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SPECIFICATIONS FOR SECOND-ORDER TURBULENCE-MOMENT MODELS IN INCOMPRESSIBLE FLOWS

NADJIB BENNOURA BOUCHIBA¹, MARTIN FERRAND¹ AND JEAN-MARC HÉRARD¹

Abstract. This paper proposes a set of specifications for the design of second-order turbulence-moment models for incompressible flows. The models are required to ensure the realisability of the Reynolds stress tensor, objectivity, stability, and compliance with the total energy inequality. We provide analytical tools to construct models that satisfy these criteria and examine several existing formulations in light of these specifications. In addition, we introduce a new model featuring a non-zero rapid term and illustrate its behaviour on a Riemann problem, highlighting the crucial role of stability in turbulence modelling.

Résumé. Cette note propose un ensemble de spécifications pour la conception de modèles de moments turbulents du second ordre appliqués aux écoulements incompressibles. Les modèles doivent garantir la réalisabilité du tenseur de Reynolds, l'objectivité, la stabilité et le respect de l'inégalité d'énergie totale. Nous présentons des outils analytiques permettant de construire des modèles conformes à ces critères et analysons plusieurs formulations existantes à la lumière de ces spécifications. Enfin, un nouveau modèle intégrant un terme rapide non nul est introduit et illustré sur un problème de Riemann, soulignant le rôle essentiel de la stabilité dans la modélisation de la turbulence.

1. INTRODUCTION

Solving the Navier–Stokes equations to predict incompressible or compressible flows is generally not computationally tractable, due to the wide range of eddy scales involved in high–Reynolds–number flows. A classical approach to circumvent this issue is to consider Reynolds–averaged equations (see e.g. [14]).

Applying the Reynolds–averaging operator to the Navier–Stokes equations yields an unclosed set of equations for the statistical moments (the mean velocity for the first moment, and the velocity covariance matrix, also called the Reynolds stress tensor, for the second), which must be closed through appropriate modelling.

In this article, we focus our analysis on second-moment turbulence closures, which consist of solving a modelled transport equation for the Reynolds stress tensor. The two main objectives of this article are, first, to establish specifications for constructing such models, and second, to analyse several propositions with respect to these specifications.

First, as a covariance matrix, the Reynolds stress must satisfy a realisability constraint (i.e., it must remain positive definite everywhere and at all times), and the same must hold for the solution of the modelled equation. Second, certain unclosed terms in the Reynolds stress tensor satisfy an objectivity criterion (i.e., they remain unchanged under a change of reference frame), and the corresponding modelled terms must do so as well. Third, to be mathematically well-posed, the global set of equations must be stable with respect to small perturbations.

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Finally, since the total kinetic energy—playing a role analogous to entropy in compressible flows—must decay, certain modelled contributions in the Reynolds stress transport equations are required to be traceless.

Let us first recall the incompressible Navier–Stokes equations:

$$\partial_t u_i + u_j \partial_j u_i + \frac{1}{\rho_0} \partial_i p = \partial_j \Sigma_{ij}, \quad (1)$$

$$\partial_i u_i = 0, \quad (2)$$

where $\partial_t(\cdot)$ and $\partial_j(\cdot)$ are, respectively, the time and space derivative in the direction i ($i \in 1, \dots, d$, where d is the dimension of space), u_i are the instantaneous velocity components, p is the pressure field, ρ_0 is the mass-density field (assumed to be constant) and Σ_{ij} are the viscous shear stress tensor components following the Stokes closure law:

$$\Sigma_{ij} = \nu_0 (\partial_j u_i + \partial_i u_j), \quad (3)$$

with ν_0 the molecular kinematic viscosity, also assumed to be constant here. In this article, we use the Einstein summation convention of multiply-mentioned indices.

Applying the Reynolds averaging operator $\overline{(\cdot)}$ to (1)–(2) gives:

$$\partial_t \bar{u}_i + \bar{u}_j \partial_j \bar{u}_i + \partial_j \overline{u'_i u'_j} + \frac{1}{\rho_0} \partial_i \bar{p} = \partial_j (\nu_0 \partial_j \bar{u}_i), \quad (4)$$

$$\partial_i \bar{u}_i = 0, \quad (5)$$

where $u'_i := u_i - \bar{u}_i$ denotes the fluctuation.

The unclosed set of the evolution equation for $R_{ij} := \overline{u'_i u'_j}$ is obtained by computing $u'_j \partial_t (u_i - \bar{u}_i) + u'_i \partial_t (u_j - \bar{u}_j)$, and averaging gives:

$$\partial_t R_{ij} + \bar{u}_k \partial_k R_{ij} + R_{ik} \partial_k \bar{u}_j + R_{kj} \partial_k \bar{u}_i + \partial_k T_{ijk} = \Theta_{ij}^p + \Theta_{ij}^\nu, \quad (6)$$

where T_{ijk} , Θ_{ij}^p and Θ_{ij}^ν are respectively the triple velocity correlation, pressure gradient – velocity correlation and viscous correlation defined as:

$$T_{ijk} := \overline{u'_i u'_j u'_k}, \quad (7)$$

$$\Theta_{ij}^p := -\frac{1}{\rho_0} \left(\overline{u'_i \partial_j p'} + \overline{u'_j \partial_i p'} \right), \quad (8)$$

$$\Theta_{ij}^\nu := \nu_0 \left(\overline{u'_i \partial_k \partial_k u'_j} + \overline{u'_j \partial_k \partial_k u'_i} \right). \quad (9)$$

The modelling problem thus consists in expressing unclosed terms T_{ijk} , Θ_{ij}^p and Θ_{ij}^ν as modelled expressions (denoted by exponent m in the following) function of the state variable and its gradient.

In section 2 we present the model specifications for second-order moment models. In section 3 we propose some tools to ensure that models are *realisable*, to impose *objectivity*, and to study *stability*. Section 4 applies this study to some well-known models such as Rotta proposition [15], and we make a new proposition with non-zero rapid pressure-strain correlation that fits the book of specification in 5.1. These models are illustrated on a one-dimensional Riemann problem in 5.2.

2. MODEL SPECIFICATIONS

First we recall that Θ^p and Θ^ν are symmetric matrix and \mathbf{T} is invariant by any index permutation.

Note that, using an integration by parts, Θ_{ij}^{ν} rewrites:

$$\Theta_{ij}^{\nu} = \partial_k (\nu_0 \partial_k R_{ij}) - 2\nu_0 \overline{\partial_k u'_i \partial_k u'_j}. \quad (10)$$

It is clear that $\varepsilon_{ij} := 2\nu_0 \overline{\partial_k u'_i \partial_k u'_j}$ is a positive semidefinite tensor, since $\forall \mathbf{n}$, $n_i \varepsilon_{ij} n_j \geq 0$. Its trace $\varepsilon := \frac{1}{2} \text{trace} \varepsilon$ is called the dissipation of the turbulent kinetic energy k , defined by:

$$k := \frac{1}{2} \text{trace} \mathbf{R}. \quad (11)$$

ε is usually used to define the integral turbulent time scale (see [14]):

$$\tau := \frac{k}{\varepsilon}. \quad (12)$$

Classically, Θ_{ij}^p is also rewritten as (see e.g. [13, 14]):

$$\Theta_{ij}^p = -\frac{1}{\rho_0} \left(\partial_j \overline{p' u'_i} + \partial_i \overline{p' u'_j} \right) + \frac{1}{\rho_0} \overline{p' (\partial_j u'_i + \partial_i u'_j)}. \quad (13)$$

Note that, using the incompressibility hypothesis in the instantaneous velocity field, $\text{trace} \Phi = 0$, where

$$\Phi_{ij} := \frac{1}{\rho_0} \overline{p' (\partial_j u'_i + \partial_i u'_j)}, \quad (14)$$

is the so-called pressure strain correlation tensor.

Historically, following [3], as the velocity fluctuation is divergence free, the pressure fluctuation p' can be formally obtained as the sum of the solutions of two Poisson problems, therefore Θ_{ij}^p is decomposed as:

$$\Theta_{ij}^p = \Theta_{ij}^{p,r}(\mathbf{R}, \tau, \nabla \bar{\mathbf{u}}) + \Theta_{ij}^{p,s}(\mathbf{R}, \tau), \quad (15)$$

where $\Theta_{ij}^{p,r} = L_{ij}^{kl}(\mathbf{R}, \tau) \partial_l \bar{u}_k$ is the so-called *rapid* term, which depends linearly on the mean velocity gradient, and $\Theta_{ij}^{p,s}$ is the *slow* term generally modelled using R_{ij} and ε only (see e.g., [13]).

2.1. Realisability/positivity

The concept of *realisability* was introduced in [17] and [21]. The Reynolds stress tensor, being the covariance matrix of velocity fluctuations, is therefore a symmetric positive half-definite matrix:

$$\forall \mathbf{n}, n_i R_{ij} n_j \geq 0. \quad (16)$$

It is equivalent to all fundamental minors of \mathbf{R} remaining positive.

The unclosed set of evolution equation of \mathbf{R} ensures that it remains realisable (see [17, 21]).

The first specification is that the solutions of the modelled transport equation of \mathbf{R} comply with the realisability requirement (providing realisable initial and boundary conditions).

2.2. Objectivity

Let us study the effect of the change of reference frame due to a time-dependent rotation and translation as [19] did (see also [6]).

We denote by \mathbf{x}^* the changed reference frame which is linked to \mathbf{x} as:

$$x_i^* = Q_{ij}(t) x_j + b_i(t), \quad (17)$$

where $\mathbf{b}(t)$ is any (constant over space) time-dependent vector and $\mathbf{Q}(t)$ is a rotation matrix ($\mathbf{Q}\mathbf{Q}^T = \mathbf{I}$).

As the velocity fluctuation is an ensemble average, it verifies:

$$u_i'^* = Q_{ik} u_k' \quad (18)$$

The Reynolds stress tensor becomes:

$$R_{ij}^* = Q_{ik} Q_{jl} R_{kl}, \quad (19)$$

and similarly, triple correlation verifies:

$$T_{ijk}^* := \overline{(u_i')^* (u_j')^* (u_k')^*} = Q_{im} Q_{jn} Q_{kp} \overline{u_m' u_n' u_p'}. \quad (20)$$

Thus, (6) rewrites (as proven in [19] – Equation (33), written here such as its LHS is composed with exact terms while its RHS is composed with terms to be modelled):

$$\begin{aligned} \partial_t R_{ij}^* + \bar{u}_k^* \partial_{k^*} R_{ij}^* + R_{ik}^* \partial_{k^*} \bar{u}_j^* + R_{kj}^* \partial_{k^*} \bar{u}_i^* - 2Q_{kl} \dot{Q}_{jl} R_{ik}^* - 2Q_{kl} \dot{Q}_{il} R_{kj}^* = -\partial_{k^*} T_{ijk}^* \\ + \Theta_{ij}^{p^*} + \Theta_{ij}^{\nu^*}, \end{aligned} \quad (21)$$

where the unclosed terms $\Theta_{ij}^{p^*}$ and $\Theta_{ij}^{\nu^*}$ verify:

$$(\Theta_{ij}^{p^*})^* := -\frac{1}{\rho_0} \left(\overline{(u_i')^* \partial_{j^*} (p')^*} + \overline{(u_j')^* \partial_{i^*} (p')^*} \right) = Q_{ik} Q_{jl} \Theta_{kl}^p, \quad (22a)$$

$$(\Theta_{ij}^{\nu^*})^* := \left(\overline{(u_i')^* \partial_{k^*} (\Sigma'_{jk})^*} + \overline{(u_j')^* \partial_{k^*} (\Sigma'_{ik})^*} \right) = Q_{ik} Q_{jl} \Theta_{kl}^\nu. \quad (22b)$$

Therefore, we add to the specifications that the modelled expressions of \mathbf{T} , Θ^p and Θ^ν must satisfy the properties (20) and (22).

2.3. Stability of perturbations near steady states

Let us consider a state variable \mathbf{W} solution of the following generic homogeneous first-order differential system:

$$\mathbf{D} \partial_t \mathbf{W} + \mathbf{A}_k \partial_k \mathbf{W} = \mathbf{0}. \quad (23)$$

Studying the linear stability of perturbations of the form

$$\mathbf{W}_0(\mathbf{x}) = \mathbf{w}_0 \exp(i(\mathbf{n} \cdot \mathbf{x})), \quad (24)$$

for all unit vector $\mathbf{n} = n_k \mathbf{e}_k$, around the steady state, consists in looking for solutions of (23) of the form $\mathbf{W}(\mathbf{x}, t) = \mathbf{w}_0 \exp(i(\mathbf{n} \cdot \mathbf{x} + \lambda t))$, where λ is solution of:

$$\det(n_k \mathbf{A}_k - \lambda \mathbf{D}) = 0. \quad (25)$$

Linear stability of the system around steady states implies **real** values for all λ solution of (25).

The *stability* requirement of the homogeneous first-order differential equations issuing from the modelled set of equations (4) – (5) – (6) with $\mathbf{W} = (\bar{\mathbf{u}}, \bar{p}/\rho_0, \mathbf{R})$ is thus added to specifications.

Remark 2.1. If \mathbf{D} is invertible, (23) rewrites

$$\partial_t \mathbf{W} + \tilde{\mathbf{A}}_k \partial_k \mathbf{W} = \mathbf{0}, \quad (26)$$

and the stability problem is to look for λ solution of $\det(n_k \tilde{\mathbf{A}}_k - \lambda \mathbf{I}) = 0$, so stability is equivalent to hyperbolicity of the system.

Note that, for the modelled set of equations (4) – (5) – (6), considering the variable:

$$\mathbf{W} = (\bar{u}_1, \dots, \bar{u}_d, \bar{p}/\rho_0, R_{11}, \dots, R_{d1}, R_{12}, \dots, R_{d2}, \dots, R_{dd}), \quad (27)$$

\mathbf{D} is not invertible due to the incompressibility constraint (5).

2.4. Total kinetic energy inequality

The transport equations of the turbulent kinetic energy $k := \frac{1}{2}\text{trace}\mathbf{R}$ and the kinetic energy of the mean field $\mathcal{E}_M := \frac{1}{2}\bar{u}_i\bar{u}_i$ write:

$$\partial_t k + \bar{u}_j \partial_j k + R_{ij} \partial_j \bar{u}_i + \frac{1}{2} \overline{\partial_j u_i' u_i' u_j'} = \frac{1}{2} \Theta_{ii}^p + \frac{1}{2} \Theta_{ii}^\nu, \quad (28)$$

$$\partial_t \mathcal{E}_M + \bar{u}_j \partial_j \mathcal{E}_M + \bar{u}_i \partial_j R_{ij} + \frac{1}{\rho_0} \partial_i (\bar{p} \bar{u}_i) - \partial_j (\nu_0 \partial_j \mathcal{E}_M) = -\nu_0 \partial_j \bar{u}_i \partial_j \bar{u}_i, \quad (29)$$

using the fact that $\partial_i u_i' = 0$ in (28) and $\partial_i \bar{u}_i = 0$ in (29).

One can note that:

$$\bar{u}_i \partial_j R_{ij} = \partial_j (\bar{u}_i R_{ij}) - R_{ij} \partial_j \bar{u}_i. \quad (30)$$

and thus the so-called production term $-R_{ij} \partial_j \bar{u}_i$ corresponds to a local transfer of energy between the mean field and the fluctuating field.

The total kinetic energy \mathcal{E} is the sum of the turbulent kinetic energy and the mean field kinetic energy:

$$\mathcal{E} := \mathcal{E}_M + k = \frac{1}{2} \left(\bar{u}_i \bar{u}_i + \overline{u_i' u_i'} \right). \quad (31)$$

Noting that:

$$\frac{1}{2} \Theta_{ii}^\nu = \partial_j (\nu_0 \partial_j k) - \nu_0 \overline{\partial_j u_i' \partial_j u_i'}, \quad (32)$$

the governing equation of \mathcal{E} reads:

$$\partial_t \mathcal{E} + \partial_j (\mathcal{E} \bar{u}_j) + \partial_j (\bar{u}_i R_{ij}) + \frac{1}{\rho_0} \partial_i (\bar{p} \bar{u}_i) - \partial_j (\nu_0 \partial_j \mathcal{E}) = -\nu_0 (\partial_j \bar{u}_i \partial_j \bar{u}_i) - \nu_0 \overline{\partial_j u_i' \partial_j u_i'} - \frac{1}{2} \overline{\partial_j u_i' u_i' u_j'} + \frac{1}{2} \Theta_{ii}^p. \quad (33)$$

The LHS may be seen as an operator acting on the state variables $(\bar{\mathbf{u}}, \bar{p}/\rho_0, \mathbf{R})$. On the RHS, the first term $-\nu_0 (\partial_j \bar{u}_i \partial_j \bar{u}_i)$ represents the kinetic energy dissipation of the mean field by molecular viscosity. The other terms of the RHS are issued from quantities that require modelling:

- The second term $\varepsilon := \nu_0 \overline{\partial_j u_i' \partial_j u_i'}$ is a positive scalar and also causes kinetic energy dissipation by molecular viscosity. It is either closed using algebraic closures ([18]) or using a transport equation ensuring positive solutions [11].
- Focusing on the triple correlation term, we note that:

$$\overline{u_i' u_i' u_j'} n_j = 0, \quad (34)$$

on wall boundaries. Hence, this correlation is sometimes neglected, sometimes relying on a Gaussian behaviour assumption ([13]).

- To ensure the decrease of the total kinetic energy, the last term Θ_{ii}^p must be negative for all strictly positive ν_0 .

Thus, if the model for Θ_{ii}^p is taken strictly negative, it must comply with the constraint:

$$\int_{\Omega} \frac{1}{2} \Theta_{ii}^p d\Omega = - \int_{\partial\Omega} \overline{p' u'_j n_j} d\Gamma, \quad (35)$$

which is null if the boundary of Ω is a wall. A relevant way to ensure this constraint is to consider:

$$\text{trace}(\Theta^p) = 0. \quad (36)$$

Remark 2.2. Let us consider a support Ω with a closed boundary $\partial\Omega$. We note:

$$E(t) = \int_{\Omega} \mathcal{E}(x, t) d\Omega \quad (37)$$

The integration of (33) on Ω gives:

$$\begin{aligned} & d_t E + \int_{\partial\Omega} \left[\mathcal{E} \overline{u_j} + \overline{u_i} R_{ij} + \frac{1}{\rho_0} \overline{p} \overline{u_j} \right] n_j d\Gamma - \int_{\Omega} \partial_i (\nu_0 \partial_i \mathcal{E}) d\Omega \\ &= - \int_{\Omega} \left[\nu_0 \left(\partial_j \overline{u_i} \partial_j \overline{u_i} + \overline{\partial_j u'_i \partial_j u'_i} \right) \right] d\Omega - \int_{\partial\Omega} \frac{1}{2} \overline{u'_i u'_i u'_j n_j} d\Gamma + \int_{\Omega} \frac{1}{2} \Theta_{ii}^p d\Omega \end{aligned} \quad (38)$$

Restricting to the family of volumes Ω_S with a wall boundary (i.e. $\forall x_S \in \partial\Omega_S, \overline{u_i} n_i|_{x_S} = u'_i n_i|_{x_S} = 0$), (38) becomes:

$$d_t E - \int_{\Omega_S} \partial_i (\nu_0 \partial_i \mathcal{E}) d\Omega_S = - \int_{\Omega_S} \left[\nu_0 \left(\partial_j \overline{u_i} \partial_j \overline{u_i} + \overline{\partial_j u'_i \partial_j u'_i} \right) \right] d\Omega_S + \int_{\Omega_S} \frac{1}{2} \Theta_{ii}^p d\Omega_S, \quad (39)$$

and thus, $\forall \nu_0 > 0$:

$$d_t E - \int_{\Omega_S} \partial_i (\nu_0 \partial_i \mathcal{E}) d\Omega_S \leq \int_{\Omega_S} \frac{1}{2} \Theta_{ii}^p d\Omega_S, \quad (40)$$

which guarantees the decrease of E when condition (36) is satisfied.

In the sequel, the condition (36) is added to the specifications.

3. SOME USEFUL TOOLS

This section presents a set of analytical tools introduced to evaluate the models and ensure their conformity with the defined specifications.

We first recall the Caley–Hamilton theorem applied to \mathbf{R} in 3D:

$$\mathbf{R}^3 - \mathcal{I}_1 \mathbf{R}^2 + \mathcal{I}_2 \mathbf{R} - \mathcal{I}_3 \mathbf{I}_3 = \mathbf{0}, \quad (41)$$

where invariants of \mathbf{R} are defined by $\mathcal{I}_1 := \text{trace} \mathbf{R}$, $\mathcal{I}_2 = ((\text{trace} \mathbf{R})^2 - \text{trace}(\mathbf{R}^2))/2$ and $\mathcal{I}_3 := \det \mathbf{R}$.

3.1. A lemma to obtain model realisability

To ensure *realisability* of the model, it is sufficient to write it in the form of Equation (42), based on [9] and [10].

Lemma 3.1. *Assume that the symmetric tensor \mathbf{R} in $\mathbb{R}^3 \times \mathbb{R}^3$ evolves as:*

$$\partial_t R_{ij} + \bar{u}_k \partial_k R_{ij} + R_{ik} \partial_k \bar{u}_j + R_{kj} \partial_k \bar{u}_i = \partial_k (\nu_0 \partial_k R_{ij}) + G_{ik} R_{kj} + R_{ik} G_{jk} + b \delta_{ij}, \quad (42)$$

and assume that $\delta_R^3 := \det \mathbf{R}$, $\mathbf{G}(\mathbf{x}, t)$ (not necessarily symmetric) in $\mathbb{R}^3 \times \mathbb{R}^3$ and $b(\mathbf{x}, t)$ in \mathbb{R} verify that:

- (i) $\delta_R^3(\mathbf{x}, t = 0) > 0$ for all $\mathbf{x} \in \Omega$,
 - (ii) $\delta_R^3(\mathbf{x}, t > 0) > 0$ for all $\mathbf{x} \in \partial\Omega^-$, where $\partial\Omega^- := \{\mathbf{x} \in \partial\Omega / \bar{\mathbf{u}} \cdot \mathbf{n} < 0\}$ and \mathbf{n} is the outward unit vector,
 - (iii) $(\text{trace}(\mathbf{G}) - \text{div}(\bar{\mathbf{u}}))$ belongs to $L^\infty(\Omega \times [0, T])$,
 - (iv) $b(\mathbf{x}, t) \geq 0$ for all $(\mathbf{x}, t) \in \Omega \times [0, T]$.
- Then $\delta_R^3(\mathbf{x}, t) > 0$ for all $(\mathbf{x}, t) \in \Omega \times [0, T]$.

Sketch of proof. Removing $\partial_k (\nu_0 \partial_k R_{ij})$ from (42), one can establish the evolution equation of δ_R^3 :

$$\partial_t \delta_R^3 + \bar{u}_j \partial_j \delta_R^3 + 2\delta_R^3 \partial_k \bar{u}_k = 2\delta_R^3 G_{kk} + b \mathcal{I}_2, \quad (43)$$

which gives positivity of δ_R^3 using conditions (i) – (ii) – (iii) – (iv) and positivity results from [11].

The final result with $\partial_k (\nu_0 \partial_k R_{ij})$ can be then obtained using a numerical argument as presented in Appendix 6 of [9], and the Lie-Trötter method (see [12]). \square

3.2. Imposing objectivity

From now on, we restrict to the modelled expression of (6), with zero triple correlations. The modelled equations read:

$$\partial_t R_{ij} + \bar{u}_k \partial_k R_{ij} + R_{ik} \partial_k \bar{u}_j + R_{kj} \partial_k \bar{u}_i = \Theta_{ij}^m, \quad (44)$$

Proposition 3.2. *To ensure that the model $\Theta^m = \Theta^{r,m}(\mathbf{R}, \tau, \nabla \bar{\mathbf{u}}) + \Theta^{s,m}(\mathbf{R}, \tau)$ verifies property (22), it is sufficient that the slow part $\Theta_{ij}^{s,m}$ is a polynomial expression of \mathbf{R} , and the rapid term $\Theta_{ij}^{r,m} = L_{ij}^{kl} \partial_l \bar{u}_k$ verifies that for all $i, j, k, l \in 1, \dots, d$,*

$$L_{ij}^{kl} = L_{ji}^{kl}, \quad (45a)$$

$$L_{ij}^{kl} = L_{ij}^{lk}, \quad (45b)$$

$$(L^*)_{ij}^{kl} = Q_{im} Q_{jn} Q_{kp} Q_{lq} L_{mn}^{pq} \quad (45c)$$

Proof. It is clear that for any power n , $(\mathbf{R}^*)^n = \mathbf{Q} \mathbf{R}^n \mathbf{Q}^T$.

To ensure objectivity of Θ^m for all states, as $\Theta^{r,m}$ is zero when $\nabla \bar{\mathbf{u}} = \mathbf{0}$, both rapid and slow part must be objective. As Θ^m is symmetric, so must be $\Theta^{r,m}$, which gives (45a).

When applying transformation (17), the mean velocity gradient reads:

$$\nabla \bar{\mathbf{u}} = \nabla^* \left(\mathbf{Q}^T \bar{\mathbf{u}}^* + \Omega \mathbf{Q}^T \mathbf{x}^* \right) \mathbf{Q} = \mathbf{Q}^T (\nabla^* \bar{\mathbf{u}}^*) \mathbf{Q} + \Omega,$$

where $\Omega := \dot{\mathbf{Q}}^T \mathbf{Q}$ is an antisymmetric matrix (the time derivative of $\mathbf{Q} \mathbf{Q}^T = \mathbf{I}$ is zero).

So (22a) applied to the rapid term gives:

$$\begin{aligned} (L^*)_{ij}^{kl} \partial_l^* \bar{u}_k^* &= Q_{im} Q_{jn} L_{mn}^{pq} \partial_q \bar{u}_p \\ &= Q_{im} Q_{jn} Q_{kp} Q_{lq} L_{mn}^{pq} \partial_l^* \bar{u}_k^* + Q_{im} Q_{jn} L_{mn}^{kl} \Omega_{kl}. \end{aligned} \quad (46)$$

Since Equation (46) holds for arbitrary $\partial_l^* \bar{u}_k^*$ and any antisymmetric tensor Ω_{kl} , the proof is thereby complete. \square

3.3. Study of the stability for any second-moment closure

Let us first start with the *non-turbulent* first-order differential system issuing from (1)-(2) (that is formally obtained setting \mathbf{R} to $\mathbf{0}$):

$$\partial_t \bar{u}_i + \bar{u}_j \partial_j \bar{u}_i + \frac{1}{\rho_0} \partial_i \bar{p} = 0, \quad (47)$$

$$\partial_i \bar{u}_i = 0. \quad (48)$$

Choosing $\mathbf{W} = (\bar{u}_1, \dots, \bar{u}_d, \bar{p}/\rho_0)^T$ (vector of dimension $d+1$), matrix of (25) gives in bloc form:

$$\mathbf{NS}_{d+1} := (n_k \mathbf{A}_k - \lambda \mathbf{D}) = \begin{pmatrix} \mathcal{U} \mathbf{I}_d & \mathbf{n} \\ \mathbf{n}^T & 0 \end{pmatrix}, \quad (49)$$

where \mathbf{I}_d is the identity matrix in dimension d , and $\mathcal{U} := \bar{\mathbf{u}} \cdot \mathbf{n} - \lambda$.

Computing the determinant by blocs gives $\det(\mathcal{U} \mathbf{I}_d) \times (0 - \mathbf{n}^T \mathbf{n} / \mathcal{U}) = -\mathcal{U}^{d-1}$. We recover that in dimension 3, $\lambda = \bar{\mathbf{u}} \cdot \mathbf{n}$ is a double eigenvalue (the other two correspond to $\pm\infty$ given by the infinite speed of sound).

Let us now investigate the stability of the global first-order differential system issuing from (4)-(5)-(6):

$$\partial_t \bar{u}_i + \bar{u}_j \partial_j \bar{u}_i + \partial_j R_{ij} + \frac{1}{\rho_0} \partial_i \bar{p} = 0, \quad (50)$$

$$\partial_i \bar{u}_i = 0, \quad (51)$$

$$\partial_t R_{ij} + \bar{u}_k \partial_k R_{ij} + R_{ik} \partial_k \bar{u}_j + R_{kj} \partial_k \bar{u}_i - \Theta_{ij}^{m,r} = 0, \quad (52)$$

where we recall that $\Theta^{m,r}$ is the so-called *rapid term* of Θ^m that depends linearly on $\nabla \mathbf{u}$. For the Rotta model [15], $\Theta^{m,r} = \mathbf{0}$. Other models involving non-zero $\Theta^{m,r}$ will be considered in Section 4.

We now consider

$$\mathbf{W} = (\bar{u}_1, \dots, \bar{u}_d, \bar{p}/\rho_0, R_{11}, \dots, R_{d1}, R_{12}, \dots, R_{d2}, \dots, R_{dd}), \quad (53)$$

with redundant components as if \mathbf{R} were not a symmetric tensor. In practice, this will help to simplify calculations.

This allows us to write the matrix $n_k \mathbf{A}_k - \lambda \mathbf{D}$ of (25) in the following bloc form of dimension $(d+1+d^2)^2$:

$$\mathbf{RANS}_{d+1+d^2} := (n_k \mathbf{A}_k - \lambda \mathbf{D}) = \left(\begin{array}{c|ccc} \mathbf{NS}_{d+1} & n_1 \mathbf{I}_d & \cdots & n_d \mathbf{I}_d \\ \hline n_k \mathbf{A}_{k1}^{Ru} & \mathbf{0}_d & & \mathbf{0}_d^T \\ \vdots & & & \\ \hline n_k \mathbf{A}_{kd}^{Ru} & \mathbf{0}_d & & \end{array} \middle| \begin{array}{c} \\ \\ \\ \mathcal{U} \mathbf{I}_{d^2} \end{array} \right), \quad (54)$$

where $n_k \mathbf{A}_{kl}^{Ru}$ are built from terms in (6) that depend on the velocity gradient, which are the so called production term and the *rapid* term that arises from Θ^m . The bloc \mathbf{NS}_{d+1} is still defined by (49) with again $\mathcal{U} := \bar{\mathbf{u}} \cdot \mathbf{n} - \lambda$.

We can now compute the determinant of \mathbf{RANS}_{d+1+d^2} by blocs:

$$\det(\mathbf{RANS}_{d+1+d^2}) = \det(\mathcal{U} \mathbf{I}_{d^2}) \times \det \begin{pmatrix} \left(\mathcal{U} \mathbf{I}_d - n_k \mathbf{A}_{kl}^{Ru} n_l / \mathcal{U} \right) & \mathbf{n} \\ \mathbf{n}^T & 0 \end{pmatrix} \quad (55)$$

Equation (55) can be further simplified, still using calculation by blocs and noting $\mathbf{A}^{Ru} := n_k \mathbf{A}_{kl}^{Ru} n_l$:

$$\begin{aligned} \det(\mathbf{RANS}_{d+1+d^2}) &= \mathcal{U}^{d^2-d+1} \det(\mathcal{U}^2 \mathbf{I}_d - \mathbf{A}^{Ru}) \times \left(0 - \mathbf{n}^T (\mathcal{U}^2 \mathbf{I}_d - \mathbf{A}^{Ru})^{-1} \mathbf{n}\right) \\ &= -\mathcal{U}^{d^2-d+1} \mathbf{n}^T \mathbf{com}(\mathcal{U}^2 \mathbf{I}_d - \mathbf{A}^{Ru})^T \mathbf{n} \end{aligned} \quad (56)$$

because $\det(\mathbf{M}) \mathbf{M}^{-1} = \mathbf{com}(\mathbf{M})^T$, where $\mathbf{com}(\mathbf{M})$ denotes the co-matrix of \mathbf{M} .

Thus, system (4)-(5)-(6) is *stable* if and only if for all unit vector \mathbf{n} :

$$\mathbf{n}^T \mathbf{com}(\mathcal{U}^2 \mathbf{I}_d - \mathbf{A}^{Ru}) \mathbf{n} = 0 \implies \mathcal{U} \in \mathbb{R} \quad (57)$$

Remark 3.3. Hyperbolicity of the evolution system

When considering the homogeneous first-order differential subsystem issuing from (50) – (52) with the state variable \mathbf{W} ($\bar{u}_1, \dots, \bar{u}_d, R_{11}, \dots, R_{d1}, R_{12}, \dots, R_{d2}, \dots, R_{dd}$) (i.e. removing (51) that makes \mathbf{D} non-invertible and removing the pressure component \bar{p}), this subsystem is *stable* if and only if it is *hyperbolic* as explained in Remark 2.1. This property can be useful if one wants to compute approximate solutions of the global system with a classical projection method introduced by [2, 20].

Then, $n_k \mathbf{A}_k - \lambda \mathbf{D}$ of (25) writes:

$$\widetilde{\mathbf{RANS}}_{d+d^2} := (n_k \mathbf{A}_k - \lambda \mathbf{D}) = \left(\begin{array}{c|ccc} \mathcal{U} \mathbf{I}_d & n_1 \mathbf{I}_d & \cdots & n_d \mathbf{I}_d \\ \hline \frac{n_k \mathbf{A}_{k1}^{Ru}}{n_k \mathbf{A}_{k1}^{Ru}} & & & \\ \vdots & & & \\ \hline \frac{n_k \mathbf{A}_{kd}^{Ru}}{n_k \mathbf{A}_{kd}^{Ru}} & & & \mathcal{U} \mathbf{I}_{d^2} \end{array} \right). \quad (58)$$

Thus, still using the computation of determinant by blocs we get:

$$\det(\widetilde{\mathbf{RANS}}_{d+d^2}) = \mathcal{U}^{d^2-d} \det(\mathcal{U}^2 \mathbf{I}_d - \mathbf{A}^{Ru}). \quad (59)$$

Thus, the subsystem is *hyperbolic* if and only if, for all unit vectors \mathbf{n} , all eigenvalues of \mathbf{A}^{Ru} are real and positive. Note that stability of the system (57) implies hyperbolicity of the subsystem.

4. EXAMINATION OF STANDARD MODELS IN LIGHT OF THE PROPOSED SPECIFICATIONS

Referring to the modelled expression (44), we begin by analysing models of type (42), which serve to ensure the *realisability* of the solution:

$$\Theta^m - \operatorname{div}(\nu_0 \nabla \mathbf{R}) = \mathbf{G} \mathbf{R} + \mathbf{R} \mathbf{G}^T + b \mathbf{I}_d. \quad (60)$$

Classically (see e. g. [14]), \mathbf{G} is decomposed into a *slow* part that does not depend on $\nabla \bar{\mathbf{u}}$ and a *rapid* part that is linear with respect to the mean velocity gradient, and using Cayley-Hamilton theorem we get:

$$\mathbf{G} = \underbrace{\frac{\varepsilon}{k} \left(\alpha_1 \mathbf{I}_d + \alpha_2 \frac{\mathbf{R}}{\mathcal{I}_1} + \alpha_3 \frac{\mathbf{R}^2}{\mathcal{I}_1^2} \right)}_{:= \mathbf{G}^s} + \underbrace{\mathbf{H} : \nabla \bar{\mathbf{u}}}_{:= \mathbf{G}^r}, \quad (61)$$

where $\mathbf{H} : \nabla \mathbf{u} = H_{ij}^{kl} \partial_l \bar{u}_k \mathbf{e}_i \otimes \mathbf{e}_j$ and \mathbf{H} is a fourth-order tensor, and $\mathcal{I}_1 := \operatorname{trace}(\mathbf{R}) = 2k$.

In this article, we rely on the proposition made in [8]:

$$H_{ij}^{kl} = \beta_1 \delta_{ij} \delta_{kl} + \beta_2 \delta_{ik} \delta_{jl} + \beta_3 \delta_{il} \delta_{jk} + \gamma_1 \delta_{ij} \frac{R_{kl}}{\mathcal{I}_1}. \quad (62)$$

In the sequel, β_1 is set to 0 as it produces a term proportional to $\operatorname{div} \bar{\mathbf{u}}$, which is null for incompressible flows. This choice gives the following fourth order tensor for the rapid term:

$$L_{ij}^{kl} = H_{ip}^{kl} R_{pj} + H_{jp}^{kl} R_{pi}. \quad (63)$$

Concerning the *objectivity* of the model, it is clear that the slow part verifies Property 3.2 and (63) verifies (45a). To ensure that (63) verifies property (45b), it is necessary and sufficient that:

$$\beta_2 = \beta_3. \quad (64)$$

Simple calculation shows that (63) verifies (45c).

Total kinetic energy inequality condition for (62) imposes:

$$\gamma_1 = -(\beta_2 + \beta_3). \quad (65)$$

Stability analysis of the system requires to specify the matrix $\mathbf{A}^{Ru} := n_k \mathbf{A}_{kl}^{Ru} n_l$ from the so-called production term and the *rapid* term of the model (62), which read:

$$\partial_t R_{il} + \bar{u}_k \partial_k R_{il} + \left(\delta_{ij} R_{kl} + R_{ik} \delta_{jl} - \left(H_{ip}^{jk} R_{pl} + R_{ip} H_{lp}^{jk} \right) \right) \partial_k \bar{u}_j = G_{ik}^s R_{kl} + R_{ik} G_{lk}^s + \dots,$$

and from simple identification we can deduce:

$$\left(\mathbf{A}_{kl}^{Ru} \right)_{ij} = \delta_{ij} R_{kl} + R_{ik} \delta_{jl} - \left(H_{ip}^{jk} R_{pl} + R_{ip} H_{lp}^{jk} \right). \quad (66)$$

4.1. The Rotta model

The Rotta model [15] writes:

$$\Theta_{\text{Rotta}}^m - \operatorname{div} (\nu_0 \nabla \mathbf{R}) = 2\alpha_1 \frac{\varepsilon}{k} \mathbf{R} - \frac{2}{3} (2\alpha_1 + 1) \varepsilon \mathbf{I}_d. \quad (67)$$

This model verifies *objectivity* condition, *realisability* condition for $\alpha_1 < -1/2$ and *total kinetic energy inequality* condition as $\operatorname{trace} \Theta_{\text{Rotta}}^m = -2\varepsilon$. Note that $-2\alpha_1$ is generally called the Rotta constant (which must be greater than 1 to satisfy assumption (iv) of Lemma 3.1, thereby guaranteeing model *realisability*).

The Rotta model corresponds to $\mathbf{H} = \mathbf{0}$ (i.e. $\beta_2 = \beta_3 = \gamma_1 = 0$), and using (66), \mathbf{A}^{Ru} tensor writes:

$$\mathbf{A}_{\text{Rotta}}^{Ru} = R_{nn} \mathbf{I}_d + \mathbf{R} \mathbf{n} \otimes \mathbf{n}, \quad (68)$$

where $R_{nn} := \mathbf{n}^T \mathbf{R} \mathbf{n} = n_k R_{kl} n_l$.

Matrix $\mathbf{A}_{\text{Rotta}}^{Ru}$ has two eigenvalues: $2R_{nn}$ (of multiplicity one, eigenspace is generated by $\operatorname{Span}(\mathbf{R} \mathbf{n})$) and R_{nn} (of multiplicity $d-1$, eigenspace is generated by \mathbf{n}^\perp). Therefore, evolution subsystem as defined in Remark 3.3 is hyperbolic (see also [9]).

For a given unit vector \mathbf{n} , Equation (57) can be written in the orthogonal reference-frame $(\mathbf{n}, \mathbf{e}_2, \mathbf{e}_3)$:

$$(1, 0, 0) \operatorname{com} \begin{pmatrix} \mathcal{U}^2 - 2R_{nn} & 0 & 0 \\ -R_{n2} & \mathcal{U}^2 - R_{nn} & 0 \\ -R_{n3} & 0 & \mathcal{U}^2 - R_{nn} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (\mathcal{U}^2 - R_{nn})^2 = 0 \implies \mathcal{U} \in \mathbb{R}, \quad (69)$$

so the system is *stable* if and only if \mathbf{R} is *realisable*.

4.2. Non-zero *rapid* pressure–strain correlation models

In this section, we analyse the model with a *rapid* contribution from the pressure–strain correlation, of type:

$$H_{ij}^{kl} = \beta_2 \delta_{ik} \delta_{jl} + \beta_3 \delta_{il} \delta_{jk} + \gamma_1 \delta_{ij} \frac{R_{kl}}{\mathcal{I}_1}. \quad (70)$$

We first study the case where $\gamma_1 = 0$, as proposed in [9], for 2D mean flows (i.e. when $\bar{u}_3 = 0$ and $R_{i3} = 0$ for $i \in 1, 2, 3$) and then for the 3D case. Then we analyse the full model imposing also the *total kinetic energy inequality*.

4.2.1. Analysis of the 2D case with $\gamma_1 = 0$

For the rapid part of the model given by (70) with $\gamma_1 = 0$, still using (66), \mathbf{A}^{Ru} tensor writes:

$$\mathbf{A}_G^{Ru} = (1 - \beta_2) (R_{nn} \mathbf{I}_d + \mathbf{R} \mathbf{n} \otimes \mathbf{n}) - \beta_3 (\mathbf{n} \otimes \mathbf{n} \mathbf{R} + \mathbf{R}). \quad (71)$$

For a given unit vector \mathbf{n} , Equation (71) can be written in the orthogonal reference-frame $(\mathbf{n}, \mathbf{e}_2)$:

$$\mathbf{A}_G^{Ru} = \begin{pmatrix} 2(1 - \beta_2 - \beta_3)R_{nn} & -2\beta_3 R_{n2} \\ (1 - \beta_2 - \beta_3)R_{n2} & (1 - \beta_2)R_{nn} - \beta_3 R_{22} \end{pmatrix}, \quad (72)$$

and Equation (57) gives:

$$(\mathcal{U}^2 - (1 - \beta_2)R_{nn} + \beta_3 R_{22}) = 0 \implies \mathcal{U} \in \mathbb{R}. \quad (73)$$

(73) is equivalent to:

$$\mathbf{n}^T ((1 - \beta_2)\mathbf{R} + \beta_3 (\text{trace}(\mathbf{R}) \mathbf{I}_2 - \mathbf{R})) \mathbf{n} \geq 0, \quad (74)$$

and we retrieve result presented in [9], Proposition 6.1.

With the *objectivity* condition (64) (that gives $\beta_2 = \beta_3$), (73) is verified for all unit vectors if:

$$\beta_2 \leq \frac{\lambda^{\min}(\mathbf{R})}{\mathcal{I}_1}, \quad (75)$$

where $\lambda^{\min}(\mathbf{R})$ is the smallest eigenvalue of \mathbf{R} .

4.2.2. Analysis of the 3D case with $\gamma_1 = 0$

Proposition 4.1. *The following model*

$$\Theta_G^m - \text{div} (\nu_0 \nabla \mathbf{R}) = 2\alpha_1 \frac{\varepsilon}{k} \mathbf{R} - \frac{2}{3} (2\alpha_1 + 1) \varepsilon \mathbf{I}_d + 2\beta_2 (\mathbf{R} \mathbf{S} + \mathbf{S} \mathbf{R}), \quad (76)$$

where the strain rate is defined as $\mathbf{S} := \frac{1}{2} (\nabla \mathbf{u} + \nabla \mathbf{u}^t)$, is realisable, objective and is stable if:

$$\alpha_1 \leq -\frac{1}{2}, \quad (77)$$

$$\beta_2 \leq \frac{\lambda^{\min}(\mathbf{R})}{\mathcal{I}_1}. \quad (78)$$

Proof. As for the Rotta model, the *slow* part is *realisable* for $\alpha_1 \leq -1/2$.

Moreover, the *rapid* part correspond to (70) with $\beta_3 = \beta_2$, so the model is *objective*.

Then, for a given unit vector \mathbf{n} , Equation (71) can be written in the orthogonal reference-frame $(\mathbf{n}, \mathbf{e}_2, \mathbf{e}_3)$:

$$\mathbf{A}_G^{Ru} = \begin{pmatrix} 2(1 - \beta_2 - \beta_3)R_{nn} & -2\beta_3 R_{n2} & -2\beta_3 R_{n3} \\ (1 - \beta_2 - \beta_3)R_{n2} & (1 - \beta_2)R_{nn} - \beta_3 R_{22} & -\beta_3 R_{23} \\ (1 - \beta_2 - \beta_3)R_{n3} & -\beta_3 R_{23} & (1 - \beta_2)R_{nn} - \beta_3 R_{33} \end{pmatrix}, \quad (79)$$

and Equation (57) gives:

$$(\mathcal{U}^2 - (1 - \beta_2)R_{nn} + \beta_3 R_{22}) (\mathcal{U}^2 - (1 - \beta_2)R_{nn} + \beta_3 R_{33}) - \beta_3^2 R_{23}^2 = 0 \implies \mathcal{U} \in \mathbb{R}. \quad (80)$$

Computing the roots of the polynomial in \mathcal{U}^2 , (80) is verified if and only if:

$$(1 - \beta_2)R_{nn} - \beta_3 \frac{R_{22} + R_{33}}{2} \geq |\beta_3| \sqrt{\left(\frac{R_{22} - R_{33}}{2}\right)^2 + R_{23}^2} \quad (81)$$

Cauchy-Schwartz inequality implies that $R_{23}^2 \leq R_{22}R_{33}$, so

$$\sqrt{\left(\frac{R_{22} - R_{33}}{2}\right)^2 + R_{23}^2} \leq \frac{R_{22} + R_{33}}{2},$$

thus, (81) is true if

$$R_{nn} \geq \beta_2 R_{nn} + \max(\beta_3, 0) (R_{22} + R_{33}),$$

which proves (78). \square

4.2.3. Analysis of the 3D case with $\gamma_1 \neq 0$

We now consider relation (65), to ensure *total kinetic energy inequality*, and we study the following model, which is proven to be *stable, objective, realisable* and verifies the *total kinetic inequality*.

Proposition 4.2. *The following model*

$$\Theta_S^m - \operatorname{div} (\nu_0 \nabla \mathbf{R}) = 2\alpha_1 \frac{\varepsilon}{k} \mathbf{R} - \frac{2}{3} (2\alpha_1 + 1) \varepsilon \mathbf{I}_d + 2\beta_2 (\mathbf{R} \mathbf{S} + \mathbf{S} \mathbf{R}) - 4\beta_2 \frac{\operatorname{trace}(\nabla \bar{\mathbf{u}} \mathbf{R})}{\mathcal{I}_1} \mathbf{R}, \quad (82)$$

is realisable, objective, stable and verifies total kinetic inequality if:

$$\alpha_1 \leq -\frac{1}{2}, \quad (83)$$

$$\beta_2 \leq \frac{\lambda^{\min}(\mathbf{R})}{\mathcal{I}_1}. \quad (84)$$

Proof. As for the Rotta model, the *slow* part is *realisable* for $\alpha_1 \leq -1/2$.

Moreover, the *rapid* part correspond to (70) with $\beta_3 = \beta_2$, so the model is *objective*, and $\gamma_1 = -(\beta_2 + \beta_3)$, so the model verifies (36) for the *total kinetic inequality*. \square

The proof of the stability of model (82) is provided in Appendix A.

5. APPLICATIONS

In this section we propose a new model using results of Section 3, and we illustrate it on Riemann problem.

5.1. Definition of a new model complying with the specifications

In the light of Section 4, we propose the following model of type (82) with:

$$\alpha_1 = -2.075, \quad (85)$$

$$\beta_2 = \frac{9}{10} \frac{\lambda^{\min}(\mathbf{R})}{\mathcal{I}_1}. \quad (86)$$

It is clear that (85) and (86) verify (83) and (84).

Note that α_1 is taken as proposed by [8] (but plays no role in cases presented in Section 5.2) and the factor $\frac{9}{10}$ for β_2 is designed to fit with rapid distortion theory from [4], where initial isotropic turbulence in sudden homogeneous shear verifies (see Eq. (3.6) of [4]):

$$-\overline{(u'_j \partial_i p' + u'_i \partial_j p')} = \frac{2k}{5} \left(\partial_j \bar{u}_i + \partial_i \bar{u}_j - \frac{2}{3} \partial_m \bar{u}_m \delta_{ij} \right). \quad (87)$$

Identifying (82) for initial isotropic turbulence for which $\lambda^{\min}(\mathbf{R})/\mathcal{I}_1 = 1/3$ gives

$$\beta_2 \left(\mathbf{R} = \frac{\mathcal{I}_1}{3} \mathbf{I}_3 \right) = \frac{3}{10}. \quad (88)$$

5.2. Illustrative applications of the model convective parts

In this section, the $(\mathbf{n}, \boldsymbol{\tau})$ coordinate system in the two-dimensional framework is considered and we denote $u_n = \mathbf{u} \cdot \mathbf{n}$, $u_\tau = \mathbf{u} \cdot \boldsymbol{\tau}$, $R_{nn} = \mathbf{n} \cdot \mathbf{R} \cdot \mathbf{n}$, $R_{n\tau} = \mathbf{n} \cdot \mathbf{R} \cdot \boldsymbol{\tau}$, $R_{\tau\tau} = \boldsymbol{\tau} \cdot \mathbf{R} \cdot \boldsymbol{\tau}$. We focus on the convective subsystem of the full model, as described in Remark 3.3, which includes all first-order derivatives of the velocity field $\bar{\mathbf{u}}$ and the Reynolds stress tensor \mathbf{R} , as defined in [5]:

$$\partial_t \bar{u}_i + \bar{u}_j \partial_j \bar{u}_i + \partial_j R_{ij} = 0, \quad (89)$$

$$\partial_t R_{ij} + \bar{u}_k \partial_k R_{ij} + R_{ik} \partial_k \bar{u}_j + R_{kj} \partial_k \bar{u}_i - \Theta_{ij}^{m,r} = 0, \quad (90)$$

5.2.1. Comparison between models

A one-dimensional Riemann problem with initial conditions summarised in Table 1, is solved in the \mathbf{n} direction. All velocities are expressed in metres per second, Reynolds stresses are expressed in metres per second squared and the characteristic size of the domain \mathcal{L} is 1 metre. Such Riemann problems have been investigated in [1] for the Rotta model. The solutions for the state variables $(u_n, u_\tau, R_{nn}, R_{nt}, R_{\tau\tau})$ are given for both the Rotta model and model (82), using the parameter value specified in (86).

The solution for the Rotta model is derived from the analytical results of [5], which rely on a linear-path approximation of shock relations. In contrast, the "exact" solution for model (82) is approximated using a converged numerical (using $h := \Delta x/\mathcal{L} = 1/10\,000$). The solutions are shown in Figure 1. For this specific test case, initial conditions are such that genuinely non-linear waves are ghost waves for the Rotta model.

It is observed that the wave packet of the convective subsystem in model (82) is entirely contained within that of the Rotta model. Furthermore, model (82) gives rise to additional intermediate states that are not present in the Rotta solution for variables u_n and R_{nn} . Realisability is also preserved for both model convective subsystems.

5.2.2. Unstable models

In this section, we revisit the same Riemann problem using again initial conditions as those presented in Table 1, but now using model (82) with an unadapted parameter. Specifically, we consider a **constant value**

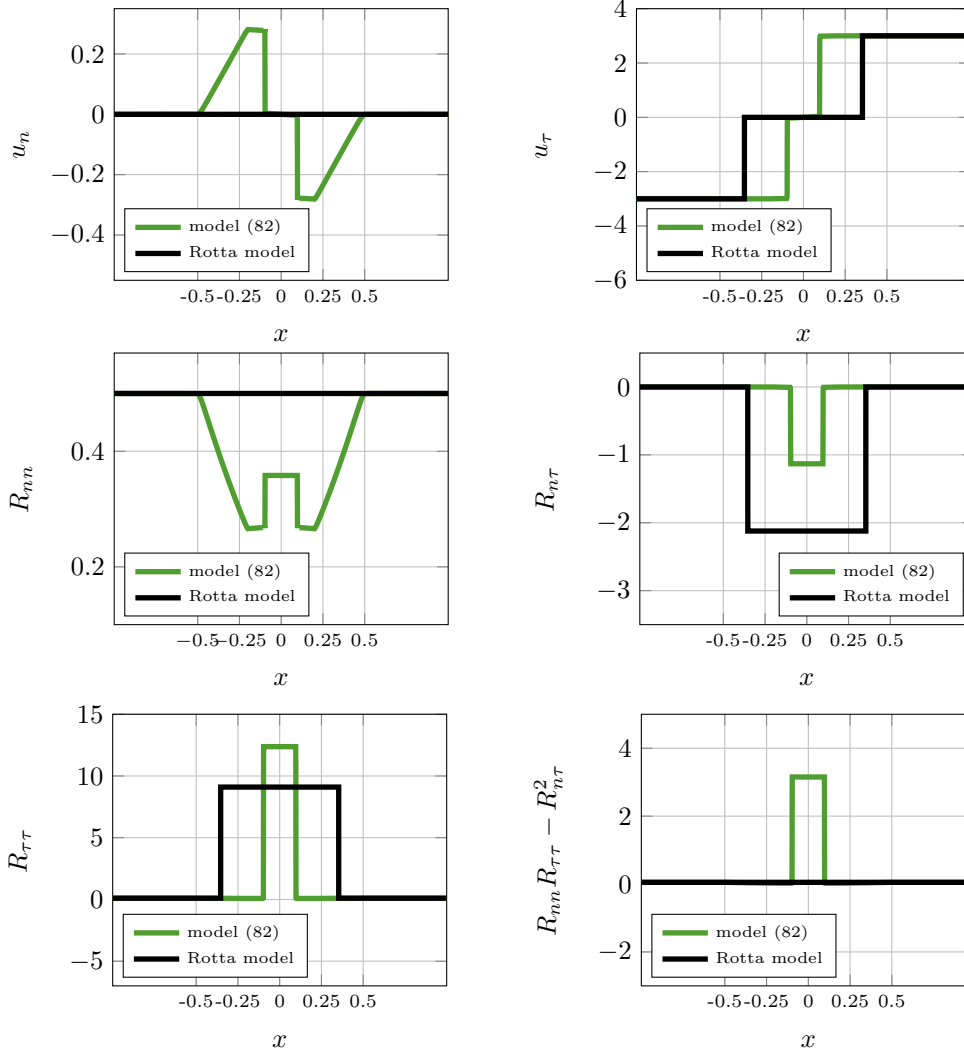


FIGURE 1. Solutions of the Riemann problem associated with Table 1 for the Rotta model and the model (82) model, using (86) (converged numerical solution) at the time $t_f = 0.5 \frac{\mathcal{L}}{\sqrt{2R_{nn}^L}}$.

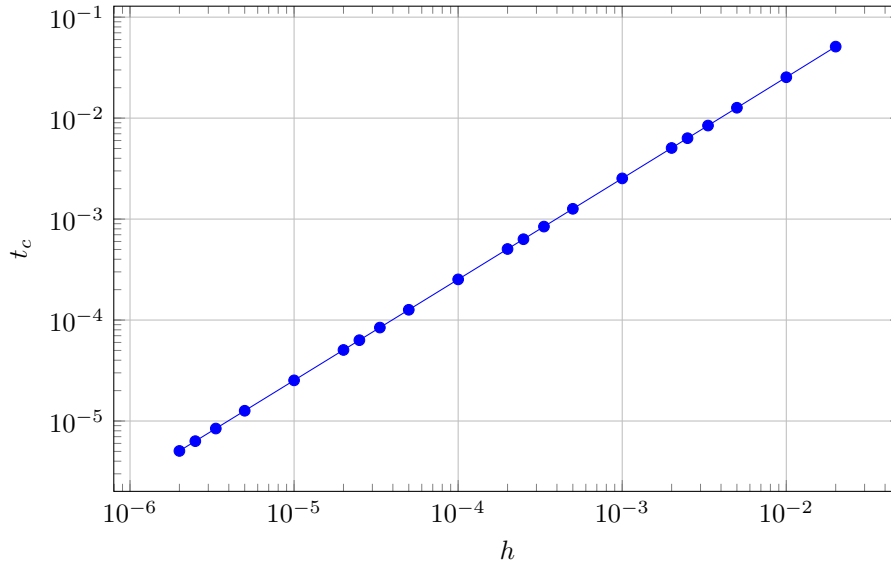
for the parameter β_2 , chosen in accordance with rapid distortion theory as proposed by [4], and set as:

$$\beta_2 = \frac{3}{10}. \quad (91)$$

A tangential shear is imposed, leading to an unstable evolution due to the violation of condition (84) for this choice of β_2 . The simulation is performed using the Rusanov scheme (see [16]). In Figure 2, we present the crash time t_c — defined as the time at which the solution loses realisability, i.e when $\det \mathbf{R} < 0$ — as a function of the mesh size h . The results are similar as the one observed in [7] (see p. 158, Fig. 8.16(b)) and reveal a linear relationship between t_c and h .

	u_n	u_τ	R_{nn}	$R_{n\tau}$	$R_{\tau\tau}$
L	0	-3	0.5	0	0.1
R	0	3	0.5	0	0.1

TABLE 1. Riemann problem initial data.

FIGURE 2. Crash time t_c as a function of the mesh size h for the model (82) with a constant $\beta_2 = 0.3$.

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A. PROOF OF STABILITY FOR MODEL (82)

Proof. For a model with *rapid* part of type (70) with non-zero γ_1 , using (66), \mathbf{A}^{Ru} tensor writes:

$$\mathbf{A}_S^{Ru} = (1 - \beta_2) (R_{nn} \mathbf{I}_d + \mathbf{R}\mathbf{n} \otimes \mathbf{n}) - \beta_3 (\mathbf{n} \otimes \mathbf{n}\mathbf{R} + \mathbf{R}) - \frac{2\gamma_1}{\mathcal{I}_1} (\mathbf{R}\mathbf{n}) \otimes (\mathbf{R}\mathbf{n}), \quad (92)$$

Thus, for a given unit vector \mathbf{n} , Equation (92) can be written in the orthogonal reference-frame $(\mathbf{n}, \mathbf{e}_2, \mathbf{e}_3)$, and Equation (57) gives:

$$P(\mathcal{U}^2) = 0 \implies \mathcal{U} \in \mathbb{R}, \quad (93)$$

where the polynomial P is defined by:

$$P(X) = \left(X - (1 - \beta_2)R_{nn} + \beta_3 R_{22} + 2\frac{\gamma_1}{\mathcal{I}_1} R_{n2}^2 \right) \left(X - (1 - \beta_2)R_{nn} + \beta_3 R_{33} + 2\frac{\gamma_1}{\mathcal{I}_1} R_{n3}^2 \right) - \left(\beta_3 R_{23} + 2\frac{\gamma_1}{\mathcal{I}_1} R_{n2} R_{n3} \right)^2. \quad (94)$$

Computing the roots of the polynomial (94), (93) is verified if and only if:

$$\begin{aligned} & (1 - \beta_2)R_{nn} - \beta_3 \frac{R_{22} + R_{33}}{2} - \gamma_1 \frac{R_{n2}^2 + R_{n3}^2}{\mathcal{I}_1} \\ & \geq \sqrt{\left(\beta_3 \frac{R_{22} - R_{33}}{2} + \gamma_1 \frac{R_{n2}^2 - R_{n3}^2}{\mathcal{I}_1}\right)^2 + \left(\beta_3 R_{23} + 2\gamma_1 \frac{R_{n2}R_{n3}}{\mathcal{I}_1}\right)^2} \end{aligned} \quad (95)$$

In the case of model (82), $\beta_3\gamma_1 = -|\beta_3\gamma_1|$ and the quantity inside the square root rewrites:

$$\begin{aligned} & \left(\beta_3 \frac{R_{22} - R_{33}}{2} + \gamma_1 \frac{R_{n2}^2 - R_{n3}^2}{\mathcal{I}_1}\right)^2 + \left(\beta_3 R_{23} + 2\gamma_1 \frac{R_{n2}R_{n3}}{\mathcal{I}_1}\right)^2 \\ & = \beta_3^2 \left(\frac{R_{22} + R_{33}}{2}\right)^2 + \gamma_1^2 \left(\frac{R_{n2}^2 + R_{n3}^2}{\mathcal{I}_1}\right)^2 + \frac{|\beta_3\gamma_1|}{\mathcal{I}_1} (R_{22} + R_{33}) (R_{n2}^2 + R_{n3}^2) \\ & \quad + \beta_3^2 (R_{23}^2 - R_{22}R_{33}) - \frac{2|\beta_3\gamma_1|}{\mathcal{I}_1} (2R_{n2}R_{n3}R_{23} + R_{n2}^2R_{22} + R_{n3}^2R_{33}), \end{aligned} \quad (96)$$

Considering u' , v' , and w' as the velocity fluctuation components in the reference frame $(\mathbf{n}, \mathbf{e}_2, \mathbf{e}_3)$, one observes that:

$$\begin{aligned} 2R_{n2}R_{n3}R_{23} + R_{n2}^2R_{22} + R_{n3}^2R_{33} & = 2\overline{u'v'}\overline{u'w'}\overline{v'w'} + \overline{u'v'^2}\overline{v'^2} + \overline{u'w'^2}\overline{w'^2} \\ & = \overline{(v'u'v' + w'u'w')^2} \geq 0. \end{aligned} \quad (97)$$

Finally, assuming $\beta_3\gamma_1 = -|\beta_3\gamma_1|$, and using (97) along with the CauchySchwarz inequality $R_{23}^2 < R_{22}R_{33}$, Equation (96) yields:

$$\left(\beta_3 \frac{R_{22} - R_{33}}{2} + \gamma_1 \frac{R_{n2}^2 - R_{n3}^2}{\mathcal{I}_1}\right)^2 + \left(\beta_3 R_{23} + 2\gamma_1 \frac{R_{n2}R_{n3}}{\mathcal{I}_1}\right)^2 \leq \left(|\beta_3| \frac{R_{22} + R_{33}}{2} + |\gamma_1| \frac{R_{n2}^2 + R_{n3}^2}{\mathcal{I}_1}\right)^2. \quad (98)$$

Similarly, if $\beta_3\gamma_1 = |\beta_3\gamma_1|$, inequality (98) can be obtained using analogous arguments.

Eventually in each case:

$$\sqrt{\left(\beta_3 \frac{R_{22} - R_{33}}{2} + \gamma_1 \frac{R_{n2}^2 - R_{n3}^2}{\mathcal{I}_1}\right)^2 + \left(\beta_3 R_{23} + 2\gamma_1 \frac{R_{n2}R_{n3}}{\mathcal{I}_1}\right)^2} \leq |\beta_3| \frac{R_{22} + R_{33}}{2} + |\gamma_1| \frac{R_{n2}^2 + R_{n3}^2}{\mathcal{I}_1}, \quad (99)$$

thus, (95) is true if:

$$R_{nn} \geq \beta_2 R_{nn} + \max(\beta_3, 0) (R_{22} + R_{33}) + 2 \max(\gamma_1, 0) \frac{R_{n2}^2 + R_{n3}^2}{\mathcal{I}_1}, \quad (100)$$

which proves stability for condition (84) with $\beta_3 = \beta_2$ and $\gamma_1 = -2\beta_2$. \square