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# Meanings of integrals and their compartmentalization: a case study

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*The many meanings of integral calculus can result in students experiencing compartmentalization of knowledge. In this case study, we examine the interplay between meanings of integral held by a graduate student in mathematics education, after taking a calculus course focusing on the notions of rate of change and accumulation. To do this, we used a task-based interview designed to elicit a conflict between antiderivative and accumulation. We suggest a novel approach to analysing knowledge compartmentalization, exposing the connections that exist and the connections that are lacking between these two meanings of integral held by the student.*

*Keywords: Integral, Accumulation, Antiderivative, Compartmentalization, Meaning.*

## Introduction

In calculus, we often use ‘integral’ as an umbrella term subsuming definite integral (with numerical bounds), indefinite integral and accumulation function. While these are all integrals, they are mathematically connected yet distinct. It is therefore important to investigate how students relate to these integrals, what connections and distinctions they form between them, and whether they can identify when each is relevant.

In the Israeli high-school curriculum, up until recently, integrals were conceptualized as either antiderivative or as area-under-a-curve. In recent years, a new curriculum has been introduced, that also includes conceptualizing integral as accumulation (Dreyfus et al., 2021). This raises the question – how do students relate accumulation to their previously held conceptualizations? In this study, we focused on antiderivative and accumulation, and investigated the connections made between them. To investigate these connections, we used Thompson’s framework for meaning (Thompson et al., 2024), in combination with compartmentalization of knowledge (Mandl et al., 1993; Shiakalli & Gagatsis, 2006; Vinner & Dreyfus, 1989).

## Theoretical Background

Calculus is often regarded as a barrier in higher education, with many students experiencing difficulties. In particular, the notion of integral presents students with many challenges (e.g., Thompson & Harel, 2021). One challenge is that there are several different conceptualizations of integral, each appropriate in different circumstances. For example, Jones (2015) found that though conceptualizations of integral as area or antiderivative are less productive for contextualized integrals than Riemann sum-based conceptualizations, US college students draw on them more frequently. Thompson et al. (2019) developed a course that conceptualizes integral as accumulation to help students develop a dynamic conception of functions, quantities and change, that centres around the fundamental theorem of calculus.

Though there exists empirical research on compartmentalization (e.g., Hausberger et al., 2021; Shiakalli & Gagatsis, 2006), a single accepted definition of the term does not exist. Mandl et al. (1993) refer to three types of compartmentalization: when a learner's previously held beliefs are not in accordance with the formal scientific knowledge that the learner acquired; when different closely interconnected concepts are stored separately, and not viewed as parts of the more complex network that makes up the problem structure; and when learners ignore knowledge from everyday experiences and deem it irrelevant to the scientific situation they face.

On the other hand, Shiakalli & Gagatsis (2006) consider learners' mathematical knowledge regarding an object as compartmentalized if the learners identify two different representations of the object as two different mathematical objects. This can result in learners struggling to transition between different representations.

Lastly, Vinner and Dreyfus (1989) define compartmentalization as a phenomenon in which a learner has two different, potentially conflicting schemes in their cognitive structure, and certain situations stimulate one scheme, while other situations stimulate the other. These schemes can even be in contradiction with one another and can result in inconsistent or incoherent learner behaviour. This can result in learners possessing relevant knowledge but not perceiving it as relevant, or in learners using two different strategies to solve the same problem getting contradicting outcomes, but without cognitive conflict arising. In other words, using schemes that are cognitively disconnected results in outcomes that are cognitively disconnected, and so, even if the outcomes are mathematically contradictory, they will not be cognitively contradictory, and will not result in cognitive conflict.

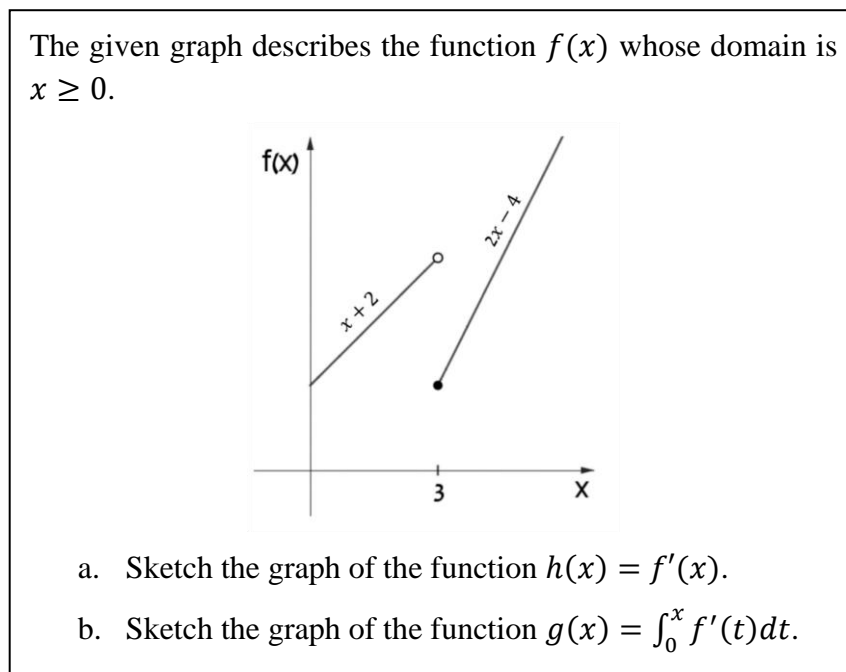
Thompson et al. (2024) builds on the theories of Piaget to define meaning-in-the-moment as the space of implications existing at the moment of understanding. In other words, a person's personal meaning is what comes to their mind in a situation, as well as what could be implied, i.e., what could easily come to mind next. The implicative nature of meaning offers a lens through which to look at personal connections between different notions. Thus, it is a suitable framework for investigating compartmentalization as well. We therefore suggest defining compartmentalization as a cognitive state where actions, operations, images, schemes or meanings that might be expected to imply one another do not. This definition is compatible with previously observed behaviours described above: if different representations do not imply one another, the learner will struggle in transitioning between them; if relevant knowledge is not easily implied by the situation, it will not be deemed relevant; if different schemes are both implied by the situation but do not imply one another, a learner using both might get contradicting outcomes, without experiencing conflict. Using a framework that accounts for all behaviours could afford a more nuanced examination of the phenomenon, and potentially expose new such behaviours.

## **Methods**

In this research we conducted an intrinsic case study (Creswell & Poth, 2018), by conducting a task-based interview with Alma, a graduate student in mathematics education, taking an analysis course centred around ideas of rate of change and accumulation. These notions were only recently introduced into Israeli high school curriculum, to be taught alongside the already established notions of antiderivative and area (Dreyfus et al., 2021). Hence, they were new to the students taking the course.

Alma was one of three students interviewed on this task; the students were chosen because they had high achievements in the course, and were thus more likely to have developed connections between their previous conceptions of integral, and accumulation. Due to space limitations, in this paper we focus only on Alma, who presented an interesting example of compartmentalization. The interview was recorded and transcribed. We also recorded all of Alma's writing and sketching.

In the task, Alma was given a non-continuous piecewise linear function. She was requested to graph the derivative of the given function, and then graph the accumulation function of that derivative [Figure 1]. Since the accumulation function  $g(x)$  differs from the given function  $f(x)$ , this task is designed to induce conflict between conceptualizations of integral as antiderivative and as



**Figure 1: The task**

accumulation. The given function has a derivative for  $[0,3)$  and  $(3, \infty)$ , and the derivative has no antiderivative in the original domain  $[0, \infty)$ , since an antiderivative function cannot be well-defined at  $x = 3$ . However, the accumulation function in the domain  $[0, \infty)$  exists and is different from  $f(x)$ . In the interviews we focused on two differences between  $f(x)$  and  $g(x)$ . First,  $f(0) \neq g(0)$ . Second,  $f(x)$  has a discontinuity at  $x = 3$  whereas  $g(x)$  is continuous throughout.

We analysed the data in accordance with Thompson et al.'s definition of personal (in-the-moment) meaning (2024), looking for the implications brought to Alma's mind, how 'easily' they are brought to her mind, and what potential 'triggers' brought them to her mind. Due to the implicative nature of meaning, to identify Alma's personal meanings we paid attention to her distinct language, i.e., terms and phrases first brought up and used by her and thus indicative of her own implications. Ergo, we interpreted all mathematical terms and notations she used, particularly those used initially, intuitively, immediately; those that are unique. Since meaning is related to scheme, we examined Alma's claims and justifications, i.e., how she used the implied terms and notations to make sense of the

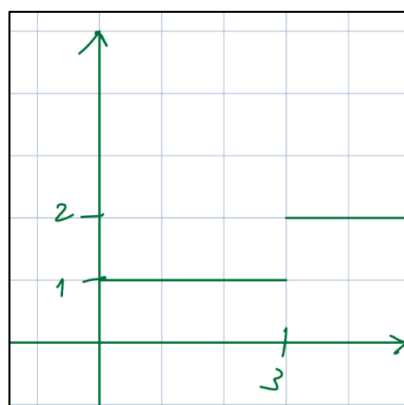
mathematics. We used this to expose the connections between the mathematical notions that comprise the meaning and possibly elucidate the structure of the scheme.

Thompson et al. (2024) refer to the space of implications as what is easily brought to mind. We interpret ‘easily’ as a scale – some implications are more easily brought to mind than others. We use several criteria to determine this. First, if an implication is recurring, if it is repeated at several points throughout the interview, then it is presumably more easily implied than an implication that occurs sparsely. Second, what is brought up immediately is likely more easily implied than what is brought up afterwards. Moreover, notions that require interviewer involvement to arise are less easily implied by the task than those that arise without intervention. Additionally, notions that are only brought up after the interviewee “gets stuck”, deems their current scheme unhelpful, or encounters a cognitive conflict are even less easily implied than the previous ones.

As stated, we also paid attention to the possible ‘triggers’ that brought each notion to mind. These can be mathematical triggers, i.e., a mathematical notation or notion brings to mind another notion, a scheme, a claim etc. ‘Triggers’ can also be external, e.g., a social norm, a recent previous experience. To determine the possible ‘triggers’ we asked ourselves several questions: Which mathematical stimuli appeared just before the implication arose? What was the goal of the reasoning in which the implication appeared? In reference to which question\given\statement did the implication appear?

## Findings

Alma has high achievements in the course, but low confidence. She began the interview exclaiming that she does not like analysis, and upon reading the task added “oh, you just killed me”. Though she displayed a lack of confidence, Alma sketched an appropriate graph for the derivative [Figure 2], however she did not clearly address  $x = 3$ . Her sketch of  $f'(x)$  was accompanied by consideration of linearly increasing functions, and ratios. The term ‘Rate of Change’ was not mentioned.



**Figure 2: Alma’s sketch of  $f'(x)$**

As Alma read the second part of the question, where she was asked to graph the accumulation function, her initial response was “I don’t know how to do this. Sorry”. However, she immediately followed with the statement “I mean, I want to say that it’s this” while pointing at the graph of  $f(x)$ . This is indicative of Alma’s implication for integral as antiderivative, though she struggles to elaborate, and upon prompting by the interviewer only adds: “simply because this is the connection between  $f'$  and  $h$ , the integral of ... this is the connection between  $h$  and the integral of  $h$ .”

From Alma's use of qualifiers in her statement – “I want to say that”, it appears that this is her own most salient implication for integral. Further probing by the interviewer revealed what was holding Alma back:

- Alma: Look, if you asked me this a month and a half ago [before the analysis course started], I would just tell you it's that.
- Interviewer: Okay.
- Alma: It's just that because of [the professor] (sighs and laughs) we can't use the word integral. Just kidding, it's not forbidden, but we just like have to explain it in different terms.

Thus, we learn that Alma's immediate (presumably stable) implication for integral is “the opposite of derivative”, and using the word integral is expressing this notion. Alma had also been exposed to conceptions of integral as accumulation in the course, however she struggled to connect these conceptions with the meaning she already possessed. This is further evident by her statement: “... what I want is to go back to this function ( $f'(x)$ ) and to integrate it. Like, not to integrate it, to ...”. In this statement she references two different actions, one of which is integrating a function, and the other she does not name, but states it is not integration. The statement was made immediately after referencing the course, and immediately before reasoning that she expected the integral to start from (0,0). Since she justified this expectation by claiming that “at zero it doesn't have an accumulation yet”, it appears that the second action referenced was accumulation.

Ergo, Alma seems to have difficulty in conceptualizing integral as accumulation. It appears that while accumulation is brought to her mind when reasoning about integrals, it is ‘less easily’ brought to mind than antiderivative, to the point where several times in the interview she even distinguished verbally between ‘integral’ and ‘accumulation’. While accumulation was brought to her mind during the task, it appears that the trigger was in large part the context of the interview as part of the course.

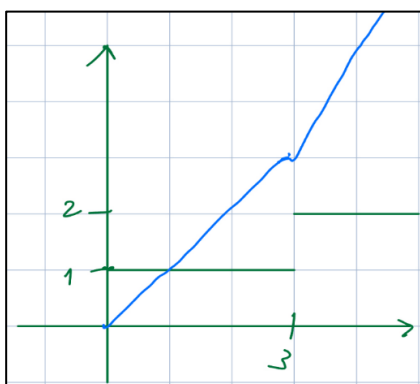
As Alma attempted to reason using accumulation, she developed a cognitive conflict between her accumulation reasoning and her antiderivative reasoning. This is due to  $f(0) = 2$  while  $g(0) = 0$ . Alma reasoned that at  $x = 0$  the accumulation function should equal zero, since “... at  $x = 0$  it doesn't have an accumulation yet”. However, her antiderivative reasoning suggested to her that  $g(x)$  and  $f(x)$  should be one and the same, as previously observed.

- Alma: Why do I want to start from zero? (9 second silence) Look, the graph here is given from  $x = 0$ , right? So that means that if I want to check the accumulation of this function, then at zero it doesn't have an accumulation yet. It's the starting point of my accumulation, that's why I want ... that's why there is this gap that I don't understand. ... conceptually I don't know how to settle it with... with the fact that when I start to sum here, because like, I can draw the integral of this function without this constant (circles 2), without the 2 in the given function ( $f(x)$ ).

Alma considered that the difference might be due to an integration constant – “what does it mean? Is it some kind of plus  $c$ ? What? Don't remember, oh, it's this plus two”. While she suggested an integration constant as a possible resolution to the conflict, her use of the word integral as distinct from accumulation suggests that she struggled connecting her conceptualization for integral as antiderivative with her conceptualization for integral as accumulation, even though both conceptualizations were implied.

As the interview continued, Alma proceeded to sketch a graph of  $g(x)$  using accumulation considerations, starting at zero, and continuous at  $x = 3$  (Figure 3). After sketching her graph, Alma gestured at the graph of  $f(x)$  and exclaimed: “and then I don’t know what this is”. Thus, Alma still had not resolved the conflict. Moreover, Alma states “... it is almost forced upon me. I can only suppose that there’s some piece of information that I don’t, I don’t understand all the way, and it’s not here (gestures at the graph she drew)”. It appears that for Alma, while both  $f(x)$  and  $g(x)$  are perceived as integrals of  $h(x)$ ,  $g(x)$  lacks some information that exists in  $f(x)$ . Ergo, this statement supports the hypothesis that for Alma there is a hierarchy between antiderivative and accumulation that favors the former. Furthermore, Alma explicitly stated: “right now there’s like two types of attempts to think about things mixed in my head”.

As she explained her conflict, Alma skirted around a possible resolution, however, she kept returning to the possibility of her sketches being mistaken. Interestingly, Alma first mentioned rate of change only at this point in the interview, suggesting it is less easily brought to her mind.



**Figure 3: Alma’s sketch of  $g(x)$  (blue)**

Alma: ... And I know these constants have some role, and yes, the negative four, it moved us down, but I can’t say how right now, what I drew wrong, or in the derivative itself, or maybe the derivative is drawn okay, but we know that the derivative, it actually only expresses rate of change, and there are a lot of different functions that can have the same rate of change, but something about them is different, and that thing is elusive to me right now.

To eliminate the doubt Alma displayed in the graph she drew of  $g(x)$ , the interviewer informed Alma that it is in fact correct. This led Alma to state:

Alma: ... rate of change alone, presumably, is not enough to describe an integral. Or this is an integral and this is ... I don’t know. Umm ... I need to think about it more.

Thus, it appears that Alma was contemplating two different solutions to her conflict. The first, that a given rate of change is not enough to ‘describe an integral’. Here Alma was likely using integral in the sense of ‘primitive function’, closely connected with antiderivative. The second, that only one of the functions is in fact the integral of  $h(x)$ . Finally, she settled on the explanation that “there are two integrals here for the same rate of change”. Additionally, Alma could not explain to the interviewer the difference between  $\int_0^x f'(t)dt$  and  $\int f'(x)dx$ , claiming that to her they are the same.

What was immediately and most easily brought to Alma’s mind by integrals is ideas of antiderivative and primitive function. This is in fact so salient to her, that she even struggled to explain it in terms

other than ‘integral’ or ‘integrating’. Thus, at times she used ‘integral’ as synonymous with  $f(x)$ . Consequently, this meaning is both dominant and laconic. However, due to the course, Alma also developed another meaning – accumulation. While her accumulative reasoning might be intended to satisfy the interviewer to some extent, accumulation is brought to her mind independently, and is explicitly used in her reasoning during the task.

Though both meanings are brought to mind by ‘ $g(x) = \int_0^x f'(t)dt$ ’, they do not seem to be brought to mind by one another. Furthermore, Alma didn’t seem to know when each meaning is relevant. Thus, Alma’s reasoning with antiderivative is independent from her reasoning with accumulation, resulting in compartmentalization.

## **Discussion and Conclusion**

In examining Alma’s meanings, we found that she can employ reasoning related to both antiderivative and accumulation, however she cannot determine which is relevant in the situation. The immediacy of her use of antiderivative reasoning, and the apparent use of accumulation reasoning to satisfy the interviewer suggest a hierarchy of implications that favours antiderivative reasoning, which is the less relevant one in the given situation. This suggests compartmentalization between the two notions.

Another aspect suggesting compartmentalization is the apparent incompatibility between Alma’s conceptions of antiderivative and accumulation. Though both notions were implied by the task, and both implied by ‘integral’, the connection Alma makes between both notions is mainly in name, and not in substance. Consequently, Alma does not recognize the requested function specifically as an accumulation function, but as an integral in a generic sense. This could also explain her struggle with referring to accumulation as integration, since she formed no meaningful connection between the meaning she held for integration prior to the course and accumulation.

We claim that though Alma experiences cognitive conflict due to her contradicting results, this behaviour still suggests compartmentalization, as the emergence of conflict is due to her tenuous and superficial connection between accumulation and antiderivative. Moreover, while she does recognize the relevant knowledge as relevant, we claim her inability to inhibit irrelevant knowledge is also a consequence of this compartmentalization. Thus, our suggested framework affords a more nuanced examination of compartmentalization, that enables the identification of more complex behaviours.

Like the students in Jones’ survey (2015), Alma invokes less relevant knowledge. Our analysis of her meaning suggests the possibility that the low frequency of Riemann sum-based conceptualizations could be a result of weak connections between them and antiderivative and area conceptualizations. These results also suggest that in attending to meaning when teaching the different manifestations of integral, it is important to attend not only to the meanings we want students to develop, but also to the meanings they currently possess, and to the connections and implications we want to exist between them, otherwise students’ knowledge could be compartmentalized. The approach suggested by Thompson et al. (2019), where a single notion of integral as an accumulation function is presented, could be a step in this direction; however, this is an issue for further research.

## Acknowledgment

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