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Uniqueness of adapted solutions to scalar BSDEs with Peano-type generators*

Shengjun Fan^a, Ying Hu^b, Shanjian Tang^c

^a*School of Mathematics, China University of Mining and Technology, Xuzhou 221116, China*

^b*Univ. Rennes, CNRS, IRMAR-UMR6625, F-35000, Rennes, France*

^c*Department of Finance and Control Sciences, School of Mathematical Sciences, Fudan University, Shanghai 200433, China*

Abstract

It is well known that a Backward Stochastic Differential Equation (BSDE) with a Peano-type generator has infinite many solutions when the terminal value is zero. In this paper, we study the uniqueness of adapted solutions of such a BSDE when the terminal value is positive. Two methods are developed. The first one is to connect the BSDE to an optimal stochastic control problem: under suitable integrability of the terminal value, with a verification argument, we prove that the first component of the adapted solution pair is the value process for some given stochastic optimal control problem. The second one appeals to a change of variables, with a flavor of more analyses: by a change of variables, the original BSDE is reduced to a convex quadratic BSDE, and then using the θ -difference method, we give a sharp result in some special case which can be applied to the BSDE defined by Kreps-Porteus utility.

Keywords: Backward stochastic differential equation, Existence and uniqueness, Peano-type, θ -difference method, Kreps-Porteus utility.

2010 MSC: 60H10

1. Introduction

Define $\mathbb{R} := (-\infty, +\infty)$, $\mathbb{R}_+ := [0, +\infty)$ and $\mathbb{R}_{++} := (0, +\infty)$. Given a real $T > 0$ and a positive integer d . Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with augmented filtration $(\mathcal{F}_t)_{t \in [0, T]}$ generated by a d -dimensional standard Brownian motion $(B_t)_{t \in [0, T]}$, and $\mathcal{F} = \mathcal{F}_T$. The equalities and inequalities between random elements are always understood to hold $\mathbb{P} - a.s.$. We are concerned with the following scalar backward stochastic differential equation (BSDE for short):

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s, \quad t \in [0, T], \quad (1.1)$$

where ξ is an \mathcal{F}_T -measurable real-valued random variable, called the terminal value, and for each $(y, z) \in \mathbb{R} \times \mathbb{R}^{1 \times d}$, the stochastic process $g(\cdot, \cdot, y, z) : \Omega \times [0, T] \rightarrow \mathbb{R}$ is (\mathcal{F}_t) -progressively measurable (the

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Email addresses: shengjunfan@cumt.edu.cn (Shengjun Fan), ying.hu@univ-rennes1.fr (Ying Hu), sjtang@fudan.edu.cn (Shanjian Tang)

function g is called the generator). An adapted solution of BSDE (1.1) is a pair of (\mathcal{F}_t) -progressively measurable processes $(Y_t, Z_t)_{t \in [0, T]}$ taking values in $\mathbb{R} \times \mathbb{R}^{1 \times d}$ such that $\mathbb{P} - a.s.$, $t \mapsto Y_t$ is continuous, $t \mapsto |g(t, Y_t, Z_t)| + |Z_t|^2$ is integrable, and (1.1) is fulfilled. Pardoux and Peng [32] established the first existence and uniqueness result on the square-integrable adapted solution of a nonlinear BSDE under the Lipschitz condition on the generator g and the square-integrable condition on the terminal value ξ . Since then, BSDEs have received a lot of attentions and gradually become a powerful tool in various fields such as mathematical finance, nonlinear expectation, PDE, stochastic control and game, and so on. The reader is referred to [12, 33, 31, 27, 7, 8, 34, 22, 23, 26, 37, 15, 18] for details among others.

The existence of a square-integrable adapted solution for BSDE (1.1) was given in Lepeltier and San Martin [29] when the terminal value ξ is square-integrable, and the generator g is only continuous and has a linear growth in both unknown variables (y, z) . See also Fan and Jiang [19] for the case of the L^p ($p > 1$) solution. By virtue of the localization method of Briand and Hu [5] along with the a priori estimate technique and the comparison theorem for adapted solutions of BSDEs, existence were given for more general situations that the generator g may have a one-sided linear/superlinear growth in the state variable y and a super-linear/sub-quadratic/quadratic growth in the state variable z . Relevant results are available in for example [27, 5, 3, 13, 14, 15, 17, 18]. It is well known that under the above-mentioned conditions, BSDE (1.1) can have multiple adapted solutions when the terminal value is zero. For example, for each $c \in [0, T]$, the pair of processes

$$(Y_t, Z_t) := \frac{1}{4} ([(c-t)^+]^2, 0), \quad t \in [0, T]$$

is an adapted solution of the following BSDE:

$$Y_t = \int_t^T \sqrt{|Y_s|} \, ds - \int_t^T Z_s \, dB_s, \quad t \in [0, T]. \quad (1.2)$$

The main reason lies in that the following Ordinary Differential Equation (ODE)

$$\begin{cases} y'(t) = -\sqrt{y(t)}, & t \in [0, T]; \\ y(T) = 0 \end{cases}$$

has infinite many solutions $y_t = [(c-t)^+]^2/4$ on the time interval $[0, T]$. To the best of our knowledge, Peano first explored this kind of phenomenon of multiple solutions of ODEs (see for example Herrmann [21], Crippa [9] and Delarue and Flandoli [10]). We call the generator like \sqrt{y} being of Peano type.

To obtain the uniqueness of an adapted solution of BSDE (1.1), some further conditions have to be imposed on the generator g . The uniform Lipschitz condition used in Pardoux and Peng [32] is just the classical one. Weaker conditions than the uniform Lipschitz condition were subsequently explored and verified during the last three decades. Among these conditions we would like to especially mention Mao's non-Lipschitz condition, the monotonicity condition and the one-sided Osgood condition with respect to the state variable y , and the uniform continuity condition and the convexity/concavity condition with respect to the state variable z . The reader is referred to [30, 31, 27, 4, 6, 24, 25, 20, 11, 3, 2, 13, 1, 16, 15, 17, 18] for more details. We mention that in order to verify the validity of these conditions, some novel ideas and tools were also constantly presented and developed, such as Gronwall's and Bihari's inequalities, Girsanov's transform, the θ -difference method and the test function technique.

The generator of Peano type like \sqrt{y} goes beyond of all works mentioned in the last paragraph. As a result, the previous BSDE (1.2) admits multiple adapted solutions. Moreover, for a nonnegative and nonzero terminal value, it is possible that a BSDE with a Peano-type generator still admits multiple adapted solutions. For example, let $\tau \in (0, T)$ be a given stopping time, $A \in \mathcal{F}_\tau$ be a given event such that $\mathbb{P}(A) \in (0, 1)$, $\lambda \in [0, 1]$ be a constant, $\eta_\lambda := \lambda\tau + (1 - \lambda)T$ be an \mathcal{F}_τ -measurable random variable, and $(Y_t^\lambda, Z_t^\lambda)_{t \in [0, \tau]}$ be the adapted square-integrable maximal solution of the following BSDE:

$$Y_t^\lambda = \left(1 + \frac{T - \tau}{2}\right)^2 \mathbf{1}_A + \left[\frac{(\eta_\lambda - \tau)^+}{2}\right]^2 \mathbf{1}_{A^c} + \int_t^\tau \sqrt{Y_s^\lambda} ds - \int_t^\tau Z_s^\lambda dB_s, \quad t \in [0, \tau].$$

Then, by setting for each $t \in [0, T]$,

$$\bar{Y}_t^\lambda := \left\{ \left(1 + \frac{T - t}{2}\right)^2 \mathbf{1}_A + \left[\frac{(\eta_\lambda - t)^+}{2}\right]^2 \mathbf{1}_{A^c} \right\} \mathbf{1}_{t > \tau} + Y_t^\lambda \mathbf{1}_{t \leq \tau} \quad \text{and} \quad \bar{Z}_t^\lambda := Z_t^\lambda \mathbf{1}_{t \leq \tau},$$

we can directly verify that $(\bar{Y}_t^\lambda, \bar{Z}_t^\lambda)_{t \in [0, T]}$ is a square-integrable adapted solution of the following BSDE

$$Y_t = \mathbf{1}_A + \int_t^T \sqrt{Y_s} ds - \int_t^T Z_s dB_s, \quad t \in [0, T].$$

Note the constant λ can be arbitrarily chosen in $[0, 1]$. The last BSDE admits infinite many square-integrable adapted solutions. Now, let us take a look at the ODE with a Peano-type generator when the terminal value is positive. By virtue of the variable separation method, it is straightforward to verify that for any given $c > 0$, the following ODE

$$\begin{cases} y'(t) = -\sqrt{y(t)}, & t \in [0, T]; \\ y(T) = c \end{cases}$$

does have a unique solution $y_t = (\sqrt{c} + T/2 - t/2)^2$ on the time interval $[0, T]$. Then, a question is naturally asked: for any given square-integrable terminal value $\xi \in \mathbb{R}_{++}$, is the square-integrable solution $(Y_t, Z_t)_{t \in [0, T]}$ such that $Y \in \mathbb{R}_{++}$ of the following BSDE

$$Y_t = \xi + \int_t^T \sqrt{Y_s} ds - \int_t^T Z_s dB_s, \quad t \in [0, T]. \quad (1.3)$$

necessarily unique? When $\xi \geq c > 0$ for some constant c , it is not difficult to see that the answer is affirmative. In fact, by taking the conditional mathematical expectation we have $Y_t \geq c$ for any adapted square-integrable positive solution $(Y_t, Z_t)_{t \in T}$ of BSDE (1.3). Then, (Y, Z) has to be the unique square-integrable adapted solution of the following BSDE with a Lipschitz generator:

$$Y_t = \xi + \int_t^T \left(\sqrt{Y_s} \mathbf{1}_{\{Y_s \geq c\}} + \sqrt{c} \mathbf{1}_{\{Y_s < c\}} \right) ds - \int_t^T Z_s dB_s, \quad t \in [0, T].$$

But, when $\text{ess inf } \xi = 0$, it seems that the answer of the previous question can not be easily obtained by any existing results. The objective of the present paper is to solve this problem. Two kinds of more complicate scenarios are further investigated. To the best of our knowledge, this is the first study on the uniqueness of the adapted solutions of BSDEs with generators of Peano type.

More specifically, in Section 3 of the present paper we apply a verification method to study uniqueness of the adapted solutions of BSDEs with generators of Peano type when the terminal value ξ is positive.

In fact, it is proved that under some integrability conditions on the terminal value, the first component Y of the adapted solution pair is indeed the value process for some given stochastic optimal control problem. See Theorem 3.2 for more details. The second is an analytical approach. After a change of variables, the original BSDE is shown to be equivalent to one convex BSDE with quadratic growth in Section 4. Then we use a θ -difference method to get a sharp result in some special case which can be applied to the BSDE defined by Kreps-Porteus utility. See Theorem 4.1 for more details. The proofs are quite involved. The whole ideas consist in employing the classical dual theory on the convex functional and the θ -difference method employed initially in Briand and Hu [6]. The Fenchel-Moreau theorem, Hölder's inequality, Jensen's inequality, Itô-Tanaka's formula, Girsanov's transform and Doob's maximal inequality on sub-martingales play crucial roles in our proof. For more clarity, let us take BSDE (1.3) for an example to quickly illustrate our whole idea.

We first introduce the verification method used in Theorem 3.2. First of all, from the duality,

$$\sqrt{y} = \inf_{q>0} \left\{ qy + \frac{1}{4q} \right\}, \quad y \in \mathbb{R}_+. \quad (1.4)$$

Then we define some admissible set \mathcal{H} which is a subset of (\mathcal{F}_t) -progressively measurable \mathbb{R}_{++} -valued processes and will be given precisely later, and for any $q \in \mathcal{H}$, define $(Y_t^q, Z_t^q)_{t \in [0, T]}$ as the unique square-integrable adapted solution of the following BSDE:

$$Y_t^q = \xi + \int_t^T \left(q_s Y_s^q + \frac{1}{4q_s} \right) ds - \int_t^T Z_s^q dB_s, \quad t \in [0, T]. \quad (1.5)$$

Finally, suppose that both ξ and $1/\sqrt{\xi}$ are square-integrable and $(Y_t, Z_t)_{t \in [0, T]}$ is any square-integrable adapted solution of BSDE (1.3) such that $Y \in \mathbb{R}_{++}$, and we prove that for each $t \in [0, T]$,

$$Y_t = \operatorname{ess\,inf}_{q \in \mathcal{H}} Y_t^q, \quad (1.6)$$

which gives uniqueness of the adapted solution of BSDE (1.3). In order to verify (1.6), we choose

$$\mathcal{H} := \left\{ (q_t)_{t \in [0, T]} : \begin{array}{l} q \text{ is an } (\mathcal{F}_t)\text{-progressively measurable } \mathbb{R}_{++}\text{-valued process} \\ \text{such that } \mathbb{E} \left[e^{2 \int_0^T q_s ds} \right] < +\infty \text{ and } \mathbb{E} \left[\left(\int_0^T \frac{1}{q_t} dt \right)^2 \right] < +\infty \end{array} \right\}.$$

For any $q \in \mathcal{H}$, by Itô's formula we can directly verify that BSDE (1.5) admits a unique square-integrable adapted solution (Y^q, Z^q) such that

$$Y_t^q = \mathbb{E} \left[e^{\int_t^T q_s ds} \xi + \int_t^T e^{\int_t^s q_u du} \frac{1}{4q_s} ds \middle| \mathcal{F}_t \right], \quad t \in [0, T].$$

And, in light of (1.3)-(1.5), by a comparison argument we can prove that for each $q \in \mathcal{H}$ and $t \in [0, T]$,

$$Y_t \leq Y_t^q.$$

Now, from the duality we pick

$$q_t^* := \frac{1}{2\sqrt{Y_t}}, \quad t \in [0, T]. \quad (1.7)$$

It remains to prove that $q^* \in \mathcal{H}$ and $Y = Y^{q^*}$. For this purpose, we first give some a priori estimate on Y . It follows from Itô's formula that

$$2\sqrt{Y_t} = 2\sqrt{\xi} + \int_t^T \left(1 + \frac{|Z_s|^2}{4(Y_s)^{3/2}} \right) ds - \int_t^T \frac{Z_s}{\sqrt{Y_s}} dB_s, \quad t \in [0, T].$$

Thus, for each $t \in [0, T]$,

$$2\sqrt{Y_t} \geq \mathbb{E} \left[2\sqrt{\xi} \mid \mathcal{F}_t \right] + T - t$$

and then

$$Y_t \geq \left(\mathbb{E} \left[\sqrt{\xi} \mid \mathcal{F}_t \right] + \frac{T-t}{2} \right)^2 \geq \left(\eta + \frac{T-t}{2} \right)^2 \quad (1.8)$$

with

$$\eta := \min_{t \in [0, T]} \mathbb{E} \left[\sqrt{\xi} \mid \mathcal{F}_t \right].$$

From this a priori estimate, we can now finish the desired proof. First, by (1.7) we know that q^* is an (\mathcal{F}_t) -progressively measurable \mathbb{R}_{++} -valued process and $\mathbb{E} \left[\left(\int_0^T 1/q_t^* dt \right)^2 \right] < +\infty$. Then, by the a priori estimate (1.8) we obtain

$$\int_0^T \frac{1}{\sqrt{Y_t}} dt \leq \int_0^T \frac{2}{2\eta + T - t} dt = 2 \ln \left(\frac{2\eta + T}{2\eta} \right) = \ln \left(1 + \frac{T}{2 \min_{t \in [0, T]} \mathbb{E} \left[\sqrt{\xi} \mid \mathcal{F}_t \right]} \right)^2$$

and, in light of Jensen's inequality and Doob's maximal inequality,

$$\begin{aligned} \mathbb{E} \left[\exp \left(\int_0^T \frac{1}{\sqrt{Y_t}} dt \right) \right] &\leq \mathbb{E} \left[\max_{t \in [0, T]} \left(1 + \frac{T}{2\mathbb{E} \left[\sqrt{\xi} \mid \mathcal{F}_t \right]} \right)^2 \right] \\ &\leq \mathbb{E} \left[\max_{t \in [0, T]} \left(\mathbb{E} \left[1 + \frac{T}{2\sqrt{\xi}} \mid \mathcal{F}_t \right] \right)^2 \right] \\ &\leq 4\mathbb{E} \left[\left(1 + \frac{T}{2\sqrt{\xi}} \right)^2 \right] < +\infty. \end{aligned} \quad (1.9)$$

Combining (1.7) and (1.9) yields that $\mathbb{E} \left[e^{2 \int_0^T q_s^* ds} \right] < +\infty$. Hence, $q^* \in \mathcal{H}$. Finally, in light of the duality (1.4) along with (1.7), we have

$$\sqrt{Y_s} = q_s^* Y_s + \frac{1}{4q_s^*}, \quad s \in [0, T]. \quad (1.10)$$

Moreover, in light of (1.3) and (1.10), from the uniqueness of the square-integrable adapted solution of BSDE (1.5) it follows that $Y = Y^{q^*}$. Thus, we have proved that the positive square-integrable adapted solution of BSDE (1.3) is unique when $1/\xi$ is integrable. We remark that this verification method is applied in this paper to a more general situation that the generator g contains not only a nonnegative term being of Peano type in the state variable y , but also a nonnegative term satisfying the monotonicity condition in y and a real-valued term being uniformly Lipschitz continuous in the state variables (y, z) .

In the sequel, let us further illustrate the θ -difference method used in Theorem 4.1. Suppose that $(Y_t, Z_t)_{t \in [0, T]}$ is a square-integrable adapted solution of BSDE (1.3) such that $Y \in \mathcal{S}^p(\mathbb{R}_{++})$ for some $p > 1$. Define $\bar{\xi} := 2\sqrt{\xi}$ and for each $t \in [0, T]$,

$$\bar{Y}_t := 2\sqrt{Y_t} \quad \text{and} \quad \bar{Z}_t := \frac{Z_t}{\sqrt{Y_t}}.$$

By virtue of Itô's formula, we can conclude that (Y, Z) is an adapted solution of BSDE (1.3) such that $Y \in \mathcal{S}^p(\mathbb{R}_{++})$ if and only if (\bar{Y}, \bar{Z}) is an adapted solution of the following BSDE:

$$\bar{Y}_t = \bar{\xi} + \int_t^T \left(1 + \frac{|\bar{Z}_s|^2}{2\bar{Y}_s} \right) ds - \int_t^T \bar{Z}_s dB_s, \quad t \in [0, T] \quad (1.11)$$

such that $\bar{Y} \in \mathcal{S}^{2p}(\mathbb{R}_{++})$. Note that the generator $g(y) := \sqrt{y}$ of BSDE (1.3) is a concave function defined on \mathbb{R}_+ , while the generator $\bar{g}(y, z) := 1 + |z|^2/2y$ of BSDE (1.11) is a joint convex function defined on $\mathbb{R}_{++} \times \mathbb{R}^{1 \times d}$. Then, for each $\theta \in (0, 1)$ and $(y_1, y_2, z_1, z_2) \in \mathbb{R}_{++} \times \mathbb{R}_{++} \times \mathbb{R}^{1 \times d} \times \mathbb{R}^{1 \times d}$, we have

$$\begin{aligned} & \mathbf{1}_{\{y_1 > \theta y_2\}} (\bar{g}(y_1, z_1) - \theta \bar{g}(y_2, z_2)) \\ &= \mathbf{1}_{\{y_1 > \theta y_2\}} \left[\bar{g} \left(\theta y_2 + (1 - \theta) \frac{y_1 - \theta y_2}{1 - \theta}, \theta z_2 + (1 - \theta) \frac{z_1 - \theta z_2}{1 - \theta} \right) - \theta \bar{g}(y_2, z_2) \right] \\ &\leq (1 - \theta) \mathbf{1}_{\{y_1 > \theta y_2\}} \bar{g} \left(\left(\frac{y_1 - \theta y_2}{1 - \theta} \right)^+, \frac{z_1 - \theta z_2}{1 - \theta} \right). \end{aligned}$$

Based on the above observation, we can employ the θ -difference method as in Briand and Hu [6], Fan and Hu [15], and Fan, Hu and Tang [17, 18] to verify that BSDE (1.11) admits a unique adapted solution (Y, Z) such that $Y \in \mathcal{S}^{2p}(\mathbb{R}_{++})$. As a result, (Y, Z) is the unique adapted solution of BSDE (1.3) such that $\bar{Y} \in \mathcal{S}^p(\mathbb{R}_{++})$. This method is applied to a slightly general situation in this paper. See BSDE (4.1) and Theorem 4.1 in Section 4 for more details. We remark that although in Section 4, the BSDE is special, but the result is sharper since no integrability condition is imposed on $1/\xi$.

The rest of this paper is organized as follows. Some notations, spaces and two technical lemmas used later are presented in Section 2. In Section 3 we state and prove Theorem 3.2 as the first main result, with the help of a verification argument. Finally in Section 4, using the θ -difference method, we study the uniqueness of the adapted solution of a special BSDE, and give Theorem 4.1. Several remarks and examples are included to illustrate both theorems. See Theorems 3.1, 3.3 and 4.2 along with examples 3.5 and 4.3 for details. In particular, the uniqueness of the Kreps-Porteus utility of an endowment $\xi \in L^p(\mathbb{R}_{++})$ is demonstrated in Theorem 4.3.

2. Preliminaries

2.1. Notations and spaces

For $a, b \in \mathbb{R}$, define $a \wedge b := \min\{a, b\}$, $a^+ := \max\{a, 0\}$, and $\text{sgn}(x) := \mathbf{1}_{x > 0} - \mathbf{1}_{x \leq 0}$ with $\mathbf{1}_A$ being the indicator function of set A . For each $p > 1$, denote by $L^p(\mathcal{F}_t; \mathbb{R})$ for $t \in [0, T]$ the total of \mathcal{F}_t -measurable \mathbb{R} -valued random variables ξ satisfying $\mathbb{E}[|\xi|^p] < +\infty$ (in particular, $L^p(\mathcal{F}_T; \mathbb{R})$ is also denoted by $L^p(\mathbb{R})$ for simplicity), $\mathcal{L}^p(\mathbb{R})$ the total of (\mathcal{F}_t) -progressively measurable \mathbb{R} -valued processes $(X_t)_{t \in [0, T]}$ satisfying

$$\|X\|_{\mathcal{L}^p(\mathbb{R})} := \left\{ \mathbb{E} \left[\left(\int_0^T |X_t| dt \right)^p \right] \right\}^{\frac{1}{p}} < +\infty,$$

$\mathcal{S}^p(\mathbb{R})$ the total of (\mathcal{F}_t) -adapted continuous \mathbb{R} -valued processes $(Y_t)_{t \in [0, T]}$ satisfying

$$\|Y\|_{\mathcal{S}^p(\mathbb{R})} := \left(\mathbb{E} \left[\sup_{t \in [0, T]} |Y_t|^p \right] \right)^{\frac{1}{p}} < +\infty,$$

and $\mathcal{M}^p(\mathbb{R}^{1 \times d})$ the total of (\mathcal{F}_t) -progressively measurable $\mathbb{R}^{1 \times d}$ -valued processes $(Z_t)_{t \in [0, T]}$ satisfying

$$\|Z\|_{\mathcal{M}^p(\mathbb{R}^{1 \times d})} := \left\{ \mathbb{E} \left[\left(\int_0^T |Z_t|^2 dt \right)^{p/2} \right] \right\}^{\frac{1}{p}} < +\infty.$$

If the \mathbb{R} is replaced with \mathbb{R}_+ or \mathbb{R}_{++} in any one of the above spaces, then it represents the total of those elements in the original space taking values in \mathbb{R}_+ or \mathbb{R}_{++} .

Denote by \mathbf{S} the totality of concave functions $\rho(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\rho(0) = 0$, $\rho(x) > 0$ for $x > 0$ and $\rho'(\cdot)$ is a continuous function on $(0, +\infty)$ taking values in \mathbb{R}_+ . Note that for $\rho(\cdot) \in \mathbf{S}$, we have

$$\forall \lambda \in (0, 1), \forall x \geq 0, \quad \rho(\lambda x) = \rho(\lambda x + (1 - \lambda)0) \geq \lambda \rho(x) + (1 - \lambda)\rho(0) = \lambda \rho(x)$$

and then the function $\rho(x)/x$, $x > 0$ is nonincreasing, i.e., for each $x, y > 0$,

$$\text{if } x \leq y, \quad \text{then } \frac{\rho(x)}{x} = \frac{\rho(\frac{x}{y}y)}{x} \geq \frac{\rho(y)}{y}. \quad (2.1)$$

Hence, $\rho(x) \leq \rho(1)x$ for $x \geq 1$, and then $\rho(\cdot)$ is of linear growth, i.e.,

$$\forall x \in \mathbb{R}_+, \quad \rho(x) \leq \rho(1) + \rho(1)x. \quad (2.2)$$

Furthermore, by (2.1) we have for each $x, y \in \mathbb{R}_{++}$,

$$\frac{\rho(x+y)}{x+y} \leq \frac{\rho(x)}{x}, \quad \frac{\rho(x+y)}{x+y} \leq \frac{\rho(y)}{y}$$

and

$$\rho(x+y) = \frac{\rho(x+y)}{x+y}x + \frac{\rho(x+y)}{x+y}y \leq \rho(x) + \rho(y).$$

As a result, $\rho(\cdot)$ can be dominated by itself, i.e.,

$$\forall x, y \in \mathbb{R}_+, \quad |\rho(x) - \rho(y)| \leq \rho(|x - y|). \quad (2.3)$$

2.2. Two Lemmas

By virtue of Fenchel-Moreau theorem, we can prove the following lemma which gives a dual representation of function $\rho(\cdot)$ in \mathbf{S} . It will play an important role in the proof of our main result.

Lemma 2.1. *For $\rho(\cdot) \in \mathbf{S}$, let*

$$\rho^*(q) := \sup_{x \in \mathbb{R}_+} \{\rho(x) - qx\}, \quad q \in \mathbb{R}_+. \quad (2.4)$$

Then, $\rho^(\cdot)$ is a nonincreasing convex function on \mathbb{R}_+ taking values in $\mathbb{R}_+ \cup \{+\infty\}$, and*

$$\rho(x) = \inf_{q \in \mathbb{R}_+, \rho^*(q) \in \mathbb{R}_+} \{\rho^*(q) + qx\}, \quad x \in \mathbb{R}_+. \quad (2.5)$$

Moreover, for any $x_0 \in \mathbb{R}_{++}$, by picking $q = \rho'(x_0) \in \mathbb{R}_+$ we have $\rho^(q) \in \mathbb{R}_+$ and*

$$\rho(x_0) = \rho^*(q) + qx_0. \quad (2.6)$$

Proof. By (2.4), it is straightforward to verify that $\rho^*(\cdot)$ is a nonincreasing convex function on \mathbb{R}_+ taking values in $\mathbb{R}_+ \cup \{+\infty\}$. Now, let us verify (2.5). Define

$$\bar{\rho}(x) := \begin{cases} -\rho(x), & x \geq 0; \\ +\infty, & x < 0. \end{cases} \quad (2.7)$$

Then, $\bar{\rho}(\cdot)$ is a proper convex function defined on \mathbb{R} . By Fenchel-Moreau theorem, if

$$\bar{\rho}^*(q) := \sup_{x \in \mathbb{R}} \{-qx - \bar{\rho}(x)\}, \quad q \in \mathbb{R}, \quad (2.8)$$

then

$$\bar{\rho}(x) = \sup_{q \in \mathbb{R}} \{-qx - \bar{\rho}^*(q)\}, \quad x \in \mathbb{R}. \quad (2.9)$$

It follows from (2.7) and (2.8) that

$$\bar{\rho}^*(q) = \sup_{x \geq 0} \{-qx + \rho(x)\}, \quad q \in \mathbb{R},$$

and then, in light of (2.4), $\bar{\rho}^*(q) = \rho^*(q)$ for $q \geq 0$ and $\bar{\rho}^*(q) = +\infty$ for $q < 0$. As a result, by (2.9),

$$\bar{\rho}(x) = \sup_{q \geq 0} \{-qx - \bar{\rho}^*(q)\} = \sup_{q \geq 0} \{-qx - \rho^*(q)\}, \quad x \in \mathbb{R}.$$

Hence, in light of (2.7), we obtain

$$\rho(x) = -\bar{\rho}(x) = -\sup_{q \geq 0} \{-qx - \rho^*(q)\} = \inf_{q \geq 0} \{qx + \rho^*(q)\}, \quad x \geq 0,$$

which is just (2.5). Moreover, for each $x_0 \in \mathbb{R}_{++}$, by picking $q = \rho'(x_0) = -\bar{\rho}'(x_0) \geq 0$ we have

$$\bar{\rho}(x_0) = -qx_0 - \bar{\rho}^*(q),$$

which yields $\rho^*(q) = \bar{\rho}^*(q) \in \mathbb{R}_+$ and (2.6). The proof is complete. \square

Remark 2.2. For example, for each given $\alpha \in (0, 1)$, the function $\rho(x) := x^\alpha$, $x \in \mathbb{R}_+$ belongs to \mathbf{S} . If

$$\rho^*(q) := \sup_{x \in \mathbb{R}_+} \{\rho(x) - qx\} = \sup_{x \in \mathbb{R}_+} \{x^\alpha - qx\} = (1 - \alpha) \left(\frac{\alpha}{q} \right)^{\frac{\alpha}{1-\alpha}}, \quad q \in \mathbb{R}_+,$$

then $\rho^*(\cdot)$ is a nonincreasing convex function on \mathbb{R}_+ taking values in $\mathbb{R}_+ \cup \{+\infty\}$, and

$$\rho(x) = \inf_{q \in \mathbb{R}_+, \rho^*(q) \in \mathbb{R}_+} \{\rho^*(q) + qx\} = \inf_{q \in \mathbb{R}_{++}} \left\{ (1 - \alpha) \left(\frac{\alpha}{q} \right)^{\frac{\alpha}{1-\alpha}} + qx \right\}, \quad x \in \mathbb{R}_+.$$

Moreover, for any $x_0 \in \mathbb{R}_{++}$, by picking $q = \rho'(x_0) = \alpha x_0^{\alpha-1} \in \mathbb{R}_+$ we have $\rho^*(q) = (1 - \alpha)x_0^\alpha \in \mathbb{R}_+$ and

$$\rho(x_0) = \rho^*(q) + qx_0 = x_0^\alpha.$$

It is noteworthy that for $\rho \in \mathbf{S}$, in light of (2.1) and (2.3), if $\lim_{x \rightarrow 0^+} \rho(x)/x = k$ for some $k > 0$, then $\rho(\cdot)$ must satisfy the Lipschitz condition with Lipschitz constant k , and if $\lim_{x \rightarrow 0^+} \rho(x)/x = +\infty$ and

$$\int_{0^+} \frac{1}{\rho(x)} dx := \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon} \frac{1}{\rho(x)} dx = +\infty,$$

then $\rho(\cdot)$ must satisfy the Osgood condition. In the present paper, we are interested in the other case of $\rho \in \mathbf{S}$, i.e., $\rho(\cdot)$ belongs to the following set:

$$\mathbb{S} := \left\{ \rho(\cdot) \in \mathbf{S} \mid \rho'(0+0) := \lim_{x \rightarrow 0^+} \frac{\rho(x)}{x} = +\infty \text{ and } \int_{0^+} \frac{1}{\rho(x)} dx < +\infty \right\}. \quad (2.10)$$

For example, let $k, c > 0$, $\alpha \in (0, 1)$ and $\beta \geq 1$ be four given constants, and $\varepsilon > 0$ be a sufficiently small constant. Define $\rho_1(x) := kx$, $x \in \mathbb{R}_+$; and the following nine functions:

$$\rho_2(x) := \begin{cases} x |\ln x|^\beta, & x \in [0, \varepsilon]; \\ \rho_2(\varepsilon) + \rho_2'(\varepsilon - 0)(x - \varepsilon), & x \in (\varepsilon, +\infty), \end{cases}$$

$$\rho_3(x) := \begin{cases} x |\ln x| |\ln |\ln x||^\beta, & x \in [0, \varepsilon]; \\ \rho_3(\varepsilon) + \rho_3'(\varepsilon - 0)(x - \varepsilon), & x \in (\varepsilon, +\infty), \end{cases}$$

$$\rho_4(x) := \begin{cases} x |\ln x|^\alpha, & x \in [0, \varepsilon]; \\ \rho_4(\varepsilon) + \rho_4'(\varepsilon - 0)(x - \varepsilon), & x \in (\varepsilon, +\infty), \end{cases}$$

$$\rho_5(x) := \begin{cases} x |\ln x| |\ln |\ln x||^\alpha, & x \in [0, \varepsilon]; \\ \rho_5(\varepsilon) + \rho_5'(\varepsilon - 0)(x - \varepsilon), & x \in (\varepsilon, +\infty), \end{cases} \quad \rho_6(x) := kx^\alpha, \quad x \in \mathbb{R}_+,$$

$$\rho_7(x) := \begin{cases} kx^\alpha, & x \in [0, c]; \\ kc^\alpha + k\alpha c^{\alpha-1}(x - c), & x \in (c, +\infty), \end{cases} \quad \rho_8(x) := \begin{cases} \sqrt{x}, & x \in [0, 1]; \\ -\frac{1}{4}x^2 + x + \frac{1}{4}, & x \in (1, 2); \\ \frac{5}{4}, & x \in [2, +\infty), \end{cases}$$

$$\rho_9(x) := \begin{cases} x^\alpha |\ln x|^k, & x \in [0, \varepsilon]; \\ \rho_9(\varepsilon) + \rho_9'(\varepsilon - 0)(x - \varepsilon), & x \in (\varepsilon, +\infty), \end{cases}$$

and

$$\rho_{10}(x) := \begin{cases} \frac{x^\alpha}{|\ln x|^k}, & x \in [0, \varepsilon]; \\ \rho_{10}(\varepsilon) + \rho_{10}'(\varepsilon - 0)(x - \varepsilon), & x \in (\varepsilon, +\infty). \end{cases}$$

Then, $\rho_1(\cdot)$ belongs to the above-mentioned first case, both $\rho_2(\cdot)$ and $\rho_3(\cdot)$ belong to the second case, and all functions $\rho_i(\cdot)$, $i = 4, \dots, 10$, belong to \mathbb{S} . Functions in \mathbb{S} will be called to be Peano-type.

We especially emphasis that it is straightforward to verify that for each $\rho(\cdot) \in \mathbb{S}$ and $h(\cdot) \in \mathbf{S}$, the function $\bar{\rho}(\cdot) := \rho(\cdot) + h(\cdot)$ must belong to \mathbb{S} .

Functions $\rho(\cdot)$ in \mathbb{S} have the following properties which will be used later, see (3.22)-(3.23) in Section 3.

Lemma 2.3. For $\rho(\cdot) \in \mathbb{S}$, let $\rho_c(x) := \rho(x) + c$ for $x \in \mathbb{R}_+$ and some constant $c \in \mathbb{R}_+$, and let

$$H_c(u) := \int_0^u \frac{1}{\rho_c(x)} dx = \int_0^u \frac{1}{c + \rho(x)} dx, \quad u \in \mathbb{R}_+.$$

Then, $H_c(\cdot)$ is well-defined and strictly increasing on \mathbb{R}_+ , $H_c(0) = 0$ and $\lim_{u \rightarrow +\infty} H_c(u) = +\infty$. Moreover, for each $k_1 \geq H_c(1)$ and $k_2, k_3 > 0$, we have

$$H_c^{-1}(k_1 + k_2) \leq 2e^{k_2(\rho(1)+c)} H_c^{-1}(k_1) \quad (2.11)$$

and

$$\rho_c(k_3 H_c^{-1}(k_1 + k_2)) \leq 2e^{k_2(\rho(1)+c)} \rho_c(k_3 H_c^{-1}(k_1)), \quad (2.12)$$

where $H_c^{-1}(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ denotes the inverse function of $H_c(\cdot)$.

Proof. By the definition (2.10) of \mathbb{S} , it is obvious that $H_c(\cdot)$ is well-defined and a strictly increasing function on \mathbb{R}_+ with $H_c(0) = 0$. In light of (2.2), we have

$$\lim_{u \rightarrow +\infty} H_c(u) = \int_0^{+\infty} \frac{1}{c + \rho(x)} dx \geq \frac{1}{\rho(1) + c} \int_0^{+\infty} \frac{1}{1+x} dx = +\infty.$$

Then the inverse function $H_c^{-1}(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ of $H_c(\cdot)$ is well-defined and is also strictly increasing on \mathbb{R}_+ . Now, fix $k_1 \geq H_c(1)$ and $k_2, k_3 > 0$, and let $u_2 = H_c^{-1}(k_1 + k_2)$ and $u_1 = H_c^{-1}(k_1)$. Then, $u_2 > u_1 \geq 1$ and in light of (2.2), we have

$$\begin{aligned} k_2 = (k_1 + k_2) - k_1 &= H_c(u_2) - H_c(u_1) = \int_{u_1}^{u_2} \frac{1}{c + \rho(x)} dx \\ &\geq \frac{1}{\rho(1) + c} \int_{u_1}^{u_2} \frac{1}{1+x} dx = \frac{1}{\rho(1) + c} \ln \frac{1+u_2}{1+u_1} = \frac{1}{\rho(1) + c} \ln \left(1 + \frac{u_2 - u_1}{1+u_1} \right), \end{aligned}$$

which yields that

$$u_2 - u_1 \leq \left(e^{k_2(\rho(1)+c)} - 1 \right) (1 + u_1) \leq 2 \left(e^{k_2(\rho(1)+c)} - 1 \right) u_1.$$

Thus, picking $C = 2e^{k_2(\rho(1)+c)} > 1$, we obtain $u_2 \leq Cu_1$. This is just (2.11). Finally, in light of (2.1) and $C > 1$, we have

$$\rho_c(k_3 H_c^{-1}(k_1 + k_2)) = \rho_c(k_3 u_2) \leq \rho_c(C k_3 u_1) \leq C \rho_c(k_3 u_1) = C \rho_c(k_3 H_c^{-1}(k_1)),$$

which is the desired conclusion (2.12). The proof is complete. \square

Remark 2.4. *The preceding proof also manifests that the conclusions of Theorem 2.3 are still true when the condition of $\rho'(0+0) = +\infty$ in (2.10) is dropped from the definition of \mathbb{S} .*

3. Main result

Let us always fix several constants $\beta, \bar{\beta}, \tilde{\beta}, \gamma \geq 0$, $\lambda > 0$ and $p > 1$ along with two (\mathcal{F}_t) -progressively measurable processes $(\alpha_t)_{t \in [0, T]}$ and $(\bar{\alpha}_t)_{t \in [0, T]}$ in $\mathcal{L}^p(\mathbb{R}_+)$. Let q be the conjugate of p , i.e., $1/p + 1/q = 1$.

3.1. Statement of the main result

Suppose that the function g can be written as

$$g(\omega, t, y, z) := f(\omega, t, y) + \bar{f}(\omega, t, y) + \tilde{f}(\omega, t, y, z), \quad (3.1)$$

where the three functions

$$\begin{aligned} f(\omega, t, y) &: \Omega \times [0, T] \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+, \\ \bar{f}(\omega, t, y) &: \Omega \times [0, T] \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+, \\ \tilde{f}(\omega, t, y, z) &: \Omega \times [0, T] \times \mathbb{R}_+ \times \mathbb{R}^{1 \times d} \longrightarrow \mathbb{R} \end{aligned}$$

are all (\mathcal{F}_t) -progressively measurable for each $(y, z) \in \mathbb{R} \times \mathbb{R}^{1 \times d}$, and satisfy the following assumptions:

(A1) $d\mathbb{P} \times dt$ -a.e., $f(\omega, t, \cdot)$ is continuous, nonnegative and concave on \mathbb{R}_+ , and has the partial differential $f'(\omega, t, y)$ at any point $y \in \mathbb{R}_{++}$ in the last argument. Moreover, there is a function $\varphi(\cdot) \in \mathbb{S}$ such that $d\mathbb{P} \times dt$ -a.e.,

- $\forall y \in \mathbb{R}_+, \alpha_t(\omega) + \varphi(y) \leq f(\omega, t, y) \leq \bar{\alpha}_t(\omega) + \beta y,$
- $\forall y \in \mathbb{R}_{++}, 0 \leq f'(\omega, t, y) \leq \lambda \varphi'(y).$

(A2) $d\mathbb{P} \times dt - a.e., \bar{f}(\omega, t, \cdot)$ is continuous and nonnegative on \mathbb{R}_+ , and satisfies a monotonicity condition and a linear growth condition, i.e.,

- $\forall (y_1, y_2) \in \mathbb{R}_+ \times \mathbb{R}_+, \operatorname{sgn}(y_1 - y_2)(\bar{f}(\omega, t, y_1) - \bar{f}(\omega, t, y_2)) \leq \bar{\beta}|y_1 - y_2|,$
- $\forall y \in \mathbb{R}_+, 0 \leq \bar{f}(\omega, t, y) \leq \bar{\alpha}_t(\omega) + \bar{\beta}y.$

(A3) $d\mathbb{P} \times dt - a.e., \tilde{f}(\omega, t, \cdot, \cdot)$ is continuous on $\mathbb{R}_+ \times \mathbb{R}^{1 \times d}$ taking values on $\mathbb{R}, \tilde{f}(\omega, t, 0, 0) = 0,$ and satisfies a uniform Lipschitz condition, i.e., for each $(y_1, y_2, z_1, z_2) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^{1 \times d} \times \mathbb{R}^{1 \times d},$

$$|\tilde{f}(\omega, t, y_1, z_1) - \tilde{f}(\omega, t, y_2, z_2)| \leq \tilde{\beta}|y_1 - y_2| + \gamma|z_1 - z_2|.$$

Remark 3.1. Regarding assumptions (A1)-(A3), we make the following remarks.

(i) Let $(u_t)_{t \in [0, T]}, (\bar{u}_t)_{t \in [0, T]}$ and $(\tilde{u}_t)_{t \in [0, T]}$ be three (\mathcal{F}_t) -progressively measurable nonnegative bounded processes such that $\operatorname{ess\,inf}_{t \in [0, T]} u_t \geq c_1$ and $\operatorname{ess\,inf}_{t \in [0, T]} \bar{u}_t \geq c_2$ for some two constants $c_1, c_2 > 0,$ and $\phi(\cdot)$ be a function in $\mathbb{S}.$ Then, it is not difficult to verify that the following function

$$f(\omega, t, y) := \alpha_t(\omega) + u_t(\omega)\phi(\bar{u}_t(\omega)y + \tilde{u}_t(\omega)), \quad (\omega, t, y) \in \Omega \times [0, T] \times \mathbb{R}_+$$

satisfies assumption (A1) with $\varphi(\cdot) := c_1\phi(c_2\cdot).$ In particular, any function in \mathbb{S} satisfies (A1). Consequently, a generator f satisfying (A1) has a close link with a function of Peano type.

(ii) For each $(\omega, t, y) \in \Omega \times [0, T] \times \mathbb{R}_+,$ define

$$\bar{f}_1(\omega, t, y) := (1 - \sqrt{y})^+ \quad \text{and} \quad \bar{f}_2(\omega, t, y) := 1 + \sin y + (3 - |B_t(\omega)|y)^+.$$

It is obvious that neither $g_1(\omega, t, \cdot)$ nor $g_2(\omega, t, \cdot)$ satisfy the usual uniform Lipschitz condition, while both of them satisfy assumption (A2).

(iii) We emphasize that $d\mathbb{P} \times dt - a.e.,$ the function $g(\omega, t, \cdot, \cdot)$ is not necessarily concave/convex, and that the function \tilde{f} in (A3) may take negative values.

The main result of the paper is stated as follows.

Theorem 3.2. Suppose the function g defined in (3.1) satisfies assumptions (A1)-(A3) and $\xi \in L^p(\mathbb{R}_{++}).$ Let the process $\alpha. \geq c$ for a constant $c \geq 0$ and the constants $\hat{p} > \bar{p} > q,$ and let $\tilde{p} > 1$ be the constant such that $1/p + 1/\tilde{p} = 1/\bar{p}.$ If $\frac{1}{\varphi(\xi)+c} \in L^{\hat{p}\lambda e^{2\hat{\beta}T}}(\mathbb{R}_+),$ then the following BSDE

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \quad t \in [0, T] \quad (3.2)$$

admits a unique adapted solution $(Y_t, Z_t)_{t \in [0, T]} \in \mathcal{S}^p(\mathbb{R}_{++}) \times \mathcal{M}^p(\mathbb{R}^{1 \times d}).$ Moreover, by denoting

$$f^*(\omega, t, q) := \sup_{y \in \mathbb{R}_+} (f(\omega, t, y) - qy), \quad (\omega, t, q) \in \Omega \times [0, T] \times \mathbb{R}_+, \quad (3.3)$$

we have for each $t \in [0, T]$,

$$Y_t = \operatorname{ess\,inf}_{q \in \mathcal{H}} \left(e^{-\int_0^t q_s \, ds} Y_t^{q^*} \right), \quad (3.4)$$

and there exists $q^* := f'(\cdot, Y) \in \mathcal{H}$ such that $Y = Y^{q^*}$, where the admissible control set is defined by

$$\mathcal{H} := \left\{ (q_t)_{t \in [0, T]} : \begin{array}{l} q \text{ is an } (\mathcal{F}_t)\text{-progressively measurable } \mathbb{R}_+ \text{-valued process} \\ \text{such that } e^{\int_0^T q_s \, ds} \in L^{\bar{p}}(\mathbb{R}_+) \text{ and } (f^*(t, q_t))_{t \in [0, T]} \in \mathcal{L}^p(\mathbb{R}_+) \end{array} \right\} \quad (3.5)$$

and $(Y_t^{q^*}, Z_t^{q^*})_{t \in [0, T]}$ is the unique adapted solution in $\mathcal{S}^{\bar{p}}(\mathbb{R}_{++}) \times \mathcal{M}^{\bar{p}}(\mathbb{R}^{1 \times d})$ of the following BSDE:

$$Y_t^{q^*} = \xi^{q^*} + \int_t^T g^{q^*}(s, Y_s^{q^*}, Z_s^{q^*}) \, ds - \int_t^T Z_s^{q^*} \, dB_s, \quad t \in [0, T] \quad (3.6)$$

with $\xi^{q^*} := e^{\int_0^T q_s \, ds} \xi$ and for each $(\omega, t, y, z) \in \Omega \times [0, T] \times \mathbb{R}_+ \times \mathbb{R}^{1 \times d}$,

$$g^{q^*}(t, y, z) := e^{\int_0^t q_s \, ds} \left[f^*(t, q_t) + \bar{f}(t, e^{-\int_0^t q_s \, ds} y) + \tilde{f}(t, e^{-\int_0^t q_s \, ds} y, e^{-\int_0^t q_s \, ds} z) \right]. \quad (3.7)$$

Remark 3.3. We make the following remarks.

(i) Suppose the function g defined in (3.1) satisfies assumptions (A1)-(A3). Let \bar{g} be defined as follows:

$$\bar{g}(\omega, t, y, z) := \begin{cases} g(\omega, t, y, z), & (\omega, t, y, z) \in \Omega \times [0, T] \times \mathbb{R}_+ \times \mathbb{R}^{1 \times d}; \\ g(\omega, t, 0, z), & (\omega, t, y, z) \in \Omega \times [0, T] \times (-\infty, 0) \times \mathbb{R}^{1 \times d}. \end{cases} \quad (3.8)$$

It is not hard to verify that for each $(y, z) \in \mathbb{R} \times \mathbb{R}^{1 \times d}$, $\bar{g}(\omega, t, y, z)$ is (\mathcal{F}_t) -progressively measurable taking values on \mathbb{R} , and $d\mathbb{P} \times dt$ - a.e.,

- $\bar{g}(\omega, t, \cdot, \cdot)$ is continuous on $\mathbb{R} \times \mathbb{R}^{1 \times d}$, and $0 \leq \alpha_t(\omega) \leq \bar{g}(\omega, t, 0, 0) \leq 2\bar{\alpha}_t(\omega)$;
- $\forall (y, z) \in \mathbb{R} \times \mathbb{R}^{1 \times d}$, $|\bar{g}(\omega, t, y, z)| \leq 2\bar{\alpha}_t(\omega) + (\beta + \bar{\beta} + \tilde{\beta})|y| + \gamma|z|$.

From some existing results such as Theorem 1 of Fan [14], Theorem 5.1 of Fan [13] and Theorem 2.3 (iv) of Fan, Hu and Tang [18], it follows that for each $\xi \in L^p(\mathbb{R})$, the following BSDE

$$Y_t = \xi + \int_t^T \bar{g}(s, Y_s, Z_s) \, ds - \int_t^T Z_s \, dB_s, \quad t \in [0, T] \quad (3.9)$$

admits an adapted solution $(Y_t, Z_t)_{t \in [0, T]} \in \mathcal{S}^p(\mathbb{R}) \times \mathcal{M}^p(\mathbb{R}^{1 \times d})$. And, by the classical comparison theorem under the Lipschitz condition we have that if $\xi > 0$, then $Y > 0$. This means that for each $\xi \in L^p(\mathbb{R}_{++})$, BSDE (3.2) admits an adapted solution $(Y_t, Z_t)_{t \in [0, T]} \in \mathcal{S}^p(\mathbb{R}_{++}) \times \mathcal{M}^p(\mathbb{R}^{1 \times d})$.

(ii) Suppose that the function f disappears in the definition (3.1) of the function g , and functions \bar{f} and \tilde{f} satisfy assumptions (A2) and (A3). Let \bar{g} be defined in (3.8). Then for each $(y, z) \in \mathbb{R} \times \mathbb{R}^{1 \times d}$, $\bar{g}(\omega, t, y, z)$ is (\mathcal{F}_t) -progressively measurable taking values on \mathbb{R} , and $d\mathbb{P} \times dt$ - a.e.,

- $\bar{g}(\omega, t, \cdot, \cdot)$ is continuous on $\mathbb{R} \times \mathbb{R}^{1 \times d}$, and $0 \leq \bar{g}(\omega, t, 0, 0) \leq \bar{\alpha}_t(\omega)$;
- $\forall (y_1, y_2, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{1 \times d}$, $\operatorname{sgn}(y_1 - y_2)(\bar{g}(\omega, t, y_1, z) - \bar{g}(\omega, t, y_2, z)) \leq (\bar{\beta} + \tilde{\beta})|y_1 - y_2|$;
- $\forall (y, z_1, z_2) \in \mathbb{R} \times \mathbb{R}^{1 \times d} \times \mathbb{R}^{1 \times d}$, $|\bar{g}(\omega, t, y, z_1) - \bar{g}(\omega, t, y, z_2)| \leq \gamma|z_1 - z_2|$;
- $\forall y \in \mathbb{R}_+$, $|\bar{g}(\omega, t, y, 0)| \leq \bar{\alpha}_t(\omega) + (\bar{\beta} + \tilde{\beta})|y|$.

It then follows from Theorem 4.2 of Briand et al. [4] that for each $\xi \in L^p(\mathbb{R})$, BSDE (3.9) admits a unique adapted solution $(Y_t, Z_t)_{t \in [0, T]} \in \mathcal{S}^p(\mathbb{R}) \times \mathcal{M}^p(\mathbb{R}^{1 \times d})$. And, by the classical comparison theorem under the Lipschitz condition we have that if $\xi > 0$, then $Y > 0$. This means that for each $\xi \in L^p(\mathbb{R}_{++})$, BSDE (3.2) admits a unique adapted solution $(Y_t, Z_t)_{t \in [0, T]} \in \mathcal{S}^p(\mathbb{R}_{++}) \times \mathcal{M}^p(\mathbb{R}^{1 \times d})$.

(iii) Suppose the function g defined in (3.1) satisfies the above assumptions (A1)-(A3). Suppose further that $\xi \in L^p(\mathbb{R}_{++})$ is bigger than or equal to some constant $\bar{c} > 0$. For each $c \in (0, \bar{c}]$, define

$$\hat{f}(\omega, t, y) := \begin{cases} f(\omega, t, y), & (\omega, t, y) \in \Omega \times [0, T] \times [c, +\infty); \\ \frac{f(\omega, t, c)}{c}y, & (\omega, t, y) \in \Omega \times [0, T] \times [0, c). \end{cases}$$

By assumption (A1) we can easily verify that $d\mathbb{P} \times dt - a.e.$, for each $(y_1, y_2) \in \mathbb{R}_+ \times \mathbb{R}_+$,

$$|\hat{f}(\omega, t, y_1) - \hat{f}(\omega, t, y_2)| \leq \lambda \varphi'(c) |y_1 - y_2|.$$

Thus, by replacing f with \hat{f} in the definition (3.1) of the function g and using a similar argument as in (ii), we can conclude that BSDE (3.2) admits at most an adapted solution $(Y_t, Z_t)_{t \in [0, T]} \in \mathcal{S}^p(\mathbb{R}_+) \times \mathcal{M}^p(\mathbb{R}^{1 \times d})$ such that $Y \geq c$. On the other hand, note that the following BSDE

$$Y_t = \bar{c} + \int_t^T (-\tilde{\beta}|Y_s| - \gamma|Z_s|)ds - \int_t^T Z_s dB_s, \quad t \in [0, T]$$

admits a unique adapted solution $(\bar{c}e^{\tilde{\beta}(t-T)}, 0)_{t \in [0, T]}$ belonging to $\mathcal{S}^p(\mathbb{R}_{++}) \times \mathcal{M}^p(\mathbb{R}^{1 \times d})$. By the nonnegativity of f and \bar{f} , the Lipschitz condition of \tilde{f} and the classical comparison theorem under the Lipschitz condition it can be checked that for any adapted solution $(Y_t, Z_t)_{t \in [0, T]}$ of BSDE (3.2), we have $Y \geq \bar{c}e^{-\tilde{\beta}T}$. Consequently, BSDE (3.2) admits a unique adapted solution $(Y_t, Z_t)_{t \in [0, T]}$ in the space $\mathcal{S}^p(\mathbb{R}_{++}) \times \mathcal{M}^p(\mathbb{R}^{1 \times d})$.

From Theorem 3.2, we immediately have the assertion (iii) of Theorem 3.3 as the following corollary.

Corollary 3.4. *Suppose the function g defined in (3.1) satisfies assumptions (A1)-(A3) and $\xi \in L^p(\mathbb{R}_{++})$. If $\xi \geq \bar{c}$ for some $\bar{c} > 0$, then BSDE (3.2) admits a unique adapted solution in $\mathcal{S}^p(\mathbb{R}_{++}) \times \mathcal{M}^p(\mathbb{R}^{1 \times d})$.*

At the end of this subsection, we present the following example.

Example 3.5. *For simplicity, we only consider the case of $\bar{f}(\cdot) = \tilde{f}(\cdot) \equiv 0$ appearing in (3.1). For each $(\omega, t, y) \in \Omega \times [0, T] \times \mathbb{R}_+$, define*

$$g_1(\omega, t, y) := \sqrt{y + |B_t(\omega)|}, \quad g_2(\omega, t, y) := \sqrt{y} + |B_t(\omega)|, \quad g_3(\omega, t, y) := \sqrt{y + T - t},$$

and

$$g_4(\omega, t, y) := \sqrt{y} + T - t, \quad g_5(\omega, t, y) := \sqrt{y + t}, \quad g_6(\omega, t, y) := \sqrt{y} + t.$$

It is easy to verify that all these functions satisfy the assumption (A1) with $\varphi(x) := \sqrt{x} \in \mathbb{S}$ and $\lambda = 1$. It then follows from Theorem 3.2 that for each $i = 1, \dots, 6$ and $\xi \in L^p(\mathbb{R}_{++})$, if $1/\xi \in L^{2\hat{p}}$ for some $\hat{p} > q$, then BSDE (3.2) with $g := g_i$ admits a unique adapted solution $(Y^i, Z^i) \in \mathcal{S}^p(\mathbb{R}_{++}) \times \mathcal{M}^p(\mathbb{R}^{1 \times d})$. For example, for any (\mathcal{F}_t) -progressively measurable bounded processes $(a_t)_{t \in [0, T]} \in \mathbb{R}$ and $(b_t)_{t \in [0, T]} \in \mathbb{R}^{1 \times d}$, the terminal value $\xi := e^{\int_0^T a_t dt |B_T| + \int_0^T b_t dB_t}$ satisfies the above-mentioned integrability condition.

On the other hand, if we first consider the BSDE on the time interval $[T/2, T]$, then from Theorem 3.2 with $\varphi(x) := \sqrt{x} \in \mathbb{S}$ and $\alpha := \sqrt{T/2} > 0$ it follows that BSDEs (3.2) with $g := g_5, g_6$ admit unique adapted solutions $(Y_t^5, Z_t^5)_{t \in [T/2, T]}$ and $(Y_t^6, Z_t^6)_{t \in [T/2, T]}$, respectively, on the time interval $[T/2, T]$ in the space $\mathcal{S}^p(\mathbb{R}_{++}) \times \mathcal{M}^p(\mathbb{R}^{1 \times d})$. Furthermore, by taking the conditional mathematical expectation it is straightforward to verify that for each $t \in [T/2, T]$,

$$Y_t^5 \geq \sqrt{\frac{T}{2}}(T-t) \quad \text{and} \quad Y_t^6 \geq \frac{T}{2}(T-t)$$

and then

$$Y_{T/2}^5 \geq \left(\frac{T}{2}\right)^{\frac{3}{2}} \quad \text{and} \quad Y_{T/2}^6 \geq \left(\frac{T}{2}\right)^2.$$

Thus, Theorem 3.4 yields that BSDEs (3.2) with generators $g := g_5, g_6$ admit also unique adapted solutions $(Y_t^5, Z_t^5)_{t \in [0, T/2]}$ and $(Y_t^6, Z_t^6)_{t \in [0, T/2]}$, respectively, on the time interval $[0, T/2]$ in the space $\mathcal{S}^p(\mathbb{R}_{++}) \times \mathcal{M}^p(\mathbb{R}^{1 \times d})$.

As a result, we conclude that for each $\xi \in L^p(\mathbb{R}_{++})$, without assuming any integrability on $1/\xi$, BSDEs (3.2) with generators $g := g_5, g_6$ admit unique adapted solutions (Y^5, Z^5) and (Y^6, Z^6) , respectively, in the space $\mathcal{S}^p(\mathbb{R}_{++}) \times \mathcal{M}^p(\mathbb{R}^{1 \times d})$.

3.2. Proof of the main result

Now, we are at a position to prove Theorem 3.2.

Proof of Theorem 3.2. The existence of an adapted solution $(Y_t, Z_t)_{t \in [0, T]} \in \mathcal{S}^p(\mathbb{R}_{++}) \times \mathcal{M}^p(\mathbb{R}^{1 \times d})$ of BSDE (3.2) has been illustrated in Theorem 3.3 (i).

Next, we further verify existence and uniqueness of the adapted solution $(Y_t^q, Z_t^q)_{t \in [0, T]} \in \mathcal{S}^{\bar{p}}(\mathbb{R}_{++}) \times \mathcal{M}^{\bar{p}}(\mathbb{R}^{1 \times d})$ of BSDE (3.6) for any $q \in \mathcal{H}$. In fact, let $q \in \mathcal{H}$ and define

$$\bar{g}^q(\omega, t, y, z) := \begin{cases} g^q(\omega, t, y, z), & (\omega, t, y, z) \in \Omega \times [0, T] \times \mathbb{R}_+ \times \mathbb{R}^{1 \times d}; \\ g^q(\omega, t, 0, z), & (\omega, t, y, z) \in \Omega \times [0, T] \times (-\infty, 0) \times \mathbb{R}^{1 \times d}. \end{cases}$$

In light of assumptions (A2) and (A3) along with the definition (3.7) of g^q , we can verify that for each $(y, z) \in \mathbb{R} \times \mathbb{R}^{1 \times d}$, $\bar{g}^q(\omega, t, y, z)$ is (\mathcal{F}_t) -progressively measurable taking values on \mathbb{R} , and $d\mathbb{P} \times dt - a.e.$,

- $\bar{g}^q(\omega, t, \cdot, \cdot)$ is continuous on $\mathbb{R} \times \mathbb{R}^{1 \times d}$, and $\bar{g}^q(\omega, t, 0, 0) \geq 0$;
- $\forall (y_1, y_2, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{1 \times d}$, $\text{sgn}(y_1 - y_2)(\bar{g}^q(\omega, t, y_1, z) - \bar{g}^q(\omega, t, y_2, z)) \leq (\bar{\beta} + \tilde{\beta})|y_1 - y_2|$;
- $\forall (y, z_1, z_2) \in \mathbb{R} \times \mathbb{R}^{1 \times d} \times \mathbb{R}^{1 \times d}$, $|\bar{g}^q(\omega, t, y, z_1) - \bar{g}^q(\omega, t, y, z_2)| \leq \gamma|z_1 - z_2|$;
- $\forall y \in \mathbb{R}_+$, $|\bar{g}^q(\omega, t, y, 0)| \leq e^{\int_0^t q_s(\omega) ds} [\bar{\alpha}_t(\omega) + f^*(\omega, t, q_t(\omega))] + (\bar{\beta} + \tilde{\beta})|y|$.

Moreover, by the definition (3.5) of \mathcal{H} we know that $e^{\int_0^T q_s ds} \in L^{\bar{p}}(\mathbb{R}_+)$ and $f^*(\cdot, q) \in \mathcal{L}^p(\mathbb{R}_+)$. It then follows from Hölder's inequality that

$$\mathbb{E}[|\xi^q|^{\bar{p}}] = \mathbb{E}\left[e^{\bar{p} \int_0^T q_s ds} |\xi|^{\bar{p}}\right] \leq \left(\mathbb{E}\left[e^{\bar{p} \int_0^T q_s ds}\right]\right)^{\frac{\bar{p}}{p}} \left(\mathbb{E}[|\xi|^p]\right)^{\frac{\bar{p}}{p}} < +\infty$$

and

$$\begin{aligned} \mathbb{E} \left[\left(\int_0^T |\bar{g}^q(t, 0, 0)| dt \right)^{\bar{p}} \right] &\leq \mathbb{E} \left[e^{\bar{p} \int_0^T q_s ds} \left(\int_0^T (\bar{\alpha}_t + f^*(t, q_t)) dt \right)^{\bar{p}} \right] \\ &\leq \left(\mathbb{E} \left[e^{\bar{p} \int_0^T q_s ds} \right] \right)^{\frac{\bar{p}}{p}} \left(\mathbb{E} \left[\left(\int_0^T (\bar{\alpha}_t + f^*(t, q_t)) dt \right)^p \right] \right)^{\frac{\bar{p}}{p}} \\ &< +\infty. \end{aligned}$$

Thus, an identical argument to (ii) of Theorem 3.3 yields that for each $q \in \mathcal{H}$, BSDE (3.6) admits a unique adapted solution $(Y_t^q, Z_t^q)_{t \in [0, T]} \in \mathcal{S}^{\bar{p}}(\mathbb{R}_{++}) \times \mathcal{M}^{\bar{p}}(\mathbb{R}^{1 \times d})$.

In the sequel, for each $q \in \mathcal{H}$, define $\hat{Y}^q := e^{-\int_0^\cdot q_s ds} Y^q$ and $\hat{Z}^q := e^{-\int_0^\cdot q_s ds} Z^q$, and let us prove that for each $t \in [0, T]$,

$$Y_t \leq \hat{Y}_t^q. \quad (3.10)$$

First, in light of (3.6) and (3.7), by Itô's formula we obtain

$$\hat{Y}_t^q = \xi + \int_t^T \hat{g}^q(s, \hat{Y}_s^q, \hat{Z}_s^q) ds - \int_t^T \hat{Z}_s^q dB_s, \quad t \in [0, T], \quad (3.11)$$

where for each $(\omega, t, y, z) \in \Omega \times [0, T] \times \mathbb{R}_+ \times \mathbb{R}^{1 \times d}$,

$$\hat{g}^q(\omega, t, y, z) := q_t(\omega)y + f^*(\omega, t, q_t(\omega)) + \bar{f}(\omega, t, y) + \tilde{f}(\omega, t, y, z).$$

On the other hand, in light of (3.3), by a similar analysis as in Theorem 2.1 we obtain

$$f(\omega, t, y) = \inf_{q \in \mathbb{R}_+; f^*(\omega, t, q) \in \mathbb{R}_+} \{f^*(\omega, t, q) + qy\}, \quad (\omega, t, y) \in \Omega \times [0, T] \times \mathbb{R}_+,$$

and then

$$f(t, Y_t) \leq q_t Y_t + f^*(t, q_t), \quad t \in [0, T].$$

Furthermore, in light of the definitions of g and \hat{g}^q , by assumptions (A2) and (A3) along with the last inequality we can deduce that

$$\begin{aligned} &\mathbf{1}_{Y_t \geq \hat{Y}_t^q} \left(g(t, Y_t, Z_t) - g^q(t, \hat{Y}_t^q, \hat{Z}_t^q) \right) \\ &= \mathbf{1}_{Y_t \geq \hat{Y}_t^q} \left[f(t, Y_t) - q_t Y_t - f^*(t, q_t) + q_t (Y_t - \hat{Y}_t^q) \right] \\ &\quad + \mathbf{1}_{Y_t \geq \hat{Y}_t^q} \left(\bar{f}(t, Y_t) - \bar{f}(t, \hat{Y}_t^q) \right) + \mathbf{1}_{Y_t \geq \hat{Y}_t^q} \left(\tilde{f}(t, Y_t, Z_t) - \tilde{f}(t, \hat{Y}_t^q, \hat{Z}_t^q) \right) \\ &\leq (q_t + \bar{\beta} + \tilde{\beta})(Y_t - \hat{Y}_t^q)^+ + \gamma \mathbf{1}_{Y_t \geq \hat{Y}_t^q} |Z_t - \hat{Z}_t^q|, \quad t \in [0, T], \end{aligned}$$

and, by letting $\hat{q}_t := (\bar{\beta} + \tilde{\beta})t + \int_0^t q_s ds$, $t \in [0, T]$ and using Itô-Tanaka's formula for $e^{\hat{q}_t} (Y_t - \hat{Y}_t^q)^+$,

$$e^{\hat{q}_t} (Y_t - \hat{Y}_t^q)^+ \leq \int_t^T e^{\hat{q}_s} \mathbf{1}_{Y_s \geq \hat{Y}_s^q} \gamma |Z_s - \hat{Z}_s^q| ds - \int_t^T e^{\hat{q}_s} \mathbf{1}_{Y_s \geq \hat{Y}_s^q} (Z_s - \hat{Z}_s^q) dB_s, \quad t \in [0, T]. \quad (3.12)$$

Now, we define a probability measure \mathbb{Q}^q which is equivalent to \mathbb{P} by

$$\frac{d\mathbb{Q}^q}{d\mathbb{P}} := \exp \left(\int_0^T \frac{\gamma(Z_s - \hat{Z}_s^q)}{|Z_s - \hat{Z}_s^q|} \mathbf{1}_{\{|Z_s - \hat{Z}_s^q| \neq 0\}} dB_s - \frac{\gamma^2}{2} \int_0^T \mathbf{1}_{\{|Z_s - \hat{Z}_s^q| \neq 0\}} ds \right).$$

Then, $d\mathbb{Q}^q/d\mathbb{P}$ admits moments of any order. According to Girsanov's theorem, the process

$$B_t^q := B_t - \int_0^t \frac{\gamma(Z_s - \hat{Z}_s^q)^\top}{|Z_s - \hat{Z}_s^q|} \mathbf{1}_{\{|Z_s - \hat{Z}_s^q| \neq 0\}} ds, \quad t \in [0, T]$$

is an (\mathcal{F}_t) -adapted standard Brownian motion, and (3.12) can be written as

$$e^{\hat{q}t}(Y_t - \hat{Y}_t^{q^*})^+ \leq - \int_t^T e^{\hat{q}s} \mathbf{1}_{Y_s \geq \hat{Y}_s^{q^*}} (Z_s - \hat{Z}_s^{q^*}) dB_s^{q^*}, \quad t \in [0, T]. \quad (3.13)$$

Finally, by the definition of \hat{q} and the integrability of Z^{q^*} we know that $e^{\hat{q} \cdot} \hat{Z}^{q^*} = e^{(\bar{\beta} + \tilde{\beta}) \cdot} Z^{q^*} \in \mathcal{M}^{\bar{p}}(\mathbb{R}^{1 \times d})$, and, by Hölder's inequality we deduce that

$$\begin{aligned} \mathbb{E} \left[\left(\int_0^T |e^{\hat{q}t} Z_t|^2 dt \right)^{\frac{\bar{p}}{2}} \right] &\leq e^{\bar{p}(\bar{\beta} + \tilde{\beta})T} \mathbb{E} \left[e^{\bar{p} \int_0^T q_s ds} \left(\int_0^T |Z_t|^2 dt \right)^{\frac{\bar{p}}{2}} \right] \\ &\leq e^{\bar{p}(\bar{\beta} + \tilde{\beta})T} \left(\mathbb{E} \left[e^{\bar{p} \int_0^T q_s ds} \right] \right)^{\frac{\bar{p}}{2}} \left(\mathbb{E} \left[\left(\int_0^T |Z_t|^2 dt \right)^{\frac{\bar{p}}{2}} \right] \right)^{\frac{\bar{p}}{2}} \\ &< +\infty \end{aligned}$$

and then

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}^{q^*}} \left[\left(\int_0^T |e^{\hat{q}t} (Z_t - \hat{Z}_t^{q^*})|^2 dt \right)^{\frac{1}{2}} \right] \\ \leq \left(\mathbb{E} \left[\left(\int_0^T |e^{\hat{q}t} (Z_t - \hat{Z}_t^{q^*})|^2 dt \right)^{\frac{\bar{p}}{2}} \right] \right)^{\frac{1}{\bar{p}}} \left(\mathbb{E} \left[\left(\frac{d\mathbb{Q}^{q^*}}{d\mathbb{P}} \right)^{\frac{\bar{p}}{\bar{p}-1}} \right] \right)^{\frac{\bar{p}-1}{\bar{p}}} < +\infty. \end{aligned}$$

Thus, (3.10) follows immediately by taking the conditional mathematical expectation with respect to \mathcal{F}_t under probability \mathbb{Q}^{q^*} on the both sides of (3.13).

In order to complete the proof of Theorem 3.2, it remains to verify that there exists a $q^* \in \mathcal{H}$ such that

$$Y = \hat{Y}^{q^*} = e^{-\int_0^\cdot q_s^* ds} Y^{q^*}. \quad (3.14)$$

The following proof will be divided into three steps.

Step 1. In this step we will prove that Y admits a desired lower bound, see (3.19) below. For each $t \in [0, T]$, we define

$$a_t := \frac{\tilde{f}(t, Y_t, Z_t) - \tilde{f}(t, 0, Z_t)}{Y_t} \mathbf{1}_{\{Y_t > 0\}} \in \mathbb{R}$$

and

$$b_t := \frac{(\tilde{f}(t, 0, Z_t) - \tilde{f}(t, 0, 0)) Z_t^\top}{|Z_t|^2} \mathbf{1}_{\{|Z_t| \neq 0\}} \in \mathbb{R}^d.$$

It follows from assumption (A3) that $|a_t| \leq \tilde{\beta}$, $|b_t| \leq \gamma$ and

$$\tilde{f}(t, Y_t, Z_t) = a_t Y_t + Z_t b_t. \quad (3.15)$$

Furthermore, define a probability measure $\bar{\mathbb{Q}}$ which is equivalent to \mathbb{P} by

$$\frac{d\bar{\mathbb{Q}}}{d\mathbb{P}} := \exp \left(\int_0^T b_s^\top B_s - \frac{1}{2} \int_0^T |b_s|^2 ds \right).$$

Then, both $\frac{d\bar{\mathbb{Q}}}{d\mathbb{P}}$ and $\frac{d\mathbb{P}}{d\bar{\mathbb{Q}}}$ admit moments of any order. According to Girsanov's theorem, the process

$$\bar{B}_t := B_t - \int_0^t b_s ds, \quad t \in [0, T]$$

is an (\mathcal{F}_t) -adapted standard Brownian motion, and in light of (3.15), BSDE (3.2) can be written as

$$Y_t = \xi + \int_t^T (f(s, Y_s) + \bar{f}(s, Y_s) + a_s Y_s) ds - \int_t^T Z_s d\bar{B}_s, \quad t \in [0, T].$$

Then, letting $\bar{\xi} := e^{\int_0^T a_s ds} \xi$, $\bar{Y}_\cdot := e^{\int_0^\cdot a_s ds} Y_\cdot$ and $\bar{Z}_\cdot := e^{\int_0^\cdot a_s ds} Z_\cdot$, by Itô's formula we obtain

$$\bar{Y}_t = \bar{\xi} + \int_t^T \bar{g}(s, \bar{Y}_s) ds - \int_t^T \bar{Z}_s d\bar{B}_s, \quad t \in [0, T] \quad (3.16)$$

with

$$\bar{g}(t, y) := e^{\int_0^t a_s ds} \left(f(t, e^{-\int_0^t a_s ds} y) + \bar{f}(t, e^{-\int_0^t a_s ds} y) \right), \quad (\omega, t, y) \in \Omega \times [0, T] \times \mathbb{R}_+.$$

From assumption (A1) along with nonnegativity of \bar{f} and boundedness of a_\cdot , it follows that $d\mathbb{P} \times dt - a.e.$, for each $t \in [0, T]$ and $y \in \mathbb{R}_+$,

$$\bar{g}(t, y) \geq e^{\int_0^t a_s ds} f(t, e^{-\int_0^t a_s ds} y) \geq e^{\int_0^t a_s ds} (\alpha_t + \varphi(y)) \geq \bar{\alpha}_t + \bar{\varphi}(y) \geq \bar{c} + \bar{\varphi}(y) \quad (3.17)$$

with

$$\bar{\alpha}_\cdot := e^{-\int_0^\cdot \beta_s ds} \alpha_\cdot \geq \bar{c} := e^{-\int_0^T \beta_s ds} c \quad \text{and} \quad \bar{\varphi}(y) := e^{-\int_0^\cdot \beta_s ds} \varphi(y) \in \mathbb{S}.$$

Now, let $\bar{\varphi}_\varepsilon(\cdot) := \bar{c} + \bar{\varphi}(\cdot)$. For each $n \geq 1$ and $t \in [0, T]$, define the following stopping time

$$\tau_n^t := \inf \left\{ u \geq t \mid \int_0^u \left| \frac{\bar{Z}_s}{\bar{\varphi}_\varepsilon(\bar{Y}_s)} \right|^2 ds \geq n \right\} \wedge T$$

and the following function

$$\Phi_\varepsilon(u) := \int_0^u \frac{1}{\bar{\varphi}_\varepsilon(y)} dy = \int_0^u \frac{1}{\bar{c} + \bar{\varphi}(y)} dy, \quad u \in \mathbb{R}_+.$$

In light of (3.16) and (3.17), using Itô's formula for $\Phi_\varepsilon(\bar{Y}_t)$ yields that for each $n \geq 1$ and $t \in [0, T]$,

$$\begin{aligned} \Phi_\varepsilon(\bar{Y}_t) &= \Phi_\varepsilon(\bar{Y}_{\tau_n^t}) + \int_t^{\tau_n^t} \left(\frac{\bar{g}(s, \bar{Y}_s)}{\bar{\varphi}_\varepsilon(\bar{Y}_s)} + \frac{\bar{\varphi}'(\bar{Y}_s)}{2(\bar{\varphi}_\varepsilon(\bar{Y}_s))^2} \right) ds - \int_t^{\tau_n^t} \frac{\bar{Z}_s}{\bar{\varphi}_\varepsilon(\bar{Y}_s)} d\bar{B}_s \\ &\geq \mathbb{E}_{\bar{\mathbb{Q}}} [\Phi_\varepsilon(\bar{Y}_{\tau_n^t}) + (\tau_n^t - t) | \mathcal{F}_t]. \end{aligned} \quad (3.18)$$

Since $\Phi_\varepsilon(0) = 0$ and for each $u > 0$, $\Phi_\varepsilon'(u) > 0$ and $\Phi_\varepsilon''(u) \leq 0$, we have $\Phi_\varepsilon(\cdot) \in \mathbf{S}$ and, in light of (2.2),

$$\Phi_\varepsilon(u) \leq \Phi_\varepsilon(1) + \Phi_\varepsilon(1)u, \quad u \in \mathbb{R}_+.$$

On the other hand, by Hölder's inequality we deduce that

$$\mathbb{E} \left[\sup_{t \in [0, T]} |\bar{Y}_t|^{\bar{p}} \right] \leq e^{\bar{p} \int_0^T \beta_s ds} \mathbb{E} \left[\sup_{t \in [0, T]} |Y_t|^{\bar{p}} \right] \leq e^{\bar{p} \int_0^T \beta_s ds} \left(\mathbb{E} \left[\sup_{t \in [0, T]} |Y_t|^p \right] \right)^{\frac{\bar{p}}{p}} < +\infty$$

and then

$$\mathbb{E}_{\bar{\mathbb{Q}}} \left[\sup_{t \in [0, T]} |\bar{Y}_t| \right] \leq \left(\mathbb{E} \left[\sup_{t \in [0, T]} |Y_t|^{\bar{p}} \right] \right)^{\frac{1}{\bar{p}}} \left(\mathbb{E} \left[\left(\frac{d\bar{\mathbb{Q}}}{d\mathbb{P}} \right)^{\frac{\bar{p}}{\bar{p}-1}} \right] \right)^{\frac{\bar{p}-1}{\bar{p}}} < +\infty.$$

In light of the last three inequalities, we can send $n \rightarrow \infty$ in (3.18) to obtain that for each $t \in [0, T]$,

$$\Phi_\varepsilon(\bar{Y}_t) \geq \mathbb{E}_{\bar{\mathbb{Q}}} [\Phi_\varepsilon(\bar{\xi}) | \mathcal{F}_t] + T - t$$

and then

$$Y_t \geq e^{-\int_0^t a_s ds} \Phi_\varepsilon^{-1} \left(\mathbb{E}_{\bar{\mathbb{Q}}} [\Phi_\varepsilon(\bar{\xi}) | \mathcal{F}_t] + T - t \right) \geq e^{-\int_0^t \beta_s ds} \Phi_\varepsilon^{-1} (\eta + T - t) \quad (3.19)$$

with

$$\eta := \min_{t \in [0, T]} \mathbb{E}_{\mathbb{Q}} [\Phi_{\bar{c}}(\bar{\xi}) | \mathcal{F}_t] \in \mathbb{R}_{++}. \quad (3.20)$$

Step 2. In this step, we pick $q^* := f'(\cdot, Y) \geq 0$ and prove that $q^* \in \mathcal{H}$. First, it follows from assumption (A1) and the definition of $\bar{\varphi}_{\bar{c}}(\cdot)$ that for each $t \in [0, T]$,

$$q_t^* = f'(t, Y_t) \leq \lambda \varphi'(Y_t) = \lambda e^{\bar{\beta}T} \bar{\varphi}'_{\bar{c}}(Y_t).$$

Then, in light of (3.19) and the definition of $\Phi_{\bar{c}}(\cdot)$ along with the nonincreasing property of $\bar{\varphi}'_{\bar{c}}(\cdot)$ and the nondecreasing property of $\bar{\varphi}_{\bar{c}}(\cdot)$, we can deduce that

$$\begin{aligned} \int_0^T q_t^* dt &\leq \lambda e^{\bar{\beta}T} \int_0^T \bar{\varphi}'_{\bar{c}}(Y_t) dt \leq \lambda e^{\bar{\beta}T} \int_0^T \bar{\varphi}'_{\bar{c}}(e^{-\bar{\beta}T} \Phi_{\bar{c}}^{-1}(\eta + T - t)) dt \\ &= \lambda e^{\bar{\beta}T} \int_{\Phi_{\bar{c}}^{-1}(\eta)}^{\Phi_{\bar{c}}^{-1}(\eta + T)} \bar{\varphi}'_{\bar{c}}(e^{-\bar{\beta}T} u) \frac{1}{\bar{\varphi}_{\bar{c}}(u)} du \\ &\leq \lambda e^{2\bar{\beta}T} \int_{\Phi_{\bar{c}}^{-1}(\eta)}^{\Phi_{\bar{c}}^{-1}(\eta + T)} \bar{\varphi}'_{\bar{c}}(e^{-\bar{\beta}T} u) \frac{1}{\bar{\varphi}_{\bar{c}}(e^{-\bar{\beta}T} u)} d(e^{-\bar{\beta}T} u) \\ &= \lambda e^{2\bar{\beta}T} \left[\ln \bar{\varphi}_{\bar{c}}(e^{-\bar{\beta}T} \Phi_{\bar{c}}^{-1}(\eta + T)) - \ln \bar{\varphi}_{\bar{c}}(e^{-\bar{\beta}T} \Phi_{\bar{c}}^{-1}(\eta)) \right]. \end{aligned}$$

By picking $\check{p} \in (\bar{p}, \hat{p})$ and using Hölder's inequality, we have

$$\begin{aligned} \mathbb{E} \left[e^{\bar{p} \int_0^T q_t^* dt} \right] &= \mathbb{E}_{\mathbb{Q}} \left[e^{\bar{p} \int_0^T q_t^* dt} \frac{d\mathbb{P}}{d\mathbb{Q}} \right] \leq \mathbb{E}_{\mathbb{Q}} \left[\left(\frac{\bar{\varphi}_{\bar{c}}(e^{-\bar{\beta}T} \Phi_{\bar{c}}^{-1}(\eta + T))}{\bar{\varphi}_{\bar{c}}(e^{-\bar{\beta}T} \Phi_{\bar{c}}^{-1}(\eta))} \right)^{\bar{p} \lambda e^{2\bar{\beta}T}} \frac{d\mathbb{P}}{d\mathbb{Q}} \right] \\ &\leq \left\{ \mathbb{E}_{\mathbb{Q}} \left[\left(\frac{\bar{\varphi}_{\bar{c}}(e^{-\bar{\beta}T} \Phi_{\bar{c}}^{-1}(\eta + T))}{\bar{\varphi}_{\bar{c}}(e^{-\bar{\beta}T} \Phi_{\bar{c}}^{-1}(\eta))} \right)^{\bar{p} \lambda e^{2\bar{\beta}T}} \right] \right\}^{\frac{\bar{p}}{\check{p}}} \left\{ \mathbb{E}_{\mathbb{Q}} \left[\left(\frac{d\mathbb{P}}{d\mathbb{Q}} \right)^{\frac{\check{p}}{\bar{p} - \check{p}}} \right] \right\}^{\frac{\bar{p} - \check{p}}{\check{p}}}. \end{aligned} \quad (3.21)$$

By Theorem 2.3, we have for $\eta \geq \Phi_{\bar{c}}(1)$,

$$\bar{\varphi}_{\bar{c}}(e^{-\bar{\beta}T} \Phi_{\bar{c}}^{-1}(\eta + T)) \leq 2e^{T(\varphi(1) + \bar{c})} \bar{\varphi}_{\bar{c}}(e^{-\bar{\beta}T} \Phi_{\bar{c}}^{-1}(\eta)). \quad (3.22)$$

Since

$$\bar{\varphi}_{\bar{c}}(e^{-\bar{\beta}T} \Phi_{\bar{c}}^{-1}(\eta)) \geq e^{-\bar{\beta}T} \bar{\varphi}_{\bar{c}}(\Phi_{\bar{c}}^{-1}(\eta)),$$

and both functions $\bar{\varphi}_{\bar{c}}(\cdot)$ and $\Phi_{\bar{c}}^{-1}(\cdot)$ are nondecreasing, in view of (3.20) and (3.22), we conclude that there is a constant $C > 0$ depending only on $(\bar{\beta}, \bar{c}, T, \varphi(1), \check{p}, \lambda)$ such that

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}} \left[\left(\frac{\bar{\varphi}_{\bar{c}}(e^{-\bar{\beta}T} \Phi_{\bar{c}}^{-1}(\eta + T))}{\bar{\varphi}_{\bar{c}}(e^{-\bar{\beta}T} \Phi_{\bar{c}}^{-1}(\eta))} \right)^{\bar{p} \lambda e^{2\bar{\beta}T}} \right] &\leq C \mathbb{E}_{\mathbb{Q}} \left[\left(\frac{1}{\bar{\varphi}_{\bar{c}}(\Phi_{\bar{c}}^{-1}(\eta))} \right)^{\bar{p} \lambda e^{2\bar{\beta}T}} \mathbf{1}_{0 < \eta < \Phi_{\bar{c}}(1)} \right] + C \\ &\leq C \mathbb{E}_{\mathbb{Q}} \left[\max_{t \in [0, T]} \left(\frac{1}{\bar{\varphi}_{\bar{c}}(\Phi_{\bar{c}}^{-1}(\mathbb{E}_{\mathbb{Q}}[\Phi_{\bar{c}}(\bar{\xi}) | \mathcal{F}_t]))} \right)^{\bar{p} \lambda e^{2\bar{\beta}T}} \right] + C. \end{aligned} \quad (3.23)$$

Since $\Phi_{\bar{c}}(\cdot), \bar{\varphi}_{\bar{c}}(\cdot) \in \mathbf{S}$, it is not difficult to verify that the function $\left(\frac{1}{\bar{\varphi}_{\bar{c}}(\Phi_{\bar{c}}^{-1}(\cdot))} \right)^{\bar{p} \lambda e^{2\bar{\beta}T}/2}$ is convex on \mathbb{R}_{++} . Then, by Jensen's inequality we have

$$\left(\frac{1}{\bar{\varphi}_{\bar{c}}(\Phi_{\bar{c}}^{-1}(\mathbb{E}_{\mathbb{Q}}[\Phi_{\bar{c}}(\bar{\xi}) | \mathcal{F}_t]))} \right)^{\bar{p} \lambda e^{2\bar{\beta}T}} \leq \left(\mathbb{E}_{\mathbb{Q}} \left[\left(\frac{1}{\bar{\varphi}_{\bar{c}}(\bar{\xi})} \right)^{\bar{p} \lambda e^{2\bar{\beta}T}/2} \middle| \mathcal{F}_t \right] \right)^2.$$

Furthermore, by Doob's maximal inequality on sub-martingales and Hölder's inequality we obtain

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}} \left[\max_{t \in [0, T]} \left(\frac{1}{\bar{\varphi}_{\bar{c}}(\Phi_{\bar{c}}^{-1}(\mathbb{E}_{\mathbb{Q}}[\Phi_{\bar{c}}(\bar{\xi})|\mathcal{F}_t])} \right)^{\bar{p}\lambda e^{2\bar{\beta}T}} \right] &\leq 4\mathbb{E}_{\mathbb{Q}} \left[\left(\frac{1}{\bar{\varphi}_{\bar{c}}(\bar{\xi})} \right)^{\bar{p}\lambda e^{2\bar{\beta}T}} \right] \\ &= 4\mathbb{E} \left[\left(\frac{1}{\bar{\varphi}_{\bar{c}}(\bar{\xi})} \right)^{\bar{p}\lambda e^{2\bar{\beta}T}} \frac{d\bar{\mathbb{Q}}}{d\mathbb{P}} \right] \leq 4 \left\{ \mathbb{E} \left[\left(\frac{1}{\bar{\varphi}_{\bar{c}}(\bar{\xi})} \right)^{\bar{p}\lambda e^{2\bar{\beta}T}} \right]^{\frac{\bar{p}}{\bar{p}-1}} \right\} \left\{ \mathbb{E} \left[\left(\frac{d\bar{\mathbb{Q}}}{d\mathbb{P}} \right)^{\frac{\bar{p}}{\bar{p}-1}} \right]^{\frac{\bar{p}-1}{\bar{p}}} \right\}. \end{aligned} \quad (3.24)$$

On the other hand, since $\bar{\varphi}(\cdot) \in \mathbf{S}$, we have

$$\bar{\varphi}_{\bar{c}}(\bar{\xi}) = e^{-\bar{\beta}T} \varphi \left(e^{\int_0^T a_s ds} \xi \right) + \bar{c} \leq e^{-\bar{\beta}T} \varphi \left(e^{\bar{\beta}T} \xi \right) + \bar{c} \leq \varphi(\xi) + \bar{c}$$

and

$$\bar{\varphi}_{\bar{c}}(\bar{\xi}) = e^{-\bar{\beta}T} \varphi \left(e^{\int_0^T a_s ds} \xi \right) + \bar{c} \geq e^{-\bar{\beta}T} \varphi \left(e^{-\bar{\beta}T} \xi \right) + \bar{c} \geq e^{-2\bar{\beta}T} \varphi(\xi) + \bar{c}.$$

Thus, if $\frac{1}{\bar{\varphi}_{\bar{c}}(\bar{\xi})} \in L^{\bar{p}\lambda e^{2\bar{\beta}T}}(\mathbb{R}_+)$, then it follows from (3.21), (3.23) and (3.24) that

$$e^{\int_0^T q_s ds} \in L^{\bar{p}}(\mathbb{R}_+).$$

It remains to show that $(f^*(t, q_t^*))_{t \in [0, T]} \in \mathcal{L}^p(\mathbb{R}_+)$. In fact, in light of (3.3) and the definition of q^* , similarly as in Theorem 2.1 we obtain that for each $t \in [0, T]$,

$$f(t, Y_t) = f^*(t, q_t^*) + q_t^* Y_t. \quad (3.25)$$

Then, in light of (3.25) and assumption (A1), we have for each $t \in [0, T]$,

$$0 \leq f^*(t, q_t^*) \leq f(t, Y_t) \leq \bar{\alpha}_t + \beta Y_t.$$

As a result, $f^*(\cdot, q^*) \in \mathcal{L}^p(\mathbb{R}_+)$ due to $\bar{\alpha} \in \mathcal{L}^p(\mathbb{R}_+)$ and $Y \in \mathcal{S}^p(\mathbb{R}_+)$.

Step 3. In this step we prove that for each $t \in [0, T]$,

$$Y_t = e^{-\int_0^t q_s^* ds} Y_t^{q^*}. \quad (3.26)$$

In fact, it follows from (3.25) that (Y, Z) solves the following BSDE:

$$Y_t = \xi + \int_t^T \left(q_s^* Y_s + f^*(s, q_s^*) + \bar{f}(s, Y_s) + \tilde{f}(s, Y_s, Z_s) \right) ds - \int_t^T Z_s dB_s, \quad t \in [0, T].$$

Define for each $t \in [0, T]$,

$$\check{Y}_t := e^{\int_0^t q_s^* ds} Y_t \quad \text{and} \quad \check{Z}_t := e^{\int_0^t q_s^* ds} Z_t.$$

Then, by Itô's formula we know that (\check{Y}, \check{Z}) solves BSDE (3.6) with $q \cdot := q^*$. On the other hand, by Hölder's inequality we deduce that, in light of $q^* \in \mathcal{H}$,

$$\mathbb{E} \left[\sup_{t \in [0, T]} |\check{Y}_t|^{\bar{p}} \right] \leq \mathbb{E} \left[e^{\bar{p} \int_0^T q_s^* ds} \sup_{t \in [0, T]} |Y_t|^{\bar{p}} \right] \leq \left(\mathbb{E} \left[e^{\bar{p} \int_0^T q_s^* ds} \right] \right)^{\frac{\bar{p}}{\bar{p}-1}} \left(\mathbb{E} \left[\sup_{t \in [0, T]} |Y_t|^{\bar{p}} \right] \right)^{\frac{\bar{p}-1}{\bar{p}}} < +\infty$$

and

$$\begin{aligned} \mathbb{E} \left[\left(\int_0^T |\check{Z}_t|^2 dt \right)^{\frac{\bar{p}}{2}} \right] &\leq \mathbb{E} \left[e^{\bar{p} \int_0^T q_s^* ds} \left(\int_0^T |Z_t|^2 dt \right)^{\frac{\bar{p}}{2}} \right] \\ &\leq \left(\mathbb{E} \left[e^{\bar{p} \int_0^T q_s^* ds} \right] \right)^{\frac{\bar{p}}{\bar{p}-1}} \left(\mathbb{E} \left[\left(\int_0^T |Z_t|^2 dt \right)^{\frac{\bar{p}}{2}} \right] \right)^{\frac{\bar{p}-1}{\bar{p}}} \\ &< +\infty, \end{aligned}$$

which means that $(\check{Y}, \check{Z}) \in \mathcal{S}^{\bar{p}}(\mathbb{R}_{++}) \times \mathcal{M}^{\bar{p}}(\mathbb{R}^{1 \times d})$. Thus, the uniqueness of the adapted solution of BSDE (3.6) in the space of $\mathcal{S}^{\bar{p}}(\mathbb{R}_{++}) \times \mathcal{M}^{\bar{p}}(\mathbb{R}^{1 \times d})$ yields that $\check{Y} = Y^{q^*}$, which is (3.26). That is to say, (3.14) holds true. The proof of Theorem 3.2 is then complete. \square

4. Sharp result on some special BSDE

In this section, we are concerned with the following BSDE of special form:

$$Y_t = \xi + \int_t^T [(k_1(s))^{1-\alpha} Y_s^\alpha + k_2(s) Y_s + k_3(s) |Z_s| + k_4(s)] ds - \int_t^T Z_s dB_s, \quad t \in [0, T], \quad (4.1)$$

where $\xi \in L^p(\mathbb{R}_{++})$ for some $p > 1$, both $\alpha \in (0, 1)$ and $c \in \mathbb{R}_{++}$ are constants, and $k_i(\cdot)$ $i = 1, 2, 3, 4$ are four (\mathcal{F}_t) -progressively measurable real-valued processes defined on $\Omega \times [0, T]$ such that

$$k_1(\cdot) \in \mathcal{L}^p(\mathbb{R}_+), \quad k_2(\cdot) \in [-c, c], \quad k_3(\cdot) \in [0, c] \quad \text{and} \quad k_4(\cdot) \in \mathcal{L}^p(\mathbb{R}_+).$$

The main result of this section is stated as follows.

Theorem 4.1. *BSDE (4.1) admits a unique adapted solution $(Y_t, Z_t)_{t \in [0, T]}$ in $\mathcal{S}^p(\mathbb{R}_{++}) \times \mathcal{M}^p(\mathbb{R}^{1 \times d})$.*

Proof. For each $x, k \in \mathbb{R}_+$, by Young's inequality we have

$$k^{1-\alpha} x^\alpha \leq (1-\alpha)k + \alpha x. \quad (4.2)$$

The existence of an adapted solution $(Y, Z)_{t \in [0, T]} \in \mathcal{S}^p(\mathbb{R}_{++}) \times \mathcal{M}^p(\mathbb{R}^{1 \times d})$ of BSDE (4.1) is available in Theorem 3.3 (i). Now, we prove the uniqueness by employing the θ -difference method used in for example Briand and Hu [6] and Fan, Hu and Tang [18].

Define $\bar{\xi} := e^{\int_0^T k_2(s) ds} \xi \in L^p(\mathbb{R}_{++})$ and

$$\bar{Y} := e^{\int_0^\cdot k_2(s) ds} Y \in \mathcal{S}^p(\mathbb{R}_{++}), \quad \bar{Z} := e^{\int_0^\cdot k_2(s) ds} Z \in \mathcal{M}^p(\mathbb{R}^{1 \times d}),$$

$$\bar{k}_1(\cdot) := e^{\int_0^\cdot k_2(s) ds} k_1(\cdot) \in \mathcal{L}^p(\mathbb{R}_+), \quad \text{and} \quad \bar{k}_4(\cdot) := e^{\int_0^\cdot k_2(s) ds} k_4(\cdot) \in \mathcal{L}^p(\mathbb{R}_+).$$

It follows from Itô's formula that (\bar{Y}, \bar{Z}) solves the following BSDE:

$$\bar{Y}_t = \bar{\xi} + \int_t^T [(\bar{k}_1(s))^{1-\alpha} \bar{Y}_s^\alpha + k_3(s) |\bar{Z}_s| + \bar{k}_4(s)] ds - \int_t^T \bar{Z}_s dB_s, \quad t \in [0, T]. \quad (4.3)$$

Let

$$H(u) := \int_0^u \frac{1}{x^\alpha} dx = \frac{1}{1-\alpha} u^{1-\alpha}, \quad u \in \mathbb{R}_+.$$

Then

$$H^{-1}(x) := [(1-\alpha)x]^{\frac{1}{1-\alpha}}, \quad x \in \mathbb{R}_+$$

is the inverse of function $H(\cdot)$. Furthermore, we define $\tilde{\xi} := H(\bar{\xi}) \in L^{\frac{p}{1-\alpha}}(\mathbb{R}_{++})$,

$$\tilde{Y} := H(\bar{Y}) = \frac{1}{1-\alpha} \bar{Y}^{1-\alpha} \in \mathcal{S}^{\frac{p}{1-\alpha}}(\mathbb{R}_{++}), \quad \tilde{Z} := H'(\bar{Y}) \bar{Z} = \frac{\bar{Z}}{\bar{Y}^\alpha}$$

and

$$\tilde{k}_4(\cdot) := \frac{1}{(1-\alpha)^{\frac{\alpha}{1-\alpha}}} \bar{k}_4(\cdot) \in \mathcal{L}^p(\mathbb{R}_+).$$

By applying Itô's formula to \tilde{Y} , it is not difficult to verify that

$$\tilde{Y}_t = \tilde{\xi} + \int_t^T g(s, \tilde{Y}_s, \tilde{Z}_s) ds - \int_t^T \tilde{Z}_s dB_s, \quad t \in [0, T], \quad (4.4)$$

where for each $(\omega, t, y, z) \in \Omega \times [0, T] \times \mathbb{R}_+ \times \mathbb{R}^{1 \times d}$,

$$g(\omega, t, y, z) := (\bar{k}_1(\omega, t))^{1-\alpha} + k_3(\omega, t)|z| + \frac{\alpha}{2(1-\alpha)} \frac{|z|^2}{y} + \tilde{k}_4(\omega, t) \frac{1}{y^{1-\alpha}}. \quad (4.5)$$

Obviously $d\mathbb{P} \times dt - a.e.$, the function $g(\omega, t, \cdot, \cdot)$ is jointly convex on $\mathbb{R}_{++} \times \mathbb{R}^{1 \times d}$.

By virtue of the θ -difference method we will show that BSDE (4.4) admits at most an adapted solution (\tilde{Y}, \tilde{Z}) such that $\tilde{Y} \in \mathcal{S}^{\bar{p}-\alpha}(\mathbb{R}_{++})$. In fact, suppose that both $(\tilde{Y}^1, \tilde{Z}^1)$ and $(\tilde{Y}^2, \tilde{Z}^2)$ are adapted solutions of BSDE (4.4) in the space $\mathcal{S}^{\bar{p}-\alpha}(\mathbb{R}_{++})$. For each $\theta \in (0, 1)$, define

$$\hat{Y}^\theta := \frac{\tilde{Y}^1 - \theta \tilde{Y}^2}{1-\theta} \quad \text{and} \quad \hat{Z}^\theta := \frac{\tilde{Z}^1 - \theta \tilde{Z}^2}{1-\theta}.$$

Observe that $d\mathbb{P} \times dt - a.e.$, for each $(y_1, y_2, z_1, z_2) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^{1 \times d} \times \mathbb{R}^{1 \times d}$,

$$\begin{aligned} & \mathbf{1}_{\{y_1 > \theta y_2\}} (g(t, y_1, z_1) - \theta g(t, y_2, z_2)) \\ &= \mathbf{1}_{\{y_1 > \theta y_2\}} \left[g\left(t, \theta y_2 + (1-\theta) \frac{y_1 - \theta y_2}{1-\theta}, \theta z_2 + (1-\theta) \frac{z_1 - \theta z_2}{1-\theta}\right) - \theta g(t, y_2, z_2) \right] \\ &\leq (1-\theta) \mathbf{1}_{\{y_1 > \theta y_2\}} g\left(t, \left(\frac{y_1 - \theta y_2}{1-\theta}\right)^+, \frac{z_1 - \theta z_2}{1-\theta}\right). \end{aligned}$$

Using Itô-Tanaka's formula, we conclude that there exists an (\mathcal{F}_t) -progressively measurable \mathbb{R}_+ -valued process $(\kappa_t^\theta)_{t \in [0, T]}$ depending on θ such that

$$(\hat{Y}_t^\theta)^+ = \tilde{\xi} + \int_t^T \left[\mathbf{1}_{\{\hat{Y}_s^\theta > 0\}} g(s, (\hat{Y}_s^\theta)^+, \hat{Z}_s^\theta) - \kappa_s^\theta \right] ds - \int_t^T dL_s^\theta - \int_t^T \mathbf{1}_{\{\hat{Y}_s^\theta > 0\}} \hat{Z}_s^\theta dB_s, \quad t \in [0, T], \quad (4.6)$$

where L^θ denotes the local time of \hat{Y}^θ at 0, which is a nonnegative nondecreasing process. Furthermore, let

$$\bar{Y}^\theta := H^{-1}((\hat{Y}^\theta)^+) = \left[(1-\alpha)(\hat{Y}^\theta)^+ \right]^{\frac{1}{1-\alpha}} \in \mathcal{S}^{\bar{p}}(\mathbb{R}_+)$$

and

$$\bar{Z}^\theta := (H^{-1})'((\hat{Y}^\theta)^+) \mathbf{1}_{\{\hat{Y}^\theta > 0\}} \hat{Z}^\theta = \frac{1}{1-\alpha} \left[(1-\alpha)(\hat{Y}^\theta)^+ \right]^{\frac{\alpha}{1-\alpha}} \hat{Z}^\theta.$$

In light of the form of (4.6) along with those arguments from (4.3) to (4.5), applying Itô's formula to \bar{Y}^θ yields that for each $t \in [0, T]$,

$$\bar{Y}_t^\theta = \bar{\xi} + \int_t^T \left[(\bar{k}_1(s))^{1-\alpha} (\bar{Y}_s^\theta)^\alpha + k_3(s) |\bar{Z}_s^\theta| + \mathbf{1}_{\{\bar{Y}_s^\theta > 0\}} \bar{k}_4(s) - \bar{\kappa}_s^\theta \right] ds - \int_t^T d\bar{L}_s^\theta - \int_t^T \bar{Z}_s^\theta dB_s, \quad (4.7)$$

where

$$\bar{\kappa}_s^\theta := \left[(1-\alpha)(\hat{Y}_s^\theta)^+ \right]^{\frac{\alpha}{1-\alpha}} \kappa_s^\theta \quad \text{and} \quad \bar{L}_s^\theta := \int_0^s \left[(1-\alpha)(\hat{Y}_r^\theta)^+ \right]^{\frac{\alpha}{1-\alpha}} dL_r^\theta.$$

In the sequel, note that $\bar{\xi} \in L^p(\mathbb{R}_{++})$, $\bar{Y}^\theta \in \mathcal{S}^{\bar{p}}(\mathbb{R}_+)$, $\bar{k}_1(\cdot) \in \mathcal{L}^p(\mathbb{R}_+)$, $k_3(\cdot) \in [0, c]$, $\bar{k}_4(\cdot) \in \mathcal{L}^p(\mathbb{R}_+)$, $\bar{\kappa}^\theta \in \mathbb{R}_+$, \bar{L}^θ is a nonnegative nondecreasing process and in light of (4.2),

$$(\bar{k}_1(s))^{1-\alpha} (\bar{Y}_s^\theta)^\alpha \leq (1-\alpha) \bar{k}_1(s) + \alpha \bar{Y}_s^\theta, \quad s \in [0, T].$$

By using a standard argument to establish a priori estimates for the adapted solution of BSDE (4.7), see Appendix for details, we deduce that there is a constant $C > 0$ independent of θ such that for each $t \in [0, T]$ and $\theta \in (0, 1)$,

$$|\tilde{Y}_t^\theta|^p \leq C \mathbb{E} \left[|\bar{\xi}|^p + \left(\int_0^T (\bar{k}_1(s) + \bar{k}_4(s)) ds \right)^p \middle| \mathcal{F}_t \right] + C,$$

and then

$$(\tilde{Y}_t^1 - \theta \tilde{Y}_t^2)^+ \leq \frac{1-\theta}{1-\alpha} \left\{ C \mathbb{E} \left[|\bar{\xi}|^p + \left(\int_0^T (\bar{k}_1(s) + \bar{k}_4(s)) ds \right)^p \middle| \mathcal{F}_t \right] + C \right\}^{\frac{1-\alpha}{p}}.$$

Sending $\theta \rightarrow 1$ in the last inequality yields $\tilde{Y}^1 \leq \tilde{Y}^2$. In the same way, we have $\tilde{Y}^2 \leq \tilde{Y}^1$, and therefore, $\tilde{Y}^2 = \tilde{Y}^1$. That is to say, BSDE (4.4) admits at most an adapted solution $(\tilde{Y}_t, \tilde{Z}_t)_{t \in [0, T]}$ such that $\tilde{Y} \in \mathcal{S}^{\frac{p}{1-\alpha}}(\mathbb{R}_{++})$.

Finally, from those arguments from (4.2) and (4.5) we conclude that BSDE (4.1) admits at most an adapted solution $(Y, Z) \in \mathcal{S}^p(\mathbb{R}_{++}) \times \mathcal{M}^p(\mathbb{R}^{1 \times d})$. The proof is complete. \square

Remark 4.2. *We have the following three remarks.*

- (i) *The conclusion of Theorem 4.1 remains true when the expression $|Z_s|$ appearing in BSDE (4.1) is replaced with $f(Z_s)$ for a positive homogeneous and convex function $f(\cdot) : \mathbb{R}^d \rightarrow \mathbb{R}_+$. In fact, in the proof of Theorem 4.1 one only needs to replace the expressions $|\bar{Z}_s|$, $k_3(\omega, t)|z|$ and $|\bar{Z}_s^\theta|$ appearing in (4.3), (4.5) and (4.7) with $f(\bar{Z}_s)$, $k_3(\omega, t)f(z)$ and $f(\bar{Z}_s^\theta)$, respectively.*
- (ii) *By virtue of the relevant result on the L^1 solution obtained for example in Briand et al. [4] and Fan, Hu and Tang [18] one can verify that when $k_3(\cdot) \equiv 0$ and $p = 1$, BSDE (4.1) admits a unique adapted solution $(Y_t, Z_t)_{t \in [0, T]}$ such that Y belongs to class (D).*
- (iii) *For the case of $\text{ess inf}_{t \in [0, T]} k_4(t) = 0$, the conclusion of Theorem 4.1 is better than that of Theorem 3.2 since the integrability condition on $1/\varphi(\xi)$ in Theorem 3.2 is moved away in Theorem 4.1. However, Theorem 4.1 is only able to tackle BSDEs with generators of certain form.*

Finally, we illustrate Theorem 4.1, by the following typical financial example.

Example 4.3. *The Kreps-Porteus (or Epstein-Zin) utility of an endowment studied in El Karoui, Peng and Quenez [12] (see also Skiadas [35], Fan [13], Xing [37], Wang, Wang and Yick [36] and Kubler, Selden and Wei [28]) can be defined by the adapted solution of the following BSDE:*

$$Y_t = \xi + \int_0^T g(Y_s) ds - \int_t^T Z_s ds, \quad t \in [0, T], \quad (4.8)$$

where $\xi \in L^p(\mathbb{R}_+)$ for some $p > 1$, and for each $y \in \mathbb{R}_+$,

$$g(y) := \frac{\rho}{\beta} \frac{c^\rho - y^\rho}{y^{\rho-1}} = \frac{\rho}{\beta} (c^\rho y^{1-\rho} - y) \quad (4.9)$$

with $\beta, c \geq 0$ and $0 \neq \rho \leq 1$. It is easily verified that whether the case of $\rho = 1$ or the case of $\rho < 0$, the generator g satisfies the monotonicity condition. Consequently, by classical results it is concluded that BSDE (4.8) admits a unique adapted solution $(Y, Z) \in \mathcal{S}^p(\mathbb{R}_+) \times \mathcal{M}^p(\mathbb{R}^{1 \times d})$.

For the case of $\rho \in (0, 1)$, the generator g is of Peano type and all previous existing results fail to answer whether BSDE (4.8) admits a unique adapted solution (Y, Z) in the space $\mathcal{S}^p(\mathbb{R}_{++}) \times \mathcal{M}^p(\mathbb{R}^{1 \times d})$ when $\xi \in L^p(\mathbb{R}_{++})$. However, Theorem 4.1 gives an affirmative answer to this question. More generally, Theorem 4.1 yields the affirmative answer when the constant c in (4.9) is replaced with a general consumption process $(c_t)_{t \in [0, T]} \in \mathcal{L}^p(\mathbb{R}_+)$.

Moreover, in light of Theorem 4.1 along with Theorem 4.2 (ii), we see that for the case of $\rho \in (0, 1)$, when $\xi \in L^1(\mathbb{R}_{++})$ and the constant c in (4.9) is replaced with a general consumption process $(c_t)_{t \in [0, T]} \in \mathcal{L}^1(\mathbb{R}_+)$, BSDE (4.8) admits a unique adapted solution (Y, Z) such that Y belongs to class (D).

Appendix A. An a priori estimate

In this section, we shall give the a priori estimate employed in the proof of Theorem 4.1.

Proposition A.1. *Suppose that $\xi \in L^p(\mathbb{R}_+)$ for some $p > 1$, $(L_t)_{t \in [0, T]}$ is an (\mathcal{F}_t) -progressively measurable nondecreasing process with $L_0 = 0$, and the function $g(\omega, t, y, z) : \Omega \times [0, T] \times \mathbb{R}_+ \times \mathbb{R}^{1 \times d}$ is (\mathcal{F}_t) -progressively measurable for each $(y, z) \in \mathbb{R}_+ \times \mathbb{R}^{1 \times d}$ such that $d\mathbb{P} \times dt$ -a.e., for each $(y, z) \in \mathbb{R}_+ \times \mathbb{R}^{1 \times d}$,*

$$g(\omega, t, y, z) \leq f_t(\omega) + \mu|y| + \lambda|z|,$$

where $f \in \mathcal{L}^p(\mathbb{R}_+)$, and $\mu, \lambda \geq 0$ are two nonnegative constants. Let $(Y_t, Z_t)_{t \in [0, T]}$ be an adapted solution of the following BSDE

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds - \int_t^T dL_s - \int_t^T Z_s dB_s, \quad t \in [0, T].$$

If $Y \in \mathcal{S}^p(\mathbb{R}_+)$, then $Z \in \mathcal{M}^p(\mathbb{R}^{1 \times d})$ and there exists a constant $C_p > 0$ depending only on p such that for each $a \geq \mu + \lambda^2$ and $t \in [0, T]$, we have

$$\mathbb{E} \left[\left(\int_0^T e^{2as} |Z_s|^2 ds \right)^{\frac{p}{2}} \middle| \mathcal{F}_t \right] \leq C_p \mathbb{E} \left[\sup_{t \in [0, T]} (e^{apt} |Y_t|^p) + \left(\int_0^T e^{as} f_s ds \right)^p \middle| \mathcal{F}_t \right].$$

Moreover, there exists a constant $\bar{C}_p > 0$ depending only on p such that for each $a \geq \mu + \lambda^2 / [1 \wedge (p - 1)]$ and $t \in [0, T]$, we have

$$\mathbb{E} \left[\sup_{t \in [0, T]} (e^{apt} |Y_t|^p) + \left(\int_0^T e^{2as} |Z_s|^2 ds \right)^{\frac{p}{2}} \middle| \mathcal{F}_t \right] \leq \bar{C}_p \mathbb{E} \left[e^{apT} |\xi|^p + \left(\int_0^T e^{as} f_s ds \right)^p \middle| \mathcal{F}_t \right].$$

Proof. Observe that under the conditions of Theorem A.1 we have for each $t \in [0, T]$,

$$\text{sgn}(Y_t)g(t, Y_t, Z_t) \leq f_t + \mu|Y_t| + \lambda|Z_t|,$$

$$2 \int_t^T Y_s dL_s \leq 0$$

and

$$p \int_t^T |Y_s|^{p-1} \text{sgn}(Y_s) dL_s \leq 0.$$

The desired conclusions follow immediately by following closely the proof procedure of Lemma 3.1 and Proposition 3.2 in Briand et al. [4]. The only difference lies in that one needs to use the conditional mathematical expectation with respect to \mathcal{F}_t instead of the original mathematical expectation. More details are omitted here. The proof is complete. \square

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