{k_n\}\|_{\ell^2}$  as  $n \rightarrow \infty$ , hence  $\{k_n\} \in \ell^2(\mathbb{N}^*; \mathbb{R})$ . The general case  $\{f_n\}, \{k_n\} \in \ell^2(\mathbb{N}^*; \mathbb{C})$  follows the same proof as

$$\begin{aligned} |Kf|^2 &= \left| \sum_{n \in \mathbb{N}^*} k_n f_n \right|^2 \\ &= \left| \sum_{n \in \mathbb{N}^*} (\Re(k_n) + i\Im(k_n))(\Re(f_n) + i\Im(f_n)) \right|^2 \\ &= \left| \sum_{n \in \mathbb{N}^*} \Re(k_n)\Re(f_n) - \Im(k_n)\Im(f_n) \right|^2 + \left| \sum_{n \in \mathbb{N}^*} \Re(k_n)\Im(f_n) + \Im(k_n)\Re(f_n) \right|^2 \end{aligned}$$

and it follows that  $|Kf| < +\infty$  is equivalent to,

$$\left| \sum_{n \in \mathbb{N}^*} \Re(k_n) \Re(f_n) - \Im(k_n) \Im(f_n) \right| < \infty \text{ and } \left| \sum_{n \in \mathbb{N}^*} \Re(k_n) \Im(f_n) + \Im(k_n) \Re(f_n) \right| < \infty.$$

Hence, it suffices to repeat the proof above for each sum with  $\Re(f_n) \equiv 0$  and  $\Im(f_n) \equiv 0$ . We therefore conclude that if  $K : H \rightarrow \mathbb{R}$  is bounded, then  $\{k_n\} \in \ell^2(\mathbb{N}^*; \mathbb{C})$ .

We now prove that  $\{k_n\} \in \ell^2(\mathbb{N}^*; \mathbb{R})$  implies that  $T$  is an Hilbert-Schmidt compact operator, and therefore not invertible. Indeed, there exists  $C > 0$ , independent of  $n \in \mathbb{N}$ , such that

$$\begin{aligned} \sum_{n \in \mathbb{N}^*} \|T\varphi_n\|^2 &= \sum_{n \in \mathbb{N}^*} \sum_{k \in \mathbb{N}} \left| \frac{k_n b_k}{\lambda_k - \lambda_n - \lambda} \right|^2 \\ &\leq C \sum_{n \in \mathbb{N}^*} |k_n|^2 \sum_{k \in \mathbb{N}} \left| \frac{1}{\lambda_k - \lambda_n - \lambda} \right|^2 \\ &< +\infty \end{aligned}$$

the convergence of the last serie coming from Lemma 4. This concludes the proof of Lemma 5. □

We have proven that  $K$  is not a bounded functional from  $H$  to  $\mathbb{R}$ , or else it would imply  $T$  is not an invertible transformation from  $H$  to itself. Despite this ill-behaviour of  $K$ , we prove that the feedback operator  $K$  is admissible.

**Lemma 6** *Let  $k_n \in \ell^\infty(\mathbb{N}^*; \mathbb{C})$ . Then  $K$  is admissible for the strongly continuous semi-group  $(e^{At})_{t \geq 0}$ , that is  $Ke^{At}f \in L^2((0, T); \mathbb{C})$ , for all  $f \in H$ .*

**Proof :** Let  $f \in H$ . Then, for all  $t \in [0, T]$ ,  $e^{At}f \in L^2((0, T); H)$  and,

$$\begin{aligned} \|Ke^{At}f\|_{L^2(0, T)}^2 &= \int_0^T |Ke^{At}f|^2 dt \\ &= \int_0^T \left| \sum_{n \in \mathbb{N}^*} k_n \langle e^{At}f, \varphi_n \rangle \right|^2 dt \\ &= \int_0^T \left| \sum_{n \in \mathbb{N}^*} k_n e^{\lambda_n t} f_n \right|^2 dt \\ &\leq \|k_n\|_{\ell^\infty}^2 \int_0^T \left( \sum_{n \in \mathbb{N}^*} e^{2\lambda_n t} \right) \left( \sum_{n \in \mathbb{N}^*} |f_n|^2 \right) dt \\ &\leq \|k_n\|_{\ell^\infty}^2 \|f\|^2 \left( \sum_{n \in \mathbb{N}^*} \frac{e^{2\lambda_n T} - 1}{2\lambda_n} \right) \\ &\leq \|k_n\|_{\ell^\infty}^2 \|f\|^2 (e^{2\omega_A T} - 1) \left( \sum_{n \in \mathbb{N}^*} \frac{1}{2\lambda_n} \right) \\ &\leq C \|k_n\|_{\ell^\infty}^2 \|f\|^2. \end{aligned}$$

□

### 3.4 Invertibility of $T$

As it can be seen from (62) the invertibility of  $T$  depends on the behavior of  $(k_n)_{n \in \mathbb{N}}$ . Recall that the  $(k_n)_{n \in \mathbb{N}}$  are defined by Corollary 1. We can show the following.

**Lemma 7** *Assume there exist  $c, C > 0$  such that  $c \leq |k_n| \leq C$ , for all  $n \in \mathbb{N}^*$ . Then  $T$  is invertible from  $H$  to itself.*

**Proof :** Notice

$$\begin{aligned}
 Tf &= \sum_{n \in \mathbb{N}^*} f_n T \varphi_n \\
 &= \sum_{n \in \mathbb{N}^*} f_n k_n (A - (\lambda_n + \lambda)I)^{-1} B \\
 &= \sum_{n \in \mathbb{N}^*} f_n k_n \sum_{k \in \mathbb{N}^*} \frac{b_k}{\lambda_k - \lambda_n - \lambda} \varphi_k \\
 &= \sum_{k \in \mathbb{N}^*} \sum_{n \in \mathbb{N}^*} f_n k_n \frac{b_k}{\lambda_k - \lambda_n - \lambda} \varphi_k \\
 &= \tau_B \check{T} \tau_K f
 \end{aligned}$$

where,

$$\tau_K : \varphi_n \mapsto k_n \varphi_n, \quad \tau_B : \varphi_n \mapsto b_n \varphi_n,$$

From the controllability assumption,  $\tau_B$  is invertible from  $H$  to itself and from the hypothesis of Lemma 7, so is  $\tau_K$ . The invertibility of  $T$  then follows from that of  $\check{T}$ . □

## 4 Quantitative estimates for the transformation

The coefficients  $k_n$  plays a fundamental role in the quantitative estimates of  $K$  and hence of  $T$ . In this section, we prove not only that the coefficients  $|k_n|$  are bounded by below away from zero, but also their behaviour with respect to  $\lambda$ . This allows, in turn, to deduce precise estimations on the norms of  $T$  and  $T^{-1}$ .

### 4.1 Exhibiting the underlying isomorphism of $T$

The expression of  $T$  with respect to  $K$  is given by (61) and the one of  $T^{-1}$  is given by,

$$T^{-1} = \tau_K^{-1} \check{T}^{-1} \tau_B^{-1}. \quad (65)$$

The norm of  $\tau_B$  and  $\tau_B^{-1}$  are easily given by the assumption on  $b_n$ . On the other hand, the norm of  $\check{T}^{-1}$  is given by Lemma 3 and Proposition 3, which provides an estimate

with respect to  $\lambda$  provided that  $\lambda$  is away from  $\mathcal{N}$ . A similar estimate also holds for  $\check{T}$ . Therefore, to have a quantitative estimate of  $T$  and  $T^{-1}$ , it suffices to estimate the norm of  $\tau_K$  and  $\tau_K^{-1}$ . This is translated in the following lemma.

**Lemma 8** *Let  $\lambda \notin \mathcal{N}$ , Then, the isomorphism  $T = T_\lambda$  satisfies*

$$\begin{aligned}\|T\| &\leq \|\check{T}\| \sup_n |b_n| \sup_n |k_n|, \\ \|T^{-1}\| &\leq \|\check{T}^{-1}\| \sup_n |b_n^{-1}| \sup_n |k_n^{-1}|.\end{aligned}$$

**Proof :** First, recall that the bound of  $\check{T}$  and  $\check{T}^{-1}$  was proven in (44) in Proposition 3. Then, from the definition of  $T$ , given by (61), and the definition of  $T^{-1}$ , given by (65), we have

$$\|T\| \leq \|\tau_B\| \|\check{T}\| \|\tau_K\|, \quad \|T^{-1}\| \leq \|\tau_K^{-1}\| \|\check{T}^{-1}\| \|\tau_B^{-1}\|$$

and since

$$\begin{aligned}\|\tau_B\| &\leq \|b_n\|_{\ell^\infty}, \quad \|\tau_K\| \leq \|k_n\|_{\ell^\infty}, \\ \|\tau_B^{-1}\| &\leq \|b_n^{-1}\|_{\ell^\infty}, \quad \|\tau_K^{-1}\| \leq \|k_n^{-1}\|_{\ell^\infty},\end{aligned}$$

the result follows easily.  $\square$

We now provide estimates on  $\|\check{T}\|$  and  $\|\check{T}^{-1}\|$  with respect to  $\lambda$  and  $\text{Dist}_\alpha(\lambda)$  (recall the definition (39)). These estimates are done in a range of spaces in anticipation of Section 6, where bounds in stronger norms are necessary to achieve finite-time stabilization. As we will see, stronger norms come from more regular spaces in which a common domain can be used to define the feedback and the well-posedness.

**Lemma 9** *Under the same hypothesis,  $\check{T}$  is an isomorphism on  $D(A^s)$ , for  $s \in (1/2\alpha - 1, 1 - 1/2\alpha)$  and we have the following estimates:*

$$\begin{aligned}\|\check{T}\|_{D(A^s)} &\leq C \left(1 + \frac{1}{\text{Dist}_\alpha(\lambda)}\right) \\ \|\check{T}^{-1}\|_{D(A^s)} &\leq C \left(1 + \frac{1}{\text{Dist}_\alpha(\lambda)}\right) e^{C\lambda^{\frac{1}{\alpha}}}.\end{aligned}$$

Accordingly, the isomorphism  $T$  satisfies, for  $s \in (1/2\alpha - 1, 1 - 1/2\alpha)$

$$\begin{aligned}\|T\|_{D(A^s)} &\leq C \left(1 + \frac{1}{\text{Dist}_\alpha(\lambda)}\right) \sup_n |b_n| \sup_n |k_n| \\ \|T^{-1}\|_{D(A^s)} &\leq C \left(1 + \frac{1}{\text{Dist}_\alpha(\lambda)}\right) \sup_n |b_n^{-1}| \sup_n |k_n^{-1}|.\end{aligned}$$

**Proof :** Using the expressions of  $\check{T}$  and  $\check{T}^{-1}$  given by (36) and (38) respectively, we have

$$\begin{aligned} \|\check{T}^{-1}f\|_{D(A^s)}^2 &= \left\| \sum_{k \in \mathbb{N}^*} f_k (\check{T}^{-1})_{kk} \varphi_k + \sum_{k \in \mathbb{N}^*} \sum_{n \in \mathbb{N}^* \setminus \{k\}} f_n (\check{T}^{-1})_{kn} \varphi_k \right\|_{D(A^s)}^2 \\ &\leq C e^{C\lambda^{\frac{1}{\alpha}}} \|f\|_{D(A^s)}^2 \\ &\quad + \sum_{k \in \mathbb{N}^*} |\lambda_k|^{2s} \left| \sum_{n \in \mathbb{N}^* \setminus \{k\}} \frac{f_n}{\lambda_n - \lambda_k - \lambda} \prod_{\substack{m \in \mathbb{N}^* \\ m \neq k}} \left(1 + \frac{\lambda}{\lambda_k - \lambda_m}\right) \prod_{\substack{p \in \mathbb{N}^* \\ p \neq n}} \left(1 - \frac{\lambda}{\lambda_n - \lambda_p}\right) \right|^2 \end{aligned}$$

We have, for  $1 < \alpha' < \alpha$ ,

$$\begin{aligned} \sum_{k \in \mathbb{N}^*} |\lambda_k|^{2s} \left| \sum_{n \in \mathbb{N}^* \setminus \{k\}} \frac{|f_n|}{|\lambda_n - \lambda_k - \lambda|} \right|^2 &= \sum_{k \in \mathbb{N}^*} |\lambda_k|^{2s} \left| \sum_{n \in \mathbb{N}^* \setminus \{k\}} \frac{|\lambda_n|^s |f_n|}{|\lambda_n|^s |\lambda_n - \lambda_k - \lambda|} \right|^2 \\ &= \sum_{k \in \mathbb{N}^*} \left( \sum_{n \in \mathbb{N}^* \setminus \{k\}} \frac{|\lambda_n|^{2s} |f_n|^2}{|\lambda_n - \lambda_k - \lambda|^{\frac{1}{\alpha'}}} \right) \left( \sum_{n \in \mathbb{N}^* \setminus \{k\}} \frac{|\lambda_k|^{2s}}{|\lambda_n|^{2s} |\lambda_n - \lambda_k - \lambda|^{2 - \frac{1}{\alpha'}}} \right). \end{aligned}$$

On one hand, using the convexity of  $x \in \mathbb{R}_+ \mapsto x^{2s}$  for  $s \geq 1/2$ ,

$$\begin{aligned} \sum_{n \in \mathbb{N}^* \setminus \{k\}} \frac{|\lambda_k|^{2s}}{|\lambda_n|^{2s} |\lambda_n - \lambda_k - \lambda|^{2 - \frac{1}{\alpha'}}} &= \sum_{n \in \mathbb{N}^* \setminus \{k\}} \frac{|\lambda_n - \lambda_k - \lambda + \lambda - \lambda_n|^{2s}}{|\lambda_n|^{2s} |\lambda_n - \lambda_k - \lambda|^{2 - \frac{1}{\alpha'}}} \\ &\leq C_s \sum_{n \in \mathbb{N}^* \setminus \{k\}} \frac{|\lambda_n - \lambda_k - \lambda|^{2s} + |\lambda - \lambda_n|^{2s}}{|\lambda_n|^{2s} |\lambda_n - \lambda_k - \lambda|^{2 - \frac{1}{\alpha'}}} \\ &\leq C_s \sum_{n \in \mathbb{N}^* \setminus \{k\}} \frac{1}{|\lambda_n|^{2s} |\lambda_n - \lambda_k - \lambda|^{2(1-s) - \frac{1}{\alpha'}}} \\ &\quad + C_s \sum_{n \in \mathbb{N}^* \setminus \{k\}} \frac{1}{|\lambda_n - \lambda_k - \lambda|^{2 - \frac{1}{\alpha'}}} \end{aligned}$$

Notice that since  $1 < \alpha' < \alpha$ , we have  $2 - \frac{1}{\alpha'} > 1$ , ensuring the convergence of the second sum. Moreover, since  $s \geq 1/2$ , the convergence of the first sum is ensured if

$$2(1-s) - \frac{1}{\alpha'} > 0$$

that is,  $s < 1 - 1/2\alpha' < 1 - 1/2\alpha$ . Finally, one deduces, since  $\frac{1}{\alpha'} > 1$ , that,

$$\sum_{n \in \mathbb{N}^* \setminus \{k\}} \frac{|\lambda_n|^{2s} |f_n|^2}{|\lambda_n - \lambda_k - \lambda|^{\frac{1}{\alpha'}}} \leq \|f\|_{D(A^s)}^2.$$

Moreover, we are able to estimate the cost of the transformation with respect to  $\lambda$ . Indeed, it is easy to see from the proof of Proposition 4 that the term  $\text{Dist}_\alpha(\lambda)$  is in every sum including terms of the form  $1/|\lambda_n - \lambda_k - \lambda|$ . Collecting these terms yields the upper bound

$$\begin{aligned} \|\check{T}^{-1}f\|_{D(A^s)}^2 &\leq C \left(1 + \frac{1}{\text{Dist}_\alpha(\lambda)^{\frac{1}{\alpha'}}}\right) \left(1 + \frac{1}{\text{Dist}_\alpha(\lambda)^{2(1-s)-\frac{1}{\alpha'}}}\right) \|f\|_{D(A^s)}^2 \\ &\leq C \left(1 + \frac{1}{\text{Dist}_\alpha(\lambda)^{2(1-s)}}\right) \|f\|_{D(A^s)}^2, \end{aligned}$$

assuming  $\text{Dist}_\alpha(\lambda) \gg 1$ . This concludes the case  $s \in [1/2, (2\alpha - 1)/2\alpha)$ . Since we have proved the result for  $s = 0$  in Proposition 4, we deduce the result for  $s \in [0, 1/2)$  by interpolation. This concludes the proof.  $\square$

## 4.2 Cost of the backstepping transformation with respect to the damping parameter $\lambda$

In this section, we prove one of the main results of this article

**Theorem 6** *For any  $N \in \mathbb{N}$ , there exists  $\lambda \in [N, N + 1]$  such that the isomorphism  $T = T(\lambda)$  satisfies the following estimate*

$$\|T\| + \|T^{-1}\| \leq Ce^{c\lambda^{\frac{1}{\alpha}}}, \quad (66)$$

where  $C$  is a constant that does not depend on  $N$  or  $\lambda$ .

From Lemma 8, Lemma 9 and Assumption 1, we have

$$\begin{aligned} \|T\| &\leq C \left(1 + \frac{1}{\text{Dist}_\alpha(\lambda)}\right) \sup_n |k_n|, \\ \|T^{-1}\| &\leq C \left(1 + \frac{1}{\text{Dist}_\alpha(\lambda)}\right) e^{c\lambda^{\frac{1}{\alpha}}} \sup_n |k_n^{-1}|. \end{aligned}$$

The quantity  $\sup |k_n|$  is easy to estimate (see Lemma 10 and Proposition 6 below). Estimating  $\sup |1/k_n|$  however, is much more subtle. The object of most of this subsection is to prove the following theorem.

**Theorem 7** *For any  $N \in \mathbb{N}$ , there exists  $\lambda \in [N, N + C]$ , where  $C > 0$  is a constant independent of  $N$ , such that the following quantitative lower bound of  $\{b_n k_n\}_{n \in \mathbb{N}}$  with respect to  $\lambda$  holds*

$$|k_n b_n| \geq e^{-c\lambda^{\frac{1}{\alpha}}}, \quad \forall n \in \mathbb{N}^*, \quad (67)$$

where  $c > 0$  is a constant independent of  $N$ , and  $\lambda$ .

We can conclude the proof of Theorem 6 by combining the estimates above, this previous theorem and from Proposition 8 below, deriving estimates of the form

$$\text{Dist}_\alpha(\lambda) \geq e^{-c\lambda^{\frac{1}{\alpha}}}$$

for values of  $\lambda$  in  $[N, N + C]$  and for any  $N \in \mathbb{N}$ .

The proof of Theorem 7 is based on four steps:

1. Decompose  $k_n b_n$  in two terms,  $F_n$  and  $J_n$  that have different behavior (see Lemma 10).
2. Show that  $J_n(\lambda) = 1$  for any  $n \in \mathbb{N}$  (see Lemma 11). This is the object of Section 4.4.
3. Use careful estimations to estimate  $F_n(\lambda)$  as a function of  $\text{Dist}_\alpha(\lambda)$  and  $\lambda$  (see Proposition 7). This is the object of Section 4.5.
4. Show that for good choice of  $\lambda$  one has a good lower bound of  $\text{Dist}_\alpha(\lambda)$  (see Proposition 8). This is done in Section 4.6.

### 4.3 Rearrangement of $k_n b_n$

We begin by a rearrangement of the expression  $k_n b_n$  to decompose it in two parts.

**Lemma 10** *The following holds for any  $n \in \mathbb{N}$*

$$k_n b_n = -\lambda F_n(\lambda) J_n(\lambda), \tag{68}$$

where

$$F_n(\lambda) := \prod_{\substack{m \in \mathbb{N}^* \\ m \neq n}} \left( 1 + \frac{\lambda}{\lambda_n - \lambda_m} \right), \tag{69}$$

$$J_n(\lambda) := \sum_{j \in \mathbb{N}} \frac{\prod_{\substack{m \in \mathbb{N}^* \\ m \neq n}} (\lambda_j - \lambda_m - \lambda)}{\prod_{\substack{m \in \mathbb{N}^* \\ m \neq j}} (\lambda_j - \lambda_m)}. \tag{70}$$

**Proof :** By definition, we know from (60) that

$$\begin{aligned}
k_n b_n &= \sum_{j \in \mathbb{N}^*} \tilde{T}_{nj}^{-1} \\
&= \sum_{j \in \mathbb{N}^*} \frac{\lambda^2}{\lambda_j - \lambda_n - \lambda} \prod_{\substack{m \in \mathbb{N}^* \\ m \neq n}} \left(1 + \frac{\lambda}{\lambda_n - \lambda_m}\right) \prod_{\substack{m \in \mathbb{N}^* \\ m \neq j}} \left(1 - \frac{\lambda}{\lambda_j - \lambda_m}\right) \\
&= \left( \prod_{\substack{m \in \mathbb{N}^* \\ m \neq n}} \left(1 + \frac{\lambda}{\lambda_n - \lambda_m}\right) \right) \left( \sum_{j \in \mathbb{N}^*} \frac{\lambda^2}{\lambda_j - \lambda_n - \lambda} \prod_{\substack{m \in \mathbb{N}^* \\ m \neq j}} \left(1 - \frac{\lambda}{\lambda_j - \lambda_m}\right) \right) \\
&= -\lambda \left( \prod_{\substack{m \in \mathbb{N}^* \\ m \neq n}} \left(1 + \frac{\lambda}{\lambda_n - \lambda_m}\right) \right) \left( \sum_{j \in \mathbb{N}^*} \frac{-\lambda}{\lambda_j - \lambda_n - \lambda} \prod_{\substack{m \in \mathbb{N}^* \\ m \neq j}} \left(1 - \frac{\lambda}{\lambda_j - \lambda_m}\right) \right) \\
&= -\lambda \left( \prod_{\substack{m \in \mathbb{N}^* \\ m \neq n}} \left(1 + \frac{\lambda}{\lambda_n - \lambda_m}\right) \right) \sum_{j \in \mathbb{N}} \frac{\prod_{\substack{m \in \mathbb{N}^* \\ m \neq n}} (\lambda_j - \lambda_m - \lambda)}{\prod_{\substack{m \in \mathbb{N}^* \\ m \neq j}} (\lambda_j - \lambda_m)} \\
&= -\lambda F_n(\lambda) J_n(\lambda).
\end{aligned}$$

where we used that

$$\begin{aligned}
&\sum_{j \in \mathbb{N}^*} \frac{-\lambda}{\lambda_j - \lambda_n - \lambda} \prod_{\substack{m \in \mathbb{N}^* \\ m \neq j}} \left(1 - \frac{\lambda}{\lambda_j - \lambda_m}\right) \\
&= \sum_{j \in \mathbb{N}^*} \frac{-\lambda}{\lambda_j - \lambda_n - \lambda} \left( \frac{\prod_{\substack{m \in \mathbb{N}^* \\ m \neq j}} (\lambda_j - \lambda_m - \lambda)}{\prod_{\substack{m \in \mathbb{N}^* \\ m \neq j}} (\lambda_j - \lambda_m)} \right) \\
&= \sum_{j \in \mathbb{N}^*} \frac{1}{\lambda_j - \lambda_n - \lambda} \left( \frac{\prod_{\substack{m \in \mathbb{N}^* \\ m \neq j}} (\lambda_j - \lambda_m - \lambda)}{\prod_{\substack{m \in \mathbb{N}^* \\ m \neq j}} (\lambda_j - \lambda_m)} \right) \\
&= \sum_{j \in \mathbb{N}^*} \left( \frac{1}{\prod_{\substack{m \in \mathbb{N}^* \\ m \neq j}} (\lambda_j - \lambda_m)} \right) \left( \frac{\prod_{\substack{m \in \mathbb{N}^* \\ m \neq j}} (\lambda_j - \lambda_m - \lambda)}{\lambda_j - \lambda_n - \lambda} \right) \\
&= \sum_{j \in \mathbb{N}} \frac{\prod_{\substack{m \in \mathbb{N}^* \\ m \neq n}} (\lambda_j - \lambda_m - \lambda)}{\prod_{\substack{m \in \mathbb{N}^* \\ m \neq j}} (\lambda_j - \lambda_m)} \tag{71}
\end{aligned}$$

□

One can note that  $F_n$  is an holomorphic function of  $\lambda \in \mathbb{C}$ , with a countable number of zeros  $\lambda = \lambda_m - \lambda_n$ ,  $m \in \mathbb{N} \setminus \{n\}$  and such that  $F_n(0) = 1$ . In the following we study the

behavior of  $F_n$  and  $J_n$ . In particular, we will show a remarkable property:  $J_n(\lambda) = 1$ , for any  $\lambda \in \mathbb{C}$  and any  $n \in \mathbb{N}^*$  (see Lemma 11). We will also show that there exists  $C, c > 0$  such that

$$|F_n(\lambda)| \geq C \text{Dist}_\alpha(\lambda) e^{-c\lambda^{\frac{1}{\alpha}}}, \quad \forall n \in \mathbb{N}^*, \forall \lambda > 1, \quad (72)$$

where  $\text{Dist}_\alpha(\lambda)$  is given by (39). Note that from Lemma 4 we also have

$$|F_n(\lambda)| \leq C e^{c\lambda^{\frac{1}{\alpha}}}. \quad (73)$$

#### 4.4 Behavior of $J_n(\lambda)$ .

We begin by drastically simplifying one of the terms introduced above.

**Lemma 11** *For  $\lambda \in \mathbb{C}$  and  $n \in \mathbb{N}^*$ , we have*

$$J_n(\lambda) = 1.$$

**Proof of Lemma 11:**

The proof is composed by two steps:

Step A: Show that a truncated version  $J_n^N(\lambda)$  of  $J_n(\lambda)$  is equal to 1 for any  $\lambda \in \mathbb{C}$ , any  $N \in \mathbb{N}^*$  and any  $1 \leq n \leq N$ .

Step B: Pass the limit in  $N$  thanks to the distribution of  $\{\lambda_n\}_{n \in \mathbb{N}^*}$ , and leveraging that  $\lambda_n \sim n^\alpha$  with  $\alpha > 1$ .

Before starting the proof of the first step, let us define the following polynomial

$$P(\lambda) := \sum_{j=1}^{N-1} 1 \cdot \left( \prod_{\substack{1 \leq m \leq N-1 \\ m \neq j}} \frac{\lambda - \lambda_m}{\lambda_j - \lambda_m} \right), \quad (74)$$

which is of order at most  $N - 2$ . This is actually the Lagrange interpolation formula such that  $P(\lambda_p) = 1, 1 \leq p \leq N - 1$ . We get, with this definition,

$$\sum_{j=1}^N \frac{\prod_{m \neq N} (\lambda_N - \lambda_m)}{\prod_{\substack{m=1 \\ m \neq j}}^N (\lambda_j - \lambda_m)} = - \sum_{j=1}^{N-1} \frac{\prod_{m=1}^{N-1} (\lambda_N - \lambda_m)}{\prod_{\substack{m=1 \\ m \neq j}}^{N-1} (\lambda_j - \lambda_m)} + 1 = 1 - P(\lambda_N) = 0. \quad (75)$$

**Step A:** Let  $\{\zeta_1, \zeta_2, \dots, \zeta_N\}$  be a given a sequence of constants satisfying  $\zeta_i \neq \zeta_j$ . Inspired by the definition of  $J_n$  in equation (70) we further define, for any  $N \in \mathbb{N}^*$  and any

$1 \leq n \leq N$ :

$$J_n^N(\lambda) := J_n^N(\lambda; \zeta_1, \zeta_2, \dots, \zeta_N) := \sum_{j=1}^N \frac{\prod_{\substack{m \leq N \\ m \neq n}} (\zeta_j - \zeta_m - \lambda)}{\prod_{\substack{m \leq N \\ m \neq j}} (\zeta_j - \zeta_m)}. \quad (76)$$

In the following we will prove that

$$J_n^N(\lambda; \zeta_1, \zeta_2, \dots, \zeta_N) = 1, \quad \forall \lambda \in \mathbb{C}, \forall n \in \{1, 2, \dots, N\}. \quad (77)$$

To do so, we immediately notice the following symmetry

$$J_{N-1}^N(\lambda; \zeta_1, \zeta_2, \dots, \zeta_{N-1}, \zeta_N) = J_N^N(\lambda; \zeta_1, \zeta_2, \dots, \zeta_{N-2}, \zeta_N, \zeta_{N-1}). \quad (78)$$

We further observe that

$$J_N^N(\lambda) = \sum_{j=1}^{N-1} \left( \frac{\prod_{m \leq N-1} (\zeta_j - \zeta_m - \lambda)}{\prod_{\substack{m \leq N \\ m \neq j}} (\zeta_j - \zeta_m)} \right) + \frac{\prod_{m \leq N-1} (\zeta_N - \zeta_m - \lambda)}{\prod_{m \leq N-1} (\zeta_N - \zeta_m)}. \quad (79)$$

By multiplying  $J_N^N(\lambda)$  with  $\prod_{m \leq N-1} (\zeta_N - \zeta_m)$  we shall prove that

$$P_N^N(\lambda) := \sum_{j=1}^{N-1} \left( \frac{\prod_{m \leq N-1} (\zeta_j - \zeta_m - \lambda) \prod_{m \leq N-1} (\zeta_N - \zeta_m)}{\prod_{\substack{m \leq N \\ m \neq j}} (\zeta_j - \zeta_m)} \right) + \prod_{m \leq N-1} (\zeta_N - \zeta_m - \lambda) - \prod_{m \leq N-1} (\zeta_N - \zeta_m),$$

equals to zero, which is equivalent to prove that

$$P_N^N(\lambda) = \sum_{j=1}^{N-1} \left( \frac{\lambda \prod_{\substack{m \leq N-1 \\ m \neq j}} (\zeta_j - \zeta_m - \lambda) \prod_{\substack{m \leq N-1 \\ m \neq j}} (\zeta_N - \zeta_m)}{\prod_{\substack{m \leq N-1 \\ m \neq j}} (\zeta_j - \zeta_m)} \right) + \prod_{m \leq N-1} (\zeta_N - \zeta_m - \lambda) - \prod_{m \leq N-1} (\zeta_N - \zeta_m) = 0.$$

Inspired by the preceding formula, for any given  $\{\zeta_1, \dots, \zeta_{N-1}, \lambda\}$  satisfying  $\zeta_i \neq \zeta_j$  we define

$$Q(\mu; \zeta_1, \dots, \zeta_{N-1}, \lambda) := \sum_{j=1}^{N-1} \left( \frac{\lambda \prod_{\substack{m \leq N-1 \\ m \neq j}} (\zeta_j - \zeta_m - \lambda) \prod_{\substack{m \leq N-1 \\ m \neq j}} (\mu - \zeta_m)}{\prod_{\substack{m \leq N-1 \\ m \neq j}} (\zeta_j - \zeta_m)} \right) + \prod_{m \leq N-1} (\mu - \zeta_m - \lambda) - \prod_{m \leq N-1} (\mu - \zeta_m), \quad (80)$$

which is clearly a polynomial of  $\mu$  of order  $N - 2$ . Now we show that  $Q(\mu) = 0$ . Indeed, by choosing  $\mu = \zeta_1$  we get that

$$\begin{aligned} Q(\zeta_1) &= \sum_{j=1}^{N-1} \left( \frac{\lambda \prod_{\substack{m \leq N-1 \\ m \neq j}} (\zeta_j - \zeta_m - \lambda) \prod_{\substack{m \leq N-1 \\ m \neq j}} (\zeta_1 - \zeta_m)}{\prod_{\substack{m \leq N-1 \\ m \neq j}} (\zeta_j - \zeta_m)} \right) \\ &\quad + \prod_{m \leq N-1} (\zeta_1 - \zeta_m - \lambda) - \prod_{m \leq N-1} (\zeta_1 - \zeta_m), \\ &= \lambda \prod_{\substack{m \leq N-1 \\ m \neq 1}} (\zeta_1 - \zeta_m - \lambda) + \prod_{m \leq N-1} (\zeta_1 - \zeta_m - \lambda) \\ &= 0. \end{aligned}$$

Thanks to the symmetry we obtain that

$$Q(\zeta_i) = 0, \forall i \in \{1, 2, \dots, N - 1\}, \quad (81)$$

which, to be combined with the fact that  $Q(\mu)$  is a polynomial of order  $N - 2$ , yields  $Q(\mu) = 0$ . In particular, by choosing  $\mu = \zeta_N$  we get

$$0 = Q(\zeta_N) = P_N^N(\lambda). \quad (82)$$

Hence, for any sequence  $\{\zeta_1, \dots, \zeta_N\}$  satisfying  $\zeta_i \neq \zeta_j$  and for any  $\lambda$  we know that  $P_N^N(\lambda) = 0$ . This concludes the proof of Step 1 and (77).

**Step B:** This step is standard and consists in passing to the limit in (77). We can apply, for instance, a dominated convergence by denoting

$$I_j^N := \frac{\lambda}{\lambda_j - \lambda_n - \lambda} \prod_{\substack{m \leq N \\ m \neq j}} \left( 1 - \frac{\lambda}{\lambda_j - \lambda_m} \right), \quad I_j := \frac{\lambda}{\lambda_j - \lambda_n - \lambda} \prod_{\substack{m \in \mathbb{N}^* \\ m \neq j}} \left( 1 - \frac{\lambda}{\lambda_j - \lambda_m} \right). \quad (83)$$

We observe that for  $j \leq N$

$$\frac{I_j^N}{I_j} = \prod_{\substack{m > N \\ m \neq j}} \left( 1 - \frac{\lambda}{\lambda_j - \lambda_m} \right)^{-1}. \quad (84)$$

Note that we have, for  $j$  given, and when  $m \rightarrow +\infty$

$$\log \left( 1 - \frac{\lambda}{\lambda_j - \lambda_m} \right) \sim -\frac{\lambda}{C|m-j|^\alpha} \quad (85)$$

and therefore

$$\sum_{m>N} \log \left( 1 - \frac{\lambda}{\lambda_j - \lambda_m} \right) \rightarrow 0, \quad N \rightarrow \infty. \quad (86)$$

This and (84) imply  $I_j^N \rightarrow I_j$  when  $N \rightarrow +\infty$  for any  $j \in \mathbb{N}^*$ . Looking again at (84) and using (29), for  $N > (\lambda/c)^{1/(\alpha-1)} - 1$ , where  $c$  is given by (29),

$$\begin{aligned} |I_j^N \mathbb{1}_{j \leq N}| &= |I_j \mathbb{1}_{j \leq N}| \prod_{\substack{m>N \\ m \neq j}} \left( 1 - \frac{\lambda}{\lambda_m - \lambda_j} \right)^{-1} \\ &\leq |I_j \mathbb{1}_{j \leq N}| \prod_{\substack{m>N \\ m \neq j}} \left( 1 - \frac{\lambda}{c|m-j|m^{\alpha-1}} \right)^{-1} \\ &\leq |I_j| \prod_{m>N} \left( 1 - \frac{\lambda}{c|m-N|m^{\alpha-1}} \right)^{-1}, \end{aligned} \quad (87)$$

where we used the fact that if  $j > N$  then  $\mathbb{1}_{j \leq N}$  vanishes. This gives,

$$|I_j^N \mathbb{1}_{j \leq N}| \leq |I_j| \prod_{k>0} \left( 1 - \frac{\lambda}{c|k|(k+N)^{\alpha-1}} \right)^{-1} \leq |I_j| \prod_{k>0} \left( 1 - \frac{1}{ck^\alpha} \right)^{-1}. \quad (88)$$

Finally, we can show that the right-hand side is integrable: from the proof of Lemma 3 and Lemma 4, we have

$$\sum_{j \in \mathbb{N}^*} |I_j| = \sum_{j \in \mathbb{N}^*} \left| \frac{\lambda}{\lambda_j - \lambda_n - \lambda} \right| \left| \prod_{\substack{m \in \mathbb{N}^* \\ m \neq j}} \left( 1 - \frac{\lambda}{\lambda_j - \lambda_m} \right) \right| < +\infty, \quad (89)$$

and

$$\prod_{k>0} \left( 1 - \frac{1}{ck^\alpha} \right)^{-1} = \exp \left( \sum_{k>0} \ln \left( 1 + \frac{1}{ck^\alpha - 1} \right) \right) < +\infty, \quad (90)$$

thus from (70) and (71), (76), (83) and (88) we conclude by dominated convergence that

$$J_n^N(\lambda) = - \sum_{j \in \mathbb{N}^*} I_j^N \mathbb{1}_{j \leq N} \rightarrow - \sum_{j \in \mathbb{N}^*} I_j = J_n(\lambda). \quad (91)$$

□

## 4.5 Behavior of $F_n$

Now, we turn to the other term of decomposition (68). Note that we have the following proposition as a direct consequence of Lemma 3.

**Proposition 6** *Let  $\lambda > 0$ . Then, there exists  $C > 0$  independent of  $\lambda$  such that,*

$$|F_n(\lambda)| \leq Ce^{C\lambda^{\frac{1}{\alpha}}}.$$

Let us recall the definition of  $\text{Dist}_\alpha(\lambda)$ ,

$$\text{Dist}_\alpha(\lambda) = \min\{|\lambda_n - \lambda_m - \lambda| : m, n \in \mathbb{N}^*\}.$$

Note that this is indeed a minimum rather than an infimum from (29). We show the following lower bound on  $|F_n(\lambda)|$ , as a function of  $\text{Dist}_\alpha(\lambda)$ .

**Proposition 7** *There exist positive constants  $c, C$  such that, for any  $\alpha > 1$ ,  $n \in \mathbb{N}^*$  and any  $\lambda > 1$ ,*

$$|F_n(\lambda)| \geq \text{Dist}_\alpha(\lambda) \cdot Ce^{-c\lambda^{\frac{1}{\alpha}}}, \quad \forall n \in \mathbb{N}^*, \forall \lambda > 1. \quad (92)$$

**Proof of Proposition 7:** Recall that, thanks to (28), for any  $n > m \geq 1$  there exist positive constants  $C, c$  such that,

$$C(n-m)n^{\alpha-1} > |\lambda_n - \lambda_m| > c(n-m)n^{\alpha-1}. \quad (93)$$

Similarly, still from (28),

$$n^\alpha \lesssim |\lambda_n| + 1 \lesssim n^\alpha, \quad \forall n \in \mathbb{N}^*. \quad (94)$$

We further have,

**Lemma 12** *There exists some effectively computable constant  $C > 0$  such that,*

$$\prod_{m \in \mathbb{N}^*} \left(1 + \frac{\lambda}{|\lambda_m|}\right) \lesssim e^{C\lambda^{\frac{1}{\alpha}}}, \quad \forall \lambda > 1. \quad (95)$$

**Proof :** Indeed, thanks to the estimate (94), it is equivalent to show the following,

$$\prod_{m \in \mathbb{N}^*} \left(1 + \frac{\lambda}{m^\alpha}\right) \lesssim e^{C\lambda^{\frac{1}{\alpha}}}, \quad \forall \lambda > 1. \quad (96)$$

The proof is similar to the proof of Lemma 3. Indeed, let

$$\begin{aligned}
\ln \left( \prod_{m \in \mathbb{N}^* \setminus \{n\}} \left( 1 + \frac{\lambda}{m^\alpha} \right) \right) &= \sum_{m \in \mathbb{N}^* \setminus \{n\}} \ln \left( 1 + \frac{\lambda}{m^\alpha} \right) \\
&\leq \log(1 + \lambda) + \int_1^{+\infty} \log \left( 1 + \frac{\lambda}{x^\alpha} \right) dx \\
&\leq \log(1 + \lambda) + x \log \left( 1 + \frac{\lambda}{x^\alpha} \right) \Big|_1^{+\infty} - \int_1^{+\infty} x d \log \left( 1 + \frac{\lambda}{x^\alpha} \right) \\
&= 2 \int_1^{+\infty} \frac{\lambda}{\lambda + x^\alpha} dx \\
&= 2 \int_1^{+\infty} \frac{1}{1 + \left( \frac{x}{\lambda^{\frac{1}{\alpha}}} \right)^\alpha} dx \\
&= 2\lambda^{\frac{1}{\alpha}} \int_{\lambda^{-\frac{1}{\alpha}}}^{+\infty} \frac{1}{1 + y^\alpha} dy \\
&\leq C\lambda^{\frac{1}{\alpha}}.
\end{aligned} \tag{97}$$

noting that  $\int_{\lambda^{-\frac{1}{\alpha}}}^{+\infty} (1 + x^\alpha)^{-1} \leq \int_0^{+\infty} (1 + x^\alpha)^{-1} < +\infty$  since  $\alpha > 1$ .  $\square$

To continue the proof of Proposition 7, we are going to proceed in four steps.

**Step 1: show that, for  $F_n(\lambda)$ , only the terms  $m \in [1, n-1]$  matter.**

Notice that

$$|F_n(\lambda)| = \prod_{m < n} \left| 1 + \frac{\lambda}{\lambda_n - \lambda_m} \right| \cdot \prod_{m > n} \left| 1 + \frac{\lambda}{\lambda_n - \lambda_m} \right| > \prod_{m < n} \left| 1 + \frac{\lambda}{\lambda_n - \lambda_m} \right|, \tag{98}$$

since  $\lambda_m - \lambda_n \leq 0$  for  $m > n$  (recall that the eigenvalues are negative). In fact  $\operatorname{Re}(\lambda_m - \lambda_n) \leq 0$  would suffice. Thus, to show Proposition 7 it suffices to show that,

$$\tilde{F}_n(\lambda) := \left| \prod_{m < n} \left( 1 + \frac{\lambda}{\lambda_n - \lambda_m} \right) \right| \gtrsim \operatorname{Dist}_\alpha(\lambda) e^{-C\lambda^{\frac{1}{\alpha}}}. \tag{99}$$

In fact, this is even an equivalence since, from Lemma 3,

$$\prod_{m > n} \left| 1 + \frac{\lambda}{\lambda_n - \lambda_m} \right| \lesssim e^{C\lambda^{\frac{1}{\alpha}}}.$$

**Step 2: case  $\lambda \in (1, c_0|\lambda_n - \lambda_{n-1}|)$  with  $c_0 < 1$  and independent from  $n$ .**

Let's assume that  $n > m$ . Since  $\lambda \leq c_0|\lambda_n - \lambda_{n-1}|$  and  $(\lambda_n)_{n \in \mathbb{N}^*}$  are real,

$$\frac{\lambda}{\lambda_m - \lambda_n} \leq \frac{\lambda}{\lambda_{n-1} - \lambda_n} \leq c_0. \tag{100}$$

Now, observe that since  $c_0 < 1$ , there exists  $k_{c_0} > 0$  depending only on this  $c_0$  such that

$$(1 - x)(1 + k_{c_0}x) \geq 1, \quad \forall x \in [0, c_0]. \quad (101)$$

One can see this since this function is concave for  $k_{c_0} > 0$  and by looking at its values at 0 and  $c_0$ . This, together with (100), implies

$$\left(1 - \frac{\lambda}{\lambda_m - \lambda_n}\right) \left(1 + \frac{k_{c_0}\lambda}{\lambda_m - \lambda_n}\right) \geq 1, \quad (102)$$

and therefore,

$$\tilde{F}_n(\lambda) \prod_{m < n} \left|1 + \frac{k_{c_0}\lambda}{\lambda_m - \lambda_n}\right| \geq 1. \quad (103)$$

Since, using Lemma 3,

$$\prod_{m < n} \left|1 + \frac{k_{c_0}\lambda}{\lambda_m - \lambda_n}\right| < C e^{C k_{c_0}^{\frac{1}{\alpha}} \lambda^{\frac{1}{\alpha}}}, \quad (104)$$

and with (103) we deduce (99).

**Step 3: case  $\lambda \in [|\lambda_n - \lambda_1|, +\infty)$ .**

First, notice that if  $\text{Dist}_\alpha(\lambda) = 0$ , then the proof of (99) is trivial. We therefore focus on the case  $\text{Dist}_\alpha(\lambda) \neq 0$ . We underline that if  $|\lambda_n - \lambda_1| = \lambda$ , then  $\lambda = \lambda_1 - \lambda_n$  and  $\text{Dist}_\alpha(\lambda) = 0$ . Therefore,  $\lambda > \lambda_1 - \lambda_n > 0$  and from the strict ordering of the eigenvalues of Assumption 1,

$$\frac{\lambda}{\lambda_m - \lambda_n} \geq \frac{\lambda}{\lambda_1 - \lambda_n} > 1.$$

Using again  $\lambda > \lambda_1 - \lambda_n > 0$ , this implies, using (93)

$$\begin{aligned} \tilde{F}_n(\lambda) &= \prod_{m < n} \left(\frac{\lambda}{\lambda_m - \lambda_n} - 1\right) > \prod_{m < n} \left(\frac{\lambda_1 - \lambda_n}{\lambda_m - \lambda_n} - 1\right) = \prod_{m < n} \left(\frac{\lambda_1 - \lambda_m}{\lambda_m - \lambda_n}\right) \\ &\geq \prod_{m < n} \frac{1}{C} \left(\frac{\lambda_1 - \lambda_m}{(n - m)n^{\alpha-1}}\right), \end{aligned} \quad (105)$$

where  $C$  is given by (93). We are now going to give an estimate on  $(\lambda_1 - \lambda_m)$ . Denote  $c := \lambda_1/\lambda_2$  satisfying  $0 < c < 1$ . As in (101), there exists  $k_c > 0$  such that,

$$(1 - x)(1 + k_c x) \geq 1, \quad \text{for all } x \in [0, c].$$

As a consequence, defining  $\tilde{k} := -\lambda_1 k_c > 0$ , we have

$$\left(1 - \frac{\lambda_1}{\lambda_m}\right) \left(1 - \frac{\tilde{k}}{\lambda_m}\right) \geq 1.$$

Therefore,

$$\prod_{m \in \mathbb{N}^*, m > 1} \left(1 - \frac{\lambda_1}{\lambda_m}\right) \left(1 + \frac{\tilde{k}}{|\lambda_m|}\right) \geq 1. \quad (106)$$

Observe that, from Lemma 12 and (94)

$$\prod_{m \in \mathbb{N}^*, m > 1} \left( 1 + \frac{\tilde{k}}{|\lambda_m|} \right) \leq C', \quad (107)$$

where  $C' > 0$  depends on  $\lambda_1$  and  $\lambda_2$  but not on  $m > 2$ ,  $n > 2$  or  $\lambda$ . Together with (106), this gives,

$$\prod_{m \in \mathbb{N}^*, m > 1} \frac{\lambda_1 - \lambda_m}{|\lambda_m|} \geq \frac{1}{C'}. \quad (108)$$

Therefore, using also (93), (105) and Stirling formula,

$$\begin{aligned} \tilde{F}_n(\lambda) &\geq \prod_{m < n} \frac{1}{C} \left( \frac{|\lambda_m|(\lambda_1 - \lambda_m)}{|\lambda_m|(n-m)n^{\alpha-1}} \right) \\ &\gtrsim \prod_{m < n} \frac{c}{C} \frac{m^\alpha}{(n-m)n^{\alpha-1}} \\ &= \frac{((n-1)!)^\alpha}{(n-1)!} \left( \frac{c}{C} \right)^{n-1} \frac{1}{n^{(\alpha-1)(n-1)}} \\ &\gtrsim e^{-cn} \\ &\gtrsim e^{-c\lambda^{\frac{1}{\alpha}}}, \end{aligned}$$

which is (99).

**Step 4: case**  $\lambda \in (c_0(\lambda_{n-1} - \lambda_n), \lambda_1 - \lambda_n)$ .

Recall that  $c_0$  is given in Step 2 and can be arbitrarily chosen as long as it is strictly smaller than 1 and does not depend on  $n$  nor  $m$ . We assume again that  $\text{Dist}_\alpha(\lambda) > 0$ , otherwise there is nothing to show.

The first thing to observe is that we only need to focus on the pairs  $(n, m)$  such that

$$m \in M_1 := \left\{ m \in \mathbb{N}^* : m < n, \frac{\lambda}{\lambda_m - \lambda_n} > c_0 \right\}.$$

Indeed, setting  $M_2 = \{1, \dots, n-1\} \setminus M_1$  one has, using (102) and Lemma 3,

$$\prod_{m \in M_2} \left( 1 + \frac{\lambda}{\lambda_n - \lambda_m} \right) \geq \left( \prod_{m \in M_2} \left( 1 + \frac{k_{c_0} \lambda}{\lambda_m - \lambda_n} \right) \right)^{-1} \gtrsim e^{-C\lambda^{\frac{1}{\alpha}}}, \quad (109)$$

where  $C$  is a constant that only depends on  $\alpha$  and  $c_0$ . Note that since  $\lambda > c_0(\lambda_{n-1} - \lambda_n)$ ,  $(n-1) \in M_1$ . In fact, there exists  $\tilde{n} \in \{1, \dots, n-1\}$  such that

$$M_1 = \{n - \tilde{n}, \dots, n-1\}$$

and  $\tilde{n}$  is given by (note that  $n \rightarrow -\lambda_n$  is a strictly increasing function)

$$\tilde{n} := \max\{\tilde{n} \in \{1, \dots, n-1\} \mid \lambda_{n-\tilde{n}} < \frac{\lambda}{c_0} + \lambda_n\}. \quad (110)$$

Thus we deduce that it suffices to show that

$$\prod_{m=n-\tilde{n}}^{n-1} \left| 1 + \frac{\lambda}{\lambda_n - \lambda_m} \right| = \prod_{p=1}^{\tilde{n}} \left| 1 - \frac{\lambda}{\lambda_{n-p} - \lambda_n} \right| \geq e^{-C\lambda^{\frac{1}{\alpha}}} \text{Dist}_\alpha(\lambda). \quad (111)$$

We start with the case  $\lambda \in (c_0(\lambda_{n-1} - \lambda_n), \lambda_{n-1} - \lambda_n)$ . Note that since  $\text{Dist}_\alpha(\lambda) > 0$ , we have in fact  $\lambda \neq \lambda_{n-1} - \lambda_n$ .

In this case, note that we always have, with  $c$  and  $C$  given by (93)

$$\begin{aligned} \prod_{p=1}^{\tilde{n}} \left| 1 - \frac{\lambda}{\lambda_{n-p} - \lambda_n} \right| &= \left| 1 - \frac{\lambda}{\lambda_{n-1} - \lambda_n} \right| \prod_{p=2}^{\tilde{n}} \left| 1 - \frac{\lambda}{\lambda_{n-p} - \lambda_n} \right| \\ &\geq \left| 1 - \frac{\lambda}{\lambda_{n-1} - \lambda_n} \right| \prod_{p=2}^{\tilde{n}} \left| 1 - \frac{\lambda_{n-1} - \lambda_n}{\lambda_{n-p} - \lambda_n} \right| \\ &= \left| 1 - \frac{\lambda}{\lambda_{n-1} - \lambda_n} \right| \prod_{p=2}^{\tilde{n}} \left| \frac{\lambda_{n-p} - \lambda_{n-1}}{\lambda_{n-p} - \lambda_n} \right| \\ &\geq \text{Dist}_\alpha(\lambda) \frac{\prod_{p=2}^{\tilde{n}} |\lambda_{n-p} - \lambda_{n-1}|}{\prod_{p=1}^{\tilde{n}} |\lambda_{n-p} - \lambda_n|} \\ &\geq \frac{\text{Dist}_\alpha(\lambda)}{C} \left(\frac{c}{C}\right)^{\tilde{n}-2} \left(\frac{n-1}{n}\right)^{(\alpha-1)\tilde{n}} \frac{(\tilde{n}-1)!}{\tilde{n}!} \\ &= \frac{\text{Dist}_\alpha(\lambda)}{C} \left(\frac{c}{C}\right)^{\tilde{n}-2} \left(1 - \frac{1}{n}\right)^{(\alpha-1)\tilde{n}} \frac{1}{\tilde{n}} \\ &\gtrsim \text{Dist}_\alpha(\lambda) e^{-c'\tilde{n}} \end{aligned}$$

where  $c'$  does not depend on  $\lambda$ ,  $\tilde{n}$  or  $n$ . To conclude, it suffices to show that  $\tilde{n} \leq \lambda^{\frac{1}{\alpha}}$ . This is done later in (125), after the case  $\lambda \in (\lambda_{n-1} - \lambda_n, \lambda_1 - \lambda_n)$ .

Let us now look at the case  $\lambda \in (\lambda_{n-1} - \lambda_n, \lambda_1 - \lambda_n)$ . We have,

$$\prod_{p=1}^{\tilde{n}} \left| 1 - \frac{\lambda}{\lambda_{n-p} - \lambda_n} \right| = \prod_{p=1}^{\tilde{n}} \frac{1}{\lambda_{n-p} - \lambda_n} \prod_{p=1}^{\tilde{n}} |\lambda_{n-p} - \lambda_n - \lambda| =: I_1 \cdot I_2,$$

Using (93), we have

$$\frac{1}{\lambda_{n-p} - \lambda_n} \geq \frac{1}{Cpn^{\alpha-1}},$$

where  $C$  is given by (93). Thus,

$$I_1 \geq \frac{1}{\tilde{n}!} \frac{1}{C^{\tilde{n}} n^{\tilde{n}(\alpha-1)}}. \quad (112)$$

Let us now look at  $I_2$ . We introduce the strictly increasing function  $f : x \rightarrow \lambda_{n-x} - \lambda_n$ , where for  $x \in \mathbb{R} \setminus \mathbb{N}$ ,  $\lambda_x$  is defined as,

$$\lambda_x := \lambda_{\lfloor x \rfloor} + (x - \lfloor x \rfloor)(\lambda_{\lfloor x \rfloor + 1} - \lambda_{\lfloor x \rfloor}) \quad (113)$$

Observe that, from the definition of  $\tilde{n}$  given in (110) and since  $\lambda < \lambda_1 - \lambda_n$ , either  $\tilde{n} < n - 1$  and therefore  $\lambda \leq c_0(\lambda_{n-(\tilde{n}+1)} - \lambda_n) < \lambda_{n-(\tilde{n}+1)} - \lambda_n$  or  $\tilde{n} = n - 1$  and therefore  $\lambda < \lambda_{n-\tilde{n}} - \lambda_n = \lambda_1 - \lambda_n$ . Similarly since,  $\lambda > \lambda_{n-1} - \lambda_n$ , we deduce that

$$f(1) < \lambda < f(\min(\tilde{n} + 1, n - 1)).$$

Therefore there exists an integer  $p_0$  such that  $p_0 \leq \min(\tilde{n} + 1, n - 1) - 1$  and  $f(p_0) < \lambda$  and  $f(p_0 + 1) \geq \lambda$ . But since  $\text{Dist}_\alpha(\lambda) > 0$ , we have  $f(p_0 + 1) \neq \lambda$ . Therefore, as  $f$  is strictly increasing,

$$\prod_{p=1}^{\tilde{n}} |\lambda_{n-p} - \lambda_n - \lambda| = \prod_{p \leq p_0} (\lambda - (\lambda_{n-p} - \lambda_n)) \prod_{\tilde{n} \geq p \geq p_0+1} ((\lambda_{n-p} - \lambda_n) - \lambda) =: I_2^- I_2^+,$$

with the convention that if  $p_0 = \tilde{n}$  then the second product is empty and  $I_2^+ = 1$ . Let us start with  $I_2^-$ . Using the definition of  $f$  and the fact that  $f(p) < \lambda$ ,  $1 \leq p \leq p_0 - 1$ ,

$$\begin{aligned} I_2^- &= (\lambda - (\lambda_{n-p_0} - \lambda_n)) \prod_{p \leq p_0-1} \lambda - (\lambda_{n-p} - \lambda_n) \\ &\geq (\lambda - f(p_0)) \prod_{p \leq p_0-1} \lambda_{n-p_0} - \lambda_{n-p} \\ &\geq c(\lambda - f(p_0)) \prod_{p \leq p_0-1} (p_0 - p)(n - p)^{\alpha-1} \\ &= c(\lambda - f(p_0))(p_0 - 1)! \prod_{p \leq p_0-1} (n - p)^{\alpha-1}, \end{aligned} \tag{114}$$

where  $c$  is given by (93). For  $I_2^+$  we have, using a similar argument,

$$\begin{aligned} I_2^+ &= ((\lambda_{n-(p_0+1)} - \lambda_n) - \lambda) \prod_{\tilde{n} \geq p \geq p_0+2} (\lambda_{n-p} - \lambda_n) - \lambda \\ &\geq (f(p_0 + 1) - \lambda) \prod_{\tilde{n} \geq p \geq p_0+2} (\lambda_{n-p} - \lambda_{n-(p_0+1)}) \\ &\geq c(\tilde{n} - p_0 - 1)!(f(p_0 + 1) - \lambda) \prod_{\tilde{n} \geq p \geq p_0+2} (n - p_0 - 1)^{\alpha-1} \\ &= c(f(p_0 + 1) - \lambda)(\tilde{n} - p_0 - 1)!(n - p_0 - 1)^{(\alpha-1)(\tilde{n}-p_0-1)} \end{aligned} \tag{115}$$

where we used (93) in the last line. Combining (114) and (115) together, we have

$$I_2 \geq c^2(f(p_0+1) - \lambda)(\lambda - f(p_0))(\tilde{n} - p_0 - 1)!(p_0 - 1)! \prod_{\tilde{n} \geq p \geq p_0+2} (n - p_0 - 1)^{\alpha-1} \prod_{p \leq p_0-1} (n - p)^{\alpha-1}. \tag{116}$$

Now observe that

$$(f(p_0 + 1) - \lambda)(\lambda - f(p_0)) \geq c \frac{\text{Dist}_\alpha(\lambda)}{2}. \tag{117}$$

Indeed,  $f(p_0 + 1) - f(p_0) = \lambda_{n-(p_0+1)} - \lambda_{n-p_0} \geq c(n - p_0)^{\alpha-1} > c$ , thus since  $\lambda \in (f(p_0), f(p_0 + 1))$  we deduce that either  $(f(p_0 + 1) - \lambda) \geq c/2$  or  $(\lambda - f(p_0)) \geq c/2$ , which

leads to (117). Hence, combining with (116)

$$\begin{aligned}
I_2 &\geq c^3 \frac{\text{Dist}_\alpha(\lambda)}{2} (\tilde{n} - p_0 - 1)! (p_0 - 1)! \prod_{\tilde{n} \geq p \geq p_0 + 2} (n - p_0 - 1)^{\alpha-1} \prod_{p \leq p_0 - 1} (n - p)^{\alpha-1} \\
&= c^3 \frac{\text{Dist}_\alpha(\lambda)}{2} (\tilde{n} - p_0 - 1)! (p_0 - 1)! (n - p_0 - 1)^{(\tilde{n} - p_0 - 1)(\alpha-1)} \frac{((n-1)!)^{\alpha-1}}{((n-p_0)!)^{\alpha-1}}.
\end{aligned} \tag{118}$$

We separate the analysis in two cases:

- Case  $\tilde{n} \leq n/2$ , in this case  $p_0 - 1 < n/2$  thus

$$\frac{(n - p_0 - 1)}{n} > \frac{1}{2},$$

hence

$$(n - p_0 - 1)^{(\tilde{n} - p_0 - 1)(\alpha-1)} \frac{((n-1)!)^{\alpha-1}}{((n-p_0)!)^{\alpha-1}} \geq \left(\frac{n}{2}\right)^{(\tilde{n}-2)(\alpha-1)}, \tag{119}$$

where we used the fact that the fraction of factorial is in fact a product of  $p_0 - 1$  terms all of which are larger than  $n/2$ . We now recall the following useful lemma:

**Lemma 13** *If  $n, m \in \mathbb{N}^*$  satisfies  $0 < n < m$ , then*

$$n!m! \geq (n+1)!(m-1)!. \tag{120}$$

Setting  $k = \lfloor \tilde{n}/2 \rfloor - 1$  and using this lemma ( $\tilde{n} - p_0 - 1 - k$  (resp.  $k - (p_0 - 1)$ ) times) combined with (119) gives

$$I_2 \geq c^3 \frac{\text{Dist}_\alpha(\lambda)}{2} k! (\tilde{n} - 1 - k)! \left(\frac{n}{2}\right)^{(\tilde{n}-2)(\alpha-1)}. \tag{121}$$

Then, assuming that  $\tilde{n}$  is even (the odd case can be done exactly similarly)

$$I_2 \geq c^3 \frac{\text{Dist}_\alpha(\lambda)}{2} \left(\frac{2}{\tilde{n}}\right)^2 \cdot \left(\frac{\tilde{n}}{2}\right)! \left(\frac{\tilde{n}}{2}\right)! \left(\frac{n}{2}\right)^{(\tilde{n}-2)(\alpha-1)},$$

and (112) together with Stirling formula,

$$\begin{aligned}
I &\geq c^3 \frac{\text{Dist}_\alpha(\lambda)}{2} \left(\frac{2}{\tilde{n}}\right)^2 \cdot \left(\frac{\tilde{n}}{2}\right)! \left(\frac{\tilde{n}}{2}\right)! \left(\frac{n}{2}\right)^{(\tilde{n}-2)(\alpha-1)} \frac{1}{\tilde{n}!} \frac{1}{C^{\tilde{n}} n^{\tilde{n}(\alpha-1)}} \\
&\gtrsim \text{Dist}_\alpha(\lambda) \frac{1}{2^{(\tilde{n}-2)(\alpha-1)} (2C)^{\tilde{n}} n^{2(\alpha-1)} \tilde{n}^2} \\
&\gtrsim \text{Dist}_\alpha(\lambda) \frac{1}{2^{\alpha\tilde{n}}} \frac{1}{C^{\tilde{n}}} \frac{1}{n^{2\alpha}} \\
&\gtrsim \text{Dist}_\alpha(\lambda) e^{-c\tilde{n}},
\end{aligned} \tag{122}$$

where  $c$  is a constant depending on  $\alpha$  (and  $C$ ) but independent of  $\lambda$ .

- Case  $\tilde{n} \in (n/2, n - 1]$ . In this case we have

$$\begin{aligned}
& (n - p_0 - 1)^{(\tilde{n} - p_0 - 1)(\alpha - 1)} \frac{((n - 1)!)^{\alpha - 1}}{((n - p_0)!)^{\alpha - 1}} \\
& \geq (\tilde{n} - p_0 - 1)^{(\tilde{n} - p_0 - 1)(\alpha - 1)} \frac{((\tilde{n} - 1)!)^{\alpha - 1}}{((\tilde{n} - p_0)!)^{\alpha - 1}} \quad (123) \\
& \gtrsim ((\tilde{n} - 1)!)^{\alpha - 1},
\end{aligned}$$

where we used that  $n \rightarrow \frac{(n - 1)!}{(n - p_0)!}$  is increasing and  $\tilde{n} \leq n - 1$ . Assuming again that  $n$  is even (the odd case can be done exactly similarly) we have similarly as previously from (118) and (112)

$$\begin{aligned}
I & \gtrsim \text{Dist}_\alpha(\lambda) \left(\frac{2}{\tilde{n}}\right)^2 \cdot \left(\frac{\tilde{n}}{2}\right)! \left(\frac{\tilde{n}}{2}\right)! (\tilde{n} - 1)!^{\alpha - 1} \frac{1}{\tilde{n}! C^{\tilde{n}} n^{\tilde{n}(\alpha - 1)}} \\
& \gtrsim \text{Dist}_\alpha(\lambda) e^{-c\tilde{n}} \quad (124)
\end{aligned}$$

We are now going to show that whenever  $\lambda \in (c_0(\lambda_{n-1} - \lambda_n), \lambda_1 - \lambda_n)$ ,  $e^{-c\tilde{n}} \gtrsim e^{-c'\lambda^{\frac{1}{\alpha}}}$ , where  $c'$  is a constant independent from  $\lambda$ . Since  $\lambda < \lambda_1 - \lambda_n < C(n - 1)n^{\alpha - 1} \leq Cn^\alpha$  and  $c_0(\lambda_{n-\tilde{n}} - \lambda_n) < \lambda$ , thus  $\lambda > cc_0n^{\alpha - 1}\tilde{n}$ . As a consequence

$$\tilde{n} \leq \frac{\lambda}{cc_0n^{\alpha - 1}} < \frac{C^{\frac{1}{\alpha}}\lambda}{cc_0\lambda^{(\alpha - 1)/\alpha}} \lesssim \lambda^{\frac{1}{\alpha}}. \quad (125)$$

Therefore

$$I \gtrsim \text{Dist}_\alpha(\lambda) e^{-c\lambda^{\frac{1}{\alpha}}}. \quad (126)$$

This concludes the proof of Proposition 7.  $\square$

## 4.6 Estimation of $\text{Dist}_\alpha(\lambda)$

To conclude on an explicit lower bound on  $F_n$  as a function of  $\lambda$ , we need an explicit estimate of  $\text{Dist}_\alpha(\lambda)$ . What we would like to show is that there are sufficiently many  $\lambda$  such that

$$\text{Dist}_\alpha(\lambda) \gtrsim e^{-c\lambda^{\frac{1}{\alpha}}}. \quad (127)$$

This is more rigorously stated in the following proposition.

**Proposition 8** *There exist positive constants  $c, C$  and a sequence  $\{\mu_N\}_{N \in \mathbb{N}^*}$  such that for any  $N \in \mathbb{N}^*$ ,*

$$\begin{aligned}
\mu_N & \in [N, N + C], \\
\text{Dist}_\alpha(\mu_N) & \gtrsim e^{-c\mu_N^{1/\alpha}}. \quad (128)
\end{aligned}$$

**Remark 7** *This inequality can be improved to stronger estimates using the explicit representation of the spectrum. For example, for the heat equation, we are even able to replace the right-hand side of the preceding inequality by 1.*

Proposition 8 is a direct consequence of the following proposition.

**Proposition 9** *For any  $N \in \mathbb{N}^*$ , there exist some numbers*

$$\mu_N \in \left\{ N + c \left( \frac{1 + 2i}{2M_N} \right) : i = 0, 1, \dots, M_N - 1 \right\} \text{ where } M_N := \left\lfloor \left( \frac{N + c}{c} \right)^{\frac{1}{\alpha-1}} \right\rfloor + 2, \quad (129)$$

with  $c$  given by (93), such that

$$|\lambda_m - \lambda_n - \mu_N| \geq \frac{c}{2M_N}, \quad \forall n, m \in \mathbb{N}^*. \quad (130)$$

**Proof of Proposition 9:** We use a contradiction argument. Suppose that this proposition does not hold. Then for any  $l_i =: N + c \left( \frac{1 + 2i}{2M_N} \right)$ ,  $i \in \{0, 1, \dots, M_N - 1\}$ , there exists at least one pair of integers  $(n_i, m_i)$  such that,

$$|\lambda_{m_i} - \lambda_{n_i} - l_i| < \frac{c}{2M_N}. \quad (131)$$

Namely,

$$\lambda_{m_i} - \lambda_{n_i} \in \left( N + c \frac{2i}{2M_N}, N + c \frac{2i + 2}{2M_N} \right). \quad (132)$$

Now we present the following lemma,

**Lemma 14** *If for some  $(n, m) \in \mathbb{N}^*$  there is,*

$$\lambda_m - \lambda_n \in (N, N + c),$$

then,

$$n \leq M_N - 2. \quad (133)$$

**Proof of Lemma 14:** By the definition of  $(n, m)$  we know that,

$$N < \lambda_m - \lambda_n < N + c,$$

thus from (93)

$$N + c > \lambda_m - \lambda_n \geq \lambda_{n-1} - \lambda_n > cn^{\alpha-1}. \quad (134)$$

Therefore, there exists  $0 < r < 1$  such that,

$$n < \left( \frac{N + c}{c} \right)^{\frac{1}{\alpha-1}} = M_N - 2 + r \leq M_N - 1. \quad (135)$$

□

Now we come back to the proof of Proposition 9. Thanks to equation (132) and Lemma 14, we know that

$$n_i \leq M_N - 2, \quad \forall i \in \{0, 1, \dots, M_N - 1\}, \quad (136)$$

which implies, since  $n_i \in \mathbb{N}^*$ , that there exist  $i_1, i_2 \in \{0, 1, \dots, M_N - 1\}$  such that  $i_1 \neq i_2$  and  $n_{i_1} = n_{i_2}$ . Since, from (132),

$$\begin{aligned} N &< \lambda_{m_{i_1}} - \lambda_{n_{i_1}} < N + c, \\ N &< \lambda_{m_{i_2}} - \lambda_{n_{i_2}} < N + c, \end{aligned}$$

and  $n_{i_1} = n_{i_2}$  we have,

$$|\lambda_{m_{i_1}} - \lambda_{m_{i_2}}| < c. \quad (137)$$

However, we know from (93) that if  $m_{i_1} \neq m_{i_2}$  then,

$$|\lambda_{m_{i_1}} - \lambda_{m_{i_2}}| > c(m_{i_1} - m_{i_2}) > c, \quad (138)$$

which, combined with (137) means that  $m_{i_1} = m_{i_2}$ . Thus, using (131),

$$|l_{i_1} - l_{i_2}| < \frac{c}{M_N}, \quad (139)$$

but from the definition of  $l_i$ , since  $i_1 \neq i_2$ ,  $|l_{i_1} - l_{i_2}| = 2c|i_1 - i_2|/2M_N \geq c/M_N$  which leads to a contradiction, and concludes the proof of Proposition 9.  $\square$

**Remark 8** *Note that this proof relies strongly on the fact that  $\alpha > 1$ ; see in particular (134).*

The lower bound of Theorem 7 then follows from the straightforward combination of Lemma 11, and Propositions 7 and 8.

## 5 Well-posedness and exponential stability of the closed-loop system

In this section, recall that, following Assumption 1 and Section 3.1, we assume without loss of generality that  $A$  is diagonalizable with simple eigenvalues. We also assume that  $\lambda \notin \mathcal{N}$ .

### 5.1 Operator equality and domain of the closed-loop operator

In keeping with the spirit of the spectral method ([25], [30], [32], [40]), a key ingredient in the study of the closed-loop operator  $A + BK : D(A + BK) \subset H \rightarrow H$  is to determine its eigenfunctions. This will allow us to describe more precisely its domain  $D(A + BK)$ , and rigorously establish the operator equality (6)

$$T(A + BK) = (A - \lambda I)T$$

on sharp spaces. We introduce the following family of non-zero functions,

$$\chi_n := (A - (\lambda_n - \lambda)I)^{-1}B, \quad \forall n \in \mathbb{N}^*.$$

First note that these functions are well defined since  $\lambda$  is chosen so that  $\lambda_n - \lambda \neq \lambda_p$  for all  $n, p \in \mathbb{N}^*$ . Moreover, since  $B \in \mathcal{H}^{-\alpha}$  and  $B \neq 0$ , by property of the resolvent, we have  $\chi_n \neq 0$  and

$$\chi_n \in H, \quad \forall n \in \mathbb{N}^*. \quad (140)$$

Moreover, they decompose simply along the Riesz basis  $\{\varphi_n\}_{n \in \mathbb{N}^*}$

$$\chi_n = \sum_{p \in \mathbb{N}^*} \frac{b_p}{\lambda_p - \lambda_n + \lambda} \varphi_p.$$

In this section, we prove that the family  $\{\chi_n\}_{n \in \mathbb{N}^*}$  consists in eigenfunctions of  $A + BK$ .

**Lemma 15** *There holds the following,*

$$\begin{aligned} \chi_n &\in D(A + BK), \\ (A + BK)\chi_n &= (\lambda_n - \lambda)\chi_n, \end{aligned} \quad \forall n \in \mathbb{N}^*. \quad (141)$$

**Proof :** First, let us prove that the feedback  $K$  defined above is well defined on the family  $\{\chi_n\}_{n \in \mathbb{N}^*}$ . Recall from the identity (59) that, for all  $n \in \mathbb{N}^*$ ,

$$\begin{aligned} 1 &= \sum_{p \in \mathbb{N}^*} \frac{k_p b_p}{\lambda_n - \lambda_p - \lambda} \\ &= - \sum_{p \in \mathbb{N}^*} \frac{k_p b_p}{\lambda_p - \lambda_n + \lambda} \\ &= - \sum_{p \in \mathbb{N}^*} k_p \langle \chi_n, \varphi_p \rangle \\ &= - \langle K, \chi_n \rangle. \end{aligned} \quad (142)$$

Thus,  $\langle K, \chi_n \rangle = -1$ ,  $\forall n \in \mathbb{N}^*$ . Since  $A\chi_n, B \in \mathcal{H}^{-\alpha} = D(A)'$ , we can compute in  $D(A)'$  (or  $\mathcal{H}^{-\alpha}$ ) for all  $n \in \mathbb{N}^*$ ,

$$\begin{aligned} (A + BK)\chi_n &= A\chi_n - B \\ &= (A - (\lambda_n - \lambda)I)\chi_n + (\lambda_n - \lambda)\chi_n - B \\ &= (A - (\lambda_n - \lambda)I)(A - (\lambda_n - \lambda)I)^{-1}B + (\lambda_n - \lambda)\chi_n - B \\ &= (\lambda_n - \lambda)\chi_n. \end{aligned}$$

Hence,

$$(A + BK)\chi_n = (\lambda_n - \lambda)\chi_n, \text{ in } D(A)'. \quad (143)$$

Now, from (140), (143) actually holds in  $H$  for all  $n \in \mathbb{N}^*$ . Hence,  $\chi_n \in D(A + BK)$ ,  $\forall n \in \mathbb{N}^*$ .  $\square$

Now, given the operator equation (6), one would expect that the backstepping transformation  $T$  should map the  $\chi_n$  to the  $\varphi_n$ , given that it *formally* conjugates  $A - \lambda I$  to  $A + BK$ . We now confirm this with the following proposition.

**Proposition 10** *The following hold,*

$$T\chi_n = \text{sgn}(\langle T\chi_n, \varphi_n \rangle) \|T\chi_n\| \varphi_n, \quad \forall n \in \mathbb{N}^*,$$

*i.e.,  $T\chi_n$  is an eigenfunction of  $A$  for the eigenvalue  $\lambda_n$ . Moreover, the  $\{T^{-1}\varphi_n\}_{n \in \mathbb{N}^*}$  form a Riesz basis of  $H$  of eigenfunctions of  $A + BK$ .*

**Proof :** Let us compute, for all  $n, p \in \mathbb{N}^*$  with  $n \neq p$ , using (33)–(34),

$$\begin{aligned} \langle (A - \lambda_n I)T\chi_n, \varphi_p \rangle_{D(A)', D(A)} &= (\lambda_p - \lambda_n) \langle T\chi_n, \varphi_p \rangle_H \\ &= (\lambda_p - \lambda_n) \left\langle \sum_{\ell \in \mathbb{N}^*} \frac{b_\ell}{\lambda_\ell - \lambda_n + \lambda} T\varphi_\ell, \varphi_p \right\rangle_H \\ &= (\lambda_p - \lambda_n) \sum_{\ell \in \mathbb{N}^*} \frac{b_\ell}{\lambda_\ell - \lambda_n + \lambda} \frac{k_\ell b_p}{\lambda_p - \lambda - \lambda_\ell} \\ &= (\lambda_p - \lambda_n) \sum_{\ell \in \mathbb{N}^*} \frac{b_p}{\lambda_\ell - \lambda_n + \lambda} \frac{k_\ell b_\ell}{\lambda_p - \lambda - \lambda_\ell} \\ &= \sum_{\ell \in \mathbb{N}^*} (\lambda_p - \lambda_\ell - \lambda + \lambda_\ell - \lambda_n + \lambda) \frac{b_p}{\lambda_\ell - \lambda_n + \lambda} \frac{k_\ell b_\ell}{\lambda_p - \lambda - \lambda_\ell} \\ &= b_p \sum_{\ell \in \mathbb{N}^*} \left( 1 + \frac{\lambda_p - \lambda_\ell - \lambda}{\lambda_\ell - \lambda_n + \lambda} \right) \frac{k_\ell b_\ell}{\lambda_p - \lambda - \lambda_\ell} \\ &= b_p \left[ \sum_{\ell \in \mathbb{N}^*} \frac{k_\ell b_\ell}{\lambda_p - \lambda - \lambda_\ell} - \sum_{\ell \in \mathbb{N}^*} \frac{k_\ell b_\ell}{\lambda_n - \lambda - \lambda_\ell} \right] \\ &= 0, \end{aligned}$$

where the last line comes from identity (59).

Hence, as  $\chi_n \neq 0$ , it is an eigenfunction of  $A$  for the eigenvalue  $\lambda_n$ . Since this eigenvalue is simple,  $T\chi_n$  is colinear to  $\varphi_n$ , and the first part of the proposition follows easily. Finally, since  $T^{-1}\varphi_n = \text{sgn}(\langle T\chi_n, \varphi_n \rangle) \chi_n / \|T\chi_n\|$ , these vectors are eigenfunctions of  $A + BK$  thanks to Lemma 15. Since  $T$  is an isomorphism from  $H$  into itself, they form a Riesz basis of  $H$ , which concludes the proof of the proposition.  $\square$

We can now describe the domain of the closed-loop operator  $A + BK$ , through its Riesz basis of eigenvectors ( $T^{-1}\varphi_n$ ), and real eigenvalues  $\lambda_n - \lambda$ . Note that this Riesz basis has a natural bi-orthonormal family, given by  $\{T^*\varphi_n\}_{n \in \mathbb{N}^*}$ , so that we have the decomposition:

$$f = \sum_{n \in \mathbb{N}^*} \langle f, T^*\varphi_n \rangle T^{-1}\varphi_n, \quad \forall f \in H.$$

This leads to the following characterization:

$$D(A + BK) = \left\{ f \in H, \quad \sum_{n \in \mathbb{N}^*} |\lambda_n - \lambda|^2 |\langle f, T^*\varphi_n \rangle|^2 < +\infty \right\}.$$

It is clear that this is equivalent to

$$D(A + BK) = \left\{ f \in H, \quad \sum_{n \in \mathbb{N}^*} |\lambda_n|^2 |\langle f, T^* \varphi_n \rangle|^2 < +\infty \right\}.$$

Moreover, for well-chosen  $\rho$ , the fractional operators  $(\rho I - (A + BK))^s$  are well-defined and bounded for  $s \leq 0$ , and densely defined for  $s > 0$  (see for example [41, Chapter IX, Section 11], or [28, Proposition 4.20]). We can then define another family of nested Banach spaces, associated to the closed-loop operator  $A + BK$ . Define first the following norms for  $s \in \mathbb{R}$ :

$$\|f\|_{s, A+BK} := \|(\rho I - (A + BK))^s f\|, \quad s \in \mathbb{R}.$$

Then, for  $s \geq 0$  we define,

$$D_s(A + BK) = \{f \in H, \quad \|f\|_{s, A+BK} < +\infty\},$$

or equivalently,

$$D_s(A + BK) := \left\{ f \in H, \quad \sum_{n \in \mathbb{N}^*} |\lambda_n|^{2s} |\langle f, T^* \varphi_n \rangle|^2 < +\infty \right\}, \quad s \geq 0,$$

and, recalling that  $\lambda_n \neq 0$ , we define a norm equivalent to  $\|\cdot\|_{s, A+BK}$  by,

$$\|f\|_{D_s(A+BK)} := \sum_{n \in \mathbb{N}^*} |\lambda_n|^{2s} |\langle f, T^* \varphi_n \rangle|^2, \quad \forall f \in D_s(A + BK). \quad (144)$$

Note that the family of functions  $(T^{-1} \varphi_n / \sqrt{|\lambda_n|^{2s}})_n$  is a Riesz basis for  $D_s(A + BK)$  (with  $s \geq 0$ ).

**Remark 9** *We stress that  $\|\cdot\|_{D_0(A+BK)}$  is equivalent to  $\|\cdot\|$  (since the  $\{T^{-1} \varphi_n\}$  form a Riesz basis of  $H$ ), but not equal.*

On the other hand, for  $s < 0$ , we define  $D_s(A + BK)$  as the completion of  $H$  with respect to  $\|\cdot\|_{s, A+BK}$ . Note also that we can define an equivalent norm on  $D_s(A + BK)$  with  $s < 0$

$$\|f\|_{D_s(A+BK)} := \sum_{n \in \mathbb{N}^*} |\lambda_n|^{2s} |\langle f, T^* \varphi_n \rangle|^2, \quad \forall f \in H.$$

Once again, for  $s_1 > s_2$  the injection  $D_{s_1}(A + BK) \rightarrow D_{s_2}(A + BK)$  is continuous and compact.

We can now establish the operator equality (6).

**Proposition 11** *There holds,*

$$D(A^s) = TD_s(A + BK), \quad \forall s \in \mathbb{R}, \quad (145)$$

and  $T : (D_s(A + BK), \|\cdot\|_{D_s(A+BK)}) \rightarrow (D(A^s), \|\cdot\|_{D(A^s)})$  is an isometry. Moreover,

$$T(A + BK) = (A - \lambda I)T \text{ on } D_s(A + BK), \quad \forall s \in \mathbb{R}. \quad (146)$$

**Proof :** We begin by proving that  $T$  can be extended to the spaces  $D_s(A + BK)$ ,  $s < 0$ . First note that for  $s \geq 0$ , one can write

$$Tf = T \sum_n \langle f, T^* \varphi_n \rangle T^{-1} \varphi_n = \sum_n \langle f, T^* \varphi_n \rangle \varphi_n, \quad \forall f \in D_s(A + BK).$$

Hence, by the definitions of these norms, for  $s \geq 0$ ,

$$\|Tf\|_{D(A^s)} = \|f\|_{D_s(A+BK)}, \quad \forall f \in D_s(A + BK). \quad (147)$$

For  $s < 0$ , we also have, from the same argument,

$$\|Tf\|_{D(A^s)} = \|f\|_{D_s(A+BK)}, \quad \forall f \in H. \quad (148)$$

Thus, since  $D(A^s)$  (resp.  $D_s(A + BK)$ ) is the completion of  $H$  for the norm  $\|\cdot\|_{D(A^s)}$  (resp.  $\|\cdot\|_{D_s(A+BK)}$ ), and  $T : (H, \|\cdot\|_{D_s(A+BK)}) \rightarrow (H, \|\cdot\|_{D(A^s)})$  (resp.  $T^{-1} : (H, \|\cdot\|_{D(A^s)}) \rightarrow (H, \|\cdot\|_{D_s(A+BK)})$ ) is bounded,  $T$  and  $T^{-1}$  can be extended continuously to elements of  $\mathcal{L}(D_s(A + BK), D(A^s))$  and  $\mathcal{L}(D(A^s), D_s(A + BK))$  respectively.

It is straightforward that these extensions are still mutual inverses, and thus, from the definitions of  $D_s(A + BK)$  and  $D(A^s)$ , we get:

$$f \in D_s(A + BK) \iff Tf \in D(A^s), \quad \forall s \in \mathbb{R},$$

which proves (145). Moreover, from (147) and (148) extended to  $f \in D_s(A + BK)$ ,  $T$  is an isometry from  $(D_s(A + BK), \|\cdot\|_{D_s(A+BK)})$  to  $(D(A^s), \|\cdot\|_{D(A^s)})$

Now, for  $n \in \mathbb{N}^*$ ,

$$\begin{aligned} T(A + BK)T^{-1}\varphi_n &= T(\lambda_n - \lambda)T^{-1}\varphi_n \\ &= (\lambda_n - \lambda)\varphi_n \\ &= (A - \lambda I)\varphi_n \\ &= (A - \lambda I)T(T^{-1}\varphi_n). \end{aligned}$$

Hence the operator equality (6) holds on Riesz basis of the spaces  $D_s(A + BK)$  for  $s \in \mathbb{R}$ . Since  $T$  is bounded on these spaces, (146) follows.  $\square$

**Remark 10** *It is important to stress the fact that we have established the operator equality in a wide array of spaces, but one should keep in mind that those spaces are the natural scale of Banach spaces associated with the closed-loop operator  $A + BK$ , and as such are specifically tailored for this purpose. In this sense, we have described the comprehensive functional range for the operator equality.*

Another important question is the relationship of the spaces  $D_s(A + BK)$  to the more intrinsic regularity spaces  $D(A^s)$ . We give some insights on this matter, showing that for a far more restricted range of spaces,  $D_s(A + BK) = D(A^s)$ .

**Proposition 12** *For  $s \in [0, 1 - 1/2\alpha)$ ,*

$$D_s(A + BK) = D(A^s).$$

**Proof :** On the one hand, we know from Lemma 9 that  $T$  is an isomorphism on  $D(A^s)$  for  $s \in [0, 1 - 1/2\alpha)$ . Combining this with Proposition 11, we get

$$D_s(A + BK) = T^{-1}(D(A^s)) = D(A^s).$$

□

## 5.2 Well-posedness and stability of the closed-loop system

Now that we have properly established the operator equality (6) in the adequate functional setting, we can use it to obtain the well-posedness and stability for the closed-loop system, in the same way as in finite dimension.

To prepare for the quantitative work of the next section, which will rely on a diverging sequence of values for the damping parameter  $\lambda$ , we now denote by  $(T_\lambda, K_\lambda)$  the back-stepping transformation and the feedback obtained in the previous sections, for a given damping parameter  $\lambda$ .

We prove well-posedness properties for the closed-loop system, first in the natural scale  $D_s(A + BK_\lambda)$ , then in the less natural (but more useful for our purposes) scale  $D(A^s)$ .

We begin with the most natural well-posedness result:

**Theorem 8** *Let  $\lambda \notin \mathcal{N}$ . For every  $s \in \mathbb{R}$ , the operator  $(A + BK_\lambda, D_{s+1}(A + BK_\lambda))$  is the generator of an exponentially stable semigroup on  $D_s(A + BK_\lambda)$ , and the following stability estimates hold,*

$$\|e^{t(A+BK_\lambda)}y_0\|_{D_s(A+BK_\lambda)} \leq C_s e^{-\lambda t} \|y_0\|_{D_s(A+BK_\lambda)}, \quad \forall y_0 \in D_s(A + BK_\lambda), \quad \forall t \geq 0, \quad (149)$$

where  $C_s$  is a positive constant that does not depend on  $\lambda$ .

**Proof :** Let  $s \in \mathbb{R}$ . Thanks to Proposition 11, the operator equality (6) rewrites as

$$(A + BK_\lambda) = T_\lambda^{-1}(A - \lambda I)T_\lambda \text{ on } T_\lambda^{-1}D(A^s) = D_s(A + BK_\lambda). \quad (150)$$

From the assumptions made on  $A$ , we know that  $(A - \lambda I, D(A^{s+1}))$  generates an exponentially stable  $C_0$ -semigroup  $S_s(t), t \geq 0$  on  $D(A^s)$ , with the following stability estimate,

$$\|S_s(t)y_0\|_{D(A^s)} \leq C_s e^{-\lambda t} \|y_0\|_{D(A^s)}, \quad \forall y_0 \in D(A^s),$$

where  $C_s$  is a positive constant that does not depend on  $\lambda$ . Now, let us define the following semigroup on  $D_s(A + BK_\lambda)$ ,

$$e^{t(A+BK_\lambda)}y := T_\lambda^{-1}S_s(t)T_\lambda y, \quad \forall y \in D_s(A + BK_\lambda).$$

This semigroup is well defined since  $T_\lambda D_s(A + BK_\lambda) = D(A^s)$ . Let us check that its infinitesimal generator is  $(A + BK_\lambda, D_{s+1}(A + BK_\lambda))$ . Let  $y \in D_s(A + BK_\lambda)$ , and compute

$$\begin{aligned} \frac{e^{t(A+BK_\lambda)}y - y}{t} &= \frac{T_\lambda^{-1}S_s(t)T_\lambda y - y}{t} \\ &= T_\lambda^{-1} \frac{S_s(t)T_\lambda y - T_\lambda y}{t}. \end{aligned}$$

By continuity of  $T_\lambda^{-1}$ , the above quantity converges in  $D_s(A + BK_\lambda)$  when  $t \rightarrow 0$  if and only if  $\frac{S_s(t)T_\lambda y - T_\lambda y}{t}$  converges in  $D(A^s)$  when  $t \rightarrow 0$ . By definition of  $S_s(t)$  and its infinitesimal generator  $(A - \lambda I, D(A^{s+1}))$ ,

$$\frac{S_s(t)T_\lambda y - T_\lambda y}{t} \xrightarrow[t \rightarrow 0]{D(A^s)} (A - \lambda I)T_\lambda y \iff T_\lambda y \in D(A^{s+1}).$$

Now, the above is equivalent to  $y \in D_{s+1}(A + BK_\lambda)$ , thanks to (145). Thus,

$$\frac{e^{t(A+BK_\lambda)}y - y}{t} \xrightarrow[t \rightarrow 0]{D_s(A+BK_\lambda)} T_\lambda^{-1}(A - \lambda I)T_\lambda y \iff y \in D_{s+1}(A + BK_\lambda).$$

Finally, we conclude using the operator equality (150):

$$\frac{e^{t(A+BK_\lambda)}y - y}{t} \xrightarrow[t \rightarrow 0]{D_s(A+BK_\lambda)} (A + BK_\lambda)y \iff y \in D_{s+1}(A + BK_\lambda),$$

hence,  $(A + BK_\lambda, D_{s+1}(A + BK_\lambda))$  is the infinitesimal generator of  $e^{t(A+BK_\lambda)}$  on  $D_s(A + BK_\lambda)$ .

Now consider  $y \in D_s(A + BK_\lambda)$ , we then have, using (147), (148) and the definition of the semigroup  $e^{t(A+BK_\lambda)}$ ,

$$\begin{aligned} \|e^{t(A+BK_\lambda)}y\|_{D_s(A+BK_\lambda)} &= \|T_\lambda^{-1}S_s(t)T_\lambda y\|_{D_s(A+BK_\lambda)} \\ &= \|S_s(t)T_\lambda y\|_{D(A^s)} \\ &\leq C_s e^{-\lambda t} \|T_\lambda y\|_{D(A^s)} \\ &= C_s e^{-\lambda t} \|y\|_{D_s(A+BK_\lambda)}, \end{aligned}$$

which ends the proof.  $\square$

We complete this stability result with the admissibility of  $K_\lambda$  (see also Lemma 6) in the functional framework associated to  $A + BK_\lambda$ .

**Proposition 13** *Let  $s > 1/2\alpha$ . For  $\lambda \notin \mathcal{N}$ ,  $K_\lambda \in (D_s(A + BK_\lambda))'$ . Thus, it is well-defined on trajectories of the semigroup  $e^{t(A+BK_\lambda)}$  on the space  $D_s(A + BK_\lambda)$ . Moreover, for any  $N \in \mathbb{N}$  there exists  $\lambda \notin \mathcal{N}$  such that  $\lambda \in [N, N + 1]$ , and*

$$\begin{aligned} |K_\lambda f| &\leq C_s e^{c\lambda^{\frac{1}{\alpha}}} \|f\|_{D_s(A+BK_\lambda)}, \\ |K_\lambda e^{t(A+BK_\lambda)} f| &\leq C'_s e^{c\lambda^{\frac{1}{\alpha}}} e^{-\lambda t} \|f\|_{D_s(A+BK_\lambda)}, \quad \forall f \in D_s(A + BK_\lambda), \quad \forall t \geq 0. \end{aligned} \tag{151}$$

where  $C_s, C'_s > 0$  depend only on  $A, B$ , and  $s$ .

**Proof :** First, from Proposition 10 and Proposition 11, we have

$$T_\lambda^{-1}\varphi_n = \text{sgn}(\langle T_\lambda \chi_n, \varphi_n \rangle) \frac{\chi_n}{\|T_\lambda \chi_n\|}, \quad \text{in } D(A + BK). \tag{152}$$

Then, from (142), we have

$$K_\lambda T_\lambda^{-1}\varphi_n = -\frac{\text{sgn}(\langle T_\lambda \chi_n, \varphi_n \rangle)}{\|T_\lambda \chi_n\|} \in \mathbb{R}.$$

Moreover, since  $T_\lambda^{-1}\varphi_n$  is an eigenvector of  $A + BK_\lambda$ , we can compute, in  $D(A)'$ ,

$$AT_\lambda^{-1}\varphi_n + (K_\lambda T_\lambda^{-1}\varphi_n)B = (\lambda_n - \lambda)T_\lambda^{-1}\varphi_n,$$

which leads to (recall that  $K_\lambda T_\lambda^{-1}\varphi_n$  is a scalar)

$$T_\lambda^{-1}\varphi_n = -(K_\lambda T_\lambda^{-1}\varphi_n)(A - (\lambda_n - \lambda)I)^{-1}B \in H.$$

Now, since  $\{T_\lambda^{-1}\varphi_n\}$  is a Riesz basis of  $H$  and  $(\varphi_n)_{n \in \mathbb{N}^*}$  is a orthonormal family of eigenvectors of  $A$ , there exists  $C > 0$  such that

$$\|T_\lambda^{-1}\| \geq \|T_\lambda^{-1}\varphi_n\|_H = \sqrt{\sum_{k \in \mathbb{N}} |\langle T_\lambda^{-1}\varphi_n, \varphi_k \rangle|^2} \geq |\langle T_\lambda^{-1}\varphi_n, \varphi_n \rangle_H| = |K_\lambda T_\lambda^{-1}\varphi_n| \frac{|b_n|}{|\lambda|}, \quad (153)$$

*i.e.*,

$$|K_\lambda T_\lambda^{-1}\varphi_n| \leq \frac{|\lambda|}{|b_n|} \|T_\lambda^{-1}\| \leq C' \lambda \|T_\lambda^{-1}\|, \quad (154)$$

where  $C' > 0$  is independent of  $n$ , from the controllability-type condition (25)

Now, let  $s > 1/2\alpha$ , and let  $f = \sum_n f_n T_\lambda^{-1}\varphi_n \in D_{1/2}(A + BK_\lambda)$  where here  $f_n := \langle f, T^*\varphi_n \rangle$ .

From the above we derive,

$$\begin{aligned} |K_\lambda f| &= \left| \sum_n f_n (K_\lambda T_\lambda^{-1}\varphi_n) \right| \\ &= \left| \sum_n f_n |\lambda_n|^s \frac{(K_\lambda T_\lambda^{-1}\varphi_n)}{|\lambda_n|^s} \right| \\ &\leq \sqrt{\sum_n |\lambda_n|^{2s} |f_n|^2} \sqrt{\sum_n \frac{|K_\lambda T_\lambda^{-1}\varphi_n|^2}{|\lambda_n|^{2s}}}. \end{aligned}$$

By Assumption 1, and more precisely from (94), and the assumption on  $s$ , we know that

$$\sum_n \frac{1}{|\lambda_n|^{2s}} < +\infty.$$

Thus, by definition of  $D_s(A + BK_\lambda)$ , and using (147) and (154), there exists  $C_s > 0$  such that

$$|K_\lambda f| \leq C_s \lambda \|T_\lambda^{-1}\| \|f\|_{D_s(A + BK_\lambda)}, \quad (155)$$

which implies that  $K_\lambda \in (D_s(A + BK_\lambda))'$ .

To get the quantitative estimates, let  $N \in \mathbb{N}$ ,  $\lambda \notin \mathcal{N}$  and  $\lambda \in [N, N + 1]$  such that Theorem 6 is satisfied. Then, (153) implies

$$C e^{c\lambda^{\frac{1}{\alpha}}} \geq \|T_\lambda^{-1}\varphi_n\| \geq |K_\lambda T_\lambda^{-1}\varphi_n| \frac{|b_n|}{\lambda},$$

which yields

$$|K_\lambda T_\lambda^{-1}\varphi_n| \leq C'' e^{c\lambda^{\frac{1}{\alpha}}} \lambda.$$

Then, (155) becomes

$$|K_\lambda f| \leq C_s''' e^{c\lambda^{\frac{1}{\alpha}}} \lambda \|f\|_{D_s(A+BK_\lambda)}.$$

Since  $\alpha > 0$ , there exists a constant  $C_\alpha > 0$  such that

$$\lambda \leq C_\alpha e^{c\lambda^{\frac{1}{\alpha}}},$$

which yields the first inequality of (151). Combining this with the fact that  $e^{t(A+BK_\lambda)}$  is a  $C^0$  semigroup on  $D_s(A+BK_\lambda)$ , and using Theorem 8, we easily obtain (151).  $\square$

As remarked in the previous section, the above theorem is given in the natural scale of spaces  $D_s(A+BK_\lambda)$ . However, they depend on  $\lambda$ , and the Riesz basis we have provided for them are implicit. To achieve finite time stabilization, we will require quantitative estimates in  $\lambda$ . This is more easily achieved in the explicit (and more practical but less natural from an operator point of view) scale of spaces  $D(A^s)$ , since they do not depend on  $\lambda$ . We start with a stability estimate for the linear closed-loop system:

**Theorem 9** *Let  $s \in [0, 1 - 1/2\alpha)$ . For  $N \in \mathbb{N}$  there exists  $\lambda \notin \mathcal{N}$  such that  $\lambda \in [N, N+1]$  and the closed-loop operator  $A+BK_\lambda$  generates an exponentially stable semigroup on  $D(A^s)$ , and there exist constants  $c_s, C_s > 0$  depending on  $A, B$ , and  $s$ , such that*

$$\|e^{t(A+BK)}y\|_{D(A^s)} \leq C_s e^{c_s \lambda^{\frac{1}{\alpha}}} e^{-\lambda t} \|y\|_{D(A^s)}. \quad (156)$$

**Proof :** Let  $s \in [0, 1 - 1/2\alpha)$ . As in the proof of Theorem 8, we define the semigroup

$$e^{t(A+BK)} := T^{-1}S_s(t)T.$$

Since  $T$  is an isomorphism on  $D(A^s)$ , and since  $S_s(t)$  is an exponentially stable semigroup on  $D(A^s)$ ,  $e^{t(A+BK)}$  is well-defined and we have the following estimate thanks to Lemma 9:

$$\begin{aligned} \|e^{t(A+BK)}y\|_{D(A^s)} &= \|T^{-1}S_s(t)Ty\|_{D(A^s)} \\ &\leq \|T^{-1}\|_{D(A^s)} e^{-\lambda t} \|Ty\|_{D(A^s)} \\ &\leq \|T^{-1}\|_{D(A^s)} \|T\|_{D(A^s)} e^{-\lambda t} \|y\|_{D(A^s)} \\ &\leq C_s e^{c_s \lambda^{\frac{1}{\alpha}}} e^{-\lambda t} \|y\|_{D(A^s)}. \end{aligned}$$

$\square$

Again, we complete the above result with an admissibility result for  $K_\lambda$ .

**Proposition 14** *Let  $s \in (1/2\alpha, 1 - 1/2\alpha)$ . For  $\lambda \notin \mathcal{N}$ ,  $K_\lambda \in D(A^s)'$ . Thus, it is well-defined on trajectories of the semigroup  $e^{t(A+BK_\lambda)}$  in the space  $D(A^s)$ . Moreover, for any  $N \in \mathbb{N}$ , there exists  $\lambda \notin \mathcal{N}$ , such that  $\lambda \in [N, N+1]$ , and*

$$\begin{aligned} |K_\lambda f| &\leq C_s e^{c\lambda^{\frac{1}{\alpha}}} \|f\|_{D(A^s)}, \\ |K_\lambda e^{t(A+BK_\lambda)}f| &\leq C_s' e^{c_s \lambda^{\frac{1}{\alpha}}} e^{-\lambda t} \|f\|_{D(A^s)}, \quad \forall y \in D(A^s), \end{aligned} \quad (157)$$

where  $c, c_s, C_s, C_s' > 0$  depend only on  $A, B$ , and  $s$ .

**Proof :** Let  $s \in (1/2\alpha, 1 - 1/2\alpha)$ . First, recall that from Proposition 12  $D(A^s) = D_s(A + BK)$ . Moreover, for  $f \in D(A^s)$ , we have

$$\|f\|_{D_s(A+BK_\lambda)} = \|Tf\|_{D(A^s)} \leq \|T\|_{D(A^s)} \|f\|_{D(A^s)}.$$

Combining this with Proposition 13 and Lemma 9, we immediately get (157). □

## 6 Finite-time stabilization

In this last section, we make use of the sharp estimations on  $\|T\|_{D(A^s)}$  and  $\|T^{-1}\|_{D(A^s)}$  with respect to  $\lambda$  obtained in Theorem 6 and 9 to deduce finite-time stabilization and small-time null-controllability.

### 6.1 Small-time null-controllability with a feedback form control

Let  $T > 0$ . We begin by building an explicit closed-loop control which achieves null-controllability in time  $T$ , using estimates from Theorem 7. From Proposition 8, for any  $N \in \mathbb{N}^*$  there exists  $\lambda(N) \notin \mathcal{N}$  such that (66) holds and,

$$\lambda(N) \in [N^\gamma, N^\gamma + C], \quad \gamma > \frac{\alpha}{\alpha - 1}, \quad (158)$$

for some  $C > 0$  independent of  $N$ . In particular,  $\lambda(N) \underset{N \rightarrow \infty}{\sim} N^\gamma \rightarrow \infty$ . Now, let  $\sigma \in (\alpha/(\alpha - 1), \gamma)$ . Then,

$$\lambda(N)^{-\frac{1}{\sigma}} \sim N^{-\frac{\gamma}{\sigma}}, \quad \forall N \in \mathbb{N}^*$$

and since  $\frac{\gamma}{\sigma} > 1$ , we have the following series convergence:

$$\sum_{N=1}^{\infty} \lambda(N)^{-\frac{1}{\sigma}} =: L_\sigma < \infty.$$

We then define

$$\begin{aligned} \delta_N &:= \frac{T}{L_\sigma} \lambda(N)^{-\frac{1}{\sigma}}, \quad \forall N \in \mathbb{N}^* \\ t_0 &:= 0, \quad t_N := \sum_{p=1}^N \delta_p, \quad \forall N \geq 1. \end{aligned} \quad (159)$$

In particular, by construction  $t_N \underset{N \rightarrow \infty}{\rightarrow} T$ .

We now define the following piecewise constant feedback control,

$$K(t) = K_{\lambda(N)} \text{ if } t \in [t_{N-1}, t_N), \quad \forall t \in [0, T), \quad \forall N \in \mathbb{N}^*, \quad (160)$$

which allows us to state the following closed-loop null-controllability result.

**Proposition 15** For any  $s \in [0, 1 - 1/2\alpha]$ , the closed-loop control system

$$\begin{cases} \frac{d}{dt}y(t) = (A + BK(t))y(t), & t > 0 \\ y(0) = y_0, \end{cases} \quad (161)$$

is well-posed in  $D(A^s)$  and satisfies  $y(T) = 0$  for any initial condition  $y_0 \in D(A^s)$ .

Moreover, if  $s > 1/2\alpha$ , the control  $K(t)y(t)$  satisfies

$$|K(t)y(t)| \xrightarrow[t \rightarrow T^-]{} 0. \quad (162)$$

**Proof :** The well-posedness of the above system is straightforward, since it can be established successively on each interval  $[t_N, t_{N+1}]$  thanks to the definition (160) of the control term, and the results of Section 5 (see Theorem 9).

To prove the null-controllability result, that is  $y(T) = 0$ , let  $s \in [0, 1 - 1/2\alpha]$  and note that estimate (156) implies that exist constants  $c_s, C_s > 0$ , independent of  $\lambda(N) > 0$ , such that, for any  $N \in \mathbb{N}^*$ , and any  $t \in [t_N, t_{N+1})$ ,

$$\|e^{t(A+BK_{\lambda(N)})}f\|_{D(A^s)} \leq C_s e^{c_s \lambda(N)^{\frac{1}{\alpha} - \lambda(N)t}} \|f\|_{D(A^s)}.$$

Now, for any  $N \in \mathbb{N}$ , given a solution  $y(t)$  of (161),

$$\|y(t_N)\|_{D(A^s)} \leq C_s e^{c_s \lambda(N)^{\frac{1}{\alpha} - \lambda(N)\delta_N}} \|y(t_{N-1})\|_{D(A^s)}, \quad \forall N \in \mathbb{N}^*.$$

Iterating this leads to

$$\begin{aligned} \|y(t)\|_{D(A^s)} &\leq C_s^N e^{\sum_{k=1}^N c_s \lambda(k)^{\frac{1}{\alpha} - \lambda(k)\delta_k}} e^{c_s \lambda(N+1)^{\frac{1}{\alpha} - \lambda(N+1)(t-t_N)}} \|y(0)\|_{D(A^s)} \\ &\leq C_s^N e^{\sum_{k=1}^N c_s \lambda(k)^{\frac{1}{\alpha} - \lambda(k)\delta_k}} e^{c_s \lambda(N+1)^{\frac{1}{\alpha}}} \|y(0)\|_{D(A^s)} \\ &= C_s^N e^{\sum_{k=1}^N c_s \lambda(k+1)^{\frac{1}{\alpha} - \lambda(k)\delta_k}} e^{c_s \lambda(1)^{\frac{1}{\alpha}}} \|y(0)\|_{D(A^s)} \\ &= e^{c'_s N + \sum_{k=1}^N c_s \lambda(k+1)^{\frac{1}{\alpha} - \lambda(k)\delta_k}} e^{c_s \lambda(1)^{\frac{1}{\alpha}}} \|y(0)\|_{D(A^s)} \\ &\leq e^{c_s \lambda(1)^{\frac{1}{\alpha}}} e^{\sum_{k=1}^N (c'_s + c_s \lambda(k+1)^{\frac{1}{\alpha} - \lambda(k)\delta_k})} \|y(0)\|_{D(A^s)}, \quad \forall t \in [t_N, t_{N+1}), \end{aligned} \quad (163)$$

where  $c'_s > 0$  is independent of  $N$ . Now, from (158) and the definition of  $\delta_k$ , we have

$$\begin{aligned} c_s \lambda(k+1)^{\frac{1}{\alpha} - \lambda(k)\delta_k} &= c_s \lambda(k+1)^{\frac{1}{\alpha}} - \frac{T}{L_\sigma} \lambda(k)^{1 - \frac{1}{\sigma}} \\ &= \lambda(k+1)^{1 - \frac{1}{\sigma}} \left( c_s \lambda(k+1)^{\frac{1-\alpha}{\alpha} + \frac{1}{\sigma}} - \left( \frac{\lambda(k)}{\lambda(k+1)} \right)^{1 - \frac{1}{\sigma}} \frac{T}{L_\sigma} \right) \end{aligned}$$

and since  $\sigma > \frac{\alpha}{\alpha - 1}$ , we have  $\frac{1 - \alpha}{\alpha} + \frac{1}{\sigma} < 0$ , so that

$$\lambda(k+1)^{\frac{1-\alpha}{\alpha} + \frac{1}{\sigma}} \xrightarrow[k \rightarrow \infty]{} 0.$$

Moreover, given the growth of  $\lambda(k)$ , it is straightforward that,

$$\frac{\lambda(k)}{\lambda(k+1)} \xrightarrow[k \rightarrow \infty]{} 1.$$

Thus, for  $k \geq 1$  large enough,

$$\begin{aligned} c'_s + c_s \lambda(k+1)^{\frac{1}{\alpha}} - \lambda(k) \delta_k &= c'_s - \frac{T}{L_\sigma} \lambda(k+1)^{1-\frac{1}{\sigma}} + o\left(\lambda(k+1)^{1-\frac{1}{\sigma}}\right) \\ &= -\frac{T}{L_\sigma} \lambda(k)^{1-\frac{1}{\sigma}} + o\left(\lambda(k)^{1-\frac{1}{\sigma}}\right). \end{aligned} \quad (164)$$

Hence, since  $\lambda(k)^{1-\frac{1}{\sigma}} \xrightarrow[k \rightarrow \infty]{} +\infty$ ,

$$\sum_{k=1}^N c'_s + \lambda(k+1)^{\frac{1}{\alpha}} - \lambda(k) \delta_k \xrightarrow[N \rightarrow \infty]{} -\infty. \quad (165)$$

Combining (163), (165) and the fact that  $t_N \xrightarrow[N \rightarrow \infty]{} T$  then yields

$$y(t) \xrightarrow[t \rightarrow T^-]{} 0.$$

Now, let  $s \geq 1/2$  and  $y_0 \in D(A^s)$ . The control  $K(t)y(t)$  is defined for all  $t \geq 0$  thanks to Proposition 13 and Proposition 12. To estimate the control term, note that, using first (157), then (163), we have

$$\begin{aligned} |K(t)y(t)| &= |K_{\lambda(N+1)}y(t)| \\ &= |K_{\lambda(N+1)}e^{(t-t_N)(A+BK_{\lambda(N+1)})}y(t_N)| \\ &\leq C_s \lambda(N+1) e^{-\lambda(N+1)(t-t_N)} e^{C\lambda(N+1)^{\frac{1}{\alpha}}} \|y(t_N)\|_{D(A^s)} \\ &\leq C_s \lambda(N+1) e^{-\lambda(N+1)(t-t_N)} e^{C\lambda(N+1)^{\frac{1}{\alpha}}} e^{c\lambda(1)^{\frac{1}{\alpha}}} e^{\sum_{k=1}^N (c' + c\lambda(k+1)^{\frac{1}{\alpha}} - \lambda(k)\delta_k)} \|y(0)\|_{D(A^s)} \\ &\leq C_s \lambda(N+1) e^{c\lambda(1)^{\frac{1}{\alpha}}} e^{\sum_{k=1}^N (c' + c''\lambda(k+1)^{\frac{1}{\alpha}} - \lambda(k)\delta_k)} \|y(0)\|_{D(A^s)} \\ &\leq C_s e^{c\lambda(1)^{\frac{1}{\alpha}}} e^{\log(\lambda(N+1)) + \sum_{k=1}^N (c' + c''\lambda(k+1)^{\frac{1}{\alpha}} - \lambda(k)\delta_k)} \|y(0)\|_{D(A^s)} \\ &\leq C_s e^{c\lambda(1)^{\frac{1}{\alpha}}} e^{\sum_{k=1}^N (\log(\lambda(k+1)) + c' + c''\lambda(k+1)^{\frac{1}{\alpha}} - \lambda(k)\delta_k)} \|y(0)\|_{D(A^s)}, \quad \forall t \in [t_N, t_{N+1}). \end{aligned}$$

Clearly, given the growth of  $\lambda(N)$  (see (158)), the log term is negligible, and we have, as in (164),

$$\log(\lambda(k+1)) + c' + c''\lambda(k+1)^{\frac{1}{\alpha}} - \lambda(k)\delta_k = -\frac{T}{L_\sigma} \lambda(k)^{1-\frac{1}{\sigma}} + o\left(\lambda(k)^{1-\frac{1}{\sigma}}\right). \quad (166)$$

Combining this with the inequality above then yields

$$|K(t)y(t)| \xrightarrow[t \rightarrow T^-]{} 0.$$

□

**Remark 11** *In the above proof, it would have been sufficient to establish the estimate (163) for the  $\|\cdot\|$  norm, since it would have implied the null-controllability result in all  $D(A^s)$  spaces. However, we establish this estimate in the range of spaces  $D(A^s)$ ,  $s \in [0, 1 - 1/2\alpha)$  because the estimates for  $s \in (1/2\alpha, 1 - 1/2\alpha)$  will play an essential role for the semi-global stabilization result in the next section.*

## 6.2 Small-time stabilization with a bounded time periodic feedback law

The above piecewise constant feedback control, though it achieves null-controllability in time  $T > 0$ , is only defined on the time interval  $[0, T]$ . This could easily be fixed by periodic extension, but an important drawback remains: from (160) and Proposition 14, the bounds on the norms  $\|K_{\lambda(N)}\|_{D(A^s)}$  diverge close to the final time  $T$ . This is an important issue in practice as the uniform stability property could fail, which would prevent a proper finite-time stabilization.

The aim of this subsection is to modify the definition of this feedback control to obtain a finite-time stabilization feedback law, according to the method developed in [15] (see also [37]).

In what follows, we will consider feedback laws satisfying the following properties:

( $\mathcal{P}_1$ )  $F(\cdot, v)$  is  $T$ -periodic:

$$F(t + T, y) = F(t, y), \quad \forall t \geq 0, \forall y \in D(A^{1/2}). \quad (167)$$

( $\mathcal{P}_2$ ) There exists  $\bar{T} \in [0, T)$  such that  $F$  satisfies the uniform estimate

$$|F(t, y)| \leq C \|y\|_{D(A^{1/2})}^{\frac{1}{2}}, \quad \forall t \in [\bar{T}, T), \forall y \in D(A^{1/2}), \quad (168)$$

for some  $C > 0$ .

( $\mathcal{P}_3$ )  $F$  is Lipschitz on the state space: there exists a continuous function  $M : [0, T) \rightarrow [0, +\infty)$  such that

$$|F(t, y_1) - F(t, y_2)| \leq M(t) \|y_1 - y_2\|_{D(A^{1/2})}, \quad \forall t \in [0, T), \forall y_1, y_2 \in D(A^{1/2}). \quad (169)$$

( $\mathcal{P}_4$ )  $F$  has a weakly compact support in space for large enough times: there exists  $\bar{T} \in [0, T)$  such that

$$\|y\|_{D(A^{1/2})} \geq M \implies F(t, y) = 0, \quad \forall t \in [\bar{T}, T), \forall y \in D(A^{1/2}). \quad (170)$$

Note that as a consequence of ( $\mathcal{P}_2$ ) and ( $\mathcal{P}_4$ ),  $F$  is uniformly bounded for large enough times: there exist  $\bar{T} \in [0, T)$  and  $C' > 0$  such that

$$|F(t, y)| \leq C', \quad \forall t \in [\bar{T}, T), \forall y \in D(A^{1/2}). \quad (171)$$

### 6.2.1 Well-posedness in $D(A^{1/2})$

In order to obtain quantitative estimates for closed-loop systems with non-linear time-varying feedbacks satisfying the properties listed above, we first prove a general regularization result for the semigroup  $e^{tA}$ :

**Proposition 16** *Let  $s_1, s_2 \in \mathbb{R}$  such that  $s_1 > s_2$ . The operator  $A$  generates a  $C_0$ -semigroup on  $D(A^{s_2})$ . Moreover, for all  $f \in D(A^{s_2})$ , and for  $t > 0$ ,  $e^{tA}f \in D(A^{s_1})$  and we have the following estimate:*

$$\|e^{tA}f\|_{s_1} \leq \left(\frac{s_1 - s_2}{t}\right)^{s_1 - s_2} e^{s_2 - s_1} \|f\|_{s_2}. \quad (172)$$

**Proof :** The definition of  $e^{tA}$  on the scale of spaces  $D(A^s)$ ,  $s \in \mathbb{R}$  can be obtained classically by the Hille-Yosida theorem. To prove the regularization estimate, note that

$$\begin{aligned} \|e^{tA}f\|_{s_1}^2 &= \sum_n |\lambda_n|^{2s_1} |\langle e^{tA}f, \varphi_n \rangle|^2 \\ &= \sum_n |\lambda_n|^{2s_1} e^{2t\lambda_n} |\langle f, \varphi_n \rangle|^2 \\ &= \sum_n e^{2t\lambda_n} |\lambda_n|^{2(s_1 - s_2)} |\lambda_n|^{2s_2} |\langle f, \varphi_n \rangle|^2. \end{aligned}$$

Recall that we have assumed  $\lambda_n < 0$  for all  $n \in \mathbb{N}$ . Thus an elementary study of the function  $x \mapsto x^{s_1 - s_2} e^{-tx}$  leads to

$$\sup_n (e^{2t\lambda_n} |\lambda_n|^{2(s_1 - s_2)}) \leq e^{2(s_2 - s_1)} \left(\frac{s_1 - s_2}{t}\right)^{2(s_1 - s_2)}.$$

Combining with the equalities above we obtain (172).  $\square$

Now we establish a well-posedness result and some estimates for a general form of time-varying feedback laws. This will ensure the soundness of the small-time stabilization result. For  $y_\tau \in H$ ,  $\tau \geq 0$ , we denote by  $\Phi(y_\tau; \tau, \cdot)$  the unique solution (if it exists) to the closed-loop system

$$\begin{cases} \frac{d}{dt}y = Ay + BF(t, y), & t \geq \tau \\ y(\tau) = y_\tau. \end{cases} \quad (173)$$

**Proposition 17** *Let  $T > 0$  and  $F : \mathbb{R}_+ \times D(A^{1/2}) \rightarrow \mathbb{C}$  satisfy properties  $(\mathcal{P}_1) - (\mathcal{P}_4)$ . For  $\tau \in [0, T)$ , and  $y_\tau \in D(A^{1/2})$ , the closed-loop system (173), has a unique solution on  $[\tau, T)$ . Moreover, for  $t \in [\tau, T)$  and  $y_\tau \in D(A^{1/2})$ ,*

$$\|\Phi(y_\tau; \tau, t)\|_{D(A^{1/2})} \leq e^{\omega(t-\tau)} \|y_\tau\|_{D(A^{1/2})} + C e^{-\frac{3\alpha+1}{4\alpha}} \|B\|_{-\frac{\alpha+1}{4\alpha}} \left(\frac{3\alpha+1}{4\alpha}\right)^{\frac{3\alpha+1}{4\alpha}} \left(\frac{4\alpha}{\alpha-1}\right) (t-\tau)^{\frac{\alpha-1}{4\alpha}}. \quad (174)$$

Finally, for  $T_1 \in (\tau, T)$ , there exists a constant  $C(T_1) > 0$  such that, for  $t \in [\tau, T_1]$  and for  $y_\tau \in D(A^{1/2})$ ,

$$\|\Phi(y_\tau; \tau, t)\|_{D(A^{1/2})} \leq C(T_1) \|y_\tau\|_{D(A^{1/2})}. \quad (175)$$

**Proof :** Let  $0 \leq \tau < T_1 < T$ . The proof of the existence of the solution relies on a classical fixed point argument on the mapping

$$\mathcal{F}(y)(t) = e^{(t-\tau)A}y_\tau + \int_\tau^t e^{(t-s)A}BF(s, y(s)) ds.$$

The fixed point argument requires that  $\mathcal{F}$  map the Banach space  $L^\infty(\tau, T_1; D(A^{1/2}))$  to itself. To that effect, the integral term must be carefully studied. First note that, given (25), we have  $B \in D(A^{-s})$  for all  $s > \frac{1}{2\alpha}$ . Define

$$s_0 := \frac{\alpha + 1}{4\alpha}.$$

Then,  $B \in D(A^{-s_0})$  and (172) implies

$$\|e^{tA}B\|_{D(A^{1/2})} \leq e^{-s_0 - \frac{1}{2}} \left(\frac{\frac{1}{2} + s_0}{t}\right)^{\frac{1}{2} + s_0} \|B\|_{-s_0},$$

and so, using (171) and noting that  $s_0 \neq 1/2$ , for all  $y \in L^\infty(\tau, T_1; D(A^{1/2}))$ ,

$$\begin{aligned} \left\| \int_\tau^t e^{(t-s)A} B F(s, y(s)) ds \right\|_{D(A^{1/2})} &\leq C' e^{-s_0 - \frac{1}{2}} \|B\|_{-s_0} \int_\tau^t \left(\frac{\frac{1}{2} + s_0}{t-s}\right)^{\frac{1}{2} + s_0} ds \\ &= C' e^{-s_0 - \frac{1}{2}} \|B\|_{-s_0} \left(\frac{1}{2} + s_0\right)^{\frac{1}{2} + s_0} \frac{(t-\tau)^{1 - \frac{1}{2} - s_0}}{1 - \frac{1}{2} - s_0} \\ &= C' e^{-\frac{3\alpha+1}{4\alpha}} \|B\|_{-s_0} \left(\frac{3\alpha+1}{4\alpha}\right)^{\frac{3\alpha+1}{4\alpha}} \left(\frac{4\alpha}{\alpha-1}\right) (t-\tau)^{\frac{\alpha-1}{4\alpha}} \\ &\leq C' e^{-\frac{3\alpha+1}{4\alpha}} \|B\|_{-s_0} \left(\frac{3\alpha+1}{4\alpha}\right)^{\frac{3\alpha+1}{4\alpha}} \left(\frac{4\alpha}{\alpha-1}\right) T^{\frac{\alpha-1}{4\alpha}}. \end{aligned} \tag{176}$$

Hence, for  $t \in [\tau, T_1]$ ,

$$\|\mathcal{F}(y)(t)\|_{D(A^{1/2})} \leq e^{\omega t} \|y_\tau\|_{D(A^{1/2})} + C' e^{-\frac{3\alpha+1}{4\alpha}} \|B\|_{-s_0} \left(\frac{3\alpha+1}{4\alpha}\right)^{\frac{3\alpha+1}{4\alpha}} \left(\frac{4\alpha}{\alpha-1}\right) T^{\frac{\alpha-1}{4\alpha}},$$

which implies that  $\mathcal{F}(y) \in L^\infty(\tau, T_1; D(A^{1/2}))$ . Together with (169), for any  $y_1, y_2 \in L^\infty(\tau, T_1; D(A^{1/2}))$ , we can then compute

$$\begin{aligned} &\left\| \int_\tau^t e^{(t-s)A} B [(F(s, y_1(s)) - F(s, y_2(s)))] ds \right\|_{D(A^{1/2})} \\ &\leq \int_\tau^t M(s) \|y_1(s) - y_2(s)\|_{D(A^{1/2})} e^{-s_0 - \frac{1}{2}} \left(\frac{\frac{1}{2} + s_0}{t-s}\right)^{\frac{1}{2} + s_0} \|B\|_{-s_0} ds \\ &\leq \sup_{[\tau, T_1]} M \|y_1 - y_2\|_{L^\infty(\tau, T_1; D(A^{1/2}))} e^{-s_0 - \frac{1}{2}} \|B\|_{-s_0} \int_\tau^t \left(\frac{\frac{1}{2} + s_0}{t-s}\right)^{\frac{1}{2} + s_0} ds \\ &\leq \sup_{[\tau, T_1]} M \|y_1 - y_2\|_{L^\infty(\tau, T_1; D(A^{1/2}))} e^{-s_0 - \frac{1}{2}} \|B\|_{-s_0} \left(\frac{1}{2} + s_0\right)^{\frac{1}{2} + s_0} \frac{(t-\tau)^{1 - \frac{1}{2} - s_0}}{1 - \frac{1}{2} - s_0} \\ &\leq \sup_{[\tau, T_1]} M e^{-\frac{3\alpha+1}{4\alpha}} \|B\|_{-s_0} \left(\frac{3\alpha+1}{4\alpha}\right)^{\frac{3\alpha+1}{4\alpha}} \left(\frac{4\alpha}{\alpha-1}\right) (t-\tau)^{\frac{\alpha-1}{4\alpha}} \|y_1 - y_2\|_{L^\infty(\tau, T_1; D(A^{1/2}))}. \end{aligned}$$

It is then clear that for  $t - \tau$  small enough,  $\mathcal{F}$  is a contraction on  $L^\infty(\tau, t; D(A^{1/2}))$ , and we can apply the usual fixed point argument to obtain the existence of a solution to (173) on  $[\tau, t]$ .

Now, the Duhamel formula combined with (171) yields, for any  $\tau \in [0, T)$ ,  $y_\tau \in D(A^{1/2})$  and  $t > \tau$  such that  $\Phi(y_\tau; \tau, \cdot)$  is defined on  $[\tau, t]$ :

$$\begin{aligned} \|\Phi(y_\tau; \tau, t)\|_{D(A^{1/2})} &\leq e^{\omega(t-\tau)} \|y_\tau\|_{D(A^{1/2})} + C e^{-\frac{3\alpha+1}{4\alpha}} \|B\|_{-s_0} \left(\frac{3\alpha+1}{4\alpha}\right)^{\frac{3\alpha+1}{4\alpha}} \left(\frac{4\alpha}{\alpha-1}\right) (t-\tau)^{\frac{\alpha-1}{4\alpha}} \\ &\leq e^{\omega T} \|y_\tau\|_{D(A^{1/2})} + C' T^{\frac{\alpha-1}{4\alpha}}. \end{aligned}$$

Therefore,  $\Phi(y_\tau; \tau, \cdot)$  is defined on  $[\tau, T)$ .

Finally, to prove (175), note that for  $\bar{T} \in [0, T)$ , we have, using again (169) together with  $(\mathcal{P}_2)$ ,

$$\begin{aligned} \|\Phi(y_\tau; \tau, t)\| &\leq e^{\omega(t-\tau)} \|y_\tau\|_{D(A^{1/2})} \\ &\quad + \left(\sup_{[\tau, T_1]} M\right) e^{-\frac{3\alpha+1}{4\alpha}} \|B\|_{-s_0} \left(\frac{3\alpha+1}{4\alpha}\right)^{\frac{3\alpha+1}{4\alpha}} \left(\frac{4\alpha}{\alpha-1}\right) \int_\tau^t \frac{1}{(t-s)^{\frac{1}{2}+s_0}} \|y(s)\|_{D(A^{1/2})} ds. \\ &\leq e^{\omega(t-\tau)} \|y_\tau\|_{D(A^{1/2})} + C'' \left(\sup_{[\tau, T_1]} M\right) \int_\tau^t \frac{1}{(t-s)^{\frac{1}{2}+s_0}} \|y(s)\|_{D(A^{1/2})} ds. \end{aligned}$$

Applying Grönwall's integral inequality, we then obtain

$$\begin{aligned} \|\Phi(y_\tau; \tau, t)\|_{D(A^{1/2})} &\leq e^{\omega(t-\tau)} \|y_\tau\|_{D(A^{1/2})} + C'' \left(\sup_{[\tau, T_1]} M\right) \int_\tau^t e^{\omega(s-\tau)} \|y_\tau\|_{D(A^{1/2})} \frac{1}{(t-s)^{\frac{1}{2}+s_0}} e^{\int_s^t \frac{1}{(t-\sigma)^{\frac{1}{2}+s_0}} d\sigma} ds \\ &= e^{\omega(t-\tau)} \|y_\tau\|_{D(A^{1/2})} + C'' \left(\sup_{[\tau, T_1]} M\right) \|y_\tau\|_{D(A^{1/2})} \int_\tau^t e^{\omega(s-\tau)} \frac{1}{(t-s)^{\frac{1}{2}+s_0}} e^{(t-s)\frac{\alpha-1}{4\alpha}} ds \\ &\leq C(T_1) \|y_\tau\|_{D(A^{1/2})}, \end{aligned}$$

where  $C(T_1) > 0$  is a constant depending on  $A, B, T_1$  and  $\tau$ . Note in particular that when  $T_1 \rightarrow T$ , it is possible to have  $C(T_1) \rightarrow \infty$  because of the  $\sup_{[\tau, T_1]} M$  term if  $M(t) \xrightarrow{t \rightarrow T} +\infty$ .

□

### 6.2.2 Definition of the feedback law

The basic strategy to modify the feedback law is to deactivate the feedback close to time  $T$  if the norm of the state is not small enough already. This ensures uniform stability of the system (which we define later on), and provides a form of uniform boundedness for the feedback law. Furthermore, in the spirit of a more practical implementation, it is appropriate to define the feedback law for  $t \geq 0$ , which we do by periodic extension with period  $T$ . Because of this modification, the global null-controllability result above becomes a semi-global stabilization result in time  $2T$ . Note that the result remains a small-time stabilization result, as  $T$  can be chosen as small as desired.

**Theorem 10** *Let  $T > 0$  and  $\Gamma > 0$ . There exists a time-varying feedback law  $F : \mathbb{R}^+ \times D(A^{1/2}) \rightarrow \mathbb{C}$  satisfying  $(\mathcal{P}_1) - (\mathcal{P}_4)$  such that the flow  $\Phi$  associated to the corresponding system (173) satisfies*

(i) *Finite-time stabilization:*

$$\|y_\tau\|_{D(A^{1/2})} \leq \Gamma \implies \Phi(y_\tau; \tau, 2T) = 0, \quad \forall \tau \in [0, T), \quad \forall y_\tau \in D(A^{1/2}). \quad (177)$$

(ii) *Uniform stability: for all  $\varepsilon > 0$ , there exists  $\eta > 0$  such that*

$$\|y\|_{D(A^{1/2})} \leq \eta \implies \|\Phi(y; t, t')\|_{D(A^{1/2})} \leq \varepsilon, \quad \forall 0 \leq t \leq t', \quad \forall y \in D(A^{1/2}). \quad (178)$$

This is the main theorem of this section. Note that  $T$  and  $\Gamma$  are arbitrary and can be chosen respectively as small and as large as desired.

**Remark 12** *For the sake of clarity, we state this theorem in the space  $D(A^{1/2})$ , but it can be checked, with an analogous proof, that it holds on the range of spaces  $D(A^s)$ ,  $s \in (1/(2\alpha), 1 - 1/(2\alpha))$ .*

**Proof :**

**Feedback design:** Introduce

$$\nu \in \left( \frac{1}{\alpha}, 1 - \frac{1}{\sigma} \right),$$

and the auxiliary sequence

$$\alpha_N := e^{-\lambda(N)^\nu},$$

where, as in Section 6.1,  $\sigma \in (\alpha/(\alpha - 1), \gamma)$  with  $\gamma$  and  $\lambda(N)$  defined by (158).

For  $n \in \mathbb{N}^*$ , define the cut-off functions :

$$h_n \in C^\infty[0, +\infty), \quad h_n(x) = 1, \quad x \in [0, \alpha_n], \quad h_n(x) = 0, \quad x \geq 2\alpha_n.$$

For  $N_c \geq 1$ , define the modified feedback:

$$F_{N_c}(t, v) = \begin{cases} K_{\lambda(N)}v & \text{if } t \in [t_{N-1}, t_N), \quad 1 \leq N < N_c \\ h_N(\|v\|_{D(A^{1/2})}) K_{\lambda(N)}v & \text{if } t \in [t_{N-1}, t_N), \quad N \geq N_c, \end{cases} \quad (179)$$

where  $\{t_N\}_{\mathbb{N}}$  is given by (159). We then extend  $F_{N_c}$  periodically to  $[0, \infty)$  so that it satisfies  $(\mathcal{P}_1)$ .

It is clear from the definition of  $h_N$ , and the regularity of the  $K_{\lambda(N)}$  (given by Proposition 14), that  $F_{N_c}$  satisfies properties  $(\mathcal{P}_3)$  and  $(\mathcal{P}_4)$ . Moreover, given the definition of  $\alpha_N$ , and Proposition 14, we have, for any  $N > N_c$  and  $t \in [t_{N-1}, t_N)$ ,

$$|F_{N_c}(t, y)| \leq |K_{\lambda(N)}y| \leq C e^{\lambda(N)^{\frac{1}{\alpha}}} \|y\|_{D(A^{1/2})} \leq 2C e^{\lambda(N)^{\frac{1}{\alpha}}} e^{-\lambda(N)^\nu},$$

and since  $\nu > 1/\alpha$ , this clearly implies  $(\mathcal{P}_2)$ .

**Semiglobal finite-time stabilization** To proceed with the proof, we must now choose the value of  $N_c$ , i.e., the time at which the cut-off term is introduced in the feedback law.

The idea is to allow situations where the cut-off de-activates the feedback term during the first period, from the initial time to  $T$ . Then, in the second period, the goal will be to guarantee that the feedback remains active on  $[T+t_{N_c-1}, 2T)$ , so that Proposition 15 applies. To that end, recall that, from (156), for  $N \geq 1$ , and  $y \in D(A^{1/2})$ ,

$$\frac{\|\Phi(y; t_{N-1}, t)\|_{D(A^{1/2})}}{\alpha_N} \leq C e^{c\lambda(N)\frac{1}{\alpha} + \lambda(N)\nu} e^{-\lambda(N)(t-t_{N-1})} \|y\|_{D(A^{1/2})}, \quad \forall t \in [t_{N-1}, t_N).$$

Now let  $\nu < \nu' < 1 - 1/\sigma$ . If  $\|y\|_{D(A^{1/2})} \leq e^{-\lambda(N)\nu'}$ , then for  $N$  large enough:

$$\frac{\|\Phi(y; t_{N-1}, t)\|_{D(A^{1/2})}}{\alpha_N} \leq e^{-\lambda(N)(t-t_{N-1})} \leq 1, \quad \forall t \in [t_{N-1}, t_N),$$

that is,  $h_n(\|\Phi(y; t_{N-1}, t)\|_{D(A^{1/2})}) = 1$  so that the feedback remains active on the whole interval  $[t_{N-1}, t_N)$ . Moreover, recalling that, from (159),

$$-\lambda(N)\delta_N = -\frac{T}{L_\sigma} \lambda(N)^{1-\frac{1}{\sigma}} = -\frac{T}{L_\sigma} \lambda(N+1)^{\nu'} \lambda(N)^{1-\frac{1}{\sigma}-\nu'} \left( \frac{\lambda(N)}{\lambda(N+1)} \right)^{\nu'},$$

and  $1 - 1/\sigma - \nu' > 0$  and

$$\left( \frac{\lambda(N)}{\lambda(N+1)} \right)^{\nu'} \xrightarrow{N \rightarrow \infty} 1,$$

for  $N$  large enough, the above inequality leads to (recall that  $\alpha_N < 1$ )

$$\|\Phi(y; t_{N-1}, t_N)\|_{D(A^{1/2})} \leq e^{-\lambda(N)\delta_N} \leq e^{-\lambda(N+1)\nu'}.$$

We can thus propagate the above reasoning and we conclude that there exists  $N_1 > 0$  such that for any for any  $N \geq N_1$ , for any  $y \in D(A^{1/2})$ ,

$$\|y\|_{D(A^{1/2})} \leq e^{-\lambda(N)\nu'} \implies \Phi(y; t_{N-1}, T) = 0. \quad (180)$$

We must now choose  $N_c \geq N_1$  so that for any  $t \in [0, T)$ ,

$$\|y\|_{D(A^{1/2})} \leq \Gamma \implies \|\Phi(y; t, t_{N_c-1} + T)\|_{D(A^{1/2})} \leq e^{-\lambda(N_c)\nu'}, \quad \forall y \in D(A^{1/2}), \quad (181)$$

which, by periodicity of  $F_{N_c}$ , and combined with (180), will prove the finite-time stabilization (177).

To proceed, we need to establish a series of inequalities to estimate the norm of the flow.

Recall that, from (163), the closed-loop system without cut-offs satisfies

$$\|y(t_{N-1})\|_{D(A^{1/2})} \leq e^{c\lambda(1)\frac{1}{\alpha}} e^{\sum_{k=1}^{N-1} (c' + c\lambda(k+1)\frac{1}{\alpha} - \lambda(k)\delta_k)} \|y(0)\|_{D(A^{1/2})}$$

From (164), we know that there exists  $N_0$  such that for  $n \geq N_0$ ,

$$c' + c''\lambda(n)\frac{1}{\alpha} - \lambda(n)\delta_n \leq 0.$$

We will thus select  $N_c > N_0 + 1$ . The period  $[0, T)$  can now be separated in three intervals.

- **First estimate. Left interval**

On the first interval  $[0, t_{N_0}]$ , since  $N_0 < N_c$ , the cut-off is not yet active and we can use the estimates on the linear system. Let  $y \in D(A^{1/2})$ ,  $0 \leq N \leq N_0 - 1$ , and  $t \in [t_N, t_{N+1})$ . Then,

$$\|\Phi(y; t, t_{N_0})\|_{D(A^{1/2})} \leq C e^{c\lambda(N+1)\frac{1}{\alpha}} e^{\sum_{n=N+2}^{N_0} c' + c''\lambda(n)\frac{1}{\alpha} - \lambda(n)\delta_n} \|y\|_{D(A^{1/2})}$$

Since  $N_0$  is fixed, we immediately get the existence of a constant  $C > 0$  (depending on  $N_0$ ) such that

$$\|\Phi(y; t, t_{N_0})\|_{D(A^{1/2})} \leq C \|y\|_{D(A^{1/2})}. \quad (182)$$

- **Second estimate. Middle interval** On the middle interval  $[t_{N_0}, t_{N_c-1}]$ , we can still use the linear estimates and the flow contracts in  $D(A^{1/2})$  norm thanks to the definition of  $N_0$ . For  $y \in D(A^{1/2})$ ,  $N_0 \leq N \leq N_c - 2$ , and  $t \in [t_N, t_{N+1})$ , we have

$$\begin{aligned} \|\Phi(y; ; t, t_{N_c-1})\|_{D(A^{1/2})} &\leq C e^{c\lambda(N+1)\frac{1}{\alpha}} G(N+1, N_c-1) \|y\|_{D(A^{1/2})} \\ &\leq C e^{c\lambda(N_c)\frac{1}{\alpha}} G(N+1, N_c-1) \|y\|_{D(A^{1/2})}, \end{aligned} \quad (183)$$

where  $G : \llbracket N_0, \infty \rrbracket^2 \rightarrow \mathbb{R}$  is defined by

$$G(N_1, N_2) = e^{\sum_{k=N_1}^{N_2} c' + c\lambda(k)\frac{1}{\alpha} - \lambda(k)\delta_k}. \quad (184)$$

- **Third estimate. Right interval**

On the third interval  $[t_{N_c-1}, T)$ , the nonlinearity of the feedback leads to the more technical estimate (174), which holds no matter whether the cut-off is active, nor when. It yields a uniform bound with an additional term that goes to 0 when  $N_c$  goes to  $\infty$ .

We can now piece all of the above together. Let  $y \in D(A^{1/2})$ , such that  $\|y\|_{D(A^{1/2})} \leq \Gamma$ . If  $t \in [0, t_{N_0}]$ , by definition of  $\Phi$ , and  $T$ -periodicity of the feedback,

$$\begin{aligned} \Phi(y; t, t_{N_c-1} + T) &= \Phi(\Phi(y; t, t_{N_0}); t_{N_0}, t_{N_c-1} + T) \\ &= \Phi(\Phi(\Phi(\Phi(y; t, t_{N_0}); t_{N_0}, t_{N_c-1}); t_{N_c-1}, T); T, T + t_{N_c-1}) \\ &= \Phi(\Phi(\Phi(\Phi(y; t, t_{N_0}); t_{N_0}, t_{N_c-1}); t_{N_c-1}, T); 0, t_{N_c-1}). \end{aligned}$$

Now, applying in turn (183) and (182), we have

$$\begin{aligned}
& \|\Phi(y; t, t_{N_c-1} + T)\|_{D(A^{1/2})} \\
& \leq C e^{c\lambda(N_c)\frac{1}{\alpha}} G(N_0 + 1, N_c - 1) \|\Phi(\Phi(\Phi(y; t, t_{N_0}); t_{N_0}, t_{N_c-1}); t_{N_c-1}, T)\|_{D(A^{1/2})} \\
& \leq C e^{c\lambda(N_c)\frac{1}{\alpha}} G(N_0 + 1, N_c - 1) \left( e^{\omega T} \|\Phi(\Phi(y; t, t_{N_0}); t_{N_0}, t_{N_c-1})\|_{D(A^{1/2})} \right. \\
& \quad \left. + C e^{-\frac{3\alpha+1}{4\alpha}} \|B\|_{-s_0} \left( \frac{3\alpha+1}{4\alpha} \right)^{\frac{3\alpha+1}{4\alpha}} \left( \frac{4\alpha}{\alpha-1} \right) (T - t_{N_c-1})^{\frac{\alpha-1}{4\alpha}} \right) \\
& \leq C e^{2c\lambda(N_c)\frac{1}{\alpha}} G(N_0 + 1, N_c - 1) \left( e^{\omega T} e^{c\lambda(N_c)\frac{1}{\alpha}} G(N_0 + 1, N_c - 1) \|\Phi(y; t, t_{N_0})\|_{D(A^{1/2})} \right. \\
& \quad \left. + C'(T - t_{N_c-1})^{\frac{\alpha-1}{4\alpha}} \right) \\
& \leq C e^{2c\lambda(N_c)\frac{1}{\alpha}} G(N_0 + 1, N_c - 1) \left( C e^{\omega T} e^{c\lambda(N_c)\frac{1}{\alpha}} G(N_0 + 1, N_c - 1) \|y\|_{D(A^{1/2})} \right. \\
& \quad \left. + C'(T - t_{N_c-1})^{\frac{\alpha-1}{4\alpha}} \right) \\
& \leq C e^{2c\lambda(N_c)\frac{1}{\alpha}} G(N_0 + 1, N_c - 1) \left( C e^{\omega T} e^{c\lambda(N_c)\frac{1}{\alpha}} G(N_0 + 1, N_c - 1) \Gamma \right. \\
& \quad \left. + C'(T - t_{N_c-1})^{\frac{\alpha-1}{4\alpha}} \right)
\end{aligned} \tag{185}$$

for some constants  $C, C'_T > 0$ .

If  $t \in [t_N, t_{N+1})$ , with  $N_0 \leq N \leq N_c - 2$ , we obtain, as above:

$$\begin{aligned}
& \|\Phi(y; t, t_{N_c-1} + T)\|_{D(A^{1/2})} \\
& \leq C e^{2c\lambda(N_c)\frac{1}{\alpha}} G(N_0 + 1, N_c - 1) \left( e^{\omega T} e^{c\lambda(N_c)\frac{1}{\alpha}} G(N_0 + 1, N_c - 1) \Gamma + C'(T - t_{N_c-1})^{\frac{\alpha-1}{4\alpha}} \right),
\end{aligned} \tag{186}$$

and finally, if  $t \in [t_N, t_{N+1})$ , with  $N \geq N_c - 1$ , we get

$$\begin{aligned}
\|\Phi(y; t, t_{N_c-1} + T)\|_{D(A^{1/2})} & \leq C e^{c\lambda(N_c)\frac{1}{\alpha}} G(N_0 + 1, N_c - 1) \left( e^{\omega T} \Gamma + C'(T - t)^{\frac{\alpha-1}{4\alpha}} \right) \\
& \leq C e^{c\lambda(N_c)\frac{1}{\alpha}} G(N_0 + 1, N_c - 1) \left( e^{\omega T} \Gamma + C'(T - t_{N_c-1})^{\frac{\alpha-1}{4\alpha}} \right)
\end{aligned} \tag{187}$$

By definition of  $G$ ,  $N_0$  and  $(\delta_n)$  (see (159) and (184)),

$$\log(G(N_0 + 1, N_c - 1)) = \sum_{k=N_0+1}^{N_c-1} c' + c'' \lambda(k)^{\frac{1}{\alpha}} - \lambda(k) \delta_k \leq c' + c'' \lambda(N_c - 1)^{\frac{1}{\alpha}} - \frac{T}{L_\sigma} \lambda(N_c - 1)^{1-\frac{1}{\sigma}}.$$

Note that we used, in particular, the fact that the sum is not empty since we chose  $N_c > N_0 + 1$ . Since  $\frac{1}{\alpha} < \nu' < 1 - \frac{1}{\sigma}$ , we clearly have

$$C e^{2c\lambda(N_c)\frac{1}{\alpha}} G(N_0 + 1, N_c - 1) e^{\lambda(N_c)\nu'} \xrightarrow{N_c \rightarrow \infty} 0.$$

Moreover, it is clear that

$$e^{\omega T} \Gamma + C'(T - t_{N_c-1})^{\frac{\alpha-1}{4\alpha}}$$

is bounded with respect to  $N_c$ , as are, a fortiori, the terms

$$C e^{\omega T} e^{c\lambda(N_c)^{\frac{1}{\alpha}}} G(N_0 + 1, N_c - 1) \Gamma + C'(T - t_{N_c-1})^{\frac{\alpha-1}{4\alpha}}$$

and

$$e^{\omega T} e^{c\lambda(N_c)^{\frac{1}{\alpha}}} G(N + 1, N_c - 1) \Gamma + C'(T - t_{N_c-1})^{\frac{\alpha-1}{4\alpha}}$$

in inequalities (185) and (186) respectively. Combining this with (185)–(187), it is possible to choose  $N_c$  large enough for (181) to hold, which in turn ensures (177).

**Uniform stability** Let  $\varepsilon > 0$ .

From (174), there exists  $T_1 \geq t_{N_c-1}$  and  $\eta_0 > 0$  such that

$$\|y\|_{D(A^{1/2})} \leq \eta_0 \implies \begin{cases} \|\Phi(y; T_1, t)\|_{D(A^{1/2})} \leq \varepsilon, \\ \Phi(y; T_1, T) = 0, \end{cases} \quad \forall t \in [T_1, T], \quad \forall y \in D(A^{1/2}). \quad (188)$$

Next, from (175), there exists a constant  $C > 0$  such that

$$\|\Phi(y_\tau; \tau, t)\|_{D(A^{1/2})} \leq C \|y_\tau\|_{D(A^{1/2})}, \quad \forall 0 \leq \tau \leq t \leq T_1, \quad \forall y_\tau \in D(A^{1/2}).$$

Thus there exists  $\eta_1 > 0$  such that

$$\|y\|_{D(A^{1/2})} \leq \eta_1 \implies \|\Phi(y; 0, t)\|_{D(A^{1/2})} \leq \min(\varepsilon, \eta_0), \quad \forall t \in [0, T_1], \quad \forall y \in D(A^{1/2}). \quad (189)$$

Combining (189) and (188), we have

$$\|y\|_{D(A^{1/2})} \leq \eta_1 \implies \begin{cases} \|\Phi(y; 0, t)\|_{D(A^{1/2})} \leq \varepsilon, \\ \Phi(y; 0, T) = 0, \end{cases} \quad \forall t \in [0, T], \quad \forall y \in D(A^{1/2}). \quad (190)$$

Now, given (174), there exists  $T_2$  and  $\eta_2 > 0$  such that:

$$\|y_\tau\|_{D(A^{1/2})} \leq \eta_2 \implies \|\Phi(y_\tau; \tau, t)\|_{D(A^{1/2})} \leq \min(\varepsilon, \eta_1), \quad \forall T_2 \leq \tau \leq t < T, \quad \forall y_\tau \in D(A^{1/2}). \quad (191)$$

Now, with  $T_2$  fixed, considering (175), there exists  $\eta_3 > 0$  such that:

$$\|y_\tau\|_{D(A^{1/2})} \leq \eta_3 \implies \|\Phi(y_\tau; \tau, t)\|_{D(A^{1/2})} \leq \min(\varepsilon, \eta_2), \quad \forall 0 \leq \tau \leq t \leq T_1, \quad \forall y_\tau \in D(A^{1/2}). \quad (192)$$

Now let  $\tau \in [0, T)$ . Then, (192) and (191) imply

$$\|y_\tau\|_{D(A^{1/2})} \leq \min(\eta_2, \eta_3) \implies \begin{cases} \|\Phi(y_\tau; \tau, t)\|_{D(A^{1/2})} \leq \varepsilon, \\ \|\Phi(y_\tau; \tau, T)\|_{D(A^{1/2})} \leq \eta_1, \end{cases} \quad , \quad \forall 0 \leq \tau \leq t < T, \quad \forall y_\tau \in D(A^{1/2}). \quad (193)$$

Using the periodicity of  $F$  (167), and combining (193) with (190), we get

$$\|y_\tau\|_{D(A^{1/2})} \leq \min(\eta_2, \eta_3) \implies \begin{cases} \|\Phi(y_\tau; \tau, t)\|_{D(A^{1/2})} \leq \varepsilon & \text{if } t \in [\tau, 2T], \\ \Phi(y_\tau; \tau, t) = 0, & \text{if } t \geq 2T, \\ \forall \tau \in [0, T), \quad \forall y_\tau \in D(A^{1/2}), \end{cases}$$

which, thanks to the periodicity of  $F$ , yields the desired uniform stability property (178).  
□

## A Technical lemmas on the spectrum of $A$

In this appendix we provide the proofs of some technical lemmas, including Lemma 2, Lemma 3, and Lemma 4.

**Proof of Lemma 2:** We easily see that

$$D_i(\lambda) = \min_{j \in \mathbb{N}^*} |\lambda_j - \lambda_i + \lambda|,$$

is well-defined since  $\inf_{j \in \mathbb{N}^*} |\lambda_j - \lambda_i + \lambda|$  is reached for a finite  $j \in \mathbb{N}^*$ . Indeed,  $\lambda_j \rightarrow -\infty$  as  $j \rightarrow \infty$  and  $j \in \mathbb{N}^* \mapsto |\lambda_j - \lambda_i + \lambda|$  is increasing with respect to  $j$  for  $\lambda_j > \lambda_i - \lambda$ .

Moreover, notice that  $j \in \mathbb{N}^* \mapsto |\lambda_j - \lambda_i + \lambda|$  may be decreasing only if there exists  $j_1 \in \mathbb{N}^*$  such that  $\lambda_j < \lambda_i - \lambda, \forall 1 \leq j \leq j_1$ . In this case, using the assumption  $\lambda_j + \lambda \neq \lambda_i, \forall i, j \in \mathbb{N}^*$

$$-\lambda_1 + \lambda_i - \lambda < \dots < -\lambda_{j_1} + \lambda_i - \lambda < 0 < -\lambda_{j_2} + \lambda_i - \lambda < \dots,$$

with  $j_2 = j_1 + 1$ . Hence,

$$D_i(\lambda) = \min\{|\lambda_{j_1} - \lambda_i + \lambda|, |\lambda_{j_2} - \lambda_i + \lambda|\}.$$

Finally, we have  $\text{Dist}_\alpha(\lambda) < D_i(\lambda)$  by definition of  $\text{Dist}_\alpha(\lambda)$  □

**Proof of Lemma 3:** First, notice that due Assumption 1 concerning the asymptotic behavior of the eigenvalues, for  $i \in \mathbb{N}$ ,

$$1 \pm \frac{\lambda}{\lambda_m - \lambda_i} \rightarrow 1, \quad m \rightarrow \infty, \quad m \neq i.$$

We first prove assertion (i). Assertion (ii) is dealt with similarly. Using the gap condition

Lemma 1,

$$\begin{aligned}
\left| \prod_{\substack{m \in \mathbb{N}^* \\ m \neq i}} \left( 1 + \frac{\lambda}{\lambda_i - \lambda_m} \right) \right| &= \prod_{\substack{m \in \mathbb{N}^* \\ m \neq i}} \left| 1 + \frac{\lambda}{\lambda_i - \lambda_m} \right| \\
&\leq \prod_{\substack{m \in \mathbb{N}^* \\ m \neq i}} \left( 1 + \left| \frac{\lambda}{\lambda_i - \lambda_m} \right| \right) \\
&\leq \prod_{\substack{m \in \mathbb{N}^* \\ m \neq i}} \left( 1 + C \frac{\lambda}{|i - m|^\alpha} \right) \\
&= \exp \left( \sum_{\substack{m \in \mathbb{N}^* \\ m \neq i}} \ln \left( 1 + C \frac{\lambda}{|i - m|^\alpha} \right) \right) \\
&\leq \exp \left( 2 \sum_{k \in \mathbb{N}^*} \ln \left( 1 + C \frac{\lambda}{k^\alpha} \right) \right).
\end{aligned}$$

Since  $\ln(1 + 1/y)$  is decreasing over  $y \in (1, \infty)$ ,

$$\begin{aligned}
\sum_{k \in \mathbb{N}^*} \ln \left( 1 + C \frac{\lambda}{k^\alpha} \right) &\leq \ln(1 + C\lambda) + \int_1^\infty \ln \left( 1 + C \frac{\lambda}{x^\alpha} \right) dx \\
&= \ln(1 + C\lambda) + x \ln \left( 1 + C \frac{\lambda}{x^\alpha} \right) \Big|_1^\infty + \alpha \int_1^\infty \frac{C\lambda}{x^\alpha + C\lambda} dx \\
&= \alpha(C\lambda)^{\frac{1}{\alpha}} \int_{(C\lambda)^{-\frac{1}{\alpha}}}^\infty \frac{1}{y^\alpha + 1} dy \\
&\leq \alpha(C\lambda)^{\frac{1}{\alpha}} \int_0^\infty \frac{1}{y^\alpha + 1} dy.
\end{aligned}$$

Hence, there exists  $C > 0$  such that, for every  $\lambda \in (0, \infty)$  and  $i \in \mathbb{N}^*$ ,

$$\left| \prod_{\substack{m \in \mathbb{N}^* \\ m \neq i}} \left( 1 + \frac{\lambda}{\lambda_i - \lambda_m} \right) \right| \leq C e^{C\lambda^{\frac{1}{\alpha}}}.$$

□

**Proof of Lemma 4:** In the sequel we only present the proof for the inequality (ii), since the inequality (i) can be treated similarly.

Let  $j \in \mathbb{N}^*$ . From Lemma 2, we may assume that there exists  $J_j \in \mathbb{N}^*$  satisfies  $|\lambda_{J_j} - \lambda_j + \lambda| = D_j(\lambda)$ . Without loss of generality, we may assume  $\lambda_j - \lambda_{J_j} - \lambda > 0$ , as a similar argument works for the case  $\lambda_{J_j} - \lambda_j + \lambda < 0$ . Under this assumption, we also know that  $\lambda_j - \lambda_{J_j-1} - \lambda \leq -D_j(\lambda) < 0$ .

Then, we write,

$$\sum_{i \in \mathbb{N}^*} \left| \frac{\lambda^2}{\lambda_j - \lambda_i - \lambda} \right| \leq \frac{2\lambda^2}{D_j(\lambda)} + \sum_{1 \leq i \leq J_j-2} \frac{\lambda^2}{\lambda_i + \lambda - \lambda_j} + \sum_{i \geq J_j+1} \frac{\lambda^2}{\lambda_j - \lambda_i - \lambda},$$

where we bounded the term  $i = J_j - 1$  using  $D_j(\lambda) \leq |\lambda_{J_j-1} - \lambda_j + \lambda|$ . Then, since  $\lambda_{J_j} - \lambda_i > 0$  for  $i > J_j$ , using the gap condition of Lemma 1,

$$\begin{aligned}
\sum_{J_j < i} \frac{\lambda^2}{\lambda_j - \lambda_i - \lambda} &= \sum_{J_j < i} \frac{\lambda^2}{\lambda_{J_j} - \lambda_i + D_j(\lambda)} \\
&\leq \sum_{J_j < i} \frac{\lambda^2}{\lambda_{J_j} - \lambda_i} \\
&\leq \sum_{J_j < i} \frac{\lambda^2}{|i - J_j|^\alpha} \\
&\leq C \sum_{k \in \mathbb{N}^*} \frac{\lambda^2}{k^\alpha} \\
&\leq C\lambda^2
\end{aligned}$$

Concerning the sum over  $i \in \{1, 2, \dots, J_j - 2\}$ , we know that

$$\begin{aligned}
\sum_{1 \leq i \leq J_j - 2} \frac{\lambda^2}{\lambda_i + \lambda - \lambda_j} &= \sum_{1 \leq i \leq J_j - 2} \frac{\lambda^2}{(\lambda_{J_j-1} + \lambda - \lambda_j) + (\lambda_i - \lambda_{J_j-1})} \\
&\leq \sum_{1 \leq i \leq J_j - 2} \frac{\lambda^2}{(\lambda_i - \lambda_{J_j-1}) + D_j(\lambda)} \\
&\leq \sum_{1 \leq i \leq J_j - 2} \frac{\lambda^2}{(\lambda_i - \lambda_{J_j-1})} \\
&\leq C \sum_{1 \leq i \leq J_j - 2} \frac{\lambda^2}{|J_j - 1 - i|^\alpha} \\
&\leq C \sum_{k \in \mathbb{N}^*} \frac{\lambda^2}{k^\alpha} \\
&\leq C\lambda^2
\end{aligned}$$

Collecting the estimates above yield the proof. □

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