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► **To cite this version:**

Ping Lin, Hatem Zaag. Feedback approximate controllability of blowup points for the heat equation with anti-interference blowup profile. 2025. <hal-05332216>

HAL Id: hal-05332216

<https://hal.science/hal-05332216v1>

Preprint submitted on 26 Oct 2025

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Feedback approximate controllability of blowup points for the heat equation with anti-interference blowup profile

Ping Lin* Hatem Zaag[†]

Abstract. This paper is concerned with a feedback approximate controllability problem of blowup points for the heat equation. We show that the system is approximately controllable for blowup points with feedback controls and the feedback operator is bounded at any time before blowup. It is also proved that the blowup profile for feedback controllability of blowup points is stable with respect to initial data. That is, suppose that the initial data has a very small perturbation, the blowup profiles also have tiny changes. More precisely, it just undergoes a tiny translation in space and time. This means that our feedback strategy is anti-interference.

Keywords. heat equation, blowup point, feedback control, approximate controllability, anti-interference blowup profile

Mathematics Subject Classification 2020. 35K20, 93B05, 93B52

1 Introduction

Control issues are almost ubiquitous. In a control problem, there is a controlled object and one (or more) control actions. According to the basic action principles of control, it can be divided into two types: open-loop control and closed-loop control (also known as feedback control). It is called open-loop control to design a series of instructions to control the object according to control requirements and external interference. The characteristic of this scheme is to control based solely on the expected results, regardless of the consequences of the control effect. Another type of method that determines the control effect based on changes in the controlled object is called closed-loop control, or feedback control. Whether it is to suppress the influence of external disturbances or reduce the influence of internal parameter changes, feedback control is far superior to open-loop control. It is worth pointing out that feedback is a special concept in control theory.

It is well known that blowup widely exists in nature and in practical applications. Some nonlinear evolution differential equations can be used to describe this phenomenon (see for

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instance [1], [4]-[6], [16]-[18], [25], [26], [28]-[30], [32]-[38], [40], [46]-[50], [55], [56], [59], [60]). Roughly speaking, blowup is a conception which means that a solution is unbounded in finite time. In certain cases, the blowup of a solution is desired. For instance, the dramatic increase in temperature leads to the ignition of a chemical reaction. However, solutions to linear partial differential equations without control generally globally exist. It is naturally interesting to find feedback controls to these equations such that the corresponding solutions blow up in finite time at some given place.

Let Ω be a bounded domain of \mathbb{R}^n with smooth boundary $\partial\Omega$. Consider the following control system,

$$\begin{cases} y_t - \Delta y = \mathbb{1}_\omega u, & x \in \Omega, t > 0, \\ y = 0, & x \in \partial\Omega, t > 0, \\ y(x, 0) = y_0(x), & x \in \Omega. \end{cases} \quad (1.1)$$

Here the control u acts on a nonempty and open subset $\omega \subset \Omega$, and $\mathbb{1}_\omega$ is the characteristic function of the set ω .

In the absence of control, we know that the solution to system (1.1) globally exists. The following definition refers to blowup time and blowup point of a solution to (1.1) with a feedback control.

Definition 1.1 (Notions of blowup time and point). *Let $T > 0$ and $a \in \Omega$. We say that T is the blowup time and a is a blowup point of the solution y to system (1.1) with a feedback control u , if $y(x, t)$ exists for all $(x, t) \in \Omega \times [0, T)$ with $|y(a_j, t_j)| \rightarrow +\infty$, as $j \rightarrow \infty$, for some sequences $a_j \rightarrow a$ and $t_j \rightarrow T$.*

In [45], Lin and Zaag proposed the following feedback exact controllability problem for blowup points.

Problem (P). *Given $a \in \Omega$, $T > 0$ and initial data $y_0 \in H_0^1(\Omega)$, can we find a feedback control u such that T is the blowup time of the solution y to (1.1), with a being the unique blowup point of y ?*

Lin and Zaag [45] studied Problem (P) and obtained the following result (see Theorem 2 in [45]).

Proposition 1.2. *For any $a \in \omega$ and any $T > 0$, there exist $T_1 \in (0, T/2)$ and $\tilde{y}_0 \in C_0^\infty(\omega)$ such that for any $y_0 \in H_0^1(\Omega)$, the solution y to (1.1) with the following feedback control*

$$u(x, t) := \begin{cases} -\mathbb{1}_\omega^* P(t)(y(t) - \tilde{y}_0)(x) - \Delta \tilde{y}_0(x), & (x, t) \in \Omega \times [0, T - T_1), \\ |y|^{p-1} y(x, t), & (x, t) \in \Omega \times [T - T_1, T), \end{cases}$$

exists on $[0, T)$, T is the blowup time of y and y has a unique blowup point a , where $p > 1$. Here, $P \in C_S([0, T - T_1]; \Sigma^+(H))$ is the unique mild solution to the following Riccati system,

$$\begin{cases} P'(t) + \Delta^* P(t) + P(t)\Delta - P(t)\mathbb{1}_\omega \mathbb{1}_\omega^* P(t) = 0, & t \in [0, T - T_1), \\ \lim_{(s,z) \rightarrow (T-T_1, z_0)} \langle P(s)z, z \rangle = +\infty, & \text{for each } z_0 \in H \text{ and } z_0 \neq 0. \end{cases} \quad (1.2)$$

Moreover, for all $R > 0$,

$$\sup_{\{|x-a| \leq R\sqrt{(T-t)|\log(T-t)|}\}} \left| (T-t)^{\frac{1}{p-1}} y(x,t) - f\left(\frac{x-a}{\sqrt{(T-t)|\log(T-t)|}}\right) \right| \rightarrow 0 \text{ as } t \rightarrow T.$$

Here $H = L^2(\Omega)$, $\Sigma^+(H)$ denotes the Banach space of all symmetric and positive operators acting in H and $C_S([0, T - T_1]; \Sigma^+(H))$ denotes the set of all mappings $S : [0, T - T_1] \rightarrow \Sigma^+(H)$ such that $S(\cdot)z_0$ is continuous on $[0, T - T_1)$ for each $z_0 \in H$. $\mathbb{1}_\omega^*$ is the adjoint operator of $\mathbb{1}_\omega$, Δ^* is the adjoint operator of Δ , and

$$f(\eta) = \left(p - 1 + \frac{(p-1)^2}{4p} |\eta|^2 \right)^{-\frac{1}{p-1}}, \quad \forall \eta \in \mathbb{R}. \quad (1.3)$$

Let $a \in \omega$, $T > 0$ and $y_0 \in H_0^1(\Omega)$. The proof of Proposition 1.2 (see the proof of Theorem 2, [45]) relies on 2 arguments:

A Nonlinear Construction Argument: We constructed a blowup solution to some parabolic equation with a localized nonlinearity. More precisely, we found a special initial data \tilde{y}_0 supported in ω such that the corresponding solution to system (1.1) (where initial time is replaced by $T - T_1$ and initial data is replaced by \tilde{y}_0) with the feedback control $u(x, t) = |y|^{p-1}y(x, t)$, $(x, t) \in \Omega \times [T - T_1, T)$ ($p > 1$), blows up in time T with $T_1 \in (0, T/2)$ at unique blowup point a , with the prescribed blowup profile;

A Linear Control Argument: We connected the given initial state y_0 to the initial data of the Construction Step. That is, we proved that the solution y to (1.1) with the feedback control $u(x, t) = -\mathbb{1}_\omega^* P(t)(y(t) - \tilde{y}_0)(x) - \Delta \tilde{y}_0(x)$, $(x, t) \in \Omega \times [0, T - T_1)$ satisfies $y(T - T_1) = \tilde{y}_0$. Combining these two steps, we could prove Proposition 1.2.

In order to state our problem studied in this paper, we need the following definition.

Definition 1.3 (Notion of feedback approximate controllability of blowup points). *Given $a \in \Omega$, $T > 0$ and $y_0 \in H_0^1(\Omega)$, we say that system (1.1) is approximately controllable for blowup points with feedback controls if for each $\varepsilon > 0$, we can find a feedback control u such that the corresponding solution y to (1.1) blows up in time T^* and has a unique blowup point a^* with $|T^* - T| < \varepsilon$ and $|a^* - a| < \varepsilon$.*

In this paper, we would like to obtain a result on feedback approximate controllability of blowup points for system (1.1) with an anti-interference blowup profile. A natural question to ask is that the feedback exact controllability of blowup points for system (1.1) has been well solved in [45], why the approximate one is to be studied? The following two reasons give the answer to this question.

(i) As we see in Proposition 1.2, the feedback operator $P(\cdot)$ which makes the solution y of system (1.1) to reach \tilde{y}_0 in time $T - T_1$ is the solution to Riccati system (1.2). Since $P(\cdot)$ satisfies the second equality of (1.2), it is indeed an unbounded operator. This is because the feedback null controllability result for linear heat equation obtained in [53] was used to prove this property, and one can not make system (1.1) be feedback null controllable by a bounded linear feedback operator due to the backward uniqueness of the solution to linear heat equation.

From point of view of application, it is better if the feedback operator could be bounded in any time before blowup. Since $P \in C_S([0, T - T_1]; \Sigma^+(H))$, we have that for any $T' \in (0, T - T_1)$, $P \in C_S([0, T']; \Sigma^+(H))$, i.e., $P(\cdot)$ is a bounded operator on $[0, T']$. To achieve this aim, we could first choose a time T' close enough to $T - T_1$ and act the feedback control $u(\cdot, t) = -\mathbb{1}_\omega^* P(t)(y(t) - \tilde{y}_0)(\cdot) - \Delta \tilde{y}_0(\cdot)$ on $[0, T']$. To avoid errors, T' should be allowed to be taken from a sufficiently small interval very close to $T - T_1$. On the other hand, as we mentioned above, the corresponding solution to system (1.1) (where initial time is replaced by $T - T_1$ and initial data is replaced by \tilde{y}_0) with the feedback control $u(x, t) = |y|^{p-1}y(x, t)$, $(x, t) \in \Omega \times [T - T_1, T)$ ($p > 1$), blows up in time T at unique blowup point a . Then, the following problem should be answered: For any $\varepsilon > 0$, when the feedback control $u(\cdot, t) = -\mathbb{1}_\omega^* P(t)(y(t) - \tilde{y}_0)(\cdot) - \Delta \tilde{y}_0(\cdot)$ acts on time $[0, T']$ with $T' \in (0, T - T_1)$ in a sufficiently small interval very close to $T - T_1$, could $y(T')$ be sufficiently approximate to \tilde{y}_0 such that when we further take the feedback control $|y|^{p-1}y$ as $t > T'$, the solution y to (1.1) could blow up in time T^* and has unique blowup point a^* , with $|T^* - T| < \varepsilon$ and $|a^* - a| < \varepsilon$? For this purpose, we naturally have to consider feedback approximate controllability of blowup points for system (1.1).

(ii) It is important to show that whether the blowup profile for feedback controllability of blowup points is stable with respect to initial data. That is, suppose that the initial data has a very small perturbation, will the blowup profiles also have tiny changes? The positive answer to this question would mean that our feedback strategy is anti-interference. This is very important in practical application. To this end, studying the case of approximate controllability is more reasonable.

The following theorem is our main result, which could give a positive answer to the above questions proposed in (i) and (ii). Theorem 1 could be viewed as a result of approximate controllability with anti-interference blowup profile, and the feedback operator stated in it is bounded at any time before blowup.

Theorem 1. *Given $a \in \omega$ and $T > 0$. Let $T_1 \in (0, T/2)$ and $\tilde{y}_0 \in C_0^\infty(\omega)$ be given in Proposition 1.2. Let $y_0 \in H_0^1(\Omega)$. Then for any $\varepsilon > 0$, there exists $\hat{\varepsilon}_1 \in (0, \min(\varepsilon/4, T_1/4))$ such that we can find $\delta \in (0, \min\{\varepsilon/4, T - T_1\})$, $\delta^* > 0$ with $(\delta - \delta^*, \delta + \delta^*) \subset (0, \min\{\varepsilon/4, T - T_1\})$ and $\varepsilon^* > 0$ such that for any $\hat{\varepsilon} \in (\delta - \delta^*, \delta + \delta^*)$ and any initial data $y_0^* \in H_0^1(\Omega)$ satisfying $\|y_0^* - y_0\|_{H_0^1(\Omega)} < \varepsilon^*$, the solution y to (1.1) with initial data y_0^* and with the following feedback control*

$$u(x, t) := \begin{cases} -\mathbb{1}_\omega^* P(t)(y(t) - \tilde{y}_0)(x) - \Delta \tilde{y}_0(x), & (x, t) \in \Omega \times [0, T - T_1 - \hat{\varepsilon}), \\ 0, & (x, t) \in \Omega \times [T - T_1 - \hat{\varepsilon}, T - T_1 - \hat{\varepsilon} + \hat{\varepsilon}_1), \\ |y|^{p-1}y(x, t), & (x, t) \in \Omega \times [T - T_1 - \hat{\varepsilon} + \hat{\varepsilon}_1, T^*) \end{cases}$$

blows up in time T^ and has unique blowup point a^* , with $|T^* - T| < \varepsilon$ and $|a^* - a| < \varepsilon$, where $p > 1$. Here, $P \in C_S([0, T - T_1]; \Sigma^+(H))$ is the unique mild solution to Riccati system (1.2).*

Moreover, for all $R > 0$,

$$\left\{ \sup_{\{|x - a^*| \leq R\sqrt{(T^* - t)|\log(T^* - t)|}\}} \left| (T^* - t)^{\frac{1}{p-1}} y(x, t) - f \left(\frac{x - a^*}{\sqrt{(T^* - t)|\log(T^* - t)|}} \right) \right| \right\} \rightarrow \alpha(1.4)$$

as $t \rightarrow T^*$, where f is defined in (1.3).

From the perspective of the purpose of control, the target of feedback controllability problem for blowup points is “infinity”, which is outside the state space of the solutions, while the targets of classical controllability problems (see for instance [2], [3], [7]-[15], [19]-[24], [27], [39], [42], [43], [51]-[53], [57], [58], [61], [62]) are within the state space. On the other hand, feedback control may be considered as a closed-loop system which plays an effective role of control.

In the past twenty years, several references studied the controllability of equations in the context of blowup. Doubova, Fernández-Cara, González-Burgos and Zuazua [15], together with Fernández-Cara and Zuazua [24] considered the controllability of weakly blowing up semilinear parabolic equations with open-loop controls. They showed that the considered systems are null and approximately controllable at any time. One can also see the recent works by Bárcena-Petisco [2], Kassab [39] and Le Balc’h [43]. In those papers, blowup occurs if no control is applied. Meanwhile, using appropriate controls, the solution can be steered to zero or as near as possible to some given target in the state space at any given time. In other words, the aim of those references was to prevent blowup thanks to controls, which is the opposite to our purpose of feedback controllability problem for blowup points, where we aim at making the solution blow up.

Recently, several references concerned the study of feedback blowup controllability. In addition to [45], Lin [44] considered the blowup controllability of the heat equation with feedback controls. As a matter of fact, Lin proved that for any initial data in $H_0^1(\Omega)$ and for any time $T > 0$, there exist a number p with $p \in (1, \infty)$ and a feedback control acting on an internal subset of the space domain such that the L^{p+1} norm of the solution to (1.1) blows up in T . Later, Han, Liu and Lin [31] studied the ordinary differential system $y'(t) = Ay(t) + Bu(t)$ in the case where (A, B) is null controllable, A and B being time independent matrices. They first concluded that this ode system is exactly blowup controllable with feedback control. Furthermore, they showed that it is also approximately blowup controllable with the feedback operator bounded at any time before blowup.

In some particular situations, one may need a sharper target for blowup. This is the case for instance in the mining process, where one may need blowup to occur at a specific space location in some given time. The problem of blowup point controllability considered in this paper and [45] could meet this need.

The idea of the proof of our main result in Theorem 1 is as follows.

Consider the following auxiliary system,

$$\begin{cases} y_t - \Delta y = \mathbb{1}_\omega |y|^{p-1} y, & x \in \Omega, t > 0, \\ y = 0, & x \in \partial\Omega, t > 0, \\ y(x, 0) = y_0(x), & x \in \Omega, \end{cases} \quad (1.5)$$

where $p > 1$ is arbitrary but fixed.

First, we will remake the proof of our result on constructing a blowup solution for system (1.5) with a prescribed profile, i.e., Theorem 1 in our paper [45], which is the following proposition. This plays an important role in establishing our main result.

Proposition 1.4. *For any $a \in \omega$, there exists $T_0 > 0$ such that for any $T \in (0, T_0)$, there exists some initial data $y_0 \in C_0^\infty(\omega)$ such that the corresponding solution y to (1.5) exists on $[0, T)$, T is the blowup time of y and y has a unique blowup point a . Moreover, for all $R > 0$,*

$$\sup_{\{|x-a| \leq R\sqrt{(T-t)|\log(T-t)|}\}} \left| (T-t)^{\frac{1}{p-1}} y(x, t) - f \left(\frac{x-a}{\sqrt{(T-t)|\log(T-t)|}} \right) \right| \rightarrow 0 \quad (1.6)$$

as $t \rightarrow T$, where f is defined in (1.3).

Just as we did in the proof of Theorem 1 in [45], thanks to some cut-off function around $a \in \omega$, we recover the \mathbb{R}^n case, however, with cut-off terms:

$$y_t - \Delta y = |y|^{p-1} y + \text{cut-off terms, with } (x, t) \in \mathbb{R}^n \times [0, T).$$

Then, we will use similar techniques to Mahmoudi, Nouaili and Zaag [46] to prove that for any $a \in \omega$, there exists $T_0 > 0$ such that for any $T \in (0, T_0)$, one can find special initial data such that the corresponding solution blows up in time T at unique blowup point a , and with the prescribed blowup profile (1.6). The control of the solution near that profile will be different, according to whether we are in the “blowup region” or the “regular region”:

- In the blowup region, near the blowup point, we reduce the question to a finite-dimensional problem. Similarity variables will be used to control the solution near the profile.
- In the regular region, far from the blowup point, we directly use standard parabolic estimates.

Note that we proceed by contradiction to solve the finite-dimensional problem and complete the proof of that step thanks to a topological argument.

However, in subsection 2.1, we will reformulate the problem and give a modified definition of shrinking set and some preliminary lemmas, which are small adaptations of those used in the proof of Theorem 1 in [45]. The order in which they appear in the proof will also change. The adaptations and the change of order would make these lemmas not only apply to construct a blowup solution corresponding to special initial data we chose in [45], but also to initial data in a small neighborhood of it. This will play an important role in proving our main result of this paper. By these lemmas, we will follow the same techniques used in [45] to prove Proposition 1.4 (see subsection 2.2).

Then, we proceed prove our main result in the following two steps.

In the first step, we will prove a stability result of the blowup profile for system (1.5).

The following theorem gives a stability result of the blowup profile with respect to initial data for system (1.5), which will be used to prove Theorem 1.

Theorem 2. *Let \hat{y} be the constructed solution in Proposition 1.4. \hat{y}_0 is its initial data in $C_0^\infty(\omega)$, \hat{T} is its blowup time and $\hat{a} \in \omega$ is its unique blowup point. Then for any $\varepsilon > 0$, there exists $\varepsilon_1 > 0$, such that for any initial data $y_0 \in W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)$ with $q > n+2$ satisfying $\|y_0 - \hat{y}_0\|_{W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)} < \varepsilon_1$, the corresponding solution y to system (1.5) blows up in finite time $T(y_0)$ and at a unique blowup point $a(y_0)$ with*

$$|T(y_0) - \hat{T}| < \varepsilon, \quad |a(y_0) - \hat{a}| < \varepsilon.$$

Moreover, for all $R > 0$,

$$\left\{ \sup_{\{|x-a(y_0)| \leq R \sqrt{|(T(y_0)-t)| \log(T(y_0)-t)}\}} \left| (T(y_0) - t)^{\frac{1}{p-1}} y(x, t) - f \left(\frac{x-a(y_0)}{\sqrt{|(T(y_0)-t)| \log(T(y_0)-t)}} \right) \right| \rightarrow 0, \quad (1.7)$$

as $t \rightarrow T(y_0)$, where f is defined in (1.3).

Theorem 2 shows the stability of the constructed solution in Proposition 1.4, with respect to perturbations in initial data. By the similar techniques used in [45] and [55], it could be proved by using geometrical interpretation of the finite-dimensional parameters, reduction to a finite-dimensional problem and continuity (see details in Section 3).

In the second step, let $a \in \omega$ and $T > 0$. Let $T_1 \in (0, T/2)$ and $\tilde{y}_0 \in C_0^\infty(\omega)$ be given in Proposition 1.2. Let $y_0 \in H_0^1(\Omega)$. Recall that in Linear Control Argument, we connect the given initial state y_0 to initial data \tilde{y}_0 of the Construction Step in nonlinear argument at time $T - T_1$ by the feedback control $u(\cdot, t) = -\mathbf{1}_\omega^* P(t)(y(t) - \tilde{y}_0)(\cdot) - \Delta \tilde{y}_0(\cdot)$, $t \in [0, T - T_1]$. However, by the second equality of (1.2), $P(\cdot)$ is indeed an unbounded operator.

Since $P \in C_S([0, T - T_1]; \Sigma^+(H))$, we could show that for initial data $y_0^* \in H_0^1(\Omega)$ with $\|y_0^* - y_0\|_{H_0^1(\Omega)}$ small enough, the corresponding solution y with the feedback control $u(\cdot, t) = -\mathbf{1}_\omega^* P(t)(y(t) - \tilde{y}_0)(\cdot) - \Delta \tilde{y}_0(\cdot)$, $t \in [0, T']$ satisfies $y(T')$ could be close enough to \tilde{y}_0 in $H_0^1(\Omega)$, where $T' \in (0, T - T_1)$ could be any time in a sufficiently small interval very close to $T - T_1$.

Furthermore, we could take 0 control to enhance the regularity of the solution y in the interval of $(T', T' + \hat{\varepsilon}_1]$, where we take $\hat{\varepsilon}_1$ small enough to ensure that $y(T' + \hat{\varepsilon}_1)$ could be close enough to \tilde{y}_0 in $W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)$ with $q > n + 2$.

Finally, by Theorem 2, we could complete the proof of Theorem 1 (see details in Section 4).

The rest of this paper is structured as follows. In Section 2, we will reconstruct a blowup solution with a prescribed profile. Section 3 will prove a stability result of the blowup profile with respect to initial data for system (1.5), i.e., Theorem 2. In section 4, we will give the proof of our main result.

2 Reconstructing a blowup solution with a prescribed profile

In this section, we will remake the proof of our result on constructing a blowup solution with a prescribed profile, i.e., Theorem 1 in our paper [45], which is Proposition 1.4 in this paper. This plays an important role in establishing our main result. Some arguments will repeat those written in [45] for the convenience of the readers and for the completeness of this paper.

Throughout this paper, we will give the proof when $n = 1$, for simplicity, bearing in mind that the proof for $n \geq 2$ is the same, with natural adaptations, as one can see from the paper [49] done for the standard heat equation.

2.1 Formulation of the problem and preliminary lemmas

Consider some $a \in \omega$ and $T > 0$. The aim of Proposition 1.4 is to show that for T small enough, there is some $y_0 \in C_0^\infty(\omega)$ such that equation (1.5) fitted with initial data y_0 has a solution $y(x, t)$ defined for all $(x, t) \in \Omega \times [0, T)$ such that y blows up at time T only at the point a , with the behavior described in (1.6).

Let $\chi_0 \in C_0^\infty(\mathbb{R}, [0, 1])$ with

$$\chi_0(\xi) = \begin{cases} 1, & |\xi| \leq 1, \\ 0, & |\xi| \geq 2. \end{cases} \quad (2.1)$$

In the regular region, we define \bar{y} by

$$\bar{y}(x, t) = y(x, t)\bar{\chi}(x), \quad x \in \Omega, \quad t \geq 0, \quad (2.2)$$

where for any $\xi \in \Omega$, $\bar{\chi}(\xi) = 1 - \chi_0\left(\frac{4(\xi-a)}{\varepsilon_0}\right)$, for some $\varepsilon_0 > 0$ will be fixed sufficiently small later. Then, \bar{y} satisfies the following equation:

$$\partial_t \bar{y} = \partial_{xx} \bar{y} + \mathbb{1}_\omega |y|^{p-1} \bar{y} - 2\bar{\chi}' \partial_x y - \bar{\chi}'' y. \quad (2.3)$$

In the blowup region, we make the following similarity variables transformation:

$$W(z, s) = (T - t)^{1/(p-1)} y(x, t) = e^{-s/(p-1)} y(a + e^{-s/2} z, T - e^{-s}) \quad (2.4)$$

with

$$x - a = (T - t)^{1/2} z, \quad T - t = e^{-s}. \quad (2.5)$$

This transformation was first introduced by Hocking and Stewartson [41] in the context of the Complex Ginzburg-Landau (CGL), which includes the semilinear heat equation as a special case. Later, it was used by Giga and Kohn in [28] to derive the well celebrated blowup results for the latter equation. Following (2.4), we see that $W(z, s)$ satisfies the following equation in $(\Omega - a)e^{s/2} \times [s_0, \infty)$, with $s_0 = -\log T$:

$$\partial_s W = \partial_z^2 W - \frac{1}{2} z \partial_z W - \frac{1}{p-1} W + |W|^{p-1} W + (\mathbb{1}_\omega - 1) |W|^{p-1} W. \quad (2.6)$$

For all $z \in \mathbb{R}$, we define

$$w(z, s) = \begin{cases} W(z, s)\chi(z, s), & z \in (\Omega - a)e^{s/2}, \\ 0, & z \in \mathbb{R} \setminus (\Omega - a)e^{s/2}, \end{cases} \quad (2.7)$$

with

$$\chi(z, s) = \chi_0\left(\frac{ze^{-s/2}}{\varepsilon_0}\right), \quad (2.8)$$

where

$$(a - 4\varepsilon_0, a + 4\varepsilon_0) \subset \omega, \quad (2.9)$$

and $\varepsilon_0 > 0$ will be fixed small enough later.

We see that whenever initial data $y_0 \in C_0^\infty(\omega)$ and y exists on $[0, T)$, it follows by the internal regularity of the heat equation that w is in the space $C^{2,1}(\mathbb{R} \times [s_0, +\infty))$ with $s_0 = -\log T$.

It can be easily obtained that

$$\partial_s w = \partial_z^2 w - \frac{1}{2}z\partial_z w - \frac{1}{p-1}w + |w|^{p-1}w + N(z, s), \quad z \in \mathbb{R}, \quad s \geq -\log T, \quad (2.10)$$

where

$$N(z, s) = \begin{cases} W\partial_s \chi - 2\partial_z W\partial_z \chi - W\partial_z^2 \chi + \frac{1}{2}zW\partial_z \chi \\ + |W|^{p-1}W(\chi - \chi^p), & z \in (\Omega - a)e^{s/2}, \quad s \geq -\log T, \\ 0, & z \in \mathbb{R} \setminus (\Omega - a)e^{s/2}, \quad s \geq -\log T. \end{cases} \quad (2.11)$$

Let

$$w = \varphi + q, \quad (2.12)$$

where

$$\varphi = f\left(\frac{z}{\sqrt{s}}\right) + \frac{\kappa}{2ps} \quad (2.13)$$

with $\kappa = (p-1)^{-\frac{1}{p-1}}$ and f is defined in (1.3). Then, q is a solution to the following equation in $\mathbb{R} \times [s_0, \infty)$,

$$\partial_s q = (\mathcal{L} + V)q + B(z, s) + R(z, s) + N(z, s), \quad (2.14)$$

where

$$\mathcal{L} = \partial_z^2 - \frac{1}{2}z\partial_z + 1, \quad V = p\varphi^{p-1} - \frac{p}{p-1}, \quad (2.15)$$

$$B(z, s) = |(\varphi + q)|^{p-1}(\varphi + q) - \varphi^p - p\varphi^{p-1}q, \quad (2.16)$$

$$R(z, s) = \partial_z^2 \varphi - \frac{1}{2}z\partial_z \varphi - \frac{1}{p-1}\varphi + \varphi^p - \partial_s \varphi, \quad (2.17)$$

$$N(z, s) = H + \partial_z G(z, s), \quad (2.18)$$

$$H(z, s) = W(\partial_s \chi + \partial_z^2 \chi + \frac{1}{2}z\partial_z \chi) + |W|^{p-1}W(\chi - \chi^p), \quad (2.19)$$

and

$$G(z, s) = -2\partial_z \chi W, \quad \partial_z G(z, s) = -2\partial_z^2 \chi W - 2\partial_z \chi \partial_z W. \quad (2.20)$$

We give a decomposition of the solution q to equation (2.14) according to the spectrum of \mathcal{L} . The operator \mathcal{L} is self-adjoint on $\mathcal{D}(\mathcal{L}) \subset L_\rho^2(\mathbb{R})$ with

$$\rho(z) = \frac{e^{-\frac{z^2}{4}}}{\sqrt{4\pi}}, \quad (2.21)$$

and

$$L^2_\rho(\mathbb{R}) = \left\{ \vartheta \in L^2_{loc}(\mathbb{R}); \|\vartheta\|_{L^2_\rho}^2 = \int |\vartheta|^2 \rho(z) dz < +\infty \right\}.$$

The spectrum of \mathcal{L} is explicitly given by

$$\text{spec}(\mathcal{L}) = \left\{ 1 - \frac{m}{2}; m \in \mathbb{N} \right\}.$$

All the eigenvalues are simple. For $1 - \frac{m}{2}$ corresponds the eigenfunction

$$h_m(z) = \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} \frac{m!}{n!(m-2n)!} (-1)^n z^{m-2n}. \quad (2.22)$$

The eigenvalues h_m satisfy the following orthogonality condition

$$\int h_n h_m \rho dz = 2^n n! \delta_{n,m}.$$

In the following, we will often use the notation

$$k_m = h_m / \|h_m\|_{L^2_\rho}^2. \quad (2.23)$$

Now, let us first introduce

$$\chi_1(z, s) = \chi_0\left(\frac{|z|}{K_0 \sqrt{s}}\right), \quad (2.24)$$

where χ_0 is defined in (2.1), $K_0 \geq 1$ will be chosen large enough. We then write

$$q = q_e + q_b, \quad (2.25)$$

where

$$q_b = q\chi_1, \quad q_e = q(1 - \chi_1). \quad (2.26)$$

From (2.1) and (2.24), we see that

$$\text{supp } q_b(s) \subset B(0, 2K_0 \sqrt{s}), \quad \text{supp } q_e(s) \subset \mathbb{R} \setminus B(0, K_0 \sqrt{s}). \quad (2.27)$$

Second, we decompose q_b as follows,

$$q_b(z, s) = \sum_{m=0}^2 q_m(s) h_m(z) + q_-(z, s), \quad (2.28)$$

where q_m is the projection of q_b on h_m , $q_-(z, s) = P_-(q_b)$, and P_- is the projection on $\{h_i; i \geq 3\}$, the negative subspace of the operator \mathcal{L} .

In order to reduce the infinite dimensional problem proposed in Proposition 1.4 into a dimensional one, we need the following definition as in [45], which is a small adaptation of the original one in [45] (see Definition 2.1 in [45]).

Definition 2.1 (Shrinking set). For all $K_0 \geq 1$, $\varepsilon_0 > 0$, $A > 0$, $\mu \in (0, 1)$, $\eta_0 \in (0, 1]$, $T > 0$ and $t \in [0, T)$, we say a function y defined in $\Omega \times [0, t]$ belongs to the set $S^*(K_0, \varepsilon_0, A, \mu, \eta_0, T, t)$ at time t , if

(i) $q(s) \in V_{K_0, A}(s)$, where $s = -\log(T - t)$, $z = e^{s/2}(x - a)$, $q(s)$ is defined in (2.4), (2.5), (2.7), (2.12) and (2.13), and $V_{K_0, A}(s)$ is the set of all functions $r \in L^\infty(\mathbb{R})$ such that

$$\begin{cases} |r_m(s)| \leq As^{-2} (m = 0, 1), & |r_2(s)| \leq A^2 s^{-2} \log s, \\ |r_-(z, s)| \leq As^{-2}(1 + |z|^3), & |r_e(z, s)| \leq A^2 s^{-1/2}, \end{cases} \quad (2.29)$$

where

$$\begin{cases} r_e(z, s) = (1 - \chi_1(z, s))r(z, s), & r_-(z, s) = P_-(\chi_1 r), \\ \text{for } m \in \mathbb{N}, & r_m(s) = \int k_m(z) \chi_1(z, s) r(z, s) \rho(z) dz. \end{cases} \quad (2.30)$$

(ii) For all $x \in \Omega$ with $|x - a| \geq \mu\varepsilon_0$, $|y(x, t)| \leq \eta_0$.

First, let us give an estimate which will be used in many places in this paper.

Since $a \in \omega$, there exists $\delta_0 \in (0, 1)$ such that for any $\varepsilon_0 \in (0, \delta_0]$, $(a - 4\varepsilon_0, a + 4\varepsilon_0) \subset \omega$. Then, for any $K_0 \geq 1$ and any $\varepsilon_0 \in (0, \delta_0]$, there exists $s_0^1(K_0, \varepsilon_0) > 0$ such that when $s_0 \geq s_0^1$, $K_0 \sqrt{s_0} < \varepsilon_0 e^{s_0/2}$.

Next, we can derive the following bounds when $y(t) \in S^*(K_0, \varepsilon_0, A, \mu, \eta_0, T, t)$ (see Lemma 2.2 in [45]). Throughout the paper, C is a constant number, which may be different according to the context.

Lemma 2.2. For all $K_0 \geq 1$, $\varepsilon_0 > 0$ and $A \geq 1$, there exists $s_0^2(K_0, \varepsilon_0, A) > 0$ such that if $s \geq s_0 \geq s_0^2$, $\mu \in (0, 1)$, $\eta_0 \in (0, 1]$, $t \in [0, T)$ and $y(t) \in S^*(K_0, \varepsilon_0, A, \mu, \eta_0, T, t)$, where $t = T - e^{-s}$, then

- (i) $\forall z \in \mathbb{R}$, $|q(z, s)| \leq CA^2 \frac{\log s}{s^2} (1 + |z|^3)$,
- (ii) $\|q(s)\|_{L^\infty} \leq CA^2 s^{-1/2}$,
- (iii) $\|W(s)\|_{L^\infty} \leq (p - 1)^{-1/(p-1)} + 2$.

By Definition 2.1, there are two types of estimates: item (i) and item (ii). In the following, we will give two different priori estimates, following two estimates. For item (i), the following lemma plays an important role in reducing the infinite dimensional problem into a finite one in the blowup region, which is a small adaptation of Proposition 4.4 in [45].

Lemma 2.3. There exists $K_0^1 \geq 1$ such that for any $K_0 \geq K_0^1$, any $\varepsilon_0 \in (0, \delta_0]$ and any $\tilde{A} > 0$, there exists $\bar{A}(\tilde{A}, K_0, \varepsilon_0) \geq 1$ such that for all $A \geq \bar{A}(\tilde{A}, K_0, \varepsilon_0)$, there exists $\bar{s}_0(\tilde{A}, A, K_0, \varepsilon_0) > 0$ such that for any $s_0 \geq \bar{s}_0(\tilde{A}, A, K_0, \varepsilon_0)$, any $\mu \in (0, 1)$ and any $\eta_0 \in (0, 1]$, for any solution of (2.14), we have the following property: if

$$\begin{cases} |q_m(s_0)| \leq As_0^{-2}, & m = 0, 1, \\ |q_2(s_0)| \leq \tilde{A}s_0^{-2} \log s_0, \\ |q_-(z, s_0)| \leq \tilde{A}(1 + |z|^3)s_0^{-2}, \\ \|q_e(s_0)\|_{L^\infty} \leq \tilde{A}s_0^{-1/2}, \end{cases} \quad (2.31)$$

if for some $s_1 \geq s_0$, we have $y(t) \in S^*(K_0, \varepsilon_0, A, \mu, \eta_0, T, t)$, whenever $t = T - e^{-s}$ with $s \in [s_0, s_1]$, then, for all $s \in [s_0, s_1]$,

$$\begin{cases} |q_2(s)| \leq A^2 s^{-2} \log s - s^{-3}, \\ |q_-(z, s)| \leq \frac{A}{2}(1 + |z|^3)s^{-2}, \\ \|q_e(s)\|_{L^\infty} \leq \frac{A^2}{2\sqrt{s}}. \end{cases} \quad (2.32)$$

For item (ii), given a small x , define $t = t_0(x)$ by

$$|x - a| = K_0 \sqrt{(T - t_0(x)) |\log(T - t_0(x))|} \quad (2.33)$$

to see that the solution is in fact ‘‘flat’’ at that time. Then, we see that the solution of (1.5) remains ‘‘flat’’ for later time. By ‘‘flat’’, we mean an almost space independent function. More precisely, we claim the following lemma, which is a small adaptation of Lemma 4.6 in [45].

Lemma 2.4. *Let $\tilde{t} \in (0, T)$. There exist $0 < \varsigma_0 < 1$ and $K_0^2 \geq 1$ such that for any $K_0 \geq K_0^2$, there exists $\widehat{\delta} \in (0, \delta_0)$ such that for any $\varepsilon_0 \in (0, \widehat{\delta}]$ and $A \geq 1$, there exists $s_0^3(K_0, \varepsilon_0, A)$ such that if $s_0 \geq s_0^3$, $\mu \in (0, 1)$ and $\eta_0 \in (0, 1]$, then for $x_0 \in \{z \in \mathbb{R}; 0 < |z - a| < \varepsilon_0\}$, $x = x_0 + \xi \sqrt{T - t_0(x_0)}$ with $|\xi| \leq |\log(T - t_0(x_0))|^{1/4}$, and for $t = t_0(x_0) + \tau(T - t_0(x_0))$, $t \in [0, \tilde{t}]$, if $y(t) \in S^*(K_0, \varepsilon_0, A, \mu, \eta_0, T, t)$, then it holds that the solution y with initial data $y_0 \in W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)$ ($q > n + 2$) to system (1.5) satisfies*

$$\forall t_0(x_0) \leq t \leq \tilde{t}, \quad \left| \frac{y(x, t)}{y^*(x_0)} - \frac{U_{K_0}(\tau)}{U_{K_0}(1)} \right| \leq \frac{C}{|\log|x_0 - a||^{\varsigma_0}},$$

where

$$y^*(x_0) = \left[\frac{(p-1)^2 |x_0 - a|^2}{8p |\log|x_0 - a||} \right]^{-\frac{1}{p-1}}, \quad (2.34)$$

and

$$U_{K_0}(\tau) = \left((p-1)(1-\tau) + \frac{(p-1)^2}{4p} K_0^2 \right)^{-\frac{1}{p-1}}. \quad (2.35)$$

Moreover, $|y(x_0, t)| \leq C_1(K_0)|y^*(x_0)|$ for all $t \in [0, \tilde{t}]$, where $C_1(K_0)$ is a constant depending on K_0 .

In order to give a priori estimate following estimate (ii) in Definition 2.1, we need the following lemma, which is similar to Proposition 4.7 in [45].

Lemma 2.5. *For any $\epsilon \in (0, 1]$, any $\varepsilon_0 \in (0, \widehat{\delta}]$ and any $\sigma_1 \geq 0$, there exist a positive constant $C_2 > 1$ independent of ϵ, ε_0 and σ_1 , $\Theta(\epsilon, \sigma_1) = \frac{\epsilon}{4e^{(\sigma_1+1)^{p-1}}}$ and $T_2(\epsilon, \varepsilon_0, \sigma_1) = \min \left\{ 1, \left(\frac{\epsilon}{4C_2 e^{(\sigma_1+1)^{p-1}} (\sigma_1+1) \left(\frac{1}{\varepsilon_0} + \frac{1}{\varepsilon_0^2} \right)} \right)^2 \right\}$ such that for any $(\mu_1, \mu_2) \in \left\{ \left(\frac{1}{4}, \frac{1}{2} \right), \left(\frac{3}{4}, \frac{7}{8} \right) \right\}$, for any $\bar{\eta} \in [0, \Theta]$ and any $\bar{t} \in [0, T_2]$, if y is the solution to system (1.5) in the time interval $[0, \bar{t}]$ with initial data $y_0 \in W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)$ ($q > n + 2$) which satisfies*

$$(i) \text{ for } x \in \Omega \text{ with } |x - a| \in [\mu_1 \varepsilon_0, \mu_2 \varepsilon_0] \text{ and } t \in [0, \bar{t}], \quad |y(x, t)| \leq \sigma_1; \quad (2.36)$$

$$(ii) \text{ for } x \in \Omega \text{ with } |x - a| \geq \mu_1 \varepsilon_0, \quad |y(x, 0)| \leq \bar{\eta}. \quad (2.37)$$

Then, for any $t \in [0, \bar{t}]$ and any $x \in \Omega$ with $|x - a| \geq \mu_2 \varepsilon_0$, it holds that

$$|y(x, t)| < \epsilon.$$

Proof. Let $\epsilon \in (0, 1]$, $\varepsilon_0 \in (0, \widehat{\delta}]$, $\sigma_1 \geq 0$ and $(\mu_1, \mu_2) \in \{(\frac{1}{4}, \frac{1}{2}), (\frac{3}{4}, \frac{7}{8})\}$. Define \mathcal{Y} by

$$\mathcal{Y}(x, t) = y(x, t)\bar{\chi}(x), \quad x \in \Omega. \quad (2.38)$$

Here $\bar{\chi}(x) = 1 - \tilde{\chi}_0(\frac{x-a}{\varepsilon_0})$, where $\tilde{\chi}_0 \in C_0^\infty(\mathbb{R}, [0, 1])$ with

$$\tilde{\chi}_0(\xi) = \begin{cases} 1, & |\xi| \leq \mu_1, \\ 0, & |\xi| \geq \mu_2. \end{cases} \quad (2.39)$$

Then, by equation (1.5), it holds that

$$\partial_t \mathcal{Y} = \partial_{xx} \mathcal{Y} + \mathbf{1}_\omega |y|^{p-1} \mathcal{Y} - 2\partial_x(\tilde{\chi}' y) + \tilde{\chi}'' y.$$

Therefore, by (ii), we can write

$$\begin{aligned} \|\mathcal{Y}(t)\|_{L^\infty(\Omega)} &\leq \left\| S(t)y(\cdot, 0)\tilde{\chi} + \int_0^t S(t-t') [\mathbf{1}_\omega |y|^{p-1} I_{|x-a| \geq \mu_1 \varepsilon_0} \mathcal{Y} \right. \\ &\quad \left. - 2\partial_x(\tilde{\chi}' I_{|x-a| \geq \mu_1 \varepsilon_0} y) + \tilde{\chi}'' I_{|x-a| \geq \mu_1 \varepsilon_0} y] dt' \right\|_{L^\infty(\Omega)}, \end{aligned} \quad (2.40)$$

where $S(t)$ is the heat kernel.

Let $\Theta(\epsilon, \sigma_1) = \frac{\epsilon}{4e^{(\sigma_1+1)^{p-1}}}$ and $\bar{\eta} \in (0, \Theta]$. We proceed by contradiction, and assume that there exists $\bar{t} \in (0, T_2]$ where $T_2(\epsilon, \varepsilon_0, \sigma_1) = \min \left\{ 1, \left(\frac{\epsilon}{4C_2 e^{(\sigma_1+1)^{p-1}} (\sigma_1+1) (\frac{1}{\varepsilon_0} + \frac{1}{\varepsilon_0^2})} \right)^2 \right\}$, such that the conclusion does not hold for all $t \in [0, \bar{t}]$, where C_2 will be defined later.

Using (ii) and the continuity of y , this means that there is $\widehat{t} \in (0, \bar{t}]$ such that the conclusion holds for all $t \in [0, \widehat{t})$ and fails at $t = \widehat{t}$. This implies that

$$\|\mathcal{Y}(\widehat{t})\|_{L^\infty(\mu_2 \varepsilon_0 \leq |x-a|, x \in \Omega)} = \epsilon. \quad (2.41)$$

Therefore, since $\tilde{\chi}'$ and $\tilde{\chi}''$ are supported by $\{x \in \Omega; \mu_1 \varepsilon_0 \leq |x - a| \leq \mu_2 \varepsilon_0\}$ and satisfy $|\tilde{\chi}'| \leq C/\varepsilon_0$, $|\tilde{\chi}''| \leq C/\varepsilon_0^2$, it holds by (i), (ii) and (2.40) that for all $t \in [0, \widehat{t}]$, we have

$$\begin{aligned} &\|\mathcal{Y}(t)\|_{L^\infty(\Omega)} \\ &\leq \bar{\eta} + (\sigma_1 + 1)^{p-1} \int_0^t \|\mathcal{Y}(t')\|_{L^\infty(\Omega)} dt' + \frac{C(\sigma_1 + 1)}{\varepsilon_0} \int_0^t \frac{dt'}{\sqrt{t-t'}} + \frac{C(\sigma_1 + 1)}{\varepsilon_0^2} \int_0^t dt' \\ &\leq \bar{\eta} + (\sigma_1 + 1)^{p-1} \int_0^t \|\mathcal{Y}(t')\|_{L^\infty(\Omega)} dt' + \frac{C(\sigma_1 + 1)}{\varepsilon_0} \sqrt{\widehat{t}} + \frac{C(\sigma_1 + 1)}{\varepsilon_0^2} \widehat{t}. \end{aligned}$$

In the above estimate, when it comes to bounding $|y| I_{\{|x-a| \geq \mu_1 \varepsilon_0\}}$, we will bound it by σ_1 if $|x - a| \leq \mu_2 \varepsilon_0$, and by $\epsilon (\leq 1)$ if $|x - a| \geq \mu_2 \varepsilon_0$. Then, by Gronwall's estimate, we have for all $t \in [0, \widehat{t}]$:

$$\begin{aligned} \|\mathcal{Y}(t)\|_{L^\infty(\Omega)} &\leq e^{(\sigma_1+1)^{p-1} t} \left(\bar{\eta} + \frac{C(\sigma_1 + 1)}{\varepsilon_0} \sqrt{\widehat{t}} + \frac{C(\sigma_1 + 1)}{\varepsilon_0^2} \widehat{t} \right) \\ &\leq e^{(\sigma_1+1)^{p-1} \widehat{t}} \bar{\eta} + C_2 e^{(\sigma_1+1)^{p-1} \widehat{t}} (\sigma_1 + 1) \sqrt{\widehat{t}} \left(\frac{1}{\varepsilon_0} + \frac{1}{\varepsilon_0^2} \right). \end{aligned}$$

Hence, if $\Theta(\epsilon, \sigma_1) = \frac{\epsilon}{4e^{(\sigma_1+1)^{p-1}}}$, $T_2(\epsilon, \epsilon_0, \sigma_1) = \min \left\{ 1, \left(\frac{\epsilon}{4C_2 e^{(\sigma_1+1)^{p-1}} (\sigma_1+1) \left(\frac{1}{\epsilon_0} + \frac{1}{\epsilon_0^2} \right)} \right)^2 \right\}$, $\bar{\eta} \in (0, \Theta)$ and $\hat{t} \leq \bar{t} \leq T_2$, we have $\|\bar{y}(t)\|_{L^\infty(\Omega)} \leq \epsilon/2$, for all $t \in [0, \hat{t}]$.

This contradicts (2.41) and completes the proof of Lemma 2.5. \square

The following transverse crossing lemma also plays an important role to obtain our main result (see Lemma 4.8 in [45]).

Lemma 2.6. *There exist $K_0^3 \geq 1$ and $A_1 \geq 1$ such that for any $K_0 \geq K_0^3$, $A \geq A_1$ and $\epsilon_0 \in (0, \delta]$, there exists $s_0^4(K_0, \epsilon_0, A)$ such that for any $s_0 \geq s_0^4$, $\mu \in (0, 1)$ and $\eta_0 \in (0, 1]$, we have the following properties: Assume there exists $\tilde{s}_* \geq s_0$ such that $y(\tilde{t}_*) \in S^*(K_0, \epsilon_0, A, \mu, \eta_0, T, \tilde{t}_*)$ with $\tilde{t}_* = T - e^{-\tilde{s}_*}$ and $(q_0, q_1)(\tilde{s}_*) \in \partial[-\frac{A}{(\tilde{s}_*)^2}, \frac{A}{(\tilde{s}_*)^2}]^2$, then there exists $\delta_1 > 0$ such that for all $\bar{\delta} \in (0, \delta_1)$, $(q_0, q_1)(\tilde{s}_* + \bar{\delta}) \notin [-\frac{A}{(\tilde{s}_* + \bar{\delta})^2}, \frac{A}{(\tilde{s}_* + \bar{\delta})^2}]^2$.*

2.2 Proof of Proposition 1.4

In this subsection, we will follow the almost same techniques with a small modification in the proof of Theorem 1 in [45] to reprove Proposition 1.4 by the lemmas stated in subsection 2.1. The idea stated in this subsection will also be used to prove the stability of blowup profile for auxiliary system (1.5) in the next section.

The proof of Proposition 1.4 can be divided into the following three steps.

Step 1. Preparation of initial data

Given $a \in \omega$, we consider initial data for system (1.5) defined for all $x \in \Omega$ by

$$\begin{aligned} y_0(x, d_0, d_1) &= T^{-\frac{1}{p-1}} \left\{ \varphi(z, s_0) \chi(8z, s_0) + \frac{A}{s_0^2} (d_0 + d_1 z) \chi_1(2z, s_0) \right\} \\ &= T^{-\frac{1}{p-1}} \left\{ \varphi\left(\frac{x-a}{\sqrt{T}}, -\log T\right) \chi_0\left(\frac{8(x-a)}{\epsilon_0}\right) \right. \\ &\quad \left. + \frac{A}{s_0^2} \left(d_0 + \frac{d_1(x-a)}{\sqrt{T}}\right) \chi_0\left(\frac{2|x-a|}{K_0 \sqrt{-T \log T}}\right) \right\}, \end{aligned} \quad (2.42)$$

where $T > 0$ will be sufficiently small, $s_0 = -\log T$, $z = \frac{x-a}{\sqrt{T}}$, χ_0 is defined in (2.1), χ is defined in (2.8) and χ_1 is defined in (2.24).

We have the following lemma, which is Lemma 3.1 in [45].

Lemma 2.7. *(Reduction for initial data) For any $K_0 \geq 1$, for any $\epsilon_0 \in (0, \delta_0]$ and for any $A \geq 1$, there exists $s_0^5(K_0, \epsilon_0, A) > 0$ such that for all $s_0 \geq s_0^5$, the following holds:*

If initial data for equation (1.5) are given by (2.42), then there exists a rectangle

$$D_T \subset [-2, 2]^2, \quad (2.43)$$

such that

(i) for all $(d_0, d_1) \in D_T$, we have $y_0 \in C_0^\infty(\omega)$ and

$$y_0(\cdot, d_0, d_1) \in S^*(K_0, \epsilon_0, A, 1/4, 0, T, 0),$$

and

$$\begin{cases} |q_2(s_0)| \leq s_0^{-2} \log s_0, \\ |q_-(z, s_0)| \leq (1 + |z|^3) s_0^{-2}, \\ |q_e(\cdot, s_0)| \leq s_0^{-1/2}. \end{cases} \quad (2.44)$$

Moreover, the restriction of $(d_0, d_1) \mapsto (q_0(s_0), q_1(s_0))$ to the boundary of D_T has degree 1.

(ii) for all $(d_0, d_1) \in D_T$ and for all $x \in \Omega$ with $|x - a| \geq \varepsilon_0/4$,

$$y_0(x, d_0, d_1) = 0.$$

Step 2. Reduction to a finite-dimensional problem

Let $K_0 \geq K_0^* := \max\{K_0^1, K_0^2, K_0^3\}$, $\varepsilon_0 \in (0, \widehat{\delta}]$, $A \geq A^* := \max\{A_1, \overline{A}(1, K_0, \varepsilon_0)\}$, $s_0 = -\log T \geq s_0^* := \max\{s_0^1, s_0^2, s_0^3, s_0^4, s_0^5, \overline{s}_0(1, A, K_0, \varepsilon_0)\}$ and $\eta_0 \in (0, 1]$. Consider $(d_0, d_1) \in D_T$. By the classical theory of parabolic equations, we can define a maximal solution y to equation (1.5) with initial data (2.42), and a maximal time $t_*(d_0, d_1) \in [0, T]$ such that

$$\forall t \in [0, t_*), y(t) \in S^*(K_0, \varepsilon_0, A, 1/2, \eta_0, T, t). \quad (2.45)$$

Then, one of the following possibilities occurs:

(1) either $t_* = T$,

(2) or $t_* < T$ and from continuity, $y(t_*) \in \partial S^*(K_0, \varepsilon_0, A, 1/2, \eta_0, T, t_*)$, in the sense that when $t = t_*$, one ‘ \leq ’ symbol in the definition of $S^*(K_0, \varepsilon_0, A, 1/2, \eta_0, T, t_*)$ is replaced by the symbol ‘ $=$ ’.

Our aim is to show that for $\mu = 1/2$, for all K_0 and A large enough, ε_0 and T small enough and for any $\eta_0 \in (0, 1]$, one can find $(d_0, d_1) \in D_T$ such that $t_*(d_0, d_1) = T$.

We argue by contradiction and assume that for all $(d_0, d_1) \in D_T$, $t_*(d_0, d_1) < T$.

On one hand, by Lemma 2.7 and Lemma 2.3, we can derive the following proposition.

Proposition 2.8. *For any $K_0 \geq K_0^*$, for any $\varepsilon_0 \in (0, \widehat{\delta}]$, for any $A \geq A^*$, for any $s_0 \geq s_0^*$ and for any $\eta_0 \in (0, 1]$, we have the following property: if $(d_0, d_1) \in D_T$ and if for $s_1 \geq s_0$, we have $\forall s \in [s_0, s_1]$, $y(t) \in S^*(K_0, \varepsilon_0, A, 1/2, \eta_0, T, t)$ with $t = T - e^{-s}$, then, for all $s \in [s_0, s_1]$,*

$$\begin{cases} |q_2(s)| \leq A^2 s^{-2} \log s - s^{-3}, \\ |q_-(z, s)| \leq \frac{A}{2} (1 + |z|^3) s^{-2}, \\ \|q_e(s)\|_{L^\infty} \leq \frac{A^2}{2\sqrt{s}}. \end{cases} \quad (2.46)$$

On the other hand, write \mathcal{R} for $\{x \in \Omega; |x - a| \geq \varepsilon_0/2\}$. We have that

$$\text{if } x \in \mathcal{R}, \text{ then } |y(x, t_*)| \leq \frac{\eta_0}{2}, \quad (2.47)$$

provided that the parameters satisfy some conditions.

Indeed, we consider $K_0 \geq K_0^*$, $\varepsilon_0 \in (0, \widehat{\delta}]$, $A \geq A^*$, $\eta_0 \in (0, 1]$, and

$$s_0 \geq s_1^* := \max \left\{ s_0^*(K_0, \varepsilon_0, A), -\log \left(T_2 \left(\frac{\eta_0}{2}, \varepsilon_0, C_1(K_0) \left[\frac{(p-1)^2 \left(\frac{\varepsilon_0}{4} \right)^2}{8p \left| \log \frac{\varepsilon_0}{4} \right|} \right]^{-\frac{1}{p-1}} \right) \right\}.$$

Applying Lemma 2.4, we see that if $y(t) \in S^*(K_0, \varepsilon_0, A, 1/2, \eta_0, T, t)$ for all $t \in [0, t_*]$, then for $x \in \Omega$ with

$$\frac{\varepsilon_0}{4} \leq |x - a| \leq \frac{\varepsilon_0}{2}, \quad \forall t \in [0, t_*],$$

$$|y(x, t)| \leq C_1(K_0) |y^*(x)| \leq C_1(K_0) \left[\frac{(p-1)^2 \left(\frac{\varepsilon_0}{4} \right)^2}{8p \left| \log \frac{\varepsilon_0}{4} \right|} \right]^{-\frac{1}{p-1}}.$$

Using Lemma 2.7, we see that for all $x \in \Omega$ with $|x - a| \geq \frac{\varepsilon_0}{4}$, $y(x, 0) = 0$.

Therefore, Lemma 2.5 applies with $(\mu_1, \mu_2) = (\frac{1}{4}, \frac{1}{2})$, $\epsilon = \frac{\eta_0}{2}$, $\bar{\eta} = 0$ and $\sigma_1 = C_1(K_0) \left[\frac{(p-1)^2 \left(\frac{\varepsilon_0}{4} \right)^2}{8p \left| \log \frac{\varepsilon_0}{4} \right|} \right]^{-\frac{1}{p-1}}$, and we see that

$$\forall x \in \Omega \text{ with } |x - a| \geq \frac{\varepsilon_0}{2}, \forall t \in [0, t_*], |y(x, t)| \leq \frac{\eta_0}{2}. \quad (2.48)$$

Hence, (2.47) holds.

Step 3. Final argument for Proposition 1.4

In this step, we shall conclude the proof of Proposition 1.4 thanks to a topological argument.

We fix $K_0 \geq K_0^*$, $\varepsilon_0 \in (0, \widehat{\delta}]$, $A \geq A^*$, $\eta_0 \in (0, 1]$ and $S_0 \geq s_1^*$. Take then $s_0 \geq S_0$ and introduce $T = e^{-s_0}$.

Note first that for any $(d_0, d_1) \in D_T$ defined in Lemma 2.7, equation (1.5) fitted with initial data (2.42) has a unique solution $y(\cdot, d_0, d_1) \in C([0, t^*), L^\infty(\Omega))$ for some maximal $t^* = t^*(d_0, d_1) \in (0, +\infty]$, thanks to a classical fixed point argument. With some classical parabolic estimates, we easily prove that $\nabla y(\cdot, d_0, d_1) \in C([0, t^*), L^\infty(\Omega))$. Using Lemma 2.7, we see that $y_0(\cdot, d_0, d_1) \in S^*(K_0, \varepsilon_0, A, 1/4, 0, T, 0)$. Therefore, we may introduce the largest time $t_* = t_*(d_0, d_1) \in [0, \min(T, t^*(d_0, d_1))]$ such that for all $t \in [0, t_*]$, $y(t, d_0, d_1) \in S^*(K_0, \varepsilon_0, A, 1/2, \eta_0, T, t)$. Then, 2 cases may occur:

- (i) either $t_*(d_0, d_1) = T$,
- (ii) or $t_*(d_0, d_1) < T$, and in that case, one of the “ \leq ” signs appearing in Definition 2.1 where $S^*(K_0, \varepsilon_0, A, 1/2, \eta_0, T, t)$ is introduced should be replaced by a “ $=$ ” sign.

We proceed in two steps:

1. we argue by contradiction and prove the existence of some $(d_0, d_1) \in D_T$ such that $t_*(d_0, d_1) = T$.

2. we consider such a $(d_0, d_1) \in D_T$ and show that the maximal existence time of $y(\cdot, d_0, d_1)$ is T , namely that $t^*(d_0, d_1) = T$. Then, we conclude the proof of Proposition 1.4.

1. A topological argument

We proceed by contradiction, and assume that for any $(d_0, d_1) \in D_T$, $t_*(d_0, d_1) < T$. By definition of $t_*(d_0, d_1)$, Proposition 2.8 applies and estimate (2.47) holds, leading to the fact the equality case mentioned in item 2 above at time $t = t_*$ concerns only the first two components mentioned in Definition 2.1, namely that

$$(q_0, q_1)(d_0, d_1, s_*(d_0, d_1)) \in \partial\left[-\frac{A}{(s_*)^2}, \frac{A}{(s_*)^2}\right]^2 \text{ where } s_* = -\log(T - t_*).$$

In particular, we can define the rescaled flow:

$$\begin{aligned} \Phi : D_T &\rightarrow \partial([-1, 1]^2), \\ (d_0, d_1) &\rightarrow \frac{s_*(d_0, d_1)^2}{A}(q_0, q_1)(d_0, d_1, s_*(d_0, d_1)). \end{aligned}$$

Then, we have the following proposition.

Proposition 2.9.

- 1) Φ is a continuous mapping from D_T to $\partial([-1, 1]^2)$.
- 2) The restriction of Φ to the boundary ∂D_T has degree 1.

From this proposition, a contradiction follows thanks to the following statement which is a consequence of Brouwer's lemma, and could be obtained directly by the proof of Theorem 12.8 in Smoller [54] (pp. 136), with a natural adaptation.

Proposition 2.10. *Consider Ψ a continuous function from the unit ball of \mathbb{R}^N to its boundary. Then, its restriction to the boundary has degree zero.*

This means that there exists $(d_0, d_1) \in D_T$ such that $t_*(d_0, d_1) = T$. By definition of t_* , this means that $y(t) \in S^*(K_0, \varepsilon_0, A, 1/2, \eta_0, T, t)$ for all $t \in [0, T)$.

2. Conclusion of the proof of Proposition 1.4

Let us summarize the result we have just proved:

If we fix $K_0 \geq K_0^*$, $\varepsilon_0 \in (0, \widehat{\delta}]$, $A \geq A^*$, $\eta_0 \in (0, 1]$, $s_0 \geq s_1^*$ and $T = e^{-s_0}$, then we see that for some $(d_0, d_1) \in D_T$, equation (1.5) with initial data (2.42) has a solution y such that for any $t \in [0, T)$, $y(t) \in S^*(K_0, \varepsilon_0, A, 1/2, \eta_0, T, t)$. By Lemma 2.2, we see that for any $s \geq -\log T$ and for any $z \in \mathbb{R}$,

$$|q(z, s)| \leq \frac{CA^2}{\sqrt{s}}.$$

By (1.3), (2.7), (2.12) and (2.13), it holds that for any $s \geq -\log T$ and for any z with $|z| \leq \varepsilon_0 e^{s/2}$, we have

$$\left|W(z, s) - f\left(\frac{z}{\sqrt{s}}\right)\right| \leq \frac{CA^2}{\sqrt{s}} + \frac{C}{s}.$$

Hence, if T is small enough, we obtain from (2.4) and (2.5) that for any $t \in [0, T)$ and for any $x \in \Omega$ with $|x - a| \leq \varepsilon_0$,

$$\left| (T - t)^{1/(p-1)} y(x, t) - f\left(\frac{x - a}{\sqrt{(T - t)|\log(T - t)|}}\right) \right| \leq \frac{C(A)}{\sqrt{|\log(T - t)|}}. \quad (2.49)$$

This implies (1.6).

This also implies that $|y(a, t)| \sim [(p - 1)(T - t)]^{-1/(p-1)}$ as $t \rightarrow T$. Hence, y blows up in finite time T , and a is a blowup point of y . In particular, this means that $t^*(d_0, d_1)$, the maximal existence time of y , is T .

It remains to prove that all points $x_0 \neq a$ are not blowup points.

We know from item (ii) in Definition 2.1 that for all $x \in \Omega$ with $|x - a| \geq \frac{\varepsilon_0}{2}$, $|y(x, t)| \leq \eta_0$. Thus, any $x_0 \in \Omega$ with $|x_0 - a| \geq \frac{\varepsilon_0}{2}$ is not a blowup point. Now, if $0 < |x_0 - a| \leq \varepsilon_0/2$, the following result from Giga and Kohn [29] allows us to conclude Proposition 1.4.

Proposition 2.11. *For all $K > 0$, there is $\eta_1 > 0$ such that if $\nu(\xi, \tau)$ solves*

$$|\nu_t - \Delta \nu| \leq K(1 + |\nu|^p), \quad (2.50)$$

and satisfies

$$|\nu(\xi, \tau)| \leq \eta_1 (T - t)^{-\frac{1}{p-1}},$$

for all $(\xi, \tau) \in B(a, r) \times [T - r^2, T)$ for some $a \in \mathbb{R}$ and $0 < r \leq 1$, then ν does not blow up at (a, T) .

Indeed, since $0 < |x_0 - a| \leq \varepsilon_0/2$, it follows from (2.49) that

$$\sup_{|x - x_0| \leq \frac{|x_0 - a|}{2}} |(T - t)^{1/(p-1)} y(x, t)| \leq \left| f\left(\frac{x_0 - a}{2\sqrt{(T - t)|\log(T - t)|}}\right) \right| + \frac{C(A)}{\sqrt{|\log(T - t)|}} \rightarrow 0,$$

as $t \rightarrow T$. By (1.5), it is easily obtained that y satisfies assumption (2.50). Therefore, applying Proposition 2.11, we see that x_0 is not a blowup point.

This completes the proof of Proposition 1.4. \square

3 Stability of blowup profile for auxiliary system

In this section, we shall prove Theorem 2, which gives a stability result of the blowup profile with respect to initial data for auxiliary system (1.5).

3.1 Formulation of the problem

Let $\hat{a} \in \omega$. From section 2, we know that there exists $K_0^* \geq 1$ such that for any $K_0 \geq K_0^*$, there exists $\hat{\delta} > 0$ with $(\hat{a} - 4\hat{\delta}, \hat{a} + 4\hat{\delta}) \subset \omega$ such that for any $K_0 \geq K_0^*$ and $\varepsilon_0 \in (0, \hat{\delta}]$, there exists $A^*(K_0, \varepsilon_0) \geq 1$ such that for any $\hat{A} \geq A^*(K_0, \varepsilon_0)$ and $\hat{\eta}_0 \in (0, 1]$, there

exists $\widehat{s}_1^*(K_0, \varepsilon_0, \widehat{A}, \widehat{\eta}_0)$ such that for any $\widehat{T} > 0$ with $\widehat{s}_0 = -\log \widehat{T} \geq \widehat{s}_1^*(K_0, \varepsilon_0, \widehat{A}, \widehat{\eta}_0)$, we could construct an initial data $\widehat{y}_0(K_0, \varepsilon_0, \widehat{A}, \widehat{T}) \in C_0^\infty(\omega)$ such that for any $t \in [0, \widehat{T})$, $\widehat{y}(t) \in S^*(K_0, \varepsilon_0, \widehat{A}, 1/2, \widehat{\eta}_0, \widehat{T}, t)$. In particular, $\widehat{q}(s) \in V_{K_0, \widehat{A}}(s)$ for all $s \geq -\log \widehat{T}$, and $|\widehat{y}(x, t)| \leq \widehat{\eta}_0$ for all $x \in \Omega$ with $|x - \widehat{a}| \geq \frac{\varepsilon_0}{2}$. Moreover, \widehat{T} is its blowup time and \widehat{a} is its unique blowup point.

Fix $K_0 \geq K_0^*$, $\widehat{A} \geq A^*$, $\varepsilon_0 \in (0, \widehat{\delta}]$, and for any $\eta_0 \in (0, 1]$ but fixed, we fix $\widehat{\eta}_0 \in \left(0, \frac{\eta_0}{8e^{(\sigma_1+1)^{p-1}}}\right)$, $\sigma_1 = C_1(K_0) \left[\frac{(p-1)^2 \left(\frac{\varepsilon_0}{4}\right)^2}{8p|\log \frac{\varepsilon_0}{4}|}\right]^{-\frac{1}{p-1}}$ with $C_1(K_0) \geq 1$ is defined in Lemma 2.4 and we fix $\widehat{s}_0 \geq \widehat{s}_1^*(K_0, \varepsilon_0, \widehat{A}, \widehat{\eta}_0)$.

Consider an initial data $y_0 \in W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)$ such that $\|\widetilde{\varepsilon}_0\|_{W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)}$ is small, where $q > n + 2$ and

$$\widetilde{\varepsilon}_0 = y_0 - \widehat{y}_0.$$

We denote by y_{y_0} the solution of equation (1.5) with initial data y_0 and $T(y_0) \leq +\infty$ its maximal time of existence.

For $(T, a) \in \mathbb{R}^+ \times \omega$, we introduce

$$W_{T,a,y_0}(z, s) = (T - t)^{1/(p-1)} y_{y_0}(x, t) = e^{-s/(p-1)} y_{y_0}(a + e^{-s/2} z, T - e^{-s}) \quad (3.1)$$

with

$$x - a = (T - t)^{1/2} z, \quad T - t = e^{-s}. \quad (3.2)$$

Define

$$w_{T,a,y_0}(z, s) = \begin{cases} W_{T,a,y_0}(z, s) \chi(z, s), & z \in (\Omega - a)e^{s/2}, \\ 0, & z \in \mathbb{R} \setminus (\Omega - a)e^{s/2}, \end{cases} \quad (3.3)$$

where χ is defined in (2.8).

Let

$$q_{T,a,y_0}(z, s) = w_{T,a,y_0}(z, s) - \varphi(z, s), \quad (3.4)$$

where φ is defined in (2.13). We then write

$$q_{T,a,y_0} = q_e + q_b, \quad (3.5)$$

where

$$q_b = q_{T,a,y_0} \chi_1, \quad q_e = q_{T,a,y_0} (1 - \chi_1). \quad (3.6)$$

Second, we decompose q_b as follows,

$$q_b(z, s) = \sum_{m=0}^2 q_m(s) h_m(z) + q_-(z, s), \quad (3.7)$$

where q_m is the projection of q_b on h_m , $q_-(z, s) = P_-(q_b)$, and P_- is the projection on $\{h_i; i \geq 3\}$, the negative subspace of the operator \mathcal{L} .

Our aim is to show that for $A > 0$, $s_0 \geq \widehat{s}_0 = -\log \widehat{T}$ large enough and for some $\mu \in (0, 1)$, if $\|\tilde{\varepsilon}_0\|_{W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)}$ is small enough, then $T(y_0)$ is finite, and y_{y_0} blows up at time $T(y_0)$ only at unique blowup point $a(y_0)$, with

$$T(y_0) \rightarrow \widehat{T}, \quad a(y_0) \rightarrow \widehat{a}, \quad \text{as } \tilde{\varepsilon}_0 = y_0 - \widehat{y}_0 \rightarrow 0 \text{ in } W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega), \quad (3.8)$$

$y_{y_0}(t) \in S^*(K_0, \varepsilon_0, A, \mu, \eta_0, T(y_0), t)$, which implies $q_{T(y_0), a(y_0), y_0}(s) \in V_{K_0, A}(s)$ for any $s \geq \sigma_0$ large enough and $t = T(y_0) - e^{-s}$.

In order to achieve the aim in (3.8), we will study q_{T, a, y_0} , where (T, a) is arbitrary in a small neighborhood of $(\widehat{T}, \widehat{a})$, hoping to have some hint that some $(\overline{T}(y_0), \overline{a}(y_0))$ close to $(\widehat{T}, \widehat{a})$ will correspond to the aimed $(T(y_0), a(y_0))$.

Note that all the q_{T, a, y_0} satisfy the same equation for all $(z, s) \in \mathbb{R} \times [-\log T, -\log(T - T(y_0))_+)$ (by convention, we note $-\log 0_+ = \infty$),

$$\partial_s q_{T, a, y_0} = (\mathcal{L} + V)q_{T, a, y_0} + B_{T, a, y_0}(z, s) + R(z, s) + N_{T, a, y_0}(z, s), \quad (3.9)$$

where \mathcal{L} and V are defined in (2.15), R is defined in (2.17) and

$$B_{T, a, y_0}(z, s) = |(\varphi + q_{T, a, y_0})|^{p-1}(\varphi + q_{T, a, y_0}) - \varphi^p - p\varphi^{p-1}q_{T, a, y_0}, \quad (3.10)$$

$$N_{T, a, y_0}(z, s) = H_{T, a, y_0} + \partial_z G_{T, a, y_0}(z, s), \quad (3.11)$$

$$H_{T, a, y_0}(z, s) = W_{T, a, y_0}(\partial_s \chi + \partial_z^2 \chi + \frac{1}{2}z\partial_z \chi) + |W_{T, a, y_0}|^{p-1}W_{T, a, y_0}(\chi - \chi^p), \quad (3.12)$$

$$G_{T, a, y_0}(z, s) = -2\partial_z \chi W_{T, a, y_0}, \quad \partial_z G_{T, a, y_0}(z, s) = -2\partial_z^2 \chi W_{T, a, y_0} - 2\partial_z \chi \partial_z W_{T, a, y_0}. \quad (3.13)$$

Introducing

$$\tilde{\varepsilon}(x, t) = y_{y_0}(x, t) - \widehat{y}(x, t), \quad \text{for all } x \in \Omega, \text{ and } 0 \leq t < \min(T(y_0), \widehat{T}).$$

Let

$$\tau = (T - \widehat{T})e^{s_0}, \quad \alpha = (a - \widehat{a})e^{s_0/2}, \quad \sigma_0 = s_0 - \log(1 + \tau) \text{ and } \tilde{z} = z\sqrt{1 + \tau} + \alpha. \quad (3.14)$$

For any $s_0 \in [-\log \widehat{T}, -\log(\widehat{T} - T(y_0))_+)$, $q_{T, a}$ is defined (at least) on $(-\log T, \sigma_0]$ for T near \widehat{T} .

For any $s_0 \in [-\log \widehat{T}, -\log(\widehat{T} - T(y_0))_+)$, we have that if $z \in (\Omega - a)e^{\sigma_0/2}$, then

$$\begin{aligned} q_{T, a, y_0}(z, \sigma_0) &= \overline{\psi}(s_0, y_0, T, a, z) \\ &:= (1 + \tau)^{\frac{1}{p-1}} [\widehat{q}(\tilde{z}, s_0) + \varphi(\tilde{z}, s_0) + e^{-\frac{s_0}{p-1}} \tilde{\varepsilon}(e^{-\frac{s_0}{2}} \tilde{z} + \widehat{a}, \widehat{T} - e^{-s_0}) \chi(z, \sigma_0)] - \varphi(z, \sigma_0) \\ &\quad + e^{-\frac{\sigma_0}{p-1}} \widehat{y}(ze^{-\frac{\sigma_0}{2}} + a, T - e^{-\sigma_0}) [\chi(z, \sigma_0) - \chi(\tilde{z}, s_0)]. \end{aligned} \quad (3.15)$$

If $z \notin (\Omega - a)e^{\sigma_0/2}$, then

$$q_{T,a,y_0}(z, \sigma_0) := \bar{\psi}(s_0, y_0, T, a, z) = -\varphi(z, \sigma_0). \quad (3.16)$$

In the following, $\bar{\psi}(s_0, y_0, T, a, z)$ appears as initial data for equation (3.9) at initial time $s = \sigma_0$. Consider the mapping $(T, a) \rightarrow \bar{\psi}(s_0, y_0, T, a, z)$. Similarly as in [45] and [55], we start from $\bar{\psi}(s_0, y_0, T, a, z)$ at initial time $s = \sigma_0$ including the reduction to a finite-dimensional problem, the construction procedure would work.

3.2 Behavior of “initial data” $\bar{\psi}$ for (T, a) near (\hat{T}, \hat{a})

In this subsection, we will prove the following proposition, which gives the properties of initial data $\bar{\psi}(s_0, y_0, T, a, z)$.

Recall that we have fixed $K_0 \geq K_0^*$, $\varepsilon_0 \in (0, \hat{\delta}]$, $\hat{A} \geq A^*$. Let $\eta_0 \in (0, 1]$. We fix $\hat{\eta}_0 \in (0, \frac{\eta_0}{8e^{(\sigma_1+1)^{p-1}}})$, where $\sigma_1 = C_1(K_0) \left[\frac{(p-1)^2 (\frac{\varepsilon_0}{4})^2}{8p |\log |\frac{\varepsilon_0}{4}||} \right]^{-\frac{1}{p-1}}$ with $C_1(K_0) \geq 1$. Let $b = \frac{(p-1)^2}{4p}$.

Proposition 3.1. *There exists $\bar{C} \geq 1$ such that for any $A \geq \bar{C}\hat{A}$, there exists $\bar{s}_0(A) \geq \hat{s}_1^*(K_0, \varepsilon_0, \hat{A}, \hat{\eta}_0)$ large enough, such that for any $s_0 \geq \bar{s}_0(A)$, there exists $\hat{\varepsilon}_0(s_0) > 0$ small enough such that for all $y_0 \in \{y \in W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega); \|y - \hat{y}_0\|_{W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)} \leq \hat{\varepsilon}_0(s_0)\}$, where $q > n + 2$, the following holds:*

(i) *There exists a set*

$$\bar{D}_{A,s_0,y_0} \subset \left\{ (T, a) \mid |T - \hat{T}| \leq \frac{2e^{-s_0}(p-1)A}{\kappa s_0^2}, |a - \hat{a}| \leq \frac{e^{-s_0/2}(p-1)^2 A}{b\kappa s_0} \right\}, \quad (3.17)$$

whose boundary is a Jordan curve such that the mapping

$$(T, a) \rightarrow \sigma_0^2(\bar{\psi}_0, \bar{\psi}_1)(s_0, y_0, T, a), \text{ where } \sigma_0 = s_0 - \log(1 + \tau), \quad (3.18)$$

is one-to-one from \bar{D}_{A,s_0,y_0} onto $[-A, A]^2$. Moreover, it is of degree -1 on the boundary.

(ii) *For all $(T, a) \in \bar{D}_{A,s_0,y_0}$, $\bar{\psi}(s_0, y_0, T, a) \in V_{K_0,A}(\sigma_0)$ with strict inequalities except for the first two, namely $(\bar{\psi}_0, \bar{\psi}_1)(s_0, y_0, T, a)$, in the sense that*

$$\begin{aligned} |\bar{\psi}_m| &\leq \frac{A}{\sigma_0^2}, \quad m = 0, 1, \\ |\bar{\psi}_2| &< C\hat{A}^2\sigma_0^{-2} \log \sigma_0, \\ \left\| \frac{\bar{\psi}_-(z, \sigma_0)}{1 + |z|^3} \right\|_{L^\infty(\mathbb{R})} &< \frac{C\hat{A}}{\sigma_0^2}, \\ \left\| \bar{\psi}_e(z, \sigma_0) \right\|_{L^\infty(\mathbb{R})} &< \frac{C\hat{A}^2}{\sigma_0^{\frac{1}{2}}}, \end{aligned}$$

for some positive constant $C > 0$.

(iii) *For all $(T, a) \in \bar{D}_{A_0,s_0,y_0}$ and for all $x \in \Omega$ with $|x - a| \geq 3\varepsilon_0/4$, $|y_{y_0}(x, T - e^{-\sigma_0})| \leq 2\hat{\eta}_0$.*

Define

$$\bar{g}(z, \sigma_0) = (1 + \tau)^{\frac{1}{p-1}} g(\tilde{z}, s_0), \quad (3.19)$$

where $\tilde{z} = z\sqrt{1 + \tau} + \alpha$, $\sigma_0 = s_0 - \log(1 + \tau)$, $g(s_0) \in L^\infty(\mathbb{R})$. In order to prove Proposition 3.1, we need the following lemma, which is an adaptation of Lemma 6.4 in [55].

Lemma 3.2. *(Expansions of $\bar{g}(z, \sigma_0)$) There exists $\tilde{s}_0 > 0$ large enough such that for any $s_0 \geq \tilde{s}_0$, and $(\tau, \alpha) \in \mathbb{R}^2$ satisfying*

$$|\tau| \leq \frac{1}{16}, \quad |\alpha| \leq \frac{1}{4} \quad (3.20)$$

such that if $g(s_0) \in L^\infty(\mathbb{R})$ and \bar{g} is given by (3.19), then

- (i) $(\frac{\sigma_0}{s_0})^2 \bar{g}_0(\sigma_0) = g_0(s_0)(1 + \frac{\tau}{p-1}) + O(\alpha g_1(s_0)) + O((|\tau| + \alpha^2)g_2(s_0)) + O((e^{-s_0} + \tau^2 + \frac{|\tau|}{s_0} + |\tau\alpha| + |\alpha|^3)\|g(s_0)\|_{L^\infty})$;
- (ii) $(\frac{\sigma_0}{s_0})^2 \bar{g}_1(\sigma_0) = g_1(s_0)(1 + \frac{(p+1)\tau}{2(p-1)}) + 2\alpha g_2(s_0) + O((|\tau| + \alpha^2)g_3(s_0)) + O((e^{-s_0} + \tau^2 + \frac{|\tau|}{s_0} + |\tau\alpha| + |\alpha|^3)\|g(s_0)\|_{L^\infty})$;
- (iii) $\bar{g}_2(\sigma_0) = g_2(s_0) + O(\alpha g_3(s_0)) + O(\alpha^2 g_4(s_0)) + O((e^{-s_0} + |\tau| + |\alpha|^3)\|g(s_0)\|_{L^\infty})$;
- (iv) $\sigma_0^{-2} \partial_\tau [\sigma_0^2 \bar{g}_0](\sigma_0) = \frac{g_0(s_0)}{p-1} + O(g_2(s_0)) + O((s_0^{-1} + |\tau| + |\alpha|)\|g(s_0)\|_{L^\infty})$;
- (v) $\partial_\tau [(\frac{\sigma_0}{s_0})^2 \bar{g}_1](\sigma_0) = O(g_1(s_0)) + O(g_3(s_0)) + O((s_0^{-1} + |\tau| + |\alpha|)\|g(s_0)\|_{L^\infty})$;
- (vi) $\partial_\alpha \bar{g}_0(\sigma_0) = O(g_1(s_0)) + O(\alpha g_2(s_0)) + O((e^{-s_0} + |\tau| + |\alpha|^2)\|g(s_0)\|_{L^\infty})$;
- (vii) $\partial_\alpha \bar{g}_1(\sigma_0) = 2g_2(s_0) + O(\alpha g_3(s_0)) + O((e^{-s_0} + |\tau| + |\alpha|^2)\|g(s_0)\|_{L^\infty})$;
- (viii) $\|\frac{\bar{g}_-(z, \sigma_0)}{1+|z|^3}\|_{L^\infty(\mathbb{R})} \leq C\|\frac{g_-(z, s_0)}{1+|z|^3}\|_{L^\infty(\mathbb{R})} + C|\alpha|(|g_1(s_0)| + |g_2(s_0)|) + O((s_0^{-2} + |\tau| + |\alpha|^3)\|g(s_0)\|_{L^\infty})$;
- (ix) $\|\bar{g}_e(z, \sigma_0)\|_{L^\infty(\mathbb{R})} \leq \|g_e(\tilde{z}, s_0)\|_{L^\infty(\mathbb{R})} + C(|\tau| + |\alpha|)\|g(s_0)\|_{L^\infty(\mathbb{R})}$.

By Lemma 3.2, we can obtain the following lemma.

Lemma 3.3. *(Expansions of modes for (T, a) close to (\hat{T}, \hat{a})) For $s_0 > 0$ large enough, there exists $C_0(s_0) > 0$ such that for any*

$$\|\tilde{\varepsilon}_0\|_{W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)} \leq \frac{1}{C_0}, \quad |\tau| \leq \frac{1}{16}, \quad |\alpha| \leq \frac{1}{4}, \quad (3.21)$$

where $q > n + 2$, we have the following inequalities:

$$\begin{aligned} & (i) \left| \frac{\sigma_0^2}{s_0^2} \bar{\psi}_0(s_0, y_0, T, a) - \hat{q}_0(s_0) - \frac{\kappa\tau}{p-1} \right| \\ & \leq C\|\tilde{\varepsilon}_0\|_{W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)} + C\frac{\hat{A}}{s_0^2}|\tau| + C\frac{\hat{A}}{s_0^2}|\alpha| + C\frac{\hat{A}^2}{s_0^{\frac{1}{2}}}e^{-s_0} + C\frac{\hat{A}^2}{s_0^{\frac{1}{2}}}(|\tau| + |\alpha|^3) \\ & \quad + C\frac{\hat{A}^2}{s_0^2} \log s_0(\tau + |\alpha|^2) + Ce^{-s_0} + C\tau^2 + C\frac{|\tau|}{s_0} + C\frac{\alpha^2}{s_0} + C|\alpha|e^{-s_0/2}; \end{aligned}$$

$$\begin{aligned}
(ii) \quad & \left| \frac{\sigma_0^2}{s_0^2} \bar{\psi}_1(s_0, y_0, T, a) - \hat{q}_1(s_0) + \frac{2b\kappa}{(p-1)^2} \frac{\alpha}{s_0} \right| \\
& \leq C \|\tilde{\varepsilon}_0\|_{W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)} + C \frac{\hat{A}^2}{s_0^{\frac{1}{2}}} e^{-s_0} + C \frac{\hat{A}}{s_0^2} |\tau| + C \frac{\hat{A}^2}{s_0^2} \log s_0 |\alpha| \\
& \quad + C \frac{\hat{A}^2}{s_0^{\frac{1}{2}}} \left(|\tau|^2 + |\tau\alpha| + \frac{|\tau|}{s_0} + |\alpha|^3 \right) + \left(\frac{C\hat{A}}{s_0^2} + e^{-Cs_0} \right) (|\tau| + |\alpha|^2) \\
& \quad + C \left(\frac{|\alpha|}{s_0^2} + e^{-s_0} + |\tau|^2 + |\tau\alpha| + \frac{|\tau|}{s_0} + |\alpha|^3 \right) + C |\alpha| e^{-s_0/2}; \\
(iii) \quad & |\bar{\psi}_2(s_0, y_0, T, a)| \leq C \|\tilde{\varepsilon}_0\|_{W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)} + \frac{C\hat{A}}{s_0^2} |\alpha| + \frac{C\hat{A}}{s_0^2} |\alpha|^2 + \\
& \quad + \frac{C\hat{A}^2}{s_0^{\frac{1}{2}}} (|\tau| + |\alpha|^3) + \frac{C\hat{A}^2}{s_0^{\frac{1}{2}}} e^{-s_0} + C (|\tau| + |\alpha|^3 + e^{-s_0}) + \frac{C}{s_0^2} \\
& \quad + C e^{-Cs_0} |\alpha| + C e^{-Cs_0} |\alpha|^2 + C \frac{|\tau|}{s_0^2} + \hat{A}^2 s_0^{-2} \log s_0 + C |\alpha| e^{-s_0/2}; \\
(iv) \quad & \left| \sigma_0^{-2} \partial_\tau [\sigma_0^2 \bar{\psi}_0](s_0, y_0, T, a) - \frac{\kappa}{p-1} \right| \\
& \leq C \|\tilde{\varepsilon}_0\|_{W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)} + \frac{C\hat{A}}{s_0^2} + \frac{C\hat{A}^2}{s_0^{\frac{1}{2}}} (|\tau| + |\alpha|) + C (|\tau| + |\alpha|) + \frac{C}{s_0} + \frac{C\hat{A}^2}{s_0^{3/2}} \\
& \quad + C e^{-Cs_0} + C e^{-Cs_0} |\alpha| + C \frac{\hat{A}^2}{s_0^2} \log s_0; \\
(v) \quad & \left| \partial_\tau \left[\frac{\sigma_0^2}{s_0^2} \bar{\psi}_1 \right] (s_0, y_0, T, a) \right| \\
& \leq C \|\tilde{\varepsilon}_0\|_{W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)} + \frac{C\hat{A}}{s_0^2} + \frac{C\hat{A}^2}{s_0^{\frac{1}{2}}} (s_0^{-1} + |\tau| + |\alpha|) \\
& \quad + C \left(\frac{1}{s_0} + |\tau| + |\alpha| \right) + C e^{-Cs_0} + C e^{-Cs_0} |\alpha|; \\
(vi) \quad & |\partial_\alpha \bar{\psi}_0(s_0, y_0, T, a)| \leq C \|\tilde{\varepsilon}_0\|_{W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)} + \frac{C\hat{A}}{s_0^2} + \frac{C\hat{A}^2}{s_0^2} \log s_0 |\alpha| + \\
& \quad + \frac{C\hat{A}^2}{s_0^{\frac{1}{2}}} (|\tau| + |\alpha|^2) + \frac{C\hat{A}^2}{s_0^{\frac{1}{2}}} e^{-s_0} + C \left(|\tau| + |\alpha|^2 + e^{-s_0} + \frac{|\alpha|}{s_0} \right) \\
& \quad + C e^{-Cs_0} |\alpha| + C e^{-Cs_0}; \\
(vii) \quad & \left| \partial_\alpha \bar{\psi}_1(s_0, y_0, T, a) + \frac{2b\kappa}{(p-1)^2} \frac{\alpha}{s_0} \right| \\
& \leq C \|\tilde{\varepsilon}_0\|_{W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)} + \frac{C\hat{A}}{s_0^2} |\alpha| + \frac{C\hat{A}^2}{s_0^2} \log s_0 + \\
& \quad + \frac{C\hat{A}^2}{s_0^{\frac{1}{2}}} (|\tau| + |\alpha|^2) + \frac{C\hat{A}^2}{s_0^{\frac{1}{2}}} e^{-s_0} + \frac{C}{s_0^2} + C (|\tau| + |\alpha|^2) \\
& \quad + C e^{-Cs_0} |\alpha| + C e^{-Cs_0};
\end{aligned}$$

$$\begin{aligned}
& (viii) \left\| \frac{\bar{\psi}_-(s_0, y_0, T, a)}{1 + |z|^3} \right\|_{L^\infty(\Omega)} \\
& \leq C \|\tilde{\varepsilon}_0\|_{W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)} + \frac{C\hat{A}}{s_0^2} + \frac{C\hat{A}}{s_0^2} |\alpha| + \frac{C\hat{A}^2}{s_0^{\frac{1}{2}}} (|\tau| + |\alpha|^3) + C \frac{\hat{A}^2}{s_0^2} \log s_0 |\alpha| + \frac{C\hat{A}^2}{s_0^{5/2}} \\
& \quad + C e^{-Cs_0} |\alpha| + \frac{C}{s_0} |\tau| + C \frac{|\alpha|}{s_0^2} + C \frac{|\alpha|^2}{s_0^2} + C |\tau|^2 + C e^{-s_0}; \\
& (ix) \left\| \bar{\psi}_e(s_0, y_0, T, a) \right\|_{L^\infty(\mathbb{R})} \\
& \leq C \|\tilde{\varepsilon}_0\|_{W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)} + \frac{C\hat{A}^2}{s_0^{\frac{1}{2}}} (1 + |\tau| + |\alpha|) + C \left(|\tau| + \frac{|\alpha|}{s_0^{\frac{1}{2}}} \right) + \frac{C}{s_0} + C e^{-Cs_0} |\alpha|.
\end{aligned}$$

Proof. From (3.15) and (3.16), we write

$$\bar{\psi}(s_0, y_0, T, a, z) = \bar{\psi}^1(z, \sigma_0) + \bar{\psi}^2(z, \sigma_0) + \bar{\psi}^3(z, \sigma_0) + \bar{\psi}^4(z, \sigma_0) + \bar{\psi}^5(z, \sigma_0)$$

for $z \in (\Omega - a)e^{\sigma_0/2}$, where

$$\bar{\psi}^1(z, \sigma_0) = (1 + \tau)^{\frac{1}{p-1}} e^{-\frac{s_0}{p-1}} \tilde{\varepsilon}(e^{-\frac{s_0}{2}} \tilde{z} + \hat{a}, \hat{T} - e^{-s_0}) \chi(z, \sigma_0); \quad (3.22)$$

$$\bar{\psi}^2(z, \sigma_0) = (1 + \tau)^{\frac{1}{p-1}} \hat{q}(\tilde{z}, s_0); \quad (3.23)$$

$$\bar{\psi}^3(z, \sigma_0) = (1 + \tau)^{\frac{1}{p-1}} \varphi(\tilde{z}, s_0); \quad (3.24)$$

$$\bar{\psi}^4(z, \sigma_0) = -\varphi(z, \sigma_0); \quad (3.25)$$

$$\bar{\psi}^5(z, \sigma_0) = e^{-\frac{\sigma_0}{p-1}} \hat{y}(ze^{-\frac{\sigma_0}{2}} + a, T - e^{-\sigma_0}) [\chi(z, \sigma_0) - \chi(\tilde{z}, s_0)], \quad (3.26)$$

and for $z \notin (\Omega - a)e^{\sigma_0/2}$,

$$\bar{\psi}^1 = \bar{\psi}^2 = \bar{\psi}^3 = \bar{\psi}^5 = 0, \quad \bar{\psi}^4(z, \sigma_0) = -\varphi(z, \sigma_0). \quad (3.27)$$

Step 1. Expansion of $\bar{\psi}^1$

When $t = \hat{T} - e^{-s_0}$, \hat{y} is meaningful in $[0, \hat{T} - e^{-s_0}]$. Thus, y_{y_0} is meaningful in $[0, \hat{T} - e^{-s_0}]$ if y_0 is close enough to \hat{y}_0 in $W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)$ with $q > n + 2$, which implies $t < \min(T(y_0), \hat{T})$ and it follows that

$$\|\tilde{\varepsilon}(\hat{T} - e^{-s_0})\|_{L^\infty(\Omega)} \leq \tilde{C}_0 \|\tilde{\varepsilon}_0\|_{W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)}, \quad (3.28)$$

whenever $\|\tilde{\varepsilon}_0\|_{W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)} \leq 1/\tilde{C}_0$ for some $\tilde{C}_0 = \tilde{C}_0(s_0)$, where $q > n + 2$.

Now, we prove that for s_0 large enough and (τ, α) satisfies (3.21), we have

$$\begin{aligned}
& |\bar{\psi}_0^1(\sigma_0)| + |\bar{\psi}_1^1(\sigma_0)| + |\bar{\psi}_2^1(\sigma_0)| + \left| \partial_\tau \left[\left(\frac{\sigma_0}{s_0} \right)^{2-\frac{1}{p}} \bar{\psi}_0^1 \right] (\sigma_0) \right| + \left| \partial_\tau \left[\left(\frac{\sigma_0}{s_0} \right)^{2-\frac{1}{p}} \bar{\psi}_1^1 \right] (\sigma_0) \right| \\
& + |\partial_\alpha \bar{\psi}_0^1(\sigma_0)| + |\partial_\alpha \bar{\psi}_1^1(\sigma_0)| + \left\| \frac{\bar{\psi}_-(z, \sigma_0)}{1 + |z|^3} \right\|_{L^\infty(\mathbb{R})} + \|\bar{\psi}_e^1(\sigma_0)\|_{L^\infty(\mathbb{R})} \\
& \leq C \|\tilde{\varepsilon}(\hat{T} - e^{-s_0})\|_{L^\infty(\Omega)} \leq C_0(s_0) \|\tilde{\varepsilon}_0\|_{W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)}. \quad (3.29)
\end{aligned}$$

Indeed, for $\bar{\psi}_0^1(\sigma_0)$, it holds that

$$\begin{aligned} |\bar{\psi}_0^1(\sigma_0)| &= \left| \int \chi_1(z, \sigma_0) \bar{\psi}^1(z, \sigma_0) \rho(z) dz \right| \\ &= \left| \int_{(\Omega-a)e^{\sigma_0/2}} \rho(z) (1+\tau)^{\frac{1}{p-1}} e^{-\frac{s_0}{p-1}} \tilde{\varepsilon}(e^{-\frac{s_0}{2}} \tilde{z} + \hat{a}, \hat{T} - e^{-s_0}) \chi_1(z, \sigma_0) \chi(z, \sigma_0) dz \right|. \end{aligned}$$

It is easy to see that $z \in (\Omega - a)e^{\sigma_0/2}$ is equivalent to $\tilde{z} \in (\Omega - \hat{a})e^{s_0/2}$. Hence,

$$|\bar{\psi}_0^1(\sigma_0)| \leq C \|\tilde{\varepsilon}(\hat{T} - e^{-s_0})\|_{L^\infty(\Omega)}. \quad (3.30)$$

Similarly, we have

$$|\bar{\psi}_1^1(\sigma_0)| \leq C \|\tilde{\varepsilon}(\hat{T} - e^{-s_0})\|_{L^\infty(\Omega)}, \quad (3.31)$$

$$|\bar{\psi}_2^1(\sigma_0)| \leq C \|\tilde{\varepsilon}(\hat{T} - e^{-s_0})\|_{L^\infty(\Omega)}, \quad (3.32)$$

and

$$\|\bar{\psi}_e^1(\sigma_0)\|_{L^\infty(\mathbb{R})} = \|(1 - \chi_1(z, \sigma_0)) \bar{\psi}^1(\sigma_0)\|_{L^\infty(\mathbb{R})} \leq C \|\tilde{\varepsilon}(\hat{T} - e^{-s_0})\|_{L^\infty(\Omega)}. \quad (3.33)$$

By the above estimates,

$$\begin{aligned} \left\| \frac{\bar{\psi}_-^1(z, \sigma_0)}{1 + |z|^3} \right\|_{L^\infty(\mathbb{R})} &= \left\| \frac{\bar{\psi}^1(z, \sigma_0) \chi_1(z, \sigma_0) - \sum_{m=0}^2 \bar{\psi}_m^1(\sigma_0) h_m(z)}{1 + |z|^3} \right\|_{L^\infty(\mathbb{R})} \\ &\leq C \|\tilde{\varepsilon}(\hat{T} - e^{-s_0})\|_{L^\infty(\Omega)}. \end{aligned} \quad (3.34)$$

For $\frac{\partial}{\partial \tau} \left[\left(\frac{\sigma_0}{s_0} \right)^2 \bar{\psi}_0^1 \right](\sigma_0)$, we have

$$\begin{aligned} \left(\frac{\sigma_0}{s_0} \right)^2 \bar{\psi}_0^1(\sigma_0) &= \left(\frac{s_0 - \log(1 + \tau)}{s_0} \right)^2 \int_{\mathbb{R}} \chi_1(z, \sigma_0) \bar{\psi}^1(z, \sigma_0) \rho(z) dz \\ &= \left(\frac{s_0 - \log(1 + \tau)}{s_0} \right)^2 (1 + \tau)^{-\frac{1}{2}} \int_{\mathbb{R}} \chi_1(z, s_0 - \log(1 + \tau)) \bar{\psi}^1(z, \sigma_0) \rho(z) d\tilde{z}. \end{aligned}$$

We see that

$$\bar{\psi}^1(z, \sigma_0) = \begin{cases} (1 + \tau)^{\frac{1}{p-1}} e^{-\frac{s_0}{p-1}} \tilde{\varepsilon}(e^{-\frac{s_0}{2}} \tilde{z} + \hat{a}, \hat{T} - e^{-s_0}) \chi(z, \sigma_0), & z \in (\Omega - a)e^{\sigma_0/2}, \\ 0, & \text{otherwise.} \end{cases}$$

However, we can take s_0 large enough such that $|a - \hat{a}| = |\alpha e^{-\frac{s_0}{2}}| \leq e^{-\frac{s_0}{2}} < \frac{\varepsilon_0}{4}$. Thus, since $(\hat{a} - 4\hat{\delta}, \hat{a} + 4\hat{\delta}) \subset \omega$, $\varepsilon_0 \in (0, \hat{\delta}]$ and $\chi(z, \sigma_0)$ is 0 outside a small neighborhood of a when $|a - \hat{a}| < \frac{\varepsilon_0}{4}$, it is easy to check that $\bar{\psi}^1(z, \sigma_0)$ is smooth in \mathbb{R} .

Then, we have that for s_0 large enough,

$$\begin{aligned}
\frac{\partial}{\partial \tau} \left[\left(\frac{\sigma_0}{s_0} \right)^{2-p} \psi_0 \right] (\sigma_0) &= -2 \frac{1}{s_0(1+\tau)} \frac{s_0 - \log(1+\tau)}{s_0} \int_{(\Omega-a)e^{\sigma_0/2}} \rho(z) (1+\tau)^{\frac{1}{p-1}} e^{-\frac{s_0}{p-1}} \\
&\cdot \tilde{\varepsilon} \left(e^{-\frac{s_0}{2}} (z\sqrt{1+\tau} + \alpha) + \hat{a}, \hat{T} - e^{-s_0} \right) \chi_1(z, s_0 - \log(1+\tau)) \chi(z, \sigma_0) dz \\
&- \frac{1}{2} (1+\tau)^{-1} \left(\frac{s_0 - \log(1+\tau)}{s_0} \right)^2 \int_{(\Omega-a)e^{\sigma_0/2}} \rho(z) \chi_1(z, s_0 - \log(1+\tau)) (1+\tau)^{\frac{1}{p-1}} e^{-\frac{s_0}{p-1}} \\
&\cdot \tilde{\varepsilon} \left(e^{-\frac{s_0}{2}} (z\sqrt{1+\tau} + \alpha) + \hat{a}, \hat{T} - e^{-s_0} \right) \chi(z, \sigma_0) dz \\
&+ \frac{1}{\sqrt{4\pi}} \left(\frac{s_0 - \log(1+\tau)}{s_0} \right)^2 \int_{(\Omega-a)e^{\sigma_0/2}} \rho(z) \frac{z^2}{4} (1+\tau)^{\frac{1}{p-1}-1} \chi_1(z, s_0 - \log(1+\tau)) e^{-\frac{s_0}{p-1}} \\
&\cdot \tilde{\varepsilon} \left(e^{-\frac{s_0}{2}} (z\sqrt{1+\tau} + \alpha) + \hat{a}, \hat{T} - e^{-s_0} \right) \chi(z, \sigma_0) dz \\
&- \frac{1}{2} \left(\frac{s_0 - \log(1+\tau)}{s_0} \right)^2 \int_{(\Omega-a)e^{\sigma_0/2}} \rho(z) z (1+\tau)^{\frac{1}{p-1}-1} \chi_{1z}(z, s_0 - \log(1+\tau)) e^{-\frac{s_0}{p-1}} \\
&\cdot \tilde{\varepsilon} \left(e^{-\frac{s_0}{2}} (z\sqrt{1+\tau} + \alpha) + \hat{a}, \hat{T} - e^{-s_0} \right) \chi(z, \sigma_0) dz \\
&- \frac{1}{2} \left(\frac{s_0 - \log(1+\tau)}{s_0} \right)^2 \int_{(\Omega-a)e^{\sigma_0/2}} \rho(z) (1+\tau)^{\frac{1}{p-1}-1} \chi_1(z, s_0 - \log(1+\tau)) e^{-\frac{s_0}{p-1}} \\
&\cdot \tilde{\varepsilon} \left(e^{-\frac{s_0}{2}} (z\sqrt{1+\tau} + \alpha) + \hat{a}, \hat{T} - e^{-s_0} \right) \chi_z(z, \sigma_0) z dz \\
&+ \left(\frac{s_0 - \log(1+\tau)}{s_0} \right)^2 \int_{(\Omega-a)e^{\sigma_0/2}} \rho(z) \frac{1}{p-1} (1+\tau)^{\frac{1}{p-1}-1} \chi_1(z, s_0 - \log(1+\tau)) e^{-\frac{s_0}{p-1}} \\
&\cdot \tilde{\varepsilon} \left(e^{-\frac{s_0}{2}} (z\sqrt{1+\tau} + \alpha) + \hat{a}, \hat{T} - e^{-s_0} \right) \chi(z, \sigma_0) dz \\
&+ \left(\frac{s_0 - \log(1+\tau)}{s_0} \right)^2 \int_{(\Omega-a)e^{\sigma_0/2}} \rho(z) (1+\tau)^{\frac{1}{p-1}} e^{-\frac{s_0}{p-1}} \\
&\cdot \tilde{\varepsilon} \left(e^{-\frac{s_0}{2}} (z\sqrt{1+\tau} + \alpha) + \hat{a}, \hat{T} - e^{-s_0} \right) \chi_1(z, \sigma_0) \frac{\chi_{\sigma_0}(z, \sigma_0)}{-(1+\tau)} dz \\
&+ \left(\frac{s_0 - \log(1+\tau)}{s_0} \right)^2 \int_{(\Omega-a)e^{\sigma_0/2}} \rho(z) (1+\tau)^{\frac{1}{p-1}} e^{-\frac{s_0}{p-1}} \\
&\cdot \tilde{\varepsilon} \left(e^{-\frac{s_0}{2}} (z\sqrt{1+\tau} + \alpha) + \hat{a}, \hat{T} - e^{-s_0} \right) \chi(z, \sigma_0) \frac{\chi_{1\sigma_0}(z, \sigma_0)}{-(1+\tau)} dz, \tag{3.35}
\end{aligned}$$

from which and the following equalities,

$$\chi_{1\sigma_0}(z, \sigma_0) = -\chi'_0 \left(\frac{|z|}{K_0 \sqrt{\sigma_0}} \right) \frac{|z|}{K_0} \frac{\sigma_0^{-3/2}}{2}, \quad \chi_{1z}(z, \sigma_0) = \chi'_0 \left(\frac{|z|}{K_0 \sqrt{\sigma_0}} \right) \frac{1}{K_0 \sqrt{\sigma_0}}, \tag{3.36}$$

$$\chi_{\sigma_0}(z, \sigma_0) = -\chi'_0 \left(\frac{ze^{-\sigma_0/2}}{\varepsilon_0} \right) \frac{z}{\varepsilon_0} \frac{e^{-\sigma_0/2}}{2}, \quad \chi_z(z, \sigma_0) = \chi'_0 \left(\frac{ze^{-\sigma_0/2}}{\varepsilon_0} \right) \frac{e^{-\sigma_0/2}}{\varepsilon_0}, \tag{3.37}$$

we have that for s_0 large enough,

$$\left| \frac{\partial}{\partial \tau} \left[\left(\frac{\sigma_0}{s_0} \right)^{2-p} \psi_0 \right] (\sigma_0) \right| \leq C \|\tilde{\varepsilon}_0(\hat{T} - e^{-s_0})\|_{L^\infty(\Omega)}. \tag{3.38}$$

Similarly to the above inequality, we have that for s_0 large enough,

$$\left| \frac{\partial}{\partial \tau} \left[\left(\frac{\sigma_0}{s_0} \right)^{2-p} \psi_1 \right] (\sigma_0) \right| \leq C \|\tilde{\varepsilon}_0(\widehat{T} - e^{-s_0})\|_{L^\infty(\Omega)}, \quad (3.39)$$

$$|\partial_\alpha \bar{\psi}_0^1(\sigma_0)| \leq C \|\tilde{\varepsilon}_0(\widehat{T} - e^{-s_0})\|_{L^\infty(\Omega)}, \quad (3.40)$$

and

$$|\partial_\alpha \bar{\psi}_1^1(\sigma_0)| \leq C \|\tilde{\varepsilon}_0(\widehat{T} - e^{-s_0})\|_{L^\infty(\Omega)}. \quad (3.41)$$

Then, by (3.30)-(3.34), (3.38)-(3.41) and (3.28), we can get (3.29).

Step 2. Expansion of $\bar{\psi}^2$

Since for any $t \in [0, \widehat{T}]$, $\widehat{y}(t) \in S^*(K_0, \varepsilon_0, \widehat{A}, 1/2, \widehat{\eta}_0, \widehat{T}, t)$, where $\widehat{q}(s) \in V_{K_0, \widehat{A}}(s)$, for all $s \geq -\log \widehat{T}$, and $|\widehat{y}(x, t)| \leq \widehat{\eta}_0$ for all $x \in \Omega$ with $|x - \widehat{a}| \geq \frac{\varepsilon_0}{2}$. In particular, for $s_0 \geq -\log \widehat{T}$, we have

$$\begin{cases} |\widehat{q}_m(s_0)| \leq \widehat{A} s_0^{-2} (m = 0, 1), & |\widehat{q}_2(s_0)| \leq \widehat{A}^2 s_0^{-2} \log s_0, \\ |\widehat{q}_-(\tilde{z}, s_0)| \leq \widehat{A} s_0^{-2} (1 + |\tilde{z}|^3), & |\widehat{q}_e(\tilde{z}, s_0)| \leq \widehat{A}^2 s_0^{-1/2}, \end{cases} \quad (3.42)$$

and

$$\|\widehat{q}(s_0)\|_{L^\infty} \leq C \widehat{A}^2 s_0^{-1/2}. \quad (3.43)$$

(i) For s_0 large enough,

$$\begin{aligned} \left| \left(\frac{\sigma_0}{s_0} \right)^{2-p} \bar{\psi}_0^2(\sigma_0) - \widehat{q}_0(s_0) \right| &\leq \frac{C \widehat{A}}{s_0^2} |\tau| + \frac{C \widehat{A}}{s_0^2} |\alpha| + \frac{C \widehat{A}^2 e^{-s_0}}{s_0^{\frac{1}{2}}} + \frac{C \widehat{A}^2}{s_0^{\frac{1}{2}}} (|\tau| + |\alpha|^3) \\ &\quad + C (|\tau| + |\alpha|^2) \frac{\widehat{A}^2}{s_0^2} \log s_0. \end{aligned} \quad (3.44)$$

Indeed, by (3.23) and (3.27),

$$\bar{\psi}^2(z, \sigma_0) = \begin{cases} (1 + \tau)^{\frac{1}{p-1}} \widehat{q}(\tilde{z}, s_0), & \tilde{z} \in (\Omega - \widehat{a})e^{s_0/2}, \\ 0, & \tilde{z} \notin (\Omega - \widehat{a})e^{s_0/2}, \end{cases} \quad (3.45)$$

we have by Lemma 3.2 (i) that for s_0 large enough,

$$\begin{aligned} \left| \left(\frac{\sigma_0}{s_0} \right)^{2-p} \bar{\psi}_0^2(\sigma_0) - \widehat{q}_0(s_0) \right| &= \left| \widehat{q}_0(s_0) \frac{\tau}{p-1} + O(\alpha \widehat{q}_1(s_0)) \right. \\ &\quad \left. + O((|\tau| + \alpha^2) \widehat{q}_2(s_0)) + O\left((e^{-s_0} + \tau^2 + \frac{|\tau|}{s_0} + |\tau\alpha| + |\alpha|^3) \|g^1(s_0)\|_{L^\infty} \right) \right|, \end{aligned} \quad (3.46)$$

where

$$g^1(\tilde{z}, s_0) = \begin{cases} \widehat{q}(\tilde{z}, s_0), & \tilde{z} \in (\Omega - \widehat{a})e^{s_0/2} (\iff z \in (\Omega - a)e^{\sigma_0/2}), \\ 0, & \tilde{z} \notin (\Omega - \widehat{a})e^{s_0/2}. \end{cases} \quad (3.47)$$

Hence, by (3.42), (3.43) and (3.46), (3.44) hold.

(ii) For s_0 large enough,

$$\begin{aligned}
& \left| \left(\frac{\sigma_0}{s_0} \right)^2 \bar{\psi}_1^2(\sigma_0) - \widehat{q}_1(s_0) \right| \\
& \leq \frac{C\widehat{A}^2 e^{-s_0}}{s_0^{\frac{1}{2}}} + \frac{C\widehat{A}^2}{s_0^{\frac{1}{2}}} \left(|\tau|^2 + |\alpha|^3 + |\tau\alpha| + \frac{|\tau|}{s_0} \right) + \frac{C\widehat{A}}{s_0^2} |\tau| + \frac{C\widehat{A}^2}{s_0^2} \log s_0 |\alpha| \\
& \quad + \left(\frac{C\widehat{A}}{s_0^2} + e^{-Cs_0} \right) (|\tau| + |\alpha|^2).
\end{aligned} \tag{3.48}$$

Indeed, since k_0, k_1, k_2 are all orthogonal to k_3 , we have

$$\begin{aligned}
& |\widehat{q}_3(s_0)| \\
& = \left| \int_{\mathbb{R}} \widehat{q}_-(\tilde{z}, s_0) k_3(\tilde{z}) \chi_1(\tilde{z}, s_0) \rho(\tilde{z}) d\tilde{z} + \int_{\mathbb{R}} \sum_{m=0}^2 q_m(s_0) h_m(\tilde{z}) k_3(\tilde{z}) \chi_1(\tilde{z}, s_0) \rho(\tilde{z}) d\tilde{z} \right| \\
& \leq C \frac{\widehat{A}}{s_0^2} + C e^{-Cs_0}.
\end{aligned} \tag{3.49}$$

Similarly, we have

$$|\widehat{q}_4(s_0)| \leq C \frac{\widehat{A}}{s_0^2} + C e^{-Cs_0}. \tag{3.50}$$

Then, we have by Lemma 3.2 (ii) that for s_0 large enough,

$$\begin{aligned}
& \left| \left(\frac{\sigma_0}{s_0} \right)^2 \bar{\psi}_1^2(\sigma_0) - \widehat{q}_1(s_0) \right| \\
& = \left| \widehat{q}_1(s_0) \frac{(p+1)\tau}{2(p-1)} + 2\alpha \widehat{q}_2(s_0) + O((|\tau| + \alpha^2) \widehat{q}_3(s_0)) \right. \\
& \quad \left. + O\left(\left(e^{-s_0} + \tau^2 + \frac{|\tau|}{s_0} + |\tau\alpha| + |\alpha|^3 \right) \|g^2(s_0)\|_{L^\infty} \right) \right|.
\end{aligned} \tag{3.51}$$

Hence, by (3.42), (3.43), (3.47) and (3.49), (3.48) holds.

(iii) For s_0 large enough,

$$\begin{aligned}
|\bar{\psi}_2^2(\sigma_0)| & \leq \frac{C\widehat{A}^2 e^{-s_0}}{s_0^{\frac{1}{2}}} + \frac{C\widehat{A}^2}{s_0^{\frac{1}{2}}} (|\tau| + |\alpha|^3) + \frac{C\widehat{A}}{s_0^2} |\alpha| + \frac{C\widehat{A}}{s_0^2} |\alpha|^2 \\
& \quad + C e^{-Cs_0} |\alpha| + C e^{-Cs_0} |\alpha|^2 + \frac{\widehat{A}^2}{s_0^2} \log s_0.
\end{aligned} \tag{3.52}$$

Indeed, by Lemma 3.2 (iii) that for s_0 large enough,

$$|\bar{\psi}_2^2(\sigma_0)| = \left| \widehat{q}_2(s_0) + O(\alpha \widehat{q}_3(s_0)) + O(\alpha^2 \widehat{q}_4(s_0)) + O((e^{-s_0} + |\tau| + |\alpha|^3) \|g^2(s_0)\|_{L^\infty}) \right|. \tag{3.53}$$

Hence, by (3.42), (3.43), (3.47), (3.49) and (3.50), (3.52) holds.

(iv) For s_0 large enough,

$$|\sigma_0^{-2} \partial_\tau [\sigma_0^2 \bar{\psi}_0^2](\sigma_0)| \leq \frac{C\widehat{A}^2}{s_0^{\frac{1}{2}}} (|\tau| + |\alpha|) + \frac{C\widehat{A}}{s_0^2} + \frac{C\widehat{A}^2}{s_0^{3/2}} + \frac{C\widehat{A}^2}{s_0^2} \log s_0. \quad (3.54)$$

Indeed, by Lemma 3.2 (iv),

$$|\sigma_0^{-2} \partial_\tau [\sigma_0^2 \bar{\psi}_0^2](\sigma_0)| = \left| \frac{\widehat{q}_0(s_0)}{p-1} + O(\widehat{q}_2(s_0)) + O((s_0^{-1} + |\tau| + |\alpha|) \|g^2(s_0)\|_{L^\infty}) \right|.$$

Hence, by (3.42), (3.43) and (3.47), (3.54) holds.

(v) For s_0 large enough,

$$|\sigma_0^{-2} \partial_\tau [\sigma_0^2 \bar{\psi}_1^2](\sigma_0)| \leq \frac{C\widehat{A}^2}{s_0^{\frac{1}{2}}} (s_0^{-1} + |\tau| + |\alpha|) + \frac{C\widehat{A}}{s_0^2} + Ce^{-Cs_0}. \quad (3.55)$$

Indeed, by Lemma 3.2 (v),

$$\begin{aligned} |\sigma_0^{-2} \partial_\tau [\sigma_0^2 \bar{\psi}_1^2](\sigma_0)| &= \left| \sigma_0^{-2} s_0^2 \partial_\tau \left[\left(\frac{\sigma_0}{s_0} \right)^2 \bar{\psi}_1^2 \right](\sigma_0) \right| \\ &= \left| \frac{s_0^2}{\sigma_0^2} [O(\widehat{q}_1(s_0)) + O(\widehat{q}_3(s_0)) + O((s_0^{-1} + |\tau| + |\alpha|) \|g^2(s_0)\|_{L^\infty})] \right|. \end{aligned}$$

Hence, by (3.42), (3.43), (3.47) and (3.49), (3.55) holds.

(vi) For s_0 large enough,

$$|\partial_\alpha \bar{\psi}_0^2(\sigma_0)| \leq \frac{C\widehat{A}}{s_0^2} + \frac{C\widehat{A}^2 e^{-s_0}}{s_0^{\frac{1}{2}}} + \frac{C\widehat{A}^2}{s_0^{\frac{1}{2}}} (|\tau| + |\alpha|^2) + C \frac{\widehat{A}^2}{s_0^2} \log s_0 |\alpha|. \quad (3.56)$$

Indeed, by Lemma 3.2 (vi),

$$|\partial_\alpha \bar{\psi}_0^2(\sigma_0)| = |O(\widehat{q}_1(s_0)) + O(\alpha \widehat{q}_2(s_0)) + O((e^{-s_0} + |\tau| + |\alpha|^2) \|g^2(s_0)\|_{L^\infty})|. \quad (3.57)$$

Hence, by (3.42), (3.43) and (3.47), (3.56) holds.

(vii) For s_0 large enough,

$$|\partial_\alpha \bar{\psi}_1^2(\sigma_0)| \leq \frac{C\widehat{A}}{s_0^2} |\alpha| + \frac{C\widehat{A}^2 e^{-s_0}}{s_0^{\frac{1}{2}}} + \frac{C\widehat{A}^2}{s_0^{\frac{1}{2}}} (|\tau| + |\alpha|^2) + C \frac{\widehat{A}^2}{s_0^2} \log s_0 + Ce^{-Cs_0} |\alpha|. \quad (3.58)$$

Indeed, by Lemma 3.2 (vii),

$$|\partial_\alpha \bar{\psi}_1^2(\sigma_0)| = |2\widehat{q}_2(s_0) + O(\alpha \widehat{q}_3(s_0)) + O((e^{-s_0} + |\tau| + |\alpha|^2) \|g^2(s_0)\|_{L^\infty})|. \quad (3.59)$$

Hence, by (3.42), (3.43), (3.47) and (3.49), (3.58) holds.

(viii) For s_0 large enough,

$$\begin{aligned} & \left\| \frac{\overline{\psi}_-^2(z, \sigma_0)}{1 + |z|^3} \right\|_{L^\infty(\mathbb{R})} \\ & \leq \frac{C\widehat{A}}{s_0^2}(|\alpha| + 1) + \frac{C\widehat{A}^2}{s_0^{\frac{5}{2}}} + \frac{C\widehat{A}^2}{s_0^{\frac{1}{2}}}(|\tau| + |\alpha|^3) + C\frac{\widehat{A}^2}{s_0^2} \log s_0 |\alpha|. \end{aligned} \quad (3.60)$$

Indeed, by Lemma 3.2 (viii),

$$\begin{aligned} \left\| \frac{\overline{\psi}_-^2(z, \sigma_0)}{1 + |z|^3} \right\|_{L^\infty(\mathbb{R})} & \leq C \left\| \frac{\widehat{q}_-(\tilde{z}, s_0)}{1 + |\tilde{z}|^3} \right\|_{L^\infty(\mathbb{R})} + C|\alpha|(|\widehat{q}_1(s_0)| + |\widehat{q}_2(s_0)|) \\ & \quad + O((s_0^{-2} + |\tau| + |\alpha|^3)\|g^2(s_0)\|_{L^\infty}). \end{aligned} \quad (3.61)$$

Hence, by (3.42), (3.43) and (3.47), (3.60) holds.

(ix) For s_0 large enough, by Lemma 3.2 (ix), (3.42), (3.43) and (3.47),

$$\begin{aligned} \|\overline{\psi}_e^2(\sigma_0)\|_{L^\infty(\mathbb{R})} & \leq \|\widehat{q}_e(\tilde{z}, s_0)\|_{L^\infty(\mathbb{R})} + C(|\tau| + |\alpha|)\|g^2(s_0)\|_{L^\infty} \\ & \leq \frac{C\widehat{A}^2}{s_0^{\frac{1}{2}}}(|\tau| + |\alpha|) + \frac{C\widehat{A}^2}{s_0^{1/2}}. \end{aligned} \quad (3.62)$$

Step 3. Expansion of $\overline{\psi}^3$ and $\overline{\psi}^4$

It is easy to check that

$$\begin{aligned} \varphi(z, s) & = \left(p - 1 + \frac{(p-1)^2 z^2}{4p s} \right)^{-\frac{1}{p-1}} + \frac{(p-1)^{-\frac{1}{p-1}}}{2ps} \\ & = \kappa - \frac{\kappa}{4ps}(|z|^2 - 2) + O\left(\frac{|z|^4}{s^2}\right). \end{aligned} \quad (3.63)$$

Recalling that $k_2(z) = \frac{z^2-2}{8}$, it follows that

$$\begin{aligned} \int \varphi(z, s) k_2(z) \rho(z) dz & = \int \left[\kappa - \frac{\kappa}{4ps}(|z|^2 - 2) + O\left(\frac{|z|^4}{s^2}\right) \right] \frac{z^2 - 2}{8} \rho(z) dz \\ & = -\frac{\kappa}{4ps} + O(s^{-2}), \end{aligned} \quad (3.64)$$

and we have that for s_0 large enough,

$$\varphi_0(s) = \kappa + O(s^{-2}), \quad \varphi_2(s) = -\frac{\kappa}{4ps} + O(s^{-2}), \quad (3.65)$$

$$\varphi_1(s) = \varphi_3(s) = 0, \quad \varphi_4(s) = O(s^{-2}), \quad |\varphi(s)| \leq C. \quad (3.66)$$

Define

$$g^2(\tilde{z}, s_0) = \begin{cases} \varphi(\tilde{z}, s_0), & \tilde{z} \in (\Omega - \widehat{a})e^{s_0/2}, \\ 0, & \tilde{z} \notin (\Omega - \widehat{a})e^{s_0/2}. \end{cases} \quad (3.67)$$

Then if s_0 is large enough, we have

$$g_i^2(s_0) = \varphi_i(s_0), i = 0, 1, 2, 3, 4, \|g^2(s_0)\| \leq C. \quad (3.68)$$

Let

$$\eta(z, \sigma_0) = \begin{cases} 0, & z \in (\Omega - a)e^{\sigma_0/2}, \\ (1 + \tau)^{\frac{1}{p-1}} \varphi(\tilde{z}, s_0), & z \notin (\Omega - a)e^{\sigma_0/2}. \end{cases} \quad (3.69)$$

By (3.24) and (3.27), it holds that $\bar{\psi}^3(z, \sigma_0) + \eta(z, \sigma_0) = (1 + \tau)^{\frac{1}{p-1}} \varphi(\tilde{z}, s_0)$, $z \in \mathbb{R}$.

Since $|a - \hat{a}| = |\alpha|e^{-s_0/2} \leq e^{-s_0/2}$, $(\hat{a} - 4\hat{\delta}, \hat{a} + 4\hat{\delta}) \subset \omega$ and $\hat{\varepsilon}_0 \in (0, \hat{\delta}]$, we have that when s_0 is large enough, which implies σ_0 is large enough, if $z \notin (\Omega - a)e^{\sigma_0/2}$, then $|z| \geq 2K_0\sqrt{\sigma_0}$. Thus,

$$\eta\chi_1(z, \sigma_0) = 0.$$

Hence, we have that for s_0 large enough,

$$\bar{\psi}_0^3(\sigma_0) = \int_{\mathbb{R}} (1 + \tau)^{\frac{1}{p-1}} \varphi(z\sqrt{1 + \tau} + \alpha, \sigma_0 + \log(1 + \tau)) \chi_1(z, \sigma_0) \rho(z) dz.$$

On the other hand, by (3.25) and (3.27),

$$\bar{\psi}_0^4(\sigma_0) = - \int_{\mathbb{R}} \varphi(z, \sigma_0) \chi_1(z, \sigma_0) \rho(z) dz.$$

Thus, we can get that for s_0 large enough,

$$\begin{aligned} & \left| \left(\frac{\sigma_0}{s_0} \right)^2 \bar{\psi}_0^3(\sigma_0) + \left(\frac{\sigma_0}{s_0} \right)^2 \bar{\psi}_0^4(\sigma_0) - \frac{\kappa\tau}{p-1} \right| \\ & \leq C e^{-\sigma_0} + C \frac{|\tau|}{\sigma_0} + C \frac{\alpha^2}{\sigma_0} + C \tau^2. \end{aligned} \quad (3.70)$$

Remember $b = \frac{(p-1)^2}{4p}$, $\frac{2b\kappa}{(p-1)^2} = \frac{\kappa}{2p}$. By Lemma 3.2 (ii), (3.65), (3.66) and (3.68), for s_0 large enough,

$$\begin{aligned} & \left| \left(\frac{\sigma_0}{s_0} \right)^2 \bar{\psi}_1^3(\sigma_0) + \frac{2b\kappa\alpha}{(p-1)^2 s_0} \right| \\ & = \left| \left(\frac{\sigma_0}{s_0} \right)^2 \bar{\psi}_1^3(\sigma_0) + \frac{\kappa}{2p} \frac{\alpha}{s_0} \right| \\ & \leq O\left(\frac{|\alpha|}{s_0^2}\right) + O(e^{-s_0} + \tau^2 + \frac{|\tau|}{s_0} + |\tau\alpha| + |\alpha|^3) \|\varphi(s_0)\|_{L^\infty} \\ & \leq C \left(\frac{|\alpha|}{s_0^2} + e^{-s_0} + \tau^2 + \frac{|\tau|}{s_0} + |\tau\alpha| + |\alpha|^3 \right). \end{aligned} \quad (3.71)$$

By Lemma 3.2 (iii), (3.65), (3.66) and (3.68), for s_0 large enough,

$$\begin{aligned} & \bar{\psi}_2^3(\sigma_0) \\ & = - \frac{\kappa}{4ps_0} + O\left(\frac{1}{s_0^2}\right) + O\left(\frac{\alpha^2}{s_0^2}\right) + O(e^{-s_0} + |\tau| + |\alpha|^3) \|\varphi(s_0)\|_{L^\infty} \\ & = - \frac{\kappa}{4ps_0} + O\left(\frac{1}{s_0^2}\right) + O(e^{-s_0} + |\tau| + |\alpha|^3). \end{aligned} \quad (3.72)$$

By Lemma 3.2 (iv), (3.65), (3.66) and (3.68), for s_0 large enough,

$$\begin{aligned}\sigma_0^{-2}\partial_\tau[\sigma_0^2\bar{\psi}_0^3](\sigma_0) &= \frac{\kappa}{p-1} + O\left(\frac{1}{s_0^2}\right) + O\left(\frac{1}{s_0}\right) + O\left(\frac{1}{s_0^2}\right) + O\left(\frac{1}{s_0} + |\tau| + |\alpha|\right)\|\varphi(s_0)\|_{L^\infty} \\ &= \frac{\kappa}{p-1} + O\left(\frac{1}{s_0}\right) + O(|\tau| + |\alpha|).\end{aligned}\quad (3.73)$$

By Lemma 3.2 (v), (3.66) and (3.68), for s_0 large enough,

$$\begin{aligned}|s_0^{-2}\partial_\tau[\sigma_0^2\bar{\psi}_1^3](\sigma_0)| &= O\left(\frac{1}{s_0} + |\tau| + |\alpha|\right)\|\varphi(s_0)\|_{L^\infty} \\ &\leq C\left(\frac{1}{s_0} + |\tau| + |\alpha|\right).\end{aligned}\quad (3.74)$$

By Lemma 3.2 (vi), (3.65), (3.66) and (3.68), for s_0 large enough,

$$\begin{aligned}|\partial_\alpha\bar{\psi}_0^3(\sigma_0)| &= |O(\alpha\varphi_2(s_0)) + O(e^{-s_0} + |\tau| + |\alpha|^2)\|\varphi(s_0)\|_{L^\infty}| \\ &\leq C\left(e^{-s_0} + |\tau| + |\alpha|^2 + \frac{|\alpha|}{s_0}\right).\end{aligned}\quad (3.75)$$

By Lemma 3.2 (vii), (3.65), (3.66) and (3.68), for s_0 large enough,

$$\begin{aligned}\partial_\alpha\bar{\psi}_1^3(\sigma_0) &= 2\varphi_2(s_0) + O(e^{-s_0} + |\tau| + |\alpha|^2)\|\varphi(s_0)\|_{L^\infty} \\ &= -\frac{\kappa}{2p}\frac{1}{s_0} + O\left(\frac{1}{s_0^2}\right) + O(e^{-s_0} + |\tau| + |\alpha|^2) \\ &= -\frac{\kappa}{2p}\frac{1}{s_0} + O\left(\frac{1}{s_0^2}\right) + O(|\tau| + |\alpha|^2) \\ &= -\frac{2b\kappa}{(p-1)^2}\frac{1}{s_0} + O\left(\frac{1}{s_0^2}\right) + O(|\tau| + |\alpha|^2).\end{aligned}\quad (3.76)$$

Now for the remaining components, we need more refined estimates, for s_0 large enough and for all $z \in (\Omega - a)e^{\sigma_0/2}$,

$$|\bar{\psi}^3(z, \sigma_0) - \varphi(z, s_0)| \leq C\left(|\tau| + \frac{|\alpha|}{s_0}\right)(1 + |z|^3).\quad (3.77)$$

Indeed, for $z \in (\Omega - a)e^{\sigma_0/2}$,

$$\varphi_z(z, s) = -\frac{1}{p-1}\left(p-1 + b\frac{z^2}{s}\right)^{-\frac{1}{p-1}-1}\frac{2z}{4ps}(p-1)^2.\quad (3.78)$$

Then, we have that for s_0 large enough,

$$\begin{aligned}&|\bar{\psi}^3(z, \sigma_0) - \varphi(z, s_0)| \\ &= |(1 + \tau)^{\frac{1}{p-1}}\varphi(\tilde{z}, s_0) - \varphi(z, s_0)| \\ &= |(1 + \tau)^{\frac{1}{p-1}}\varphi(z\sqrt{(1 + \tau)} + \alpha, s_0) - (1 + \tau)^{\frac{1}{p-1}}\varphi(z, s_0) + (1 + \tau)^{\frac{1}{p-1}}\varphi(z, s_0) - \varphi(z, s_0)|\end{aligned}$$

$$\begin{aligned}
&\leq (1+\tau)^{\frac{1}{p-1}} |\varphi(z\sqrt{(1+\tau)} + \alpha, s_0) - \varphi(z, s_0)| + |[(1+\tau)^{\frac{1}{p-1}} - 1]\varphi(z, s_0)| \\
&\leq C \frac{1}{s_0} |z + \theta(z\sqrt{(1+\tau)} + \alpha - z)| |z\sqrt{(1+\tau)} + \alpha - z| + C|\tau| \\
&\leq C \frac{1}{s_0} (|z| + |z\tau| + |\alpha|)(|\tau z| + |\alpha|) + C|\tau| \\
&\leq C \left(|\tau| + \frac{|\alpha|}{s_0} \right) (1 + |z|^3). \tag{3.79}
\end{aligned}$$

For $z \notin (\Omega - a)e^{\sigma_0/2}$, $\bar{\psi}^3(z, \sigma_0) - \varphi(z, s_0) = -\varphi(z, s_0)$.

On the other hand, for s_0 large enough and for all $z \in (\Omega - a)e^{\sigma_0/2}$, we could obtain

$$|\bar{\psi}^3(z, \sigma_0) - \varphi(z, s_0)| \leq C \left(|\tau| + \frac{|\alpha|}{\sqrt{s_0}} \right). \tag{3.80}$$

For all s large enough and for all $z \in \mathbb{R}$, it holds by (3.63) that

$$\left| \varphi(z, s) - \kappa + \frac{\kappa}{4ps}(z^2 - 2) \right| \leq C \frac{|z|^4}{s^2}, \tag{3.81}$$

and we have

$$|\partial_s \varphi(z, s)| \leq C \min \left(\frac{1}{s}, \frac{1 + |z|^2}{s^2} \right). \tag{3.82}$$

Since $\bar{\psi}^4(z, \sigma_0) = -\varphi(z, \sigma_0)$ with $\sigma_0 = s_0 - \log(1 + \tau)$, it holds that for s_0 large enough,

$$\left| \frac{1}{\sigma_0} - \frac{1}{s_0} \right| \leq \frac{C}{s_0^2} |s_0 - \sigma_0| = \left| \frac{C}{s_0^2} \log(1 + \tau) \right| \leq \frac{C}{s_0^2} |\tau|.$$

Then, it follows that by (3.63), for s_0 large enough,

$$\begin{aligned}
&|\bar{\psi}^4(z, \sigma_0) + \kappa - \frac{\kappa}{4ps_0} h_2(z)| \\
&\leq C \frac{|z|^4}{s_0^2} + C \frac{|\tau|(1 + |z|^2)}{s_0^2}. \tag{3.83}
\end{aligned}$$

Moreover, we have that for s_0 large enough,

$$|\bar{\psi}^4(z, \sigma_0) + \varphi(z, s_0)| \leq C|\tau| \min \left(\frac{1}{s_0}, \frac{1 + |z|^2}{s_0^2} \right). \tag{3.84}$$

Thus, for s_0 large enough,

$$\left(\frac{\sigma_0}{s_0} \right)^2 \bar{\psi}_0^4(\sigma_0) = -\kappa + O(s_0^2) + O(\tau s_0^{-1}), \tag{3.85}$$

$$\bar{\psi}_1^4(\sigma_0) = 0, \quad \bar{\psi}_2^4(\sigma_0) = \frac{\kappa}{4ps_0} + O(s_0^{-2}) + O(\tau s_0^{-2}), \tag{3.86}$$

$$|s_0^{-2} \partial_\tau [\sigma_0^2 \bar{\psi}_0^4](\sigma_0)| \leq \frac{C}{s_0}, \quad (3.87)$$

and

$$\partial_\tau [\sigma_0^2 \bar{\psi}_1^4](\sigma_0) = 0, \quad \partial_\alpha \bar{\psi}_0^4(\sigma_0) = \partial_\alpha \bar{\psi}_1^4(\sigma_0) = 0. \quad (3.88)$$

In order to estimate $\bar{\psi}_-^3(z, \sigma_0) + \bar{\psi}_-^4(z, \sigma_0)$, we will use and easily check the following estimate:

$$\left\| \frac{v_-(z, \sigma_0)}{1 + |z|^3} \right\|_{L^\infty(\mathbb{R})} \leq C \left\| \frac{v(z, \sigma_0)}{1 + |z|^3} \right\|_{L^\infty(\mathbb{R})}. \quad (3.89)$$

For $z \notin (\Omega - a)e^{\sigma_0/2}$, it holds that for s_0 large enough,

$$\left| \frac{\bar{\psi}^3(z, \sigma_0) + \bar{\psi}^4(z, \sigma_0)}{1 + |z|^3} \right| = \frac{|-\varphi(z, \sigma_0)|}{1 + |z|^3} \leq C e^{-s_0}. \quad (3.90)$$

For $z \in (\Omega - a)e^{\sigma_0/2}$, we introduce

$$\Psi(z, s_0, \tau, \alpha) = \begin{cases} \bar{\psi}^3(z, \sigma_0) + \bar{\psi}^4(z, \sigma_0) \\ = (1 + \tau)^{\frac{1}{p-1}} \varphi(\tilde{z}, s_0) - \varphi(z, \sigma_0), & z \in (\Omega - a)e^{\sigma_0/2}, \\ 0, & z \notin (\Omega - a)e^{\sigma_0/2}, \end{cases} \quad (3.91)$$

and

$$\Psi_1(z, \sigma_0) = \begin{cases} 0, & z \in (\Omega - a)e^{\sigma_0/2}, \\ -\varphi(z, \sigma_0), & z \notin (\Omega - a)e^{\sigma_0/2}, \end{cases} \quad (3.92)$$

where

$$\varphi(z, s) = \left[p - 1 + \frac{(p-1)^2 z^2}{4p s} \right]^{-\frac{1}{p-1}} + \frac{\kappa}{2ps}, \quad \tilde{z} = z\sqrt{1 + \tau} + \alpha, \quad \sigma_0 = s_0 - \log(1 + \tau).$$

Note that $\Psi(z, s_0, 0, 0) = 0$. Making a Taylor expansion, we may write

$$\begin{aligned} & \Psi(z, s_0, \tau, \alpha) - \tau \Psi_\tau(z, s_0, 0, 0) - \alpha \Psi_\alpha(z, s_0, 0, 0) \\ &= \frac{1}{2} \Psi_{\alpha\alpha}(z, s_0, \theta(z, s_0)\tau, \theta(z, s_0)\alpha) \alpha^2 + \frac{1}{2} \Psi_{\tau\tau}(z, s_0, \theta(z, s_0)\tau, \theta(z, s_0)\alpha) \tau^2 \\ & \quad + \Psi_{\tau\alpha}(z, s_0, \theta(z, s_0)\tau, \theta(z, s_0)\alpha) \tau \alpha, \end{aligned} \quad (3.93)$$

where $0 < \theta(z, s_0) < 1$.

First, for $z \in (\Omega - a)e^{\sigma_0/2}$,

$$\begin{aligned}
& \Psi_\tau(z, s_0, \tau, \alpha) \\
&= \frac{1}{p-1} (1+\tau)^{\frac{1}{p-1}-1} \varphi(z\sqrt{1+\tau} + \alpha, s_0) \\
&\quad - \frac{p-1}{2p} (1+\tau)^{\frac{1}{p-1}} \left[p-1 + \frac{(p-1)^2}{4p} \frac{(z\sqrt{1+\tau} + \alpha)^2}{s_0} \right]^{-\frac{1}{p-1}-1} \frac{(z\sqrt{1+\tau} + \alpha)}{s_0} \times \\
&\quad \times \frac{z}{2\sqrt{1+\tau}} \\
&\quad + \frac{p-1}{4p} \left[p-1 + \frac{(p-1)^2}{4p} \frac{z^2}{s_0 - \log(1+\tau)} \right]^{-\frac{1}{p-1}-1} \frac{z^2}{(s_0 - \log(1+\tau))^2} \frac{1}{1+\tau} \\
&\quad - \frac{\kappa}{2p(s_0 - \log(1+\tau))^2} \frac{1}{1+\tau}, \tag{3.94}
\end{aligned}$$

which implies for $z \in (\Omega - a)e^{\sigma_0/2}$,

$$\begin{aligned}
& \Psi_\tau(z, s_0, 0, 0) \\
&= \frac{1}{p-1} \varphi(z, s_0) - \frac{p-1}{2p} \left[p-1 + \frac{(p-1)^2}{4p} \frac{z^2}{s_0} \right]^{-\frac{1}{p-1}-1} \frac{z^2}{2s_0} \\
&\quad + \frac{p-1}{4p} \left[p-1 + \frac{(p-1)^2}{4p} \frac{z^2}{s_0} \right]^{-\frac{1}{p-1}-1} \frac{z^2}{s_0^2} - \frac{\kappa}{2ps_0^2}. \tag{3.95}
\end{aligned}$$

For $z \notin (\Omega - a)e^{\sigma_0/2}$,

$$\Psi_\tau(z, s_0, 0, 0) = 0. \tag{3.96}$$

In (3.95), we see four terms. Using the usual trick, we can bound the projection on the negative part (after cut-off) of the 2nd, 3rd and 4th term by

$$\frac{C}{s_0} (1 + |z|^3),$$

which after multiplication by τ , should give us a convenient rate.

Let us then try to bound the projection on the negative part (after cut-off) of $\varphi(z, s_0)$, namely

$$[\varphi(z, s_0)]_-(z, \sigma_0) = \chi_1(z, \sigma_0) \varphi(z, s_0) - \sum_{j=0}^2 [\varphi(z, s_0)]_j(\sigma_0) h_j(\sigma_0).$$

Note first that

$$\chi_1(z, \sigma_0) = 1 + O\left(\frac{|z|^2}{\sigma_0}\right),$$

and

$$\varphi(z, s_0) = \kappa + O\left(\frac{|z|^2}{s_0}\right) + \frac{\kappa}{2ps_0}.$$

Recalling that φ is even in z , this implies that for s_0 large enough,

$$\begin{aligned}
[\varphi(z, s_0)]_0(\sigma_0) &= \int \chi_1(z, \sigma_0) k_0(z) \varphi(z, s_0) \rho(z) dz \\
&= \int \chi_1(z, \sigma_0) \left[\kappa + O\left(\frac{|z|^2}{s_0}\right) + \frac{\kappa}{2ps_0} \right] \rho(z) dz \\
&= \int [(\chi_1(z, \sigma_0) - 1)\kappa + \kappa] \rho(z) dz + \int \chi_1(z, \sigma_0) \left[O\left(\frac{|z|^2}{s_0}\right) + \frac{\kappa}{2ps_0} \right] \rho(z) dz \\
&= \kappa + O\left(\frac{1}{s_0}\right), \tag{3.97}
\end{aligned}$$

$$[\varphi(z, s_0)]_1(\sigma_0) = 0,$$

and by (3.64), for s_0 large enough,

$$\begin{aligned}
[\varphi(z, s_0)]_2(\sigma_0) &= \int \chi_1(z, \sigma_0) k_2(z) \varphi(z, s_0) \rho(z) dz \\
&= \int [(\chi_1(z, \sigma_0) - 1)k_2(z) \varphi(z, s_0) \rho(z) dz + \int k_2(z) \varphi(z, s_0) \rho(z) dz \\
&= \int [(\chi_1(z, \sigma_0) - 1)k_2(z) \varphi(z, s_0) \rho(z) dz - \frac{\kappa}{4ps_0} + O\left(\frac{1}{s_0}\right) \\
&= O\left(\frac{1}{s_0}\right). \tag{3.98}
\end{aligned}$$

This way, for s_0 large enough,

$$\begin{aligned}
&[\varphi(z, s_0)]_-(z, \sigma_0) \\
&= \left[1 + O\left(\frac{|z|^2}{s_0}\right) \right] \left[\kappa + O\left(\frac{|z|^2}{s_0}\right) + \frac{\kappa}{2ps_0} \right] - \left(\kappa + O\left(\frac{1}{s_0}\right) \right) h_0(z) - O\left(\frac{1}{s_0}\right) h_2(z) \\
&= \kappa + O\left(\frac{|z|^2}{s_0}\right) - \kappa + O\left(\frac{1}{s_0}\right) + O\left(\frac{1 + |z|^2}{s_0}\right) \\
&= O\left(\frac{1 + |z|^2}{s_0}\right). \tag{3.99}
\end{aligned}$$

Thus, for s_0 large enough,

$$|[\varphi(z, s_0)]_-(z, \sigma_0)| \leq \frac{C}{s_0} (1 + |z|^2) \leq \frac{C}{s_0} (1 + |z|^3),$$

from which we have that for s_0 large enough,

$$|\tau(\Psi_\tau)_-(z, s_0, 0, 0)| \leq \frac{C|\tau|}{s_0} (1 + |z|^3). \tag{3.100}$$

Second, for $z \in (\Omega - a)e^{\sigma_0/2}$,

$$\begin{aligned}
&\Psi_\alpha(z, s_0, \tau, \alpha) \\
&= -\frac{1}{p-1} (1 + \tau)^{\frac{1}{p-1}} \left[p - 1 + \frac{(p-1)^2 (z\sqrt{1+\tau} + \alpha)^2}{4ps_0} \right]^{-\frac{1}{p-1}-1} \frac{(p-1)^2 (z\sqrt{1+\tau} + \alpha)}{2ps_0}. \tag{3.101}
\end{aligned}$$

Hence, for $z \in (\Omega - a)e^{\sigma_0/2}$,

$$\Psi_\alpha(z, s_0, 0, 0) = -\frac{1}{p-1}(1+\tau)^{\frac{1}{p-1}} \left[p-1 + \frac{(p-1)^2 z^2}{4p s_0} \right]^{-\frac{1}{p-1}-1} \frac{(p-1)^2 z}{2ps_0}. \quad (3.102)$$

Write $\bar{\varphi} = \left[p-1 + \frac{(p-1)^2 z^2}{4p s_0} \right]^{-\frac{1}{p-1}-1} z$. Then $\bar{\varphi} = (p-1)^{-\frac{1}{p-1}-1} z + O\left(\frac{|z|^2}{s_0}\right)z$.

It is easy to check that $\bar{\varphi}_0(\sigma_0) = \bar{\varphi}_2(\sigma_0) = 0$, and for s_0 large enough,

$$\begin{aligned} \bar{\varphi}_1 &= \int (\chi_1 - 1 + 1)(p-1)^{-\frac{1}{p-1}-1} \frac{z^2}{2} \rho dz + \int \chi_1(z, \sigma_0) O\left(\frac{|z|^2}{s_0}\right) \frac{z^2}{2} \rho dz \\ &= \int (p-1)^{-\frac{1}{p-1}-1} \left(\frac{z^2-2}{2}\right) \rho dz + \int (p-1)^{-\frac{1}{p-1}-1} \rho dz + O\left(\frac{1}{s_0}\right) \\ &= (p-1)^{-\frac{1}{p-1}-1} + O\left(\frac{1}{s_0}\right). \end{aligned} \quad (3.103)$$

Thus, for s_0 large enough,

$$\begin{aligned} &(\Psi_\alpha)(z, s_0, 0, 0)_-(z, \sigma_0) \\ &= \left[1 + O\left(\frac{|z|^2}{s_0}\right) \right] \left[- (1+\tau)^{\frac{1}{p-1}} \frac{(p-1)}{2ps_0} (p-1)^{-\frac{1}{p-1}-1} z - (1+\tau)^{\frac{1}{p-1}} \frac{(p-1)}{2ps_0} O\left(\frac{|z|^2}{s_0}\right) z \right] \\ &\quad + (1+\tau)^{\frac{1}{p-1}} \frac{(p-1)}{2ps_0} \left[(p-1)^{-\frac{1}{p-1}-1} + O\left(\frac{1}{s_0}\right) \right] z \\ &= O\left(\frac{1+|z|^3}{s_0^2}\right), \end{aligned} \quad (3.104)$$

which implies that for s_0 large enough,

$$|\alpha(\Psi_\alpha)_-(z, s_0, 0, 0)| \leq \frac{C|\alpha|}{s_0^2} (1+|z|^3). \quad (3.105)$$

Further, we can check that for s_0 large enough,

$$\begin{aligned} |\Psi_{\tau\tau}(z, s_0, \theta(z, s_0)\tau, \theta(z, s_0)\alpha)\tau^2| &\leq C(1+|z|^3)\tau^2, \\ |\Psi_{\tau\alpha}(z, s_0, \theta(z, s_0)\tau, \theta(z, s_0)\alpha)\tau\alpha| &\leq C(1+|z|^3)\left(\tau^2 + \frac{\alpha^2}{s_0}\right). \end{aligned}$$

Hence, for s_0 large enough,

$$\left| \frac{1}{2}(\Psi_{\tau\tau})_-(z, s_0, \theta(z, s_0)\tau, \theta(z, s_0)\alpha)\tau^2 \right| \leq C(1+|z|^3)\tau^2, \quad (3.106)$$

$$|(\Psi_{\tau\alpha})_-(z, s_0, \theta(z, s_0)\tau, \theta(z, s_0)\alpha)\tau\alpha| \leq C(1+|z|^3)\left(\tau^2 + \frac{\alpha^2}{s_0}\right). \quad (3.107)$$

Finally, we have

$$\begin{aligned} &\Psi_{\alpha\alpha}(z, s_0, \tau, \alpha) \\ &= \frac{p}{(p-1)^2} (1+\tau)^{\frac{1}{p-1}} \left[p-1 + \frac{(p-1)^2 (z\sqrt{1+\tau} + \alpha)^2}{4p s_0} \right]^{-\frac{1}{p-1}-2} \frac{(p-1)^4 (z\sqrt{1+\tau} + \alpha)^2}{4(ps_0)^2} \\ &\quad - \frac{p-1}{2ps_0} (1+\tau)^{\frac{1}{p-1}} \left[p-1 + \frac{(p-1)^2 (z\sqrt{1+\tau} + \alpha)^2}{4p s_0} \right]^{-\frac{1}{p-1}-1} = J_1 + J_2. \end{aligned} \quad (3.108)$$

For J_1 , it holds that for s_0 large enough,

$$|J_1(z, s_0, \theta(z, s_0)\tau, \theta(z, s_0))\alpha^2| \leq C(1 + |z|^3) \frac{\alpha^2}{s_0^2},$$

which implies that for s_0 large enough,

$$|(J_1)_-(z, s_0, \theta(z, s_0)\tau, \theta(z, s_0))\alpha^2| \leq C(1 + |z|^3) \frac{\alpha^2}{s_0^2}. \quad (3.109)$$

For J_2 , we have

$$\begin{aligned} & J_2(z, s_0, \theta(z, s_0)\tau, \theta(z, s_0)) \\ &= -\frac{p-1}{2ps_0}(1+\theta\tau)^{\frac{1}{p-1}} \left[p-1 + \frac{(p-1)^2}{4p} \frac{(z\sqrt{1+\theta\tau} + \theta\alpha)^2}{s_0} \right]^{-\frac{1}{p-1}-1} \\ &= -\frac{p-1}{2ps_0}(1+\theta\tau)^{\frac{1}{p-1}} \left\{ \left[p-1 + \frac{(p-1)^2}{4p} \frac{(z\sqrt{1+\theta\tau} + \theta\alpha)^2}{s_0} \right]^{-\frac{1}{p-1}-1} \right. \\ &\quad \left. - \left[p-1 + \frac{(p-1)^2}{4p} \frac{z^2}{s_0} \right]^{-\frac{1}{p-1}-1} \right\} - \frac{p-1}{2ps_0}(1+\theta\tau)^{\frac{1}{p-1}} \left\{ \left[p-1 + \frac{(p-1)^2}{4p} \frac{z^2}{s_0} \right]^{-\frac{1}{p-1}-1} \right. \\ &\quad \left. + \frac{(p-1)^{-\frac{1}{p-1}-1}}{2s_0} \right\} + \frac{(p-1)^{-\frac{1}{p-1}}}{4ps_0^2}(1+\theta\tau)^{\frac{1}{p-1}} = J_2^1 + J_2^2 + J_2^3. \end{aligned}$$

It is easy to check that

$$|(J_2^1)_-(z, s_0, \theta(z, s_0)\tau, \theta(z, s_0))\alpha^2| \leq C(1 + |z|^3) \frac{\alpha^2}{s_0^2}, \quad (3.110)$$

and

$$|(J_2^3)_-(z, s_0, \theta(z, s_0)\tau, \theta(z, s_0))\alpha^2| \leq C(1 + |z|^3) \frac{\alpha^2}{s_0^2}, \quad (3.111)$$

and by the similar argument used to prove (3.99), we have

$$|(J_2^2)_-(z, s_0, \theta(z, s_0)\tau, \theta(z, s_0))\alpha^2| \leq C(1 + |z|^3) \frac{\alpha^2}{s_0^2}. \quad (3.112)$$

Hence, by (3.93), (3.100), (3.105)-(3.107), (3.109)-(3.112), it holds that for s_0 large enough,

$$\left\| \frac{\bar{\psi}_-^3(z, \sigma_0) + \bar{\psi}_-^4(z, \sigma_0)}{1 + |z|^3} \right\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C}{s_0} |\tau| + C \frac{|\alpha|}{s_0^2} + C \frac{|\alpha|^2}{s_0^2} + C |\tau|^2 + C e^{-s_0}. \quad (3.113)$$

When $z \in (\Omega - a)e^{\sigma_0/2}$, it holds that by (3.77) that for s_0 large enough,

$$|\bar{\psi}_e^3(z, \sigma_0) + \bar{\psi}_e^4(z, \sigma_0)| = |(1 - \chi_1)(\bar{\psi}^3(z, \sigma_0) + \bar{\psi}^4(z, \sigma_0))| \leq C \left(|\tau| + \frac{|\alpha|}{\sqrt{s_0}} \right). \quad (3.114)$$

When $z \notin (\Omega - a)e^{\sigma_0/2}$, for s_0 large enough,

$$|\bar{\psi}_e^3(z, \sigma_0) + \bar{\psi}_e^4(z, \sigma_0)| = |(1 - \chi_1)\varphi(z, \sigma_0)| \leq \frac{C}{\sigma_0}. \quad (3.115)$$

Step 4. Expansion of $\bar{\psi}^5$

(i) For $\bar{\psi}_0^5(\sigma_0)$, we have that when s_0 is large enough such that $|a - \hat{a}| = |\alpha e^{-\frac{s_0}{2}}| \leq e^{-\frac{s_0}{2}} < \frac{\varepsilon_0}{4}$,

$$|\bar{\psi}_0^5(\sigma_0)| \leq C|\alpha|e^{-s_0/2}. \quad (3.116)$$

Indeed,

$$\begin{aligned} \bar{\psi}_0^5(\sigma_0) &= \int_{\mathbb{R}} \bar{\psi}^5(\sigma_0)k_0(z)\chi_1(z, \sigma_0)\rho(z)dz \\ &= \int_{(\Omega-a)e^{\sigma_0/2}} e^{-\frac{\sigma_0}{p-1}}\hat{y}(e^{-\frac{\sigma_0}{2}}z + a, T - e^{-\sigma_0})[\chi(z, \sigma_0) - \chi(\tilde{z}, s_0)]\chi_1(z, \sigma_0)\rho(z)dz, \end{aligned}$$

where

$$\chi(z, \sigma_0) = \chi_0\left(\frac{ze^{-\sigma_0/2}}{\varepsilon_0}\right), \quad \chi(\tilde{z}, s_0) = \chi_0\left(\frac{\tilde{z}e^{-s_0/2}}{\varepsilon_0}\right) = \chi_0\left(\frac{ze^{-\sigma_0/2} + a - \hat{a}}{\varepsilon_0}\right).$$

On one hand, when $x \in \Omega$ with $|x - \hat{a}| \leq \frac{\varepsilon_0}{2}$ and s_0 is large enough such that $|a - \hat{a}| < \frac{\varepsilon_0}{4}$, we have $\chi(z, \sigma_0) - \chi(\tilde{z}, s_0) = 0$.

On the other hand, for $x \in \Omega$ with $|x - \hat{a}| \geq \varepsilon_0/2$, it holds that

$$\left| \chi_0\left(\frac{ze^{-\sigma_0/2}}{\varepsilon_0}\right) - \chi_0\left(\frac{ze^{-\sigma_0/2} + a - \hat{a}}{\varepsilon_0}\right) \right| \leq \frac{C}{\varepsilon_0}|a - \hat{a}|,$$

and

$$|\hat{y}(e^{-\frac{\sigma_0}{2}}z + a, T - e^{-\sigma_0})| = |\hat{y}(e^{-\frac{s_0}{2}}\tilde{z} + \hat{a}, \hat{T} - e^{-s_0})| \leq \hat{\eta}_0 \leq 1.$$

Hence, when s_0 is large enough such that $|a - \hat{a}| = |\alpha e^{-\frac{s_0}{2}}| \leq e^{-\frac{s_0}{2}} < \frac{\varepsilon_0}{4}$,

$$|\bar{\psi}_0^5(\sigma_0)| \leq \frac{C}{\varepsilon_0}e^{-\frac{\sigma_0}{p-1}}|a - \hat{a}| = \frac{C}{\varepsilon_0}e^{-\frac{\sigma_0}{p-1}}|a - \hat{a}|e^{s_0/2}e^{-s_0/2} \leq C|\alpha|e^{-s_0/2}. \quad (3.117)$$

(ii) Similarly to (3.116), we have that when s_0 is large enough,

$$\left| \frac{\sigma_0^2}{s_0^2}\bar{\psi}_1^5(\sigma_0) \right| \leq C|\alpha|e^{-s_0/2}, \quad (3.118)$$

and

$$|\bar{\psi}_2^5(\sigma_0)| \leq C|\alpha|e^{-s_0/2}. \quad (3.119)$$

(iii) For $\sigma_0^{-2}\partial_\tau[\sigma_0^2\bar{\psi}_0^5]$, we have that when s_0 is large enough,

$$\left| \sigma_0^{-2}\partial_\tau[\sigma_0^2\bar{\psi}_0^5] \right| \leq Ce^{-Cs_0}(1 + |\alpha|). \quad (3.120)$$

Similarly to $\bar{\psi}_0^{-1}(\sigma_0)$, it holds that for s_0 large enough,

$$\begin{aligned}
& \left(\frac{\sigma_0}{s_0}\right)^2 \bar{\psi}_0^{-5}(\sigma_0) \\
&= \left(\frac{\sigma_0}{s_0}\right)^2 \int_{\mathbb{R}} e^{-\frac{\sigma_0}{p-1}} \hat{y}(e^{-\frac{s_0}{2}} \tilde{z} + \hat{a}, \hat{T} - e^{-s_0}) \chi_1(z, \sigma_0) \rho(z) [\chi(z, \sigma_0) - \chi(\tilde{z}, s_0)] dz \\
&= \left(\frac{s_0 - \log(1 + \tau)}{s_0}\right)^2 (1 + \tau)^{-\frac{1}{2}} \int_{\mathbb{R}} \chi_1(z, s_0 - \log(1 + \tau)) e^{-\frac{\sigma_0}{p-1}} \hat{y}(e^{-\frac{s_0}{2}} \tilde{z} + \hat{a}, \hat{T} - e^{-s_0}) \\
&\quad \cdot [\chi(z, \sigma_0) - \chi(\tilde{z}, s_0)] \rho(z) d\tilde{z}. \tag{3.121}
\end{aligned}$$

Hence,

$$\begin{aligned}
& \frac{\partial}{\partial \tau} \left[\left(\frac{\sigma_0}{s_0}\right)^2 \bar{\psi}_0^{-5} \right] (\sigma_0) = -2 \frac{1}{s_0(1 + \tau)} \frac{s_0 - \log(1 + \tau)}{s_0} \int_{(\Omega-a)e^{\sigma_0/2}} \chi_1(z, s_0 - \log(1 + \tau)) \\
& e^{-\frac{\sigma_0}{p-1}} \hat{y}(e^{-\frac{s_0}{2}} \tilde{z} + \hat{a}, \hat{T} - e^{-s_0}) [\chi(z, \sigma_0) - \chi(\tilde{z}, s_0)] \rho(z) dz \\
& - \frac{1}{2} (1 + \tau)^{-1} \left(\frac{s_0 - \log(1 + \tau)}{s_0}\right)^2 \int_{(\Omega-a)e^{\sigma_0/2}} \chi_1(z, s_0 - \log(1 + \tau)) e^{-\frac{\sigma_0}{p-1}} \hat{y}(e^{-\frac{s_0}{2}} \tilde{z} \\
& + \hat{a}, \hat{T} - e^{-s_0}) [\chi(z, \sigma_0) - \chi(\tilde{z}, s_0)] \rho(z) dz \\
& + \frac{1}{(p-1)(1 + \tau)} \left(\frac{s_0 - \log(1 + \tau)}{s_0}\right)^2 \int_{(\Omega-a)e^{\sigma_0/2}} \chi_1(z, s_0 - \log(1 + \tau)) e^{-\frac{\sigma_0}{p-1}} \hat{y}(e^{-\frac{s_0}{2}} \tilde{z} \\
& + \hat{a}, \hat{T} - e^{-s_0}) [\chi(z, \sigma_0) - \chi(\tilde{z}, s_0)] \rho(z) dz \\
& - \left(\frac{s_0 - \log(1 + \tau)}{s_0}\right)^2 \int_{(\Omega-a)e^{\sigma_0/2}} (1 + \tau)^{-1} \chi_{1\sigma_0}(z, s_0 - \log(1 + \tau)) e^{-\frac{\sigma_0}{p-1}} \hat{y}(e^{-\frac{s_0}{2}} \tilde{z} \\
& + \hat{a}, \hat{T} - e^{-s_0}) [\chi(z, \sigma_0) - \chi(\tilde{z}, s_0)] \rho(z) dz \\
& - \frac{1}{2} \left(\frac{s_0 - \log(1 + \tau)}{s_0}\right)^2 \int_{(\Omega-a)e^{\sigma_0/2}} (1 + \tau)^{-1} z \chi_{1z}(z, s_0 - \log(1 + \tau)) e^{-\frac{\sigma_0}{p-1}} \hat{y}(e^{-\frac{s_0}{2}} \tilde{z} \\
& + \hat{a}, \hat{T} - e^{-s_0}) [\chi(z, \sigma_0) - \chi(\tilde{z}, s_0)] \rho(z) dz \\
& - \frac{1}{2} \left(\frac{s_0 - \log(1 + \tau)}{s_0}\right)^2 \int_{(\Omega-a)e^{\sigma_0/2}} (1 + \tau)^{-1} z \chi_1(z, s_0 - \log(1 + \tau)) e^{-\frac{\sigma_0}{p-1}} \hat{y}(e^{-\frac{s_0}{2}} \tilde{z} \\
& + \hat{a}, \hat{T} - e^{-s_0}) \chi_z(z, \sigma_0) \rho(z) dz + \left(\frac{s_0 - \log(1 + \tau)}{s_0}\right)^2 \int_{(\Omega-a)e^{\sigma_0/2}} \frac{z^2}{4} e^{-\frac{\sigma_0}{p-1}} \hat{y}(e^{-\frac{s_0}{2}} \tilde{z} \\
& + \hat{a}, \hat{T} - e^{-s_0}) \chi_1(z, \sigma_0) [\chi(z, \sigma_0) - \chi(\tilde{z}, s_0)] (1 + \tau)^{-1} \rho(z) dz \\
& - \left(\frac{s_0 - \log(1 + \tau)}{s_0}\right)^2 \int_{(\Omega-a)e^{\sigma_0/2}} (1 + \tau)^{-1} \chi_1(z, s_0 - \log(1 + \tau)) e^{-\frac{\sigma_0}{p-1}} \hat{y}(e^{-\frac{s_0}{2}} \tilde{z} \\
& + \hat{a}, \hat{T} - e^{-s_0}) \chi_{\sigma_0}(z, \sigma_0) \rho(z) dz. \tag{3.122}
\end{aligned}$$

For $|x - \hat{a}| \geq \varepsilon_0/2$, we have

$$|\hat{y}(e^{-\frac{\sigma_0}{2}} z + a, T - e^{-\sigma_0})| = |\hat{y}(e^{-\frac{s_0}{2}} \tilde{z} + \hat{a}, \hat{T} - e^{-s_0})| \leq \hat{\eta}_0 \leq 1.$$

If $|x - \hat{a}| < \frac{\varepsilon_0}{2}$ and $|a - \hat{a}| < \frac{\varepsilon_0}{4}$, then $\chi(z, \sigma_0) - \chi(\tilde{z}, s_0) = 0$, which implies $\partial_\tau [\chi(z, \sigma_0) - \chi(\tilde{z}, s_0)] = 0$. Thus, for s_0 large enough, we can get (3.120) by the similar argument used in the proof of (3.38).

(iv) For $\partial_\tau \left[\frac{\sigma_0^2}{s_0^2} \bar{\psi}_1^5 \right]$, we have that for s_0 large enough,

$$\left| \partial_\tau \left[\frac{\sigma_0^2}{s_0^2} \bar{\psi}_1^5 \right] \right| \leq C e^{-C s_0} (1 + |\alpha|). \quad (3.123)$$

Since

$$\begin{aligned} \bar{\psi}_1^5(\sigma_0) &= \int_{\mathbb{R}} k_1(z) \bar{\psi}^5(\sigma_0) \chi_1(z, \sigma_0) \rho(z) dz \\ &= \int_{(\Omega-a)e^{\sigma_0/2}} \frac{z}{2} e^{-\frac{\sigma_0}{p-1}} \hat{y}(e^{-\frac{\sigma_0}{2}} z + a, T - e^{-\sigma_0}) \chi_1(z, \sigma_0) [\chi(z, \sigma_0) - \chi(\tilde{z}, s_0)] \rho(z) dz, \end{aligned}$$

similarly to (iii), we can get (3.123).

(v) For $\partial_\alpha \bar{\psi}_0^5$ and $\partial_\alpha \bar{\psi}_1^5$, we can use the similar techniques used in the proof of (iii) to get that for s_0 large enough,

$$\left| \partial_\alpha \bar{\psi}_0^5 \right| \leq C e^{-C s_0} (1 + |\alpha|), \quad (3.124)$$

and

$$\left| \partial_\alpha \bar{\psi}_1^5 \right| \leq C e^{-C s_0} (1 + |\alpha|). \quad (3.125)$$

(vi) For $\bar{\psi}_-^5(z, \sigma_0)$, similarly to (i), we have that for s_0 large enough,

$$\begin{aligned} &\left\| \frac{\bar{\psi}_-^5(z, \sigma_0)}{1 + |z|^3} \right\|_{L^\infty(\mathbb{R})} \leq \left\| \frac{\bar{\psi}^5(z, \sigma_0)}{1 + |z|^3} \right\|_{L^\infty(\mathbb{R})} \\ &= \left\| \frac{e^{-\frac{\sigma_0}{p-1}} \hat{y}(e^{-\frac{\sigma_0}{2}} z + a, T - e^{-\sigma_0}) [\chi(z, \sigma_0) - \chi(\tilde{z}, s_0)]}{1 + |z|^3} \right\|_{L^\infty((\Omega-a)e^{\sigma_0/2})} \\ &\leq C e^{-s_0/2} |\alpha|. \end{aligned} \quad (3.126)$$

(vii) For $\bar{\psi}_e^5(z, \sigma_0)$, similarly to (i), we have that for s_0 large enough,

$$\begin{aligned} &\left\| \bar{\psi}_e^5(z, \sigma_0) \right\|_{L^\infty(\mathbb{R})} \\ &\leq \left\| e^{-\frac{\sigma_0}{p-1}} \hat{y}(e^{-\frac{\sigma_0}{2}} z + a, T - e^{-\sigma_0}) (1 - \chi_1(z, \sigma_0)) [\chi(z, \sigma_0) - \chi(\tilde{z}, s_0)] \right\|_{L^\infty((\Omega-a)e^{\sigma_0/2})} \\ &\leq C e^{-s_0/2} |\alpha|. \end{aligned} \quad (3.127)$$

Conclusion of Lemma 3.3.

For s_0 large enough, Lemma 3.3 (i) holds from (3.29), (3.44), (3.70) and (3.116);

For s_0 large enough, Lemma 3.3 (ii) holds from (3.29), (3.48), (3.71), (3.86) and (3.118);

For s_0 large enough, Lemma 3.3 (iii) holds from (3.29), (3.52), (3.72), (3.86) and (3.119);

For s_0 large enough, Lemma 3.3 (iv) holds from (3.29), (3.54), (3.73), (3.87) and (3.120);

For s_0 large enough, Lemma 3.3 (v) holds from (3.29), (3.55), (3.74), (3.88) and (3.123);
For s_0 large enough, Lemma 3.3 (vi) holds from (3.29), (3.56), (3.75), (3.88) and (3.124);
For s_0 large enough, Lemma 3.3 (vii) holds from (3.29), (3.58), (3.76), (3.88) and (3.125);
For s_0 large enough, Lemma 3.3 (viii) holds from (3.29), (3.60), (3.113) and (3.126);
For s_0 large enough, Lemma 3.3 (ix) holds from (3.29), (3.62), (3.114), (3.115) and (3.127).

Proof of Proposition 3.1.

Assuming Lemma 3.3 holds.

(i) Introducing the following change of functions and variables:

$$\tilde{\psi}_m(s_0, \tilde{\varepsilon}_0, \tilde{\tau}, \tilde{\alpha}) = \sigma_0^2 \bar{\psi}_m(s_0, y_0, T, a) \quad (3.128)$$

for $m = 0, 1$, $\tilde{\varepsilon}_0 = y_0 - \hat{y}_0$, $\tilde{\tau} = s_0^2 \tau$, $\tilde{\alpha} = s_0 \alpha$. By Lemma 3.3, it holds that whenever $\|\tilde{\varepsilon}_0\|_{W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)} \leq \frac{1}{C_0(s_0)}$, $|\tilde{\tau}| \leq \frac{s_0^2}{16}$, $|\tilde{\alpha}| \leq \frac{s_0}{4}$ and s_0 is large enough, it follows that

$$\begin{aligned} & \left| \tilde{\psi}_0(s_0, \tilde{\varepsilon}_0, \tilde{\tau}, \tilde{\alpha}) - s_0^2 \hat{q}_0(s_0) - \frac{\kappa \tilde{\tau}}{p-1} \right| \\ &= \left| \sigma_0^2 \bar{\psi}_0(s_0, y_0, T, a) - s_0^2 \hat{q}_0(s_0) - \frac{\kappa s_0^2 \tau}{p-1} \right| \\ &= s_0^2 \left| \frac{\sigma_0^2}{s_0^2} \bar{\psi}_0(s_0, y_0, T, a) - \hat{q}_0(s_0) - \frac{\kappa \tau}{p-1} \right| \\ &\leq C s_0^2 \|\tilde{\varepsilon}_0\|_{W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)} + C \frac{\hat{A}}{s_0^2} |\tilde{\tau}| + C \frac{\hat{A}}{s_0} |\tilde{\alpha}| + C \hat{A}^2 s_0^{\frac{3}{2}} e^{-s_0} + C \frac{\hat{A}^2}{s_0^{\frac{1}{2}}} |\tilde{\tau}| + C \frac{\hat{A}^2}{s_0^{\frac{3}{2}}} |\tilde{\alpha}|^3 \\ &\quad + C \hat{A}^2 \log s_0 \left(\frac{|\tilde{\tau}|}{s_0^2} + \frac{|\tilde{\alpha}|^2}{s_0^2} \right) + C e^{-s_0} s_0^2 + C \frac{\tilde{\tau}^2}{s_0^2} + C \frac{|\tilde{\tau}|}{s_0} + C \frac{\tilde{\alpha}^2}{s_0} + C |\tilde{\alpha}| s_0 e^{-s_0/2}, \quad (3.129) \end{aligned}$$

$$\begin{aligned} & \left| \tilde{\psi}_1(s_0, \tilde{\varepsilon}_0, \tilde{\tau}, \tilde{\alpha}) - s_0^2 \hat{q}_1(s_0) + \frac{2b\kappa}{(p-1)^2} \tilde{\alpha} \right| \\ &= \left| \sigma_0^2 \bar{\psi}_1(s_0, y_0, T, a) - s_0^2 \hat{q}_1(s_0) + \frac{2b\kappa}{(p-1)^2} s_0 \alpha \right| \\ &= s_0^2 \left| \frac{\sigma_0^2}{s_0^2} \bar{\psi}_1(s_0, y_0, T, a) - \hat{q}_1(s_0) + \frac{2b\kappa}{(p-1)^2} \frac{\alpha}{s_0} \right| \\ &\leq C s_0^2 \|\tilde{\varepsilon}_0\|_{W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)} + C \hat{A}^2 s_0^{\frac{3}{2}} e^{-s_0} + C \frac{\hat{A}}{s_0^2} |\tilde{\tau}| + C \frac{\hat{A}^2}{s_0^{\frac{5}{2}}} |\tilde{\tau}^2| \\ &\quad + C \frac{\hat{A}^2}{s_0^{\frac{3}{2}}} |\tilde{\alpha}|^3 + C \frac{\hat{A}^2}{s_0^2} |\tilde{\tau}| + C \frac{\hat{A}}{s_0^2} |\tilde{\tau}| + C \frac{\hat{A}}{s_0^2} |\tilde{\alpha}|^2 + C \hat{A}^2 \log s_0 \left(\frac{|\tilde{\alpha}|}{s_0} \right) \\ &\quad + C \frac{\hat{A}^2}{s_0^{\frac{3}{2}}} |\tilde{\tau} \tilde{\alpha}| + C e^{-s_0} s_0^2 + C \frac{\tilde{\tau}^2}{s_0^2} + C \frac{|\tilde{\tau}|}{s_0} + C \frac{|\tilde{\alpha}|}{s_0} \\ &\quad + C \frac{|\tilde{\tau} \tilde{\alpha}|}{s_0} + C \frac{|\tilde{\alpha}|^3}{s_0} + C |\tilde{\alpha}| s_0 e^{-s_0/2} + C |\tilde{\tau}| e^{-C s_0} + C |\tilde{\alpha}|^2 e^{-C s_0}. \quad (3.130) \end{aligned}$$

Let

$$\text{Jac}_{\tilde{\tau}, \tilde{\alpha}}(s_0, \tilde{\varepsilon}_0, \tilde{\psi}_0, \tilde{\psi}_1)(\tilde{\tau}, \tilde{\alpha}) = \begin{vmatrix} \frac{\partial \tilde{\psi}_0}{\partial \tilde{\tau}} & \frac{\partial \tilde{\psi}_0}{\partial \tilde{\alpha}} \\ \frac{\partial \tilde{\psi}_1}{\partial \tilde{\tau}} & \frac{\partial \tilde{\psi}_1}{\partial \tilde{\alpha}} \end{vmatrix} = \frac{\partial \tilde{\psi}_0}{\partial \tilde{\tau}} \frac{\partial \tilde{\psi}_1}{\partial \tilde{\alpha}} - \frac{\partial \tilde{\psi}_0}{\partial \tilde{\alpha}} \frac{\partial \tilde{\psi}_1}{\partial \tilde{\tau}}. \quad (3.131)$$

Since

$$\begin{aligned} \frac{\partial \tilde{\psi}_0}{\partial \tilde{\tau}} - \frac{\kappa}{p-1} &= \frac{\partial(\sigma_0^2 \bar{\psi}_0)}{\partial \tau} \frac{\partial \tau}{\partial \tilde{\tau}} - \frac{\kappa}{p-1} \\ &= \frac{1}{s_0^2} \frac{\partial(\sigma_0^2 \bar{\psi}_0)}{\partial \tau} - \frac{\kappa}{p-1} = \frac{\sigma_0^2}{s_0^2} \frac{1}{\sigma_0^2} \frac{\partial(\sigma_0^2 \bar{\psi}_0)}{\partial \tau} - \frac{\sigma_0^2}{s_0^2} \frac{\kappa}{p-1} \\ &\quad + \left(\frac{\sigma_0^2}{s_0^2} - 1 \right) \frac{\kappa}{p-1}, \end{aligned} \quad (3.132)$$

and

$$\left| \frac{\sigma_0^2}{s_0^2} - 1 \right| \leq \frac{C}{s_0} |\log(1 + \tau)| \leq \frac{C|\tau|}{s_0} = \frac{C|\tilde{\tau}|}{s_0^3}, \quad (3.133)$$

we have by Lemma 3.3 that for s_0 large enough, the following four inequalities holds:

$$\begin{aligned} &\left| \frac{\partial \tilde{\psi}_0}{\partial \tilde{\tau}} - \frac{\kappa}{p-1} \right| \\ &\leq C \|\tilde{\varepsilon}_0\|_{W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)} + C \frac{\hat{A}}{s_0^2} + C e^{-Cs_0} + C \frac{\hat{A}^2}{s_0^{\frac{1}{2}}} \left(\frac{1}{s_0} + \frac{|\tilde{\tau}|}{s_0^2} + \frac{|\tilde{\alpha}|}{s_0} \right) \\ &\quad + C \frac{\hat{A}^2}{s_0^2} \log s_0 + C \left(\frac{1}{s_0} + \frac{|\tilde{\tau}|}{s_0^2} + \frac{|\tilde{\alpha}|}{s_0} \right) + C \frac{|\tilde{\alpha}|}{s_0} e^{-Cs_0} + C \frac{|\tilde{\tau}|}{s_0^3}, \end{aligned} \quad (3.134)$$

$$\begin{aligned} &\left| \frac{\partial \tilde{\psi}_0}{\partial \tilde{\alpha}} \right| = \left| \frac{\partial(\sigma_0^2 \bar{\psi}_0)}{\partial \alpha} \frac{\partial \alpha}{\partial \tilde{\alpha}} \right| = \left| \frac{\partial(\bar{\psi}_0)}{\partial \alpha} \frac{\sigma_0^2}{s_0} \right| \\ &\leq C s_0 \|\tilde{\varepsilon}_0\|_{W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)} + C \frac{\hat{A}}{s_0} + \frac{C \hat{A}^2 \log s_0 |\tilde{\alpha}|}{s_0^2} + C s_0^{\frac{1}{2}} \hat{A}^2 e^{-s_0} + C s_0 e^{-s_0} \\ &\quad + C \frac{|\tilde{\alpha}|}{s_0} + C \frac{|\tilde{\tau}|}{s_0} + C \frac{\hat{A}^2}{s_0^{\frac{2}{3}}} |\tilde{\tau}| + C \frac{\hat{A}^2}{s_0^{\frac{2}{3}}} |\tilde{\alpha}|^2 + C \frac{|\tilde{\alpha}|^2}{s_0} + C |\tilde{\alpha}| e^{-Cs_0} + C s_0 e^{-Cs_0}, \end{aligned} \quad (3.135)$$

$$\begin{aligned} &\left| \frac{\partial \tilde{\psi}_1}{\partial \tilde{\tau}} \right| = \left| \frac{\partial(\sigma_0^2 \bar{\psi}_1)}{\partial \tau} \frac{\partial \tau}{\partial \tilde{\tau}} \right| = \left| \frac{\partial(\sigma_0^2 \bar{\psi}_1)}{\partial \tau} \frac{1}{s_0^2} \right| \\ &\leq C \|\tilde{\varepsilon}_0\|_{W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)} + C \frac{\hat{A}}{s_0^2} + C \frac{\hat{A}^2}{s_0^{\frac{1}{2}}} \left(\frac{1}{s_0} + \frac{|\tilde{\tau}|}{s_0^2} + \frac{|\tilde{\alpha}|}{s_0} \right) + C \left(\frac{1}{s_0} + \frac{|\tilde{\tau}|}{s_0^2} \right. \\ &\quad \left. + \frac{|\tilde{\alpha}|}{s_0} \right) + C e^{-Cs_0} + C \frac{|\tilde{\alpha}|}{s_0} e^{-Cs_0}, \end{aligned} \quad (3.136)$$

and

$$\begin{aligned}
& \left| \frac{\partial \tilde{\psi}_1}{\partial \tilde{\alpha}} + \frac{2b\kappa}{(p-1)^2} \right| = \left| \frac{\partial(\tilde{\psi}_1)}{\partial \alpha} \frac{\sigma_0^2}{s_0} + \frac{2b\kappa}{(p-1)^2} \right| \\
& = \left| \frac{\partial(\tilde{\psi}_1)}{\partial \alpha} \frac{\sigma_0^2}{s_0} + \frac{1}{s_0} \frac{\sigma_0^2}{s_0} \frac{2b\kappa}{(p-1)^2} - \frac{\sigma_0^2}{s_0^2} \frac{2b\kappa}{(p-1)^2} + \frac{2b\kappa}{(p-1)^2} \right| \\
& \leq C s_0 \|\tilde{\varepsilon}_0\|_{W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)} + C \frac{\hat{A}^2}{s_0} \log s_0 + C \frac{\hat{A}}{s_0^2} |\tilde{\alpha}| + C s_0^{\frac{1}{2}} \hat{A}^2 e^{-s_0} + C \frac{1}{s_0} + C \frac{|\tilde{\tau}|}{s_0} \\
& \quad + C \frac{\hat{A}^2}{s_0^{\frac{3}{2}}} |\tilde{\tau}| + C \frac{\hat{A}^2}{s_0^{\frac{3}{2}}} |\tilde{\alpha}|^2 + C \frac{|\tilde{\alpha}|^2}{s_0} + C |\tilde{\alpha}| e^{-C s_0} + C s_0 e^{-C s_0} + C \frac{|\tilde{\tau}|}{s_0^{\frac{3}{2}}}. \tag{3.137}
\end{aligned}$$

Consider now s_0 large, and $\tilde{\varepsilon}_0$ such that

$$\|\tilde{\varepsilon}_0\|_{W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)} \leq \hat{\varepsilon}_0(s_0) := \frac{1}{s_0^3 \tilde{C}_0(s_0)}. \tag{3.138}$$

From the above-mentioned expansions, we see that for s_0 large enough, the function

$$(\tilde{\tau}, \tilde{\alpha}) \rightarrow (\tilde{\psi}_0, \tilde{\psi}_1)(s_0, \tilde{\varepsilon}_0, \tilde{\tau}, \tilde{\alpha})$$

is C^1 -diffeomorphism from the rectangle

$$\mathcal{R}_A = \left[-\frac{2(p-1)A}{\kappa}, \frac{2(p-1)A}{\kappa} \right] \times \left[-\frac{(p-1)^2 A}{b\kappa}, \frac{(p-1)^2 A}{b\kappa} \right]$$

($\forall (\tilde{\tau}, \tilde{\alpha}) \in \mathcal{R}_A$, and $\text{Jac}_{\tilde{\tau}, \tilde{\alpha}}(s_0, \varepsilon_0, \tilde{\psi}_0, \tilde{\psi}_1) \neq 0$.) onto a set $\mathcal{E}_{A, s_0, y_0}$ which approaches (in some appropriate sense) from the rectangle

$$[s_0^2 \hat{q}_0(s_0) - 2A, s_0^2 \hat{q}_0(s_0) + 2A] \times [s_0^2 \hat{q}_1(s_0) - 2A, s_0^2 \hat{q}_1(s_0) + 2A]$$

as $s_0 \rightarrow \infty$, where $A > 0$ large enough will be fixed later.

Since $\hat{q}(s_0) \in V_{K_0, \hat{A}}(s_0)$, we have

$$|s_0^2 \hat{q}_0(s_0)| \leq \hat{A}, \quad |s_0^2 \hat{q}_1(s_0)| \leq \hat{A},$$

we clearly see that

$$[-A, A]^2 \subset \mathcal{E}_{A, s_0, y_0}$$

for $A \geq 2\hat{A}$ and s_0 large enough, hence, there exists a set $\overline{D}_{A, s_0, y_0} \subset \mathcal{R}_A$ such that

$$(\tilde{\psi}_0, \tilde{\psi}_1)(s_0, \varepsilon_0, \overline{D}_{A, s_0, y_0}) = [-A, A]^2.$$

Moreover, from (3.129) and (3.130), the function $(\tilde{\psi}_0, \tilde{\psi}_1)$ has degree -1 on the boundary of $\overline{D}_{A, s_0, y_0}$. By transformation (3.128), the conclusion of item (i) holds.

(ii) Take $(T, a) \in \overline{D}_{A, s_0, y_0}$ and let us check that $\tilde{\psi}(s_0, y_0, T, a) \in V_{K_0, A}(s_0)$. Note that $\tilde{\tau} = s_0^2 \tau$, $\tilde{\alpha} = s_0 \alpha$, $\tau = (T - \hat{T})e^{s_0}$, $\alpha = (a - \hat{a})e^{s_0/2}$ and

$$\overline{D}_{A, s_0, y_0} \subset \left\{ (T, a) \mid |T - \hat{T}| \leq \frac{2e^{-s_0}(p-1)A}{\kappa s_0^2}, \quad |a - \hat{a}| \leq \frac{e^{-s_0/2}(p-1)^2 A}{b\kappa s_0} \right\}. \tag{3.139}$$

From item (i), we know by construction that

$$|\bar{\psi}_m| \leq A\sigma_0^{-2}, \quad m = 0, 1.$$

For the other estimates to be checked, note first from (3.139), and the definition (3.14) of (τ, α) that

$$|\tau| \leq \frac{2(p-1)A}{\kappa s_0^2}, \quad |\alpha| \leq \frac{(p-1)^2 A}{b\kappa s_0}.$$

Therefore, using (3.138) and then items (iii), (viii) and (ix) of Lemma 3.3, we see that for s_0 large, we have

$$|\bar{\psi}_2| < C\hat{A}^2\sigma_0^{-2} \log \sigma_0,$$

$$\left\| \frac{\bar{\psi}_-(z, \sigma_0)}{1 + |z|^3} \right\|_{L^\infty(\mathbb{R})} < \frac{C\hat{A}}{\sigma_0^2}, \quad (3.140)$$

and

$$\left\| \bar{\psi}_e(z, \sigma_0) \right\|_{L^\infty(\mathbb{R})} < \frac{C\hat{A}^2}{\sigma_0^{\frac{1}{2}}}. \quad (3.141)$$

If $A \geq \bar{C}\hat{A}$ for some large $\bar{C} > 0$ and $s_0 \geq s_0(A)$, then we see that the conclusion follows.

(iii) Take $(T, a) \in \bar{D}_{A, s_0, y_0}$ with $A \geq \bar{C}\hat{A}$. Recall that for all $x \in \Omega$ with $|x - \hat{a}| \geq \varepsilon_0/2$, $|\hat{y}(x, t)| \leq \hat{\eta}_0$ for any $t \in [0, \hat{T})$. Then, for any $s_0 \geq \bar{s}_0(A)$ but fixed, there exists a constant $C_3 > 0$ such that $\|\hat{y}(\hat{T} - e^{-s_0})\|_{L^\infty(\Omega)} \leq C_3$. By system (1.5), we have that if $\|\tilde{\varepsilon}_0\|_{W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)}$ is small enough, then $\|y_{y_0}(t)\|_{L^\infty(\Omega)} \leq C_4$ for some $C_4 > 0$, where $t = \hat{T} - e^{-s_0} = T - e^{-\sigma_0}$, and $\|y_{y_0}(t) - \hat{y}(t)\|_{L^\infty(\Omega)} \leq C_5(s_0)\|\tilde{\varepsilon}_0\|_{W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)}$. Thus, if $C_5(s_0)\|\tilde{\varepsilon}_0\|_{W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)} \leq \hat{\eta}_0$, then for all $x \in \Omega$ with $|x - a| \geq 3\varepsilon_0/4$ and when s_0 is large enough such that $|a - \hat{a}| = |\alpha e^{-\frac{s_0}{2}}| \leq e^{-\frac{s_0}{2}} < \frac{\varepsilon_0}{4}$, which implies $|x - \hat{a}| \geq |x - a| - |a - \hat{a}| > \varepsilon_0/2$, we have $|y_{y_0}(x, T - e^{-\sigma_0})| \leq 2\hat{\eta}_0$. \square

Proof of Theorem 2. By Proposition 3.1 and lemmas in Subsection 2.1, we can use the very similar techniques we used in the proof of Proposition 1.4 in Subsection 2.2 to complete the proof of Theorem 2.

4 Proof of Theorem 1

In this section, we will use Theorem 2 and the feedback null controllability results for linear heat equations obtained in [53] to prove Theorem 1.

Proof. Consider $p > 1$. By Proposition 1.4, we conclude that for any $a \in \omega$, there exists $T_0 > 0$ such that for any $T \in (0, T_0)$, there exists some initial data $\tilde{y}_0 \in C_0^\infty(\omega)$ such that the corresponding solution y to (1.5) exists on $[0, T)$, T is the blowup time of y and y has a unique blowup point a .

Then, given $(a, T) \in \omega \times (0, +\infty)$, we take a time $T_1 \in (0, \min\{T_0, T/2\})$. Applying Proposition 1.4, we know that there exists initial data $\tilde{y}_0 \in C_0^\infty(\omega)$ such that the solution y to (1.5) blows up at time T_1 with a being its unique blowup point. Introducing

$$\mathcal{Y}^*(x, t) := y(x, t - T + T_1), \quad (x, t) \in \Omega \times [T - T_1, T),$$

we see that \mathcal{Y}^* satisfies the following equation:

$$\begin{cases} \mathcal{Y}_t^* - \Delta \mathcal{Y}^* = \mathbf{1}_\omega |\mathcal{Y}^*|^{p-1} \mathcal{Y}^*, & (x, t) \in \Omega \times (T - T_1, T), \\ \mathcal{Y}^* = 0, & (x, t) \in \partial\Omega \times (T - T_1, T), \\ \mathcal{Y}^*(x, T - T_1) = \tilde{y}_0(x), & x \in \Omega. \end{cases}$$

Moreover, \mathcal{Y}^* blows up at time T , only at the blowup point a .

On the other hand, given $y_0 \in H_0^1(\Omega)$, take the following system in consideration:

$$\begin{cases} \mathcal{Z}_t - \Delta \mathcal{Z} = \mathbf{1}_\omega v, & (x, t) \in \Omega \times (0, T - T_1), \\ \mathcal{Z} = 0, & (x, t) \in \partial\Omega \times (0, T - T_1), \\ \mathcal{Z}(x, 0) = y_0(x) - \tilde{y}_0(x), & x \in \Omega, \end{cases} \quad (4.1)$$

where $v \in L^2(0, T - T_1; H)$. Since $y_0 - \tilde{y}_0 \in H_0^1(\Omega)$, it is well known that for each $v \in L^2(0, T - T_1; H)$, there exists a unique solution \mathcal{Z} in $C([0, T - T_1]; H_0^1(\Omega))$ to (4.1).

Consider now the following optimal control problem:

$$(\mathcal{P}) \min \left\{ \int_0^{T-T_1} \|v(t)\|_H^2 dt; \mathcal{Z} \text{ satisfies (4.1), } \mathcal{Z}(T - T_1) = 0 \right\},$$

together with the Riccati system (1.2). By Theorem 2.1 in [53] and its proof, there exists a unique mild solution $P \in C_S([0, T - T_1]; \Sigma^+(H))$ to problem (1.2). Moreover, $v(t) = -\mathbf{1}_\omega^* P(t) \mathcal{Z}(t)$ defined for all $t \in [0, T - T_1]$ is the optimal feedback control for problem (\mathcal{P}) .

Set now for all $t \in [0, T - T_1]$, $\check{y}(t) := \mathcal{Z}(t) + \tilde{y}_0(x)$ and

$$u_1(t) := v(t) - \Delta \tilde{y}_0(x) = -\mathbf{1}_\omega^* P(t) (\check{y}(t) - \tilde{y}_0(x)) - \Delta \tilde{y}_0(x).$$

It holds by the construction of $\tilde{y}_0(x)$ that

$$\Delta \tilde{y}_0(x) = \mathbf{1}_\omega \Delta \tilde{y}_0(x).$$

Then, it follows that \check{y} is the solution to the following equation:

$$\begin{cases} \check{y}_t - \Delta \check{y} = \mathbf{1}_\omega u_1, & x \in \Omega \times (0, T - T_1), \\ \check{y} = 0, & x \in \partial\Omega \times (0, T - T_1), \\ \check{y}(x, 0) = y_0(x), & x \in \Omega, \end{cases} \quad (4.2)$$

with

$$\check{y}(T - T_1) = \tilde{y}_0(x).$$

Set

$$y(x, t) := \begin{cases} \tilde{y}(x, t), & (x, t) \in \Omega \times [0, T - T_1], \\ \mathcal{Y}^*(x, t), & (x, t) \in \Omega \times [T - T_1, T], \end{cases}$$

and

$$u(x, t) := \begin{cases} -\mathbf{1}_\omega^* P(t)(y(t) - \tilde{y}_0)(x) - \Delta \tilde{y}_0(x), & (x, t) \in \Omega \times (0, T - T_1), \\ |y|^{p-1} y(x, t), & (x, t) \in \Omega \times [T - T_1, T]. \end{cases}$$

Then, y is the solution to (1.1) with the feedback control u , and it follows that y blows up in T and has a as a unique blowup point.

Consider the following two equations,

$$\begin{cases} \tilde{\Phi}_t - \Delta \tilde{\Phi} = 0, & x \in \Omega, t > 0, \\ \tilde{\Phi} = 0, & x \in \partial\Omega, t > 0, \\ \tilde{\Phi}(x, 0) = \tilde{\Phi}_0(x), & x \in \Omega, \end{cases} \quad (4.3)$$

and

$$\begin{cases} \phi_t - \Delta \phi = 0, & x \in \Omega, t > 0, \\ \phi = 0, & x \in \partial\Omega, t > 0, \\ \phi(x, 0) = \tilde{y}_0(x), & x \in \Omega. \end{cases} \quad (4.4)$$

It is well known that if $\tilde{\Phi}_0 \in W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)$ with $q > n + 2$, then the solution $\tilde{\Phi} \in C([0, T]; W^{2,q}(\Omega)) \cap W_0^{1,q}(\Omega) \cap C^1([0, T]; L^q(\Omega))$ for any $T > 0$. By Sobolev embedding theorem, we have $W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega) \hookrightarrow C^\alpha(\bar{\Omega})$ and $C([0, T]; W^{2,q}(\Omega)) \cap W_0^{1,q}(\Omega) \cap C^1([0, T]; L^q(\Omega)) \hookrightarrow C^{\beta, \beta/2}(\bar{\Omega} \times (0, T))$ for some $0 < \alpha < 1$ and $0 < \beta < 1$, where $q > n + 2$.

If $\tilde{\Phi}_0 \in H_0^1(\Omega)$, then for any $t > 0$,

$$\|\tilde{\Phi}(t) - \phi(t)\|_{L^\infty(\Omega)} \leq Ct^{-\frac{n}{4}} \|\tilde{\Phi}_0 - \tilde{y}_0\|_{L^2(\Omega)} \leq Ct^{-\frac{n}{4}} \|\tilde{\Phi}_0 - \tilde{y}_0\|_{H_0^1(\Omega)}. \quad (4.5)$$

Thus, for $1 < q < \infty$,

$$\|\tilde{\Phi}(t) - \phi(t)\|_{W^{2,q}(\Omega)} \leq C \left(1 + \frac{2}{t}\right) \left\| \tilde{\Phi}\left(\frac{t}{2}\right) - \phi\left(\frac{t}{2}\right) \right\|_{L^q(\Omega)} \quad (4.6)$$

$$\leq C \left(1 + \frac{2}{t}\right) \left(\frac{t}{2}\right)^{-\frac{n}{4}} \|\tilde{\Phi}_0 - \tilde{y}_0\|_{H_0^1(\Omega)}. \quad (4.7)$$

Hence

$$\|\tilde{\Phi}(t) - \phi(t)\|_{L^\infty(\Omega) \cap W^{2,q}(\Omega)} \leq \tilde{C} \left(t^{-\frac{n}{4}} + \left(1 + \frac{2}{t}\right) \left(\frac{t}{2}\right)^{-\frac{n}{4}} \right) \|\tilde{\Phi}_0 - \tilde{y}_0\|_{H_0^1(\Omega)}, \quad (4.8)$$

for some $\tilde{C} > 0$.

Let $\varepsilon > 0$ and let $\varepsilon_1 > 0$ be given in Theorem 2 for $\varepsilon/2$. Since $\tilde{y}_0 \in C_0^\infty(\omega)$, we have that there exists a time $\hat{\varepsilon}_1 \in (0, \min(\frac{\varepsilon}{4}, \frac{T_1}{4}))$ such that

$$\|\phi(\hat{\varepsilon}_1) - \tilde{y}_0\|_{L^\infty(\Omega) \cap W^{2,q}(\Omega)} < \frac{\varepsilon_1}{2}. \quad (4.9)$$

Since $\check{y} \in C([0, T - T_1]; H_0^1(\Omega))$, there exist $\delta \in (0, \min\{\varepsilon/4, T - T_1\})$, $\delta^* > 0$ with $(\delta - \delta^*, \delta + \delta^*) \subset (0, \min\{\varepsilon/4, T - T_1\})$ such that for any $\hat{\varepsilon} \in (\delta - \delta^*, \delta + \delta^*)$,

$$\|\check{y}(T - T_1 - \hat{\varepsilon}) - \tilde{y}_0\|_{H_0^1(\Omega)} < \frac{1}{\tilde{C}\left(\hat{\varepsilon}_1^{-\frac{n}{4}} + \left(1 + \frac{2}{\hat{\varepsilon}_1}\right)\left(\frac{\hat{\varepsilon}_1}{2}\right)^{-\frac{n}{4}}\right)} \frac{\varepsilon_1}{4}. \quad (4.10)$$

Since $P \in C_S([0, T - T_1 - \delta + \delta^*]; \Sigma^+(H))$, there exists $\varepsilon^* > 0$ such that for any initial data y_0^* satisfying $\|y_0^* - y_0\|_{H_0^1(\Omega)} < \varepsilon^*$ and any $\hat{\varepsilon} \in (\delta - \delta^*, \delta + \delta^*)$, the solution y^* to (1.1) with initial data y_0^* and with the following feedback control

$$u(x, t) := -\mathbf{1}_\omega^* P(t)(y^*(t) - \tilde{y}_0)(x) - \Delta \tilde{y}_0(x), \quad (x, t) \in \Omega \times (0, T - T_1 - \delta + \delta^*)$$

satisfies

$$\|y^*(T - T_1 - \hat{\varepsilon}) - \check{y}(T - T_1 - \hat{\varepsilon})\|_{H_0^1(\Omega)} < \frac{1}{\tilde{C}\left(\hat{\varepsilon}_1^{-\frac{n}{4}} + \left(1 + \frac{2}{\hat{\varepsilon}_1}\right)\left(\frac{\hat{\varepsilon}_1}{2}\right)^{-\frac{n}{4}}\right)} \frac{\varepsilon_1}{4}, \quad (4.11)$$

from which and (4.10), it holds that

$$\|y^*(T - T_1 - \hat{\varepsilon}) - \tilde{y}_0\|_{H_0^1(\Omega)} < \frac{1}{\tilde{C}\left(\hat{\varepsilon}_1^{-\frac{n}{4}} + \left(1 + \frac{2}{\hat{\varepsilon}_1}\right)\left(\frac{\hat{\varepsilon}_1}{2}\right)^{-\frac{n}{4}}\right)} \frac{\varepsilon_1}{2}. \quad (4.12)$$

Further, we consider the following equation,

$$\begin{cases} Y_t - \Delta Y = 0, & (x, t) \in \Omega \times (T - T_1 - \hat{\varepsilon}, T - T_1 - \hat{\varepsilon} + \hat{\varepsilon}_1), \\ Y = 0, & (x, t) \in \partial\Omega \times (T - T_1 - \hat{\varepsilon}, T - T_1 - \hat{\varepsilon} + \hat{\varepsilon}_1), \\ Y(x, T - T_1 - \hat{\varepsilon}) = y^*(T - T_1 - \hat{\varepsilon})(x), & x \in \Omega, \end{cases}$$

and

$$\begin{cases} \tilde{\phi}_t - \Delta \tilde{\phi} = 0, & (x, t) \in \Omega \times (T - T_1 - \hat{\varepsilon}, T - T_1 - \hat{\varepsilon} + \hat{\varepsilon}_1), \\ \tilde{\phi} = 0, & (x, t) \in \partial\Omega \times (T - T_1 - \hat{\varepsilon}, T - T_1 - \hat{\varepsilon} + \hat{\varepsilon}_1), \\ \tilde{\phi}(x, T - T_1 - \hat{\varepsilon}) = \tilde{y}_0(x), & x \in \Omega. \end{cases}$$

By (4.8) and (4.9), it holds that

$$\begin{aligned} & \|Y(T - T_1 - \hat{\varepsilon} + \hat{\varepsilon}_1) - \tilde{\phi}(T - T_1 - \hat{\varepsilon} + \hat{\varepsilon}_1)\|_{L^\infty(\Omega) \cap W^{2,q}(\Omega)} \\ & \leq \tilde{C}\left(\hat{\varepsilon}_1^{-\frac{n}{4}} + \left(1 + \frac{2}{\hat{\varepsilon}_1}\right)\left(\frac{\hat{\varepsilon}_1}{2}\right)^{-\frac{n}{4}}\right) \|y^*(T - T_1 - \hat{\varepsilon}) - \tilde{y}_0\|_{H_0^1(\Omega)} \end{aligned} \quad (4.13)$$

and

$$\|\tilde{\phi}(T - T_1 - \hat{\varepsilon} + \hat{\varepsilon}_1) - \tilde{y}_0\|_{L^\infty(\Omega) \cap W^{2,q}(\Omega)} < \frac{\varepsilon_1}{2}, \quad (4.14)$$

from which, (4.13) and (4.12) we have

$$\|Y(T - T_1 - \hat{\varepsilon} + \hat{\varepsilon}_1) - \tilde{y}_0\|_{L^\infty(\Omega) \cap W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)} < \varepsilon_1. \quad (4.15)$$

Then by Theorem 2, the solution \bar{Y} to the following equation

$$\begin{cases} \bar{Y}_t - \Delta \bar{Y} = \mathbf{1}_\omega |\bar{Y}|^{p-1} \bar{Y}, & x \in \Omega, t > T - T_1 - \hat{\varepsilon} + \hat{\varepsilon}_1, \\ \bar{Y} = 0, & x \in \partial\Omega, t > T - T_1 - \hat{\varepsilon} + \hat{\varepsilon}_1, \\ \bar{Y}(x, T - T_1 - \hat{\varepsilon} + \hat{\varepsilon}_1) = \bar{Y}(T - T_1 - \hat{\varepsilon} + \hat{\varepsilon}_1)(x), & x \in \Omega, \end{cases}$$

blows up at time T^* , only at the blowup point a^* , $|T^* - (T - T_1 - \hat{\varepsilon} + \hat{\varepsilon}_1) - T_1| < \varepsilon/2$ and $|a^* - a| < \varepsilon/2$. Since $\hat{\varepsilon} < \varepsilon/4$ and $\hat{\varepsilon}_1 < \varepsilon/4$, we have $|T^* - T| < \varepsilon$.

Moreover, for all $R > 0$,

$$\sup_{\{|x-a^*| \leq R\sqrt{(T^*-t)|\log(T^*-t)}\}} \left| (T^* - t)^{\frac{1}{p-1}} \bar{Y}(x, t) - f\left(\frac{x - a^*}{\sqrt{(T^* - t)|\log(T^* - t)|}}\right) \right| \rightarrow 0$$

as $t \rightarrow T^*$, where f is defined in (1.3).

Set

$$y(x, t) := \begin{cases} y^*(x, t), & (x, t) \in \Omega \times [0, T - T_1 - \hat{\varepsilon}), \\ Y(x, t), & (x, t) \in \Omega \times [T - T_1 - \hat{\varepsilon}, T - T_1 - \hat{\varepsilon} + \hat{\varepsilon}_1), \\ \bar{Y}(x, t), & (x, t) \in \Omega \times [T - T_1 - \hat{\varepsilon} + \hat{\varepsilon}_1, T^*), \end{cases}$$

and

$$u(x, t) := \begin{cases} -\mathbf{1}_\omega P(t)(y(t) - \tilde{y}_0)(x) - \Delta \tilde{y}_0(x), & (x, t) \in \Omega \times [0, T - T_1 - \hat{\varepsilon}), \\ 0, & (x, t) \in \Omega \times [T - T_1 - \hat{\varepsilon}, T - T_1 - \hat{\varepsilon} + \hat{\varepsilon}_1), \\ |y|^{p-1} y(x, t), & (x, t) \in \Omega \times [T - T_1 - \hat{\varepsilon} + \hat{\varepsilon}_1, T^*). \end{cases}$$

Then from the above argument, Theorem 1 holds. \square

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