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# A non-local model for heterogeneous material flow on conveyor belts

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October 7, 2025

## Abstract

In this paper, a finite volume approximation scheme is used to solve a non-local macroscopic material flow model in two space dimensions, accounting for the presence of boundaries in the non-local terms. Based on a previous result for the scalar case, we extend the setting to a system of heterogeneous material on bounded domains. We prove the convergence of the approximate solutions constructed using the Roe scheme with dimensional splitting, where the major challenge lies in the treatment of the discontinuity occurring in the flux function. Numerical tests show a good agreement with microscopic simulations.

*Keywords:* non-local systems of conservation laws; heterogeneous material flow; Roe scheme; boundary conditions.

*2010 Mathematics Subject Classification:* 35L65, 65M12

## 1 Introduction

In this paper, we consider the Cauchy problem for a system of non-local conservation laws in two space dimensions describing the movement of  $N$  classes of objects of different sizes and corresponding maximal densities  $R_c > 0$ ,  $c = 1, \dots, N$ , on a domain  $\Omega \subseteq \mathbb{R}^2$ . The initial-boundary value problem for the particle densities  $\boldsymbol{\rho} = (\rho^1, \dots, \rho^N)$  as a function of time  $t$  and position  $(x, y) \in \Omega$  is given by

$$\begin{cases} \partial_t \rho^c + \nabla \cdot \left( \rho^c \left( \mathbf{v}^{stat}(x, y) + \mathbf{v}_c^{dyn}[r](x, y) \right) \right) = 0, & (t, x, y) \in [0, T] \times \Omega, \quad c = 1, \dots, N, \\ \rho^c(0, x, y) = \rho_o^c(x, y), & (x, y) \in \Omega, \end{cases} \quad (1.1)$$

for any  $T > 0$ , where for the weighting factor  $\varepsilon_c > 0$

$$r = \sum_{c=1}^N \alpha_c \rho^c, \quad \mathbf{v}_c^{dyn}[r] = H(\tilde{\eta} * r - r_{max}) \mathbf{I}_c[r] \quad \text{with } \mathbf{I}_c[r] = -\varepsilon_c \frac{\nabla(\eta_c * r)}{\sqrt{1 + \|\nabla(\eta_c * r)\|^2}}. \quad (1.2)$$

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Above,  $H$  denotes the Heaviside function, which becomes active whenever the maximal density  $r_{max} > 0$  (expressed in terms of the smallest object density, see `pce` concept in traffic) is exceeded, while  $\tilde{\eta}, \eta_c$  are positive, smooth mollifiers depending on the object sizes. The (weighted) total density  $r$  is defined referring to the PCE concept [15], where we set  $\alpha_c = R_1/R_c$ . Let us suppose  $c = 1$  is the class of smallest pieces, so that its maximal density  $R_1 > R_c$  for all  $c > 1$ . In this way,  $r_{max} = R_1$ .

The scalar version of 1.1 was introduced in [10], to model the flow of particles on a conveyor belt. The velocity vector field induced by the conveyor belt is denoted by  $\mathbf{v}^{stat}$  and is constant in time and the same for all classes. Below the maximal density  $r_{max}$ , parts are transported with the velocity of the conveyor belt. If the density exceeds  $r_{max}$ , the dynamic velocity vector field  $\mathbf{v}_c^{dyn}$  becomes active and models the collision of particles via the operator  $\mathbf{I}_c[r]$ . The negative gradient of the convolution  $\eta_c * r$  pushes the mass towards lower density regions. The particular choice of the collision operator  $\mathbf{I}_c(r)$  was introduced in [5] to describe crowd dynamics. In particular, the norm appearing in the denominator is the Euclidean norm in  $\mathbb{R}^2$ , so that the denominator acts as a smooth normalisation factor.

To model boundaries in the form of walls and other obstacles, we follow an approach introduced in [4] and later developed in [8, 9], by incorporating the information in the convolution kernel. For this, assume  $\Omega^c = \mathbb{R}^2 \setminus \Omega$  to be a compact set consisting of a finite number  $M \in \mathbb{N}$  of connected components  $\Omega^c = \Omega_1^c \cup \dots \cup \Omega_M^c$ . We then replace  $r$  in the non-local operator by the augmented total density  $r_\Omega = \sum_{c=1}^{N+1} \alpha_c \rho_\Omega^c$  with  $\alpha_{N+1} = 1$  and  $\rho_0^c = 0$  for  $x \in \Omega^c, c = 1, \dots, N$  and where  $\rho_\Omega^c : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ , is given by

$$\rho_\Omega^c := \begin{cases} \rho^c & \text{if } c = 1, \dots, N, \\ \sum_{l=1}^M R_{N+l} \chi_{\Omega_l^c} & \text{if } c = N + 1, \end{cases} \quad (1.3)$$

with  $R_{N+l} > 0, l = 1, \dots, M$ , big enough so that  $(\mathbf{v}^{stat}(x, y) + \mathbf{v}_c^{dyn}[r_\Omega](x, y)) \cdot \mathbf{n}(x, y) \leq 0$  for all  $(x, y) \in \partial\Omega, t \geq 0$ ,  $\mathbf{n}$  being the outward normal to  $\Omega$ . Thus, the domain  $\Omega$  is invariant and we can rewrite the Cauchy problem (1.1) on  $\mathbb{R}^2$ :

$$\begin{cases} \partial_t \rho^c + \nabla \cdot \left( \rho^c \left( \mathbf{v}^{stat}(x, y) + \mathbf{v}_c^{dyn}[r_\Omega](x, y) \right) \right) = 0, & (t, x, y) \in [0, T] \times \mathbb{R}^2, \quad c = 1, \dots, N. \\ \rho^c(0, x, y) = \rho_o^c(x, y), & (x, y) \in \mathbb{R}^2, \end{cases} \quad (1.4)$$

Conservation laws with non-local flux function have been recently introduced in the literature to describe transport phenomena accounting for non-local interaction effects among agents, such as road traffic flow [3] or pedestrian dynamics [5]. General well-posedness results have been provided by [2] in the scalar one-dimensional case, while [1, 9] deal with systems of non-local conservation laws in multi-space dimensions. Even if the results in [1] apply to our problem (1.1), estimates in [1] were obtained for general flux functions using finite volume approximate solutions constructed via Lax-Friedrichs scheme. In [14] sharp estimates for the Roe scheme were derived, which is known to give less diffusive solutions in the case of non-local problems in multi-D. The aim of the present paper is to extend the results for the Roe scheme in [14] to the system case, including the presence of obstacles. This mainly involves some further assumptions on the interaction term and some changes in the estimates in Appendix A. We recall that, more recently, an alternative approach to prove well-posedness of some non-local conservation laws, based on the method of characteristics and on Banach's

fixed-point theorem, was proposed in [11, 12]. In particular, no entropy condition is required to prove uniqueness and stability of solutions, showing that weak solutions are indeed unique. Nevertheless, these results rely on stronger regularity assumptions on the flux function, and cannot give information on the convergence of the finite volume numerical schemes studied in this paper.

We finally remark that, even if the original model proposes the use of the discontinuous Heaviside function, the stability of the numerical schemes requires a smooth approximation of it. Together with the introduction of the mollifier  $\tilde{\eta}$ , the maximal density constraint is not satisfied. Indeed, the  $\mathbf{L}^\infty$ -norm of the derivatives  $H'$  and  $\nabla\tilde{\eta}$  are included in several estimates (for instance in the CFL condition (3.13) that guarantees the **BV**-estimates), which blow up with it.

The paper is organised as follows: for a better overview, we introduce our main results in Section 2 and we postpone the proof of convergence of the approximate solutions constructed by the Roe scheme in Section 3. This includes a proof of the Lipschitz continuous dependence on the initial data in Section 3.6. Section 4 is devoted to discussion of the numerical tests.

## 2 Main results

Throughout the paper, we will denote  $\mathcal{I}(r, s) := [\min\{r, s\}, \max\{r, s\}]$ , for any  $r, s \in \mathbb{R}$ . We require the following assumptions to hold for  $c \in \{1, \dots, N\}$ :

**( $\Omega.1$ )** The domain  $\Omega \subseteq \mathbb{R}^2$  is a non-empty open set with smooth boundary  $\partial\Omega$ , so that the outward normal  $\mathbf{n}(x, y)$  is uniquely defined for all  $(x, y) \in \partial\Omega$ . Moreover, we assume that  $\Omega^c = \mathbb{R}^2 \setminus \Omega$  is a compact set consisting of a finite number  $M$  of connected components.

**( $\Omega.2$ )** For all  $k = 1, \dots, N$  the vector fields  $\mathbf{v}^{stat} + \mathbf{v}_c^{dyn}$  point inward along the boundary  $\partial\Omega$  of  $\Omega$ , i.e.  $(\mathbf{v}^{stat} + \mathbf{v}_c^{dyn}) \cdot \mathbf{n}(x, y) \leq 0$  for all  $(x, y) \in \partial\Omega, t \geq 0$ .

**( $v$ )**  $\mathbf{v}^{stat} \in \mathbf{C}^2(\mathbb{R}^2; \mathbb{R}^2)$ .

**( $H$ )** The function  $H$  is a smooth approximation of the Heaviside function, so that its derivative  $H'$  is bounded. In particular, we denote by  $L_H$  the Lipschitz constant of the function  $H$ :

$$L_H = \|H'\|_{\mathbf{L}^\infty(\mathbb{R})}, \quad (2.1)$$

**( $\eta$ )**  $\tilde{\eta}, \eta_c \in (\mathbf{C}^3 \cap \mathbf{W}^{3,\infty})(\mathbb{R}^2; \mathbb{R}^+)$  for all  $c \in \{1, \dots, N\}$ .

Recall the definition of solution to the Cauchy problem (1.1), see also [1, 2, 5].

**Definition 2.1.** Let  $\rho_o \in \mathbf{L}^\infty(\mathbb{R}^2; \mathbb{R}_+^N)$  such that  $\text{supp } \rho_o \subset \Omega$ . A map  $\rho : [0, T] \rightarrow \mathbf{L}^\infty(\mathbb{R}^2; \mathbb{R}^N)$  is a solution to (1.1) with initial datum  $\rho_o$ , if, for all  $c \in \{1, \dots, N\}$ ,  $\rho^c$  is a Kruřkov solution to

$$\begin{cases} \partial_t \rho^c + \nabla \cdot (\rho^c V^c(t, x, y)) = 0, & (t, x, y) \in [0, T] \times \mathbb{R}^2, \\ \rho^c(0, x, y) = \rho_o^c(x, y), & (x, y) \in \mathbb{R}^2, \end{cases} \quad (2.2)$$

with  $V^c(t, x, y) = \mathbf{v}^{stat}(x, y) - \varepsilon_c H(\tilde{\eta} * r - r_{\max}) \frac{\nabla(\eta_c * r_\Omega)}{\sqrt{1 + \|\nabla(\eta_c * r_\Omega)\|^2}}(x, y)$ .

Above, for the definition of Kruřkov solution we refer to [13, Definition 1].

**Theorem 2.2.** *Let  $\rho_o \in (\mathbf{L}^\infty \cap \mathbf{BV})(\Omega; \mathbb{R}_+^N)$  for  $\Omega \in \mathbb{R}^2$ . Let assumptions  $(\mathbf{v})$ ,  $(\mathbf{H})$  and  $(\boldsymbol{\eta})$  hold. Then, for all  $T > 0$ , there exists a unique entropy weak solution  $\boldsymbol{\rho} \in (\mathbf{L}^\infty \cap \mathbf{BV})([0, T] \times \Omega; \mathbb{R}_+^N)$  to problem (1.1). Moreover, the following estimates hold: for all  $t \in [0, T]$  and all  $c \in \{1, \dots, N\}$*

$$\begin{aligned} \|\boldsymbol{\rho}(t, \cdot)\|_{\mathbf{L}^1(\mathbb{R}^2; \mathbb{R}^N)} &= \|\rho_o\|_{\mathbf{L}^1(\mathbb{R}^2; \mathbb{R}^N)}, \\ \|\rho^c(t, \cdot)\|_{\mathbf{L}^\infty(\mathbb{R}^2; \mathbb{R})} &\leq \|\rho_o^c\|_{\mathbf{L}^\infty(\mathbb{R}^2; \mathbb{R})} e^{\mathcal{C}_\infty^c t}, \\ \text{TV}(\rho^c(t, \cdot)) &\leq \text{TV}(\rho_o^c) e^{2t\mathcal{K}_1^c} + \frac{2\mathcal{K}_2^c}{\mathcal{K}_1^c} \left( e^{2t\mathcal{K}_1^c} - 1 \right), \\ \|\boldsymbol{\rho}(t, \cdot) - \boldsymbol{\rho}(t - \tau, \cdot)\|_{\mathbf{L}^1(\mathbb{R}^2; \mathbb{R}^N)} &\leq 2\mathcal{C}_t(t) \tau, \quad \text{for } \tau > 0, \end{aligned}$$

where  $\mathcal{C}_\infty^c$  is defined in (3.12),  $\mathcal{K}_1^c$  is defined in (3.16),  $\mathcal{K}_2^c$  is defined in (3.17) and  $\mathcal{C}_t(t)$  is as in (3.38).

The proof of existence of solutions is based on the constructions of a converging sequence of approximate solutions and follows standard guidelines, see e.g. [1, Theorem 2.3]. In the literature, this usually relies on Lax-Friedrichs type schemes. In this paper, we derive the necessary compactness estimates for Roe scheme. The bounds presented in Theorem 2.2 are obtained by passing to the limit in the corresponding discrete bounds. Uniqueness is ensured by Proposition 3.9, which provides the Lipschitz continuous dependence estimate of solutions to (1.1) on the initial data.

### 3 Existence

We introduce the uniform mesh of width  $\Delta x$  along the  $x$ -axis and  $\Delta y$  along the  $y$ -axis, and a time step  $\Delta t$  subject to a CFL condition, specified later on. For  $k \in \mathbb{Z}$  set

$$\begin{aligned} x_k &= (k - 1/2)\Delta x, & y_k &= (k - 1/2)\Delta y, \\ x_{k+1/2} &= k\Delta x, & y_{k+1/2} &= k\Delta y, \end{aligned}$$

where  $(x_{i+\frac{1}{2}}, y_j)$  and  $(x_i, y_{j+\frac{1}{2}})$  denote the cells interfaces and  $(x_i, y_j)$  are the cells centers. Set  $N_T = \lceil T/\Delta t \rceil$  and let  $t^n = n \Delta t$ ,  $n = 0, \dots, N_T$ , be the time mesh. Set  $\lambda_x = \Delta t/\Delta x$  and  $\lambda_y = \Delta t/\Delta y$ . For the sake of shortness, sometimes we will use also the notation  $\mathbf{x}_{i,j} = (x_i, y_j)$ .

We approximate the initial datum as follows: for  $i, j \in \mathbb{Z}$  and all  $c \in \{1, \dots, N\}$

$$\rho_{i,j}^{c,0} = \frac{1}{\Delta x \Delta y} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{i-1/2}}^{y_{i+1/2}} \rho_o^c(x, y) \, dx \, dy,$$

and we define a piece-wise constant solution to (1.1) as

$$\rho_\Delta^c(t, x, y) = \rho_{i,j}^{c,n} \quad \text{for} \quad \begin{cases} t \in [t^n, t^{n+1}[, \\ x \in [x_{i-1/2}, x_{i+1/2}[, \\ y \in [y_{j-1/2}, y_{j+1/2}[, \end{cases} \quad \text{where} \quad \begin{cases} n = 0, \dots, N_T - 1, \\ i \in \mathbb{Z}, \\ j \in \mathbb{Z}. \end{cases} \quad (3.1)$$

through a modified Roe-type scheme with dimensional splitting, as in [10]:

**Algorithm 3.1.**

$$V_1(x, y, u, w) = v_1^{stat}(x, y) u + \min\{0, v_1^{stat}(x, y)\}(w - u) \quad (3.2)$$

$$F^c(u, w, J^c(t, x, y)) = J^c(t, x, y)u + \min\{0, J^c(t, x, y)\}(w - u) \quad (3.3)$$

$$V_2(x, y, u, w) = v_2^{stat}(x, y) u + \min\{0, v_2^{stat}(x, y)\}(w - u) \quad (3.4)$$

for  $n = 0, \dots, N_T - 1$

$$\begin{aligned} \rho_{i,j}^{c,n+1/2} &= \rho_{i,j}^{c,n} - \lambda_x \left[ V_1(x_{i+1/2,j}, \rho_{i,j}^{c,n}, \rho_{i+1,j}^{c,n}) - V_1(x_{i-1/2,j}, \rho_{i-1,j}^{c,n}, \rho_{i,j}^{c,n}) \right. \\ &\quad \left. + F^c(\rho_{i,j}^{c,n}, \rho_{i+1,j}^{c,n}, J_1^{c,n}(x_{i+1/2,j})) - F^c(\rho_{i-1,j}^{c,n}, \rho_{i,j}^{c,n}, J_1^{c,n}(x_{i-1/2,j})) \right] \end{aligned} \quad (3.5)$$

$$\begin{aligned} \rho_{i,j}^{c,n+1} &= \rho_{i,j}^{c,n+1/2} - \lambda_y \left[ V_2(x_{i,j+1/2}, \rho_{i,j}^{c,n}, \rho_{i,j+1}^{c,n}) - V_2(x_{i,j-1/2}, \rho_{i,j-1}^{c,n}, \rho_{i,j}^{c,n}) \right. \\ &\quad \left. + F^c(\rho_{i,j}^{c,n}, \rho_{i,j+1}^{c,n}, J_2^{c,n}(x_{i,j+1/2})) - F^c(\rho_{i,j-1}^{c,n}, \rho_{i,j}^{c,n}, J_2^{c,n}(x_{i,j-1/2})) \right] \end{aligned} \quad (3.6)$$

end

Above, we set  $\mathbf{v}_c^{dyn}(\rho^{1,n}, \dots, \rho^{N,n})(x, y) = (J_1^{c,n}(x, y), J_2^{c,n}(x, y))$  and the convolution products are computed through the following quadrature formula, for  $i = 1, 2$ ,

$$\left( \partial_i \eta_c * \left( \sum_{k=1}^{N+1} \alpha_c \rho_{\Omega}^k \right) \right) (x_i, y_j) = \Delta x \Delta y \sum_{h, \ell \in \mathbb{Z}} \left( \sum_{k=1}^{N+1} \alpha_c \rho_{\Omega, h, \ell}^k \right) \partial_i \eta_c(x_{i-h}, y_{j-\ell}), \quad (3.7)$$

where  $\partial_1 \eta_c = \partial_x \eta_c$  and  $\partial_2 \eta_c = \partial_y \eta_c$  and  $\rho^k$  is the density of class  $k$ . Remark that the choice of evaluating the numerical flux at  $t^n$  for both fractional steps allows to save computational time, because the convolution products (3.7) are computed only once per time step.

### 3.1 Positivity

In the case of positive initial datum, we prove that under a suitable CFL condition the approximate solution to (1.1) constructed via the Algorithm 3.1 remains positive.

**Lemma 3.2. (Positivity)** *Let  $\rho_o \in \mathbf{L}^\infty(\Omega; \mathbb{R}_+^N)$ . Let  $(\mathbf{v})$  and  $(\boldsymbol{\eta})$  hold. Assume that*

$$\lambda_x \leq \frac{1}{2(\varepsilon_{max} + \|v_1^{stat}\|_{\mathbf{L}^\infty})}, \quad \lambda_y \leq \frac{1}{2(\varepsilon_{max} + \|v_2^{stat}\|_{\mathbf{L}^\infty})}. \quad (3.8)$$

where  $\varepsilon_{max} := \max_c \varepsilon_c$ . Then, for all  $c \in \{1, \dots, N\}$ ,  $t > 0$  and  $(x, y) \in \mathbb{R}^2$ , the piece-wise constant approximate solution  $\rho_\Delta^c$  constructed through Algorithm 3.1 is such that  $\rho_\Delta^c(t, x, y) \geq 0$ .

**Proof.** Fix  $n$  between 0 and  $N_T - 1$  and assume that  $\rho_{i,j}^{c,n} \geq 0$  for all  $i, j \in \mathbb{Z}$ . Consider (3.5), with the notation (3.2) and (3.3), and observe that:

$$\begin{aligned} &V_1(x_{i+1/2,j}, \rho_{i,j}^{c,n}, \rho_{i+1,j}^{c,n}) + F^c(\rho_{i,j}^{c,n}, \rho_{i+1,j}^{c,n}, J_1^{c,n}(x_{i+1/2,j})) \\ &\leq v_{i+1/2} \rho_{i,j}^{c,n} + J_1^{c,n}(x_{i+1/2,j}) \rho_{i,j}^{c,n} \leq \left( \|v_1^{stat}\|_{\mathbf{L}^\infty} + \varepsilon_c \right) \rho_{i,j}^{c,n}, \\ &V_1(x_{i-1/2,j}, \rho_{i-1,j}^{c,n}, \rho_{i,j}^{c,n}) + F^c(\rho_{i-1,j}^{c,n}, \rho_{i,j}^{c,n}, J_1^{c,n}(x_{i-1/2,j})) \end{aligned}$$

$$\geq v_{i-1/2} \rho_{i,j}^{c,n} + J_1^{c,n}(x_{i-1/2,j}) \rho_{i,j}^{c,n} \geq - \left( \|v_1^{stat}\|_{\mathbf{L}^\infty} + \varepsilon_c \right) \rho_{i,j}^{c,n},$$

where we used the notation

$$v_{i\pm 1/2} = v_1^{stat}(x_{i\pm 1/2,j}). \quad (3.9)$$

Therefore, by (3.5),

$$\rho_{i,j}^{c,n+1/2} \geq \rho_{i,j}^{c,n} - 2 \lambda_x \left( \|v_1^{stat}\|_{\mathbf{L}^\infty} + \varepsilon_c \right) \rho_{i,j}^{c,n} \geq 0,$$

thanks to the CFL condition (3.8). Starting from (3.6), an analogous argument shows that  $\rho_{i,j}^{c,n+1} \geq 0$ , concluding the proof.  $\square$

### 3.2 $\mathbf{L}^1$ bound

The following result on the  $\mathbf{L}^1$  bound follows from the conservation property of the Roe scheme.

**Lemma 3.3. ( $\mathbf{L}^1$  bound)** *Let  $\rho_o \in \mathbf{L}^\infty(\Omega; \mathbb{R}_+^N)$ . Let  $(\mathbf{v})$ ,  $(\boldsymbol{\eta})$  and (3.8) hold. Then, for all  $c \in \{1, \dots, N\}$ ,  $t > 0$  and  $(x, y) \in \mathbb{R}^2$ ,  $\rho_\Delta^c$  constructed through Algorithm 3.1 satisfies*

$$\|\rho_\Delta^c(t, \cdot, \cdot)\|_{\mathbf{L}^1(\mathbb{R}^2)} = \|\rho_o^c\|_{\mathbf{L}^1(\mathbb{R}^2)}. \quad (3.10)$$

### 3.3 $\mathbf{L}^\infty$ bound

**Lemma 3.4. ( $\mathbf{L}^\infty$  bound)** *Let  $\rho_o \in \mathbf{L}^\infty(\mathbb{R}^2; \mathbb{R}_+^N)$ . Let  $(\mathbf{v})$ ,  $(\boldsymbol{\eta})$  and (3.8) hold. Then, for all  $c \in \{0, \dots, N\}$ ,  $t > 0$  and  $(x, y) \in \mathbb{R}^2$ ,  $\rho_\Delta^c$  constructed through Algorithm 3.1 satisfies*

$$\|\rho_\Delta^c(t, \cdot, \cdot)\|_{\mathbf{L}^\infty(\mathbb{R}^2)} \leq \|\rho_o^c\|_{\mathbf{L}^\infty} e^{C_\infty^c t}, \quad (3.11)$$

where

$$C_\infty^c = \left\| \partial_x v_1^{stat} \right\|_{\mathbf{L}^\infty} + \left\| \partial_y v_2^{stat} \right\|_{\mathbf{L}^\infty} + 2 \varepsilon_c \left( 2 \left\| \nabla^2 \eta_c \right\|_{\mathbf{L}^\infty} + L_H \|\nabla \tilde{\eta}_c\|_{\mathbf{L}^\infty} \right) \|r_{\Omega,o}\|_{\mathbf{L}^1}. \quad (3.12)$$

**Proof.** Fix  $c \in \{1, \dots, N\}$ . Setting  $r_{\Omega,o} = \sum_{k=1}^{N+1} \alpha_k \rho_{\Omega,o}^k$  and exploiting the notation introduced in (3.9), we observe that (3.5) attains its maximum for  $v_{i+1/2} < 0$ ,  $v_{i-1/2} \geq 0$ ,  $J_1^{c,n}(x_{i+1/2,j}) < 0$  and  $J_1^{c,n}(x_{i-1/2,j}) \geq 0$ . In this case

$$\begin{aligned} \rho_{i,j}^{c,n+1/2} &\leq \rho_{i,j}^{c,n} - \lambda_x \left( \rho_{i+1,j}^{c,n} v_{i+1/2} + J_1^{c,n}(x_{i+1/2,j}) \rho_{i+1,j}^{c,n} \right) \\ &\quad + \lambda_x \left( \rho_{i-1,j}^{c,n} v_{i-1/2} + J_1^{c,n}(x_{i-1/2,j}) \rho_{i-1,j}^{c,n} \right), \end{aligned}$$

where we used the positivity of  $\rho_{i,j}^{c,n}$  and discarded all the terms giving a negative contribution. Moreover, since  $v_{i+1/2} < 0$  and  $v_{i-1/2} \geq 0$ ,

$$\begin{aligned} \lambda_x \left( -\rho_{i+1,j}^{c,n} v_{i+1/2} + \rho_{i-1,j}^{c,n} v_{i-1/2} \right) &\leq \lambda_x \|\rho^{c,n}\|_{\mathbf{L}^\infty} \left( -v_{i+1/2} + v_{i-1/2} \right) \\ &= \lambda_x \|\rho^{c,n}\|_{\mathbf{L}^\infty} (-\Delta x) \partial_x v_1^{stat}(\hat{x}_i) \end{aligned}$$

with  $\hat{x}_i \in ]x_{i-1/2}, x_{i+1/2}[$ . In a similar way, since  $J_1^{c,n}(x_{i+1/2}) < 0$  and  $J_1^{c,n}(x_{i-1/2}) \geq 0$ , we get

$$\begin{aligned} & \lambda_x \left( -J_1^{c,n}(x_{i+1/2}) \rho_{i+1,j}^{c,n} + J_1^{c,n}(x_{i-1/2}) \rho_{i-1,j}^{c,n} \right) \\ & \leq \lambda_x \|\rho^{c,n}\|_{\mathbf{L}^\infty} \left( -J_1^{c,n}(x_{i+1/2,j}) + J_1^{c,n}(x_{i-1/2,j}) \right) \\ & \leq \lambda_x \|\rho^{c,n}\|_{\mathbf{L}^\infty} \varepsilon_c \Delta x \left( 2 \|\nabla^2 \eta_c\|_{\mathbf{L}^\infty} + L_H \|\nabla \tilde{\eta}_c\|_{\mathbf{L}^\infty} \right) \|r_\Omega^n\|_{\mathbf{L}^1}, \end{aligned}$$

thanks to (A.2). Therefore,

$$\rho_{i,j}^{c,n+1/2} \leq \|\rho^{c,n}\|_{\mathbf{L}^\infty} \left[ 1 + \Delta t \left( \|\partial_x v_1^{stat}\|_{\mathbf{L}^\infty} + \varepsilon_c \left( 2 \|\nabla^2 \eta_c\|_{\mathbf{L}^\infty} + L_H \|\nabla \tilde{\eta}_c\|_{\mathbf{L}^\infty} \right) \|r_{\Omega,o}\|_{\mathbf{L}^1} \right) \right].$$

In a similar way we get

$$\rho_{i,j}^{c,n+1} \leq \|\rho^{c,n+1/2}\|_{\mathbf{L}^\infty} \left[ 1 + \Delta t \left( \|\partial_y v_2^{stat}\|_{\mathbf{L}^\infty} + \varepsilon_c \left( 2 \|\nabla^2 \eta_c\|_{\mathbf{L}^\infty} + L_H \|\nabla \tilde{\eta}_c\|_{\mathbf{L}^\infty} \right) \|r_{\Omega,o}\|_{\mathbf{L}^1} \right) \right].$$

An iterative argument completes the proof.  $\square$

### 3.4 BV bound

**Proposition 3.5. (BV estimate in space)** *Let  $\rho_o \in (\mathbf{L}^\infty \cap \mathbf{BV})(\Omega; \mathbb{R}_+^N)$ . Let  $(\mathbf{v})$ ,  $(\mathbf{H})$ , and  $(\boldsymbol{\eta})$  hold. Assume that*

$$\lambda_x \leq \frac{1}{3(\varepsilon_{max} L_H + \|v_1^{stat}\|_{\mathbf{L}^\infty})}, \quad \lambda_y \leq \frac{1}{3(\varepsilon_{max} L_H + \|v_2^{stat}\|_{\mathbf{L}^\infty})}. \quad (3.13)$$

where  $\varepsilon_{max} := \max_c \varepsilon_c$ . Then, for all  $c \in \{1, \dots, N\}$ ,  $t > 0$ ,  $\rho_\Delta^c$  constructed through Algorithm 3.1 satisfies the following estimate: for all  $n = 0, \dots, N_T$ ,

$$\sum_{i,j \in \mathbb{Z}} \left( \Delta y \left| \rho_{i+1,j}^{c,n} - \rho_{i,j}^{c,n} \right| + \Delta x \left| \rho_{i,j+1}^{c,n} - \rho_{i,j}^{c,n} \right| \right) \leq \mathcal{C}_x^c(t^n), \quad (3.14)$$

where

$$\mathcal{C}_x^c(t) = e^{2t\mathcal{K}_1^c} \sum_{i,j \in \mathbb{Z}} \left( \Delta x \left| \rho_{i,j+1}^{c,0} - \rho_{i,j}^{c,0} \right| + \Delta y \left| \rho_{i+1,j}^{c,0} - \rho_{i,j}^{c,0} \right| \right) + \frac{2\mathcal{K}_2^c}{\mathcal{K}_1^c} \left( e^{2t\mathcal{K}_1^c} - 1 \right), \quad (3.15)$$

with

$$\mathcal{K}_1^c = 6 \left( \|\nabla \mathbf{v}^{stat}\|_{\mathbf{L}^\infty} + \varepsilon_c \left( 2 \|\nabla^2 \eta_c\|_{\mathbf{L}^\infty} + L_H \|\nabla \tilde{\eta}_c\|_{\mathbf{L}^\infty} \right) \|r_{\Omega,o}\|_{\mathbf{L}^1} \right), \quad (3.16)$$

$$\mathcal{K}_2^c = \left( 4\varepsilon_c \left( c_1^c \|r_{\Omega,o}\|_{\mathbf{L}^1} + c_2^c \|r_{\Omega,o}\|_{\mathbf{L}^1}^2 \right) + 3 \|\nabla^2 \mathbf{v}^{stat}\|_{\mathbf{L}^\infty} \right) \|\rho_o^c\|_{\mathbf{L}^1}, \quad (3.17)$$

and  $c_1^c, c_2^c$  are defined in (A.6).

**Remark 3.6.** Observe that the CFL conditions (3.13) are stricter than (3.8).

**Proof.** We follow the idea of [1, Lemma 2.7] as done in [9]. First consider for  $c \in \{1, \dots, N\}$  the term

$$\sum_{i,j \in \mathbb{Z}} \Delta y \left| \rho_{i+1,j}^{c,n+1/2} - \rho_{i,j}^{c,n+1/2} \right|.$$

In particular, fixing  $i, j \in \mathbb{Z}$  and omitting the dependencies on  $y_j$  for the sake of simplicity and setting  $r_{\Omega,o} = \sum_{k=1}^{N+1} \alpha_k \rho_{\Omega,o}^k$ , by (3.5) we get

$$\begin{aligned} \rho_{i+1}^{c,n+1/2} - \rho_i^{c,n+1/2} &= \rho_{i+1}^{c,n} - \rho_i^{c,n} - \lambda_x \left[ V_1(x_{i+3/2}, \rho_{i+1}^{c,n}, \rho_{i+2}^{c,n}) + F^c(\rho_{i+1}^{c,n}, \rho_{i+2}^{c,n}, J_1^{c,n}(x_{i+3/2})) \right. \\ &\quad - V_1(x_{i+1/2}, \rho_i^{c,n}, \rho_{i+1}^{c,n}) - F^c(\rho_i^{c,n}, \rho_{i+1}^{c,n}, J_1^{c,n}(x_{i+1/2})) \\ &\quad - V_1(x_{i+1/2}, \rho_i^{c,n}, \rho_{i+1}^{c,n}) - F^c(\rho_i^{c,n}, \rho_{i+1}^{c,n}, J_1^{c,n}(x_{i+1/2})) \\ &\quad \left. + V_1(x_{i-1/2}, \rho_{i-1}^{c,n}, \rho_i^{c,n}) + F^c(\rho_{i-1}^{c,n}, \rho_i^{c,n}, J_1^{c,n}(x_{i-1/2})) \right] \\ &\quad \pm \lambda_x \left[ V_1(x_{i+3/2}, \rho_i^{c,n}, \rho_{i+1}^{c,n}) + F^c(\rho_i^{c,n}, \rho_{i+1}^{c,n}, J_1^{c,n}(x_{i+3/2})) \right. \\ &\quad \left. - V_1(x_{i+1/2}, \rho_{i-1}^{c,n}, \rho_i^{c,n}) - F^c(\rho_{i-1}^{c,n}, \rho_i^{c,n}, J_1^{c,n}(x_{i+1/2})) \right] \\ &= \mathcal{A}_{i,j}^{c,n} - \lambda_x \mathcal{B}_{i,j}^{c,n}, \end{aligned}$$

where we set

$$\begin{aligned} \mathcal{A}_{i,j}^{c,n} &= \rho_{i+1}^{c,n} - \rho_i^{c,n} - \lambda_x \left[ V_1(x_{i+3/2}, \rho_{i+1}^{c,n}, \rho_{i+2}^{c,n}) + F^c(\rho_{i+1}^{c,n}, \rho_{i+2}^{c,n}, J_1^{c,n}(x_{i+3/2})) \right. \\ &\quad - V_1(x_{i+1/2}, \rho_i^{c,n}, \rho_{i+1}^{c,n}) - F^c(\rho_i^{c,n}, \rho_{i+1}^{c,n}, J_1^{c,n}(x_{i+1/2})) \\ &\quad + V_1(x_{i+1/2}, \rho_{i-1}^{c,n}, \rho_i^{c,n}) + F^c(\rho_{i-1}^{c,n}, \rho_i^{c,n}, J_1^{c,n}(x_{i+1/2})) \\ &\quad \left. - V_1(x_{i+3/2}, \rho_i^{c,n}, \rho_{i+1}^{c,n}) - F^c(\rho_i^{c,n}, \rho_{i+1}^{c,n}, J_1^{c,n}(x_{i+3/2})) \right], \\ \mathcal{B}_{i,j}^{c,n} &= V_1(x_{i+3/2}, \rho_i^{c,n}, \rho_{i+1}^{c,n}) + F^c(\rho_i^{c,n}, \rho_{i+1}^{c,n}, J_1^{c,n}(x_{i+3/2})) \\ &\quad - V_1(x_{i+1/2}, \rho_{i-1}^{c,n}, \rho_i^{c,n}) - F^c(\rho_{i-1}^{c,n}, \rho_i^{c,n}, J_1^{c,n}(x_{i+1/2})) \\ &\quad + V_1(x_{i-1/2}, \rho_{i-1}^{c,n}, \rho_i^{c,n}) + F^c(\rho_{i-1}^{c,n}, \rho_i^{c,n}, J_1^{c,n}(x_{i-1/2})) \\ &\quad - V_1(x_{i+1/2}, \rho_i^{c,n}, \rho_{i+1}^{c,n}) - F^c(\rho_i^{c,n}, \rho_{i+1}^{c,n}, J_1^{c,n}(x_{i+1/2})). \end{aligned}$$

For the sake of shortness, introduce the following notation

$$H_{h,\ell}^{c,n}(u, w) = V_1(x_{h,\ell}, u, w) + F^c(u, w, J_1^{c,n}(x_{h,\ell})), \quad (3.18)$$

so that, dropping the  $j$  dependencies,  $\mathcal{A}_{i,j}^{k,n}$  reads

$$\begin{aligned} \mathcal{A}_{i,j}^{c,n} &= \rho_{i+1}^{c,n} - \rho_i^{c,n} - \lambda_x \left[ H_{i+3/2}^{c,n}(\rho_{i+1}^{c,n}, \rho_{i+2}^{c,n}) - H_{i+1/2}^{c,n}(\rho_i^{c,n}, \rho_{i+1}^{c,n}) \right. \\ &\quad \left. + H_{i+1/2}^{c,n}(\rho_{i-1}^{c,n}, \rho_i^{c,n}) - H_{i+3/2}^{c,n}(\rho_i^{c,n}, \rho_{i+1}^{c,n}) \right] \\ &= \rho_{i+1}^{c,n} - \rho_i^{c,n} - \lambda_x \frac{H_{i+3/2}^{c,n}(\rho_{i+1}^{c,n}, \rho_{i+2}^{c,n}) - H_{i+3/2}^{c,n}(\rho_{i+1}^{c,n}, \rho_{i+1}^{c,n})}{\rho_{i+2}^{c,n} - \rho_{i+1}^{c,n}} (\rho_{i+2}^{c,n} - \rho_{i+1}^{c,n}) \\ &\quad - \lambda_x \frac{H_{i+3/2}^{c,n}(\rho_{i+1}^{c,n}, \rho_{i+1}^{c,n}) - H_{i+3/2}^{c,n}(\rho_i^{c,n}, \rho_{i+1}^{c,n})}{\rho_{i+1}^{c,n} - \rho_i^{c,n}} (\rho_{i+1}^{c,n} - \rho_i^{c,n}) \end{aligned}$$

$$\begin{aligned}
& + \lambda_x \frac{H_{i+1/2}^{c,n}(\rho_i^{c,n}, \rho_{i+1}^{c,n}) - H_{i+1/2}^{c,n}(\rho_i^{c,n}, \rho_i^{c,n})}{\rho_{i+1}^{c,n} - \rho_i^{c,n}} (\rho_{i+1}^{c,n} - \rho_i^{c,n}) \\
& + \lambda_x \frac{H_{i+1/2}^{c,n}(\rho_i^{c,n}, \rho_i^{c,n}) - H_{i+1/2}^{c,n}(\rho_{i-1}^{c,n}, \rho_i^{c,n})}{\rho_i^{c,n} - \rho_{i-1}^{c,n}} (\rho_i^{c,n} - \rho_{i-1}^{c,n}) \\
& = \delta_{i+1}^{c,n} (\rho_{i+2}^{c,n} - \rho_{i+1}^{c,n}) + \vartheta_i^{c,n} (\rho_i^{c,n} - \rho_{i-1}^{c,n}) + (1 - \delta_i^{c,n} - \vartheta_{i+1}^{c,n}) (\rho_{i+1}^{c,n} - \rho_i^{c,n}),
\end{aligned}$$

where

$$\delta_i^{c,n} = \begin{cases} -\lambda_x \frac{H_{i+1/2}^{c,n}(\rho_i^{c,n}, \rho_{i+1}^{c,n}) - H_{i+1/2}^{c,n}(\rho_i^{c,n}, \rho_i^{c,n})}{\rho_{i+1}^{c,n} - \rho_i^{c,n}} & \text{if } \rho_i^{c,n} \neq \rho_{i+1}^{c,n}, \\ 0 & \text{if } \rho_i^{c,n} = \rho_{i+1}^{c,n}, \end{cases} \quad (3.19)$$

$$\vartheta_i^{c,n} = \begin{cases} \lambda_x \frac{H_{i+1/2}^{c,n}(\rho_i^{c,n}, \rho_i^{c,n}) - H_{i+1/2}^{c,n}(\rho_{i-1}^{c,n}, \rho_i^{c,n})}{\rho_i^{c,n} - \rho_{i-1}^{c,n}} & \text{if } \rho_i^{c,n} \neq \rho_{i-1}^{c,n}, \\ 0 & \text{if } \rho_i^{c,n} = \rho_{i-1}^{c,n}. \end{cases} \quad (3.20)$$

Exploiting (3.18), observe that, whenever  $\rho_i^{c,n} \neq \rho_{i+1}^{c,n}$ ,

$$\begin{aligned}
\delta_i^{c,n} & = -\frac{\lambda_x}{\rho_{i+1}^{c,n} - \rho_i^{c,n}} \left[ V_1(x_{i+1/2}, \rho_i^{c,n}, \rho_{i+1}^{c,n}) + F^c(\rho_i^{c,n}, \rho_{i+1}^{c,n}, J_1^{c,n}(x_{i+1/2})) \right. \\
& \quad \left. - V_1(x_{i+1/2}, \rho_i^{c,n}, \rho_i^{c,n}) - F^c(\rho_i^{c,n}, \rho_i^{c,n}, J_1^{c,n}(x_{i+1/2})) \right] \\
& = -\frac{\lambda_x}{\rho_{i+1}^{c,n} - \rho_i^{c,n}} \left[ \min \{0, v_1^{stat}(x_{i+1/2})\} (\rho_{i+1}^{c,n} - \rho_i^{c,n}) + \min \{0, J_1^{c,n}(x_{i+1/2})\} (\rho_{i+1}^{c,n} - \rho_i^{c,n}) \right] \\
& = -\lambda_x \left( \min \{0, v_1^{stat}(x_{i+1/2})\} + \min \{0, J_1^{c,n}(x_{i+1/2})\} \right).
\end{aligned}$$

By (3.13) we get

$$\delta_i^{c,n} \in \left[0, \frac{1}{3}\right].$$

In a similar way one can prove that  $\vartheta_i^{c,n} \in [0, 1/3]$ . Thus,

$$\sum_{i,j \in \mathbb{Z}} |\mathcal{A}_{i,j}^{c,n}| \leq \sum_{i,j \in \mathbb{Z}} |\rho_{i+1,j}^{c,n} - \rho_{i,j}^{c,n}|. \quad (3.21)$$

We pass now to  $\mathcal{B}_{i,j}^{c,n}$ . Consider separately the terms involving  $V_1$  and those involving  $F^c$ . Observe that the maps

$$x \mapsto \min \{0, v_1^{stat}(x)\}, \quad x \mapsto \min \{0, J_1^{c,n}(x)\}$$

are Lipschitz continuous, with constant respectively  $\|\partial_x v_1^{stat}\|_{\mathbf{L}^\infty}$  and

$\varepsilon_c \left( 2 \|\nabla^2 \eta_c\|_{\mathbf{L}^\infty} + L_H \|\nabla \tilde{\eta}_c\|_{\mathbf{L}^\infty} \right) \|r_{\Omega,o}\|_{\mathbf{L}^1}$ . Exploiting (3.2) we get:

$$V_1(x_{i+3/2}, \rho_i^{c,n}, \rho_{i+1}^{c,n}) - V_1(x_{i+1/2}, \rho_{i-1}^{c,n}, \rho_i^{c,n}) + V_1(x_{i-1/2}, \rho_{i-1}^{c,n}, \rho_i^{c,n}) - V_1(x_{i+1/2}, \rho_i^{c,n}, \rho_{i+1}^{c,n})$$

$$\begin{aligned}
&= v_1^{stat}(x_{i+3/2})\rho_i^{c,n} + \min \left\{ 0, v_1^{stat}(x_{i+3/2}) \right\} (\rho_{i+1}^{c,n} - \rho_i^{c,n}) \\
&\quad - v_1^{stat}(x_{i+1/2})\rho_i^{c,n} - \min \left\{ 0, v_1^{stat}(x_{i+1/2}) \right\} (\rho_{i+1}^{c,n} - \rho_i^{c,n}) \\
&\quad + v_1^{stat}(x_{i-1/2})\rho_{i-1}^{c,n} + \min \left\{ 0, v_1^{stat}(x_{i-1/2}) \right\} (\rho_i^{c,n} - \rho_{i-1}^{c,n}) \\
&\quad - v_1^{stat}(x_{i+1/2})\rho_{i-1}^{c,n} - \min \left\{ 0, v_1^{stat}(x_{i+1/2}) \right\} (\rho_i^{c,n} - \rho_{i-1}^{c,n}) \\
&\quad \pm \left( v_1^{stat}(x_{i-1/2}) - v_1^{stat}(x_{i+1/2}) \right) \rho_i^{c,n} \\
&= \left( v_1^{stat}(x_{i+3/2}) - 2v_1^{stat}(x_{i+1/2}) + v_1^{stat}(x_{i-1/2}) \right) \rho_i^{c,n} \\
&\quad + \left( v_1^{stat}(x_{i-1/2}) - v_1^{stat}(x_{i+1/2}) \right) (\rho_{i-1}^{c,n} - \rho_i^{c,n}) \\
&\quad + \left( \min \left\{ 0, v_1^{stat}(x_{i+3/2}) \right\} - \min \left\{ 0, v_1^{stat}(x_{i+1/2}) \right\} \right) (\rho_{i+1}^{c,n} - \rho_i^{c,n}) \\
&\quad + \left( \min \left\{ 0, v_1^{stat}(x_{i-1/2}) \right\} - \min \left\{ 0, v_1^{stat}(x_{i+1/2}) \right\} \right) (\rho_i^{c,n} - \rho_{i-1}^{c,n}) \\
&\leq 2(\Delta x)^2 \left\| \partial_{xx}^2 v_1^{stat} \right\|_{\mathbf{L}^\infty} |\rho_i^{c,n}| + \Delta x \left\| \partial_x v_1^{stat} \right\|_{\mathbf{L}^\infty} \left( \left| \rho_{i+1}^{c,n} - \rho_i^{c,n} \right| + 2 \left| \rho_i^{c,n} - \rho_{i-1}^{c,n} \right| \right), \tag{3.22}
\end{aligned}$$

since

$$\begin{aligned}
v_1^{stat}(x_{i+3/2}) - 2v_1^{stat}(x_{i+1/2}) + v_1^{stat}(x_{i-1/2}) &= \Delta x \partial_x v_1^{stat}(\xi_{i+1}) - \Delta x \partial_x v_1^{stat}(\xi_i) \\
&= \Delta x (\xi_{i+1} - \xi_i) \partial_{xx}^2 v_1^{stat}(\zeta_{i+1/2}),
\end{aligned}$$

with  $\xi_i \in ]x_{i-1/2}, x_{i+1/2}[$  and  $\zeta_{i+1/2} \in ]\xi_i, \xi_{i+1}[$ . Similarly, exploiting (3.3) we obtain

$$\begin{aligned}
&F^c(\rho_i^{c,n}, \rho_{i+1}^{c,n}, J_1^{c,n}(x_{i+3/2})) - F^c(\rho_{i-1}^{c,n}, \rho_i^{c,n}, J_1^{c,n}(x_{i+1/2})) \\
&\quad + F^c(\rho_{i-1}^{c,n}, \rho_i^{c,n}, J_1^{c,n}(x_{i-1/2})) - F^c(\rho_i^{c,n}, \rho_{i+1}^{c,n}, J_1^{c,n}(x_{i+1/2})) \\
&= J_1^{c,n}(x_{i+3/2})\rho_i^{c,n} + \min \left\{ 0, J_1^{c,n}(x_{i+3/2}) \right\} (\rho_{i+1}^{c,n} - \rho_i^{c,n}) \\
&\quad - J_1^{c,n}(x_{i+1/2})\rho_i^{c,n} - \min \left\{ 0, J_1^{c,n}(x_{i+1/2}) \right\} (\rho_{i+1}^{c,n} - \rho_i^{c,n}) \\
&\quad + J_1^{c,n}(x_{i-1/2})\rho_{i-1}^{c,n} + \min \left\{ 0, J_1^{c,n}(x_{i-1/2}) \right\} (\rho_i^{c,n} - \rho_{i-1}^{c,n}) \\
&\quad - J_1^{c,n}(x_{i+1/2})\rho_{i-1}^{c,n} - \min \left\{ 0, J_1^{c,n}(x_{i+1/2}) \right\} (\rho_i^{c,n} - \rho_{i-1}^{c,n}) \\
&\quad \pm \left( J_1^{c,n}(x_{i-1/2}) - J_1^{c,n}(x_{i+1/2}) \right) \rho_i^{c,n} \\
&= \left( J_1^{c,n}(x_{i+3/2}) - 2J_1^{c,n}(x_{i+1/2}) + J_1^{c,n}(x_{i-1/2}) \right) \rho_i^{c,n} \\
&\quad + \left( J_1^{c,n}(x_{i-1/2}) - J_1^{c,n}(x_{i+1/2}) \right) (\rho_{i-1}^{c,n} - \rho_i^{c,n}) \\
&\quad + \left( \min \left\{ 0, J_1^{c,n}(x_{i+3/2}) \right\} - \min \left\{ 0, J_1^{c,n}(x_{i+1/2}) \right\} \right) (\rho_{i+1}^{c,n} - \rho_i^{c,n}) \\
&\quad + \left( \min \left\{ 0, J_1^{c,n}(x_{i-1/2}) \right\} - \min \left\{ 0, J_1^{c,n}(x_{i+1/2}) \right\} \right) (\rho_i^{c,n} - \rho_{i-1}^{c,n}) \\
&\leq 2\varepsilon_c \Delta x C_c |\rho_i^{c,n}| \tag{3.23} \\
&\quad + \varepsilon_c \Delta x \left( 2 \left\| \nabla^2 \eta_c \right\|_{\mathbf{L}^\infty} + L_H \left\| \nabla \tilde{\eta}_c \right\|_{L^\infty} \right) \|r_{\Omega, o}\|_{\mathbf{L}^1} \left( \left| \rho_{i+1}^{c,n} - \rho_i^{c,n} \right| + 2 \left| \rho_i^{c,n} - \rho_{i-1}^{c,n} \right| \right),
\end{aligned}$$

where we used (A.2) and (A.4), with the notation (A.6). Collecting together (3.22) and (3.23) we therefore obtain

$$\begin{aligned} \left| \mathcal{B}_{i,j}^{c,n} \right| &\leq 2 (\Delta x)^2 \left( \left\| \partial_{xx}^2 v_1^{stat} \right\|_{\mathbf{L}^\infty} + \varepsilon_c C_c \right) |\rho_i^{c,n}| + \Delta x \left( \left\| \partial_x v_1^{stat} \right\|_{\mathbf{L}^\infty} \right. \\ &\quad \left. + \varepsilon_c \left( 2 \left\| \nabla^2 \eta_c \right\|_{\mathbf{L}^\infty} + L_H \left\| \nabla \tilde{\eta}_c \right\|_{\mathbf{L}^\infty} \right) \left\| r_{\Omega,o} \right\|_{\mathbf{L}^1} \right) \left( \left| \rho_{i+1}^{c,n} - \rho_i^{c,n} \right| + 2 \left| \rho_i^{c,n} - \rho_{i-1}^{c,n} \right| \right), \end{aligned}$$

so that

$$\begin{aligned} \sum_{i,j \in \mathbb{Z}} \lambda_x \left| \mathcal{B}_{i,j}^{c,n} \right| &\leq 2 \Delta t \left( \left\| \partial_{xx}^2 v_1^{stat} \right\|_{\mathbf{L}^\infty} + \varepsilon_c C_c \right) \Delta x \sum_{i,j \in \mathbb{Z}} |\rho_i^{c,n}| \tag{3.24} \\ &\quad + 3 \Delta t \left( \left\| \partial_x v_1^{stat} \right\|_{\mathbf{L}^\infty} + \varepsilon_c \left( 2 \left\| \nabla^2 \eta_c \right\|_{\mathbf{L}^\infty} + L_H \left\| \nabla \tilde{\eta}_c \right\|_{\mathbf{L}^\infty} \right) \left\| r_{\Omega,o} \right\|_{\mathbf{L}^1} \right) \sum_{i,j \in \mathbb{Z}} \left| \rho_{i+1}^{c,n} - \rho_i^{c,n} \right|. \end{aligned}$$

Therefore, by (3.21) and (3.24), using also Lemma 3.3

$$\begin{aligned} &\sum_{i,j \in \mathbb{Z}} \Delta y \left| \rho_{i+1,j}^{c,n+1/2} - \rho_{i,j}^{c,n+1/2} \right| \\ &\leq \sum_{i,j \in \mathbb{Z}} \Delta y \left( \left| \mathcal{A}_{i,j}^{c,n} \right| + \lambda_x \left| \mathcal{B}_{i,j}^{c,n} \right| \right) \\ &\leq \left[ 1 + 3 \Delta t \left( \left\| \partial_x v_1^{stat} \right\|_{\mathbf{L}^\infty} + \varepsilon_c \left( 2 \left\| \nabla^2 \eta_c \right\|_{\mathbf{L}^\infty} + L_H \left\| \nabla \tilde{\eta}_c \right\|_{\mathbf{L}^\infty} \right) \left\| r_{\Omega,o} \right\|_{\mathbf{L}^1} \right) \right] \tag{3.25} \\ &\quad \cdot \sum_{i,j \in \mathbb{Z}} \Delta y \left| \rho_{i+1}^{c,n} - \rho_i^{c,n} \right| + 2 \Delta t \left[ \left\| \partial_{xx}^2 v_1^{stat} \right\|_{\mathbf{L}^\infty} + \varepsilon_c C_c \right] \left\| \rho_o^c \right\|_{\mathbf{L}^1}. \end{aligned}$$

Now pass to the term

$$\sum_{i,j \in \mathbb{Z}} \Delta x \left| \rho_{i,j+1}^{c,n+1/2} - \rho_{i,j}^{c,n+1/2} \right|.$$

Fix  $i, j \in \mathbb{Z}$  and exploit (3.5) again to get

$$\begin{aligned} &\rho_{i,j+1}^{c,n+1/2} - \rho_{i,j}^{c,n+1/2} \\ &= \rho_{i,j+1}^{c,n} - \rho_{i,j}^{c,n} - \lambda_x \left[ V_1(x_{i+1/2,j+1}, \rho_{i,j+1}^{c,n}, \rho_{i+1,j+1}^{c,n}) + F^c(\rho_{i,j+1}^{c,n}, \rho_{i+1,j+1}^{c,n}, J_1^{c,n}(x_{i+1/2,j+1})) \right. \\ &\quad - V_1(x_{i-1/2,j+1}, \rho_{i-1,j+1}^{c,n}, \rho_{i,j+1}^{c,n}) - F^c(\rho_{i-1,j+1}^{c,n}, \rho_{i,j+1}^{c,n}, J_1^{c,n}(x_{i-1/2,j+1})) \\ &\quad - V_1(x_{i+1/2,j}, \rho_{i,j}^{c,n}, \rho_{i+1,j}^{c,n}) - F^c(\rho_{i,j}^{c,n}, \rho_{i+1,j}^{c,n}, J_1^{c,n}(x_{i+1/2,j})) \\ &\quad \left. + V_1(x_{i-1/2,j}, \rho_{i-1,j}^{c,n}, \rho_{i,j}^{c,n}) + F^c(\rho_{i-1,j}^{c,n}, \rho_{i,j}^{c,n}, J_1^{c,n}(x_{i-1/2,j})) \right] \\ &\quad \pm \lambda_x \left[ V_1(x_{i+1/2,j+1}, \rho_{i,j}^{c,n}, \rho_{i+1,j}^{c,n}) + F^c(\rho_{i,j}^{c,n}, \rho_{i+1,j}^{c,n}, J_1^{c,n}(x_{i+1/2,j+1})) \right. \\ &\quad \left. - V_1(x_{i-1/2,j+1}, \rho_{i-1,j}^{c,n}, \rho_{i,j}^{c,n}) - F^c(\rho_{i-1,j}^{c,n}, \rho_{i,j}^{c,n}, J_1^{c,n}(x_{i-1/2,j+1})) \right] \\ &= \mathcal{D}_{i,j}^{c,n} + \lambda_x \mathcal{E}_{i,j}^{c,n}, \end{aligned}$$

where we set

$$\begin{aligned} \mathcal{D}_{i,j}^{c,n} &= \rho_{i,j+1}^{c,n} - \rho_{i,j}^{c,n} \\ &\quad - \lambda_x \left[ V_1(x_{i+1/2,j+1}, \rho_{i,j+1}^{c,n}, \rho_{i+1,j+1}^{c,n}) + F^c(\rho_{i,j+1}^{c,n}, \rho_{i+1,j+1}^{c,n}, J_1^{c,n}(x_{i+1/2,j+1})) \right. \\ &\quad - V_1(x_{i+1/2,j+1}, \rho_{i,j}^{c,n}, \rho_{i+1,j}^{c,n}) - F^c(\rho_{i,j}^{c,n}, \rho_{i+1,j}^{c,n}, J_1^{c,n}(x_{i+1/2,j+1})) \\ &\quad + V_1(x_{i-1/2,j+1}, \rho_{i-1,j}^{c,n}, \rho_{i,j}^{c,n}) + F^c(\rho_{i-1,j}^{c,n}, \rho_{i,j}^{c,n}, J_1^{c,n}(x_{i-1/2,j+1})) \\ &\quad \left. - V_1(x_{i-1/2,j+1}, \rho_{i-1,j+1}^{c,n}, \rho_{i,j+1}^{c,n}) - F^c(\rho_{i-1,j+1}^{c,n}, \rho_{i,j+1}^{c,n}, J_1^{c,n}(x_{i-1/2,j+1})) \right] \end{aligned}$$

and

$$\begin{aligned} \mathcal{E}_{i,j}^{c,n} &= V_1(x_{i+1/2,j}, \rho_{i,j}^{c,n}, \rho_{i+1,j}^{c,n}) + F^c(\rho_{i,j}^{c,n}, \rho_{i+1,j}^{c,n}, J_1^{c,n}(x_{i+1/2,j})) \\ &\quad - V_1(x_{i+1/2,j+1}, \rho_{i,j}^{c,n}, \rho_{i+1,j}^{c,n}) - F^c(\rho_{i,j}^{c,n}, \rho_{i+1,j}^{c,n}, J_1^{c,n}(x_{i+1/2,j+1})) \\ &\quad + V_1(x_{i-1/2,j+1}, \rho_{i-1,j}^{c,n}, \rho_{i,j}^{c,n}) + F^c(\rho_{i-1,j}^{c,n}, \rho_{i,j}^{c,n}, J_1^{c,n}(x_{i-1/2,j+1})) \\ &\quad - V_1(x_{i-1/2,j}, \rho_{i-1,j}^{c,n}, \rho_{i,j}^{c,n}) - F^c(\rho_{i-1,j}^{c,n}, \rho_{i,j}^{c,n}, J_1^{c,n}(x_{i-1/2,j})). \end{aligned}$$

Similarly as before, rearrange  $\mathcal{D}_{i,j}^{k,n}$ , exploiting the notation (3.18):

$$\begin{aligned} \mathcal{D}_{i,j}^{c,n} &= \rho_{i,j+1}^{c,n} - \rho_{i,j}^{c,n} - \lambda_x \left[ H_{i+1/2,j+1}^{c,n}(\rho_{i,j+1}^{c,n}, \rho_{i+1,j+1}^{c,n}) - H_{i+1/2,j+1}^{c,n}(\rho_{i,j}^{c,n}, \rho_{i+1,j}^{c,n}) \right. \\ &\quad \left. + H_{i-1/2,j+1}^{c,n}(\rho_{i-1,j}^{c,n}, \rho_{i,j}^{c,n}) - H_{i-1/2,j+1}^{c,n}(\rho_{i-1,j+1}^{c,n}, \rho_{i,j+1}^{c,n}) \right] \\ &\quad \pm \lambda_x H_{i+1/2,j+1}^{c,n}(\rho_{i,j}^{c,n}, \rho_{i+1,j+1}^{c,n}) \pm \lambda_x H_{i-1/2,j+1}^{c,n}(\rho_{i-1,j}^{c,n}, \rho_{i,j+1}^{c,n}) \\ &= \rho_{i,j+1}^{c,n} - \rho_{i,j}^{c,n} \\ &\quad - \lambda_x \frac{H_{i+1/2,j+1}^{c,n}(\rho_{i,j+1}^{c,n}, \rho_{i+1,j+1}^{c,n}) - H_{i+1/2,j+1}^{c,n}(\rho_{i,j}^{c,n}, \rho_{i+1,j+1}^{c,n})}{\rho_{i,j+1}^{c,n} - \rho_{i,j}^{c,n}} (\rho_{i,j+1}^{c,n} - \rho_{i,j}^{c,n}) \\ &\quad - \lambda_x \frac{H_{i+1/2,j+1}^{c,n}(\rho_{i,j}^{c,n}, \rho_{i+1,j+1}^{c,n}) - H_{i+1/2,j+1}^{c,n}(\rho_{i,j}^{c,n}, \rho_{i+1,j}^{c,n})}{\rho_{i+1,j+1}^{c,n} - \rho_{i+1,j}^{c,n}} (\rho_{i+1,j+1}^{c,n} - \rho_{i+1,j}^{c,n}) \\ &\quad + \lambda_x \frac{H_{i-1/2,j+1}^{c,n}(\rho_{i-1,j}^{c,n}, \rho_{i,j+1}^{c,n}) - H_{i-1/2,j+1}^{c,n}(\rho_{i-1,j}^{c,n}, \rho_{i,j}^{c,n})}{\rho_{i,j+1}^{c,n} - \rho_{i,j}^{c,n}} (\rho_{i,j+1}^{c,n} - \rho_{i,j}^{c,n}) \\ &\quad + \lambda_x \frac{H_{i-1/2,j+1}^{c,n}(\rho_{i-1,j+1}^{c,n}, \rho_{i,j+1}^{c,n}) - H_{i-1/2,j+1}^{c,n}(\rho_{i-1,j}^{c,n}, \rho_{i,j+1}^{c,n})}{\rho_{i-1,j+1}^{c,n} - \rho_{i-1,j}^{c,n}} (\rho_{i-1,j+1}^{c,n} - \rho_{i-1,j}^{c,n}) \\ &= (1 - \kappa_{i,j}^{c,n} - \nu_{i,j}^{c,n})(\rho_{i,j+1}^{c,n} - \rho_{i,j}^{c,n}) + \nu_{i+1,j}^{c,n}(\rho_{i+1,j+1}^{c,n} - \rho_{i+1,j}^{c,n}) + \kappa_{i-1,j}^{c,n}(\rho_{i-1,j+1}^{c,n} - \rho_{i-1,j}^{c,n}), \end{aligned}$$

where

$$\begin{aligned} \kappa_{i,j}^{c,n} &= \begin{cases} \lambda_x \frac{H_{i+1/2,j+1}^{c,n}(\rho_{i,j+1}^{c,n}, \rho_{i+1,j+1}^{c,n}) - H_{i+1/2,j+1}^{c,n}(\rho_{i,j}^{c,n}, \rho_{i+1,j+1}^{c,n})}{\rho_{i,j+1}^{c,n} - \rho_{i,j}^{c,n}} & \text{if } \rho_{i,j+1}^{c,n} \neq \rho_{i,j}^{c,n}, \\ 0 & \text{if } \rho_{i,j+1}^{c,n} = \rho_{i,j}^{c,n}, \end{cases} \\ \nu_{i,j}^{c,n} &= \begin{cases} -\lambda_x \frac{H_{i-1/2,j+1}^{c,n}(\rho_{i-1,j}^{c,n}, \rho_{i,j+1}^{c,n}) - H_{i-1/2,j+1}^{c,n}(\rho_{i-1,j}^{c,n}, \rho_{i,j}^{c,n})}{\rho_{i,j+1}^{c,n} - \rho_{i,j}^{c,n}} & \text{if } \rho_{i,j+1}^{c,n} \neq \rho_{i,j}^{c,n}, \\ 0 & \text{if } \rho_{i,j+1}^{c,n} = \rho_{i,j}^{c,n}. \end{cases} \end{aligned}$$

As for  $\delta_i^{c,n}$  (3.19) and  $\vartheta_i^{c,n}$  (3.20), it is immediate to prove that  $\kappa_{i,j}^{c,n}, \nu_{i,j}^{c,n} \in \left[0, \frac{1}{3}\right]$  for all  $i, j \in \mathbb{Z}$ . Hence,

$$\sum_{i,j \in \mathbb{Z}} |\mathcal{D}_{i,j}^{c,n}| \leq \sum_{i,j \in \mathbb{Z}} \left| \rho_{i,j+1}^{c,n} - \rho_{i,j}^{c,n} \right|. \quad (3.26)$$

Pass now to  $\mathcal{E}_{i,j}^{c,n}$ : we can proceed analogously to  $\mathcal{B}_{i,j}^{c,n}$ , treating separately the terms involving  $V_1$  and those involving  $F^c$ . First, by (3.2),

$$\begin{aligned} & V_1(x_{i+1/2,j}, \rho_{i,j}^{c,n}, \rho_{i+1,j}^{c,n}) - V_1(x_{i+1/2,j+1}, \rho_{i,j}^{c,n}, \rho_{i+1,j}^{c,n}) \\ & + V_1(x_{i-1/2,j+1}, \rho_{i-1,j}^{c,n}, \rho_{i,j}^{c,n}) - V_1(x_{i-1/2,j}, \rho_{i-1,j}^{c,n}, \rho_{i,j}^{c,n}) \\ = & v_1^{\text{stat}}(x_{i+1/2,j}) \rho_{i,j}^{c,n} + \min \left\{ 0, v_1^{\text{stat}}(x_{i+1/2,j}) \right\} (\rho_{i+1,j}^{c,n} - \rho_{i,j}^{c,n}) \\ & - v_1^{\text{stat}}(x_{i+1/2,j+1}) \rho_{i,j}^{c,n} - \min \left\{ 0, v_1^{\text{stat}}(x_{i+1/2,j+1}) \right\} (\rho_{i+1,j}^{c,n} - \rho_{i,j}^{c,n}) \\ & + v_1^{\text{stat}}(x_{i-1/2,j+1}) \rho_{i-1,j}^{c,n} + \min \left\{ 0, v_1^{\text{stat}}(x_{i-1/2,j+1}) \right\} (\rho_{i,j}^{c,n} - \rho_{i-1,j}^{c,n}) \\ & - v_1^{\text{stat}}(x_{i-1/2,j}) \rho_{i-1,j}^{c,n} - \min \left\{ 0, v_1^{\text{stat}}(x_{i-1/2,j}) \right\} (\rho_{i,j}^{c,n} - \rho_{i-1,j}^{c,n}) \\ & \pm \left( v_1^{\text{stat}}(x_{i-1/2,j+1}) - v_1^{\text{stat}}(x_{i-1/2,j}) \right) \rho_{i,j}^{c,n} \\ = & \left( v_1^{\text{stat}}(x_{i+1/2,j}) - v_1^{\text{stat}}(x_{i+1/2,j+1}) - v_1^{\text{stat}}(x_{i-1/2,j}) + v_1^{\text{stat}}(x_{i-1/2,j+1}) \right) \rho_{i,j}^{c,n} \\ & + \left( v_1^{\text{stat}}(x_{i-1/2,j+1}) - v_1^{\text{stat}}(x_{i-1/2,j}) \right) (\rho_{i-1,j}^{c,n} - \rho_{i,j}^{c,n}) \\ & + \left( \min \left\{ 0, v_1^{\text{stat}}(x_{i+1/2,j}) \right\} - \min \left\{ 0, v_1^{\text{stat}}(x_{i+1/2,j+1}) \right\} \right) (\rho_{i+1,j}^{c,n} - \rho_{i,j}^{c,n}) \\ & + \left( \min \left\{ 0, v_1^{\text{stat}}(x_{i-1/2,j+1}) \right\} - \min \left\{ 0, v_1^{\text{stat}}(x_{i-1/2,j}) \right\} \right) (\rho_{i,j}^{c,n} - \rho_{i-1,j}^{c,n}) \\ \leq & \Delta x \Delta y \left\| \partial_{xy}^2 v_1^{\text{stat}} \right\|_{\mathbf{L}^\infty} \left| \rho_{i,j}^{c,n} \right| + \Delta y \left\| \partial_y v_1^{\text{stat}} \right\|_{\mathbf{L}^\infty} \left( \left| \rho_{i+1,j}^{c,n} - \rho_{i,j}^{c,n} \right| + 2 \left| \rho_{i,j}^{c,n} - \rho_{i-1,j}^{c,n} \right| \right), \quad (3.27) \end{aligned}$$

since

$$\begin{aligned} & v_1^{\text{stat}}(x_{i+1/2,j}) - v_1^{\text{stat}}(x_{i+1/2,j+1}) - v_1^{\text{stat}}(x_{i-1/2,j}) + v_1^{\text{stat}}(x_{i-1/2,j+1}) \\ = & \Delta x \partial_x v_1^{\text{stat}}(\xi_i, y_j) - \Delta x \partial_x v_1^{\text{stat}}(\xi_i, y_{j+1}) \\ = & - \Delta x \Delta y \partial_{xy}^2 v_1^{\text{stat}}(\xi_i, \zeta_{j+1/2}), \end{aligned}$$

with  $\xi_i \in ]x_{i-1/2}, x_{i+1/2}[$  and  $\zeta_{j+1/2} \in ]y_j, y_{j+1}[$ . In a similar way, by (3.3),

$$\begin{aligned} & F^c(\rho_{i,j}^{c,n}, \rho_{i+1,j}^{c,n}, J_1^{c,n}(x_{i+1/2,j})) - F^c(\rho_{i,j}^{c,n}, \rho_{i+1,j}^{c,n}, J_1^{c,n}(x_{i+1/2,j+1})) \\ & + F^c(\rho_{i-1,j}^{c,n}, \rho_{i,j}^{c,n}, J_1^{c,n}(x_{i-1/2,j+1})) - F^c(\rho_{i-1,j}^{c,n}, \rho_{i,j}^{c,n}, J_1^{c,n}(x_{i-1/2,j})) \\ = & J_1^{c,n}(x_{i+1/2,j}) \rho_{i,j}^{c,n} + \min \left\{ 0, J_1^{c,n}(x_{i+1/2,j}) \right\} (\rho_{i+1,j}^{c,n} - \rho_{i,j}^{c,n}) \\ & - J_1^{c,n}(x_{i+1/2,j+1}) \rho_{i,j}^{c,n} - \min \left\{ 0, J_1^{c,n}(x_{i+1/2,j+1}) \right\} (\rho_{i+1,j}^{c,n} - \rho_{i,j}^{c,n}) \end{aligned}$$

$$\begin{aligned}
& + J_1^{c,n}(x_{i-1/2,j+1}) \rho_{i-1,j}^{c,n} + \min \left\{ 0, J_1^{c,n}(x_{i-1/2,j+1}) \right\} \left( \rho_{i,j}^{c,n} - \rho_{i-1,j}^{c,n} \right) \\
& - J_1^{c,n}(x_{i-1/2,j}) \rho_{i-1,j}^{c,n} - \min \left\{ 0, J_1^{c,n}(x_{i-1/2,j}) \right\} \left( \rho_{i,j}^{c,n} - \rho_{i-1,j}^{c,n} \right) \\
& \pm \left( J_1^{c,n}(x_{i-1/2,j+1}) - J_1^{c,n}(x_{i-1/2,j}) \right) \rho_{i,j}^{c,n} \\
& = \left( J_1^{c,n}(x_{i+1/2,j}) - J_1^{c,n}(x_{i+1/2,j+1}) - J_1^{c,n}(x_{i-1/2,j}) + J_1^{c,n}(x_{i-1/2,j+1}) \right) \rho_{i,j}^{c,n} \\
& + \left( J_1^{c,n}(x_{i-1/2,j+1}) - J_1^{c,n}(x_{i-1/2,j}) \right) \left( \rho_{i-1,j}^{c,n} - \rho_{i,j}^{c,n} \right) \\
& + \left( \min \left\{ 0, J_1^{c,n}(x_{i+1/2,j}) \right\} - \min \left\{ 0, J_1^{c,n}(x_{i+1/2,j+1}) \right\} \right) \left( \rho_{i+1,j}^{c,n} - \rho_{i,j}^{c,n} \right) \\
& + \left( \min \left\{ 0, J_1^{c,n}(x_{i-1/2,j+1}) \right\} - \min \left\{ 0, J_1^{c,n}(x_{i-1/2,j}) \right\} \right) \left( \rho_{i,j}^{c,n} - \rho_{i-1,j}^{c,n} \right) \\
& \leq 2 \varepsilon_c \Delta x \Delta y C_c \left| \rho_{i,j}^{c,n} \right| \tag{3.28} \\
& + \varepsilon_c \Delta y \left( 2 \left\| \nabla^2 \eta_c \right\|_{\mathbf{L}^\infty} + L_H \left\| \nabla \tilde{\eta}_c \right\|_{\mathbf{L}^\infty} \right) \left\| r_{\Omega,o} \right\|_{\mathbf{L}^1} \left( \left| \rho_{i+1,j}^{c,n} - \rho_{i,j}^{c,n} \right| + 2 \left| \rho_{i,j}^{c,n} + \rho_{i-1,j}^{c,n} \right| \right),
\end{aligned}$$

where we used (A.3) and (A.5), with the notation (A.6). Therefore, collecting together (3.27) and (3.28), we get

$$\begin{aligned}
\left| \mathcal{E}_{i,j}^{c,n} \right| & \leq \Delta x \Delta y \left( \left\| \partial_{xy}^2 v_1^{stat} \right\|_{\mathbf{L}^\infty} + 2 \varepsilon_c C_c \right) \left| \rho_{i,j}^{c,n} \right| \\
& + \Delta y \left( \left\| \partial_y v_1^{stat} \right\|_{\mathbf{L}^\infty} + \varepsilon_c \left( 2 \left\| \nabla^2 \eta_c \right\|_{\mathbf{L}^\infty} + L_H \left\| \nabla \tilde{\eta}_c \right\|_{\mathbf{L}^\infty} \right) \left\| r_{\Omega,o} \right\|_{\mathbf{L}^1} \right) \\
& \left( \left| \rho_{i+1,j}^{c,n} - \rho_{i,j}^{c,n} \right| + 2 \left| \rho_{i,j}^{c,n} + \rho_{i-1,j}^{c,n} \right| \right),
\end{aligned}$$

so that

$$\begin{aligned}
\sum_{i,j \in \mathbb{Z}} \lambda_x \left| \mathcal{E}_{i,j}^{c,n} \right| & \leq 3 \lambda_x \Delta y \left( \left\| \partial_y v_1^{stat} \right\|_{\mathbf{L}^\infty} + \varepsilon_c \left( 2 \left\| \nabla^2 \eta_c \right\|_{\mathbf{L}^\infty} + L_H \left\| \nabla \tilde{\eta}_c \right\|_{\mathbf{L}^\infty} \right) \left\| r_{\Omega,o} \right\|_{\mathbf{L}^1} \right) \\
& \cdot \sum_{i,j \in \mathbb{Z}} \left| \rho_{i+1,j}^{c,n} - \rho_{i,j}^{c,n} \right| + \Delta t \left( \left\| \partial_{xy}^2 v_1^{stat} \right\|_{\mathbf{L}^\infty} + 2 \varepsilon_c C_c \right) \Delta y \sum_{i,j \in \mathbb{Z}} \rho_{i,j}^{c,n}. \tag{3.29}
\end{aligned}$$

Hence, by (3.26) and (3.29), using also Lemma 3.3, we obtain

$$\begin{aligned}
& \sum_{i,j \in \mathbb{Z}} \Delta x \left| \rho_{i,j+1}^{c,n+1/2} - \rho_{i,j}^{c,n+1/2} \right| \\
& \leq \sum_{i,j \in \mathbb{Z}} \Delta x \left( \left| \mathcal{D}_{i,j}^{c,n} \right| + \lambda_x \left| \mathcal{E}_{i,j}^{c,n} \right| \right) \\
& \leq \sum_{i,j \in \mathbb{Z}} \Delta x \left| \rho_{i,j+1}^{c,n} - \rho_{i,j}^{c,n} \right| + \Delta t \left( \left\| \partial_{xy}^2 v_1^{stat} \right\|_{\mathbf{L}^\infty} + 2 \varepsilon_c C_c \right) \left\| \rho_o^c \right\|_{\mathbf{L}^1}. \tag{3.30} \\
& + 3 \Delta t \left( \left\| \partial_y v_1^{stat} \right\|_{\mathbf{L}^\infty} + \varepsilon_c \left( 2 \left\| \nabla^2 \eta_c \right\|_{\mathbf{L}^\infty} + L_H \left\| \nabla \tilde{\eta}_c \right\|_{\mathbf{L}^\infty} \right) \left\| r_{\Omega,o} \right\|_{\mathbf{L}^1} \right) \sum_{i,j \in \mathbb{Z}} \Delta y \left| \rho_{i+1,j}^{c,n} - \rho_{i,j}^{c,n} \right|
\end{aligned}$$

Setting

$$K_1^c = 3 \left( \left\| \partial_x v_1^{stat} \right\|_{\mathbf{L}^\infty} + \left\| \partial_y v_1^{stat} \right\|_{\mathbf{L}^\infty} + \varepsilon_c \left( 4 \left\| \nabla^2 \eta_c \right\|_{\mathbf{L}^\infty} + 2L_H \left\| \nabla \tilde{\eta}_c \right\|_{\mathbf{L}^\infty} \right) \left\| r_{\Omega, o} \right\|_{\mathbf{L}^1} \right), \quad (3.31)$$

$$K_2^c = \left( 4\varepsilon_c C_c + 2 \left\| \partial_{xx}^2 v_1^{stat} \right\|_{\mathbf{L}^\infty} + \left\| \partial_{xy}^2 v_1^{stat} \right\|_{\mathbf{L}^\infty} \right) \left\| \rho_o^c \right\|_{\mathbf{L}^1}, \quad (3.32)$$

by (3.25) and (3.30) we conclude

$$\begin{aligned} & \sum_{i,j \in \mathbb{Z}} \left( \Delta y \left| \rho_{i+1,j}^{c,n+1/2} - \rho_{i,j}^{c,n+1/2} \right| + \Delta x \left| \rho_{i,j+1}^{c,n+1/2} - \rho_{i,j}^{c,n+1/2} \right| \right) \\ & \leq (1 + \Delta t K_1^c) \sum_{i,j \in \mathbb{Z}} \left( \Delta x \left| \rho_{i,j+1}^{c,n} - \rho_{i,j}^{c,n} \right| + \Delta y \left| \rho_{i+1,j}^{c,n} - \rho_{i,j}^{c,n} \right| \right) + \Delta t K_2^c. \end{aligned}$$

Analogous computations yield

$$\begin{aligned} & \sum_{i,j \in \mathbb{Z}} \left( \Delta y \left| \rho_{i+1,j}^{c,n+1} - \rho_{i,j}^{c,n+1} \right| + \Delta x \left| \rho_{i,j+1}^{c,n+1} - \rho_{i,j}^{c,n+1} \right| \right) \\ & \leq (1 + \Delta t K_3^c) \sum_{i,j \in \mathbb{Z}} \left( \Delta x \left| \rho_{i,j+1}^{c,n+1/2} - \rho_{i,j}^{c,n+1/2} \right| + \Delta y \left| \rho_{i+1,j}^{c,n+1/2} - \rho_{i,j}^{c,n+1/2} \right| \right) + \Delta t K_4^c, \end{aligned}$$

where

$$K_3^c = 3 \left( \left\| \partial_x v_2^{stat} \right\|_{\mathbf{L}^\infty} + \left\| \partial_y v_2^{stat} \right\|_{\mathbf{L}^\infty} + \varepsilon_c \left( 4 \left\| \nabla^2 \eta_c \right\|_{\mathbf{L}^\infty} + 2L_H \left\| \nabla \tilde{\eta}_c \right\|_{\mathbf{L}^\infty} \right) \left\| r_{\Omega, o} \right\|_{\mathbf{L}^1} \right), \quad (3.33)$$

$$K_4^c = \left( 4\varepsilon_c C_c + 2 \left\| \partial_{yy}^2 v_2^{stat} \right\|_{\mathbf{L}^\infty} + \left\| \partial_{xy}^2 v_2^{stat} \right\|_{\mathbf{L}^\infty} \right) \left\| \rho_o^c \right\|_{\mathbf{L}^1}. \quad (3.34)$$

Observe that, using the notation (3.16) and (3.17),

$$K_1^c, K_3^c \leq \mathcal{K}_1^c, \quad K_2^c, K_4^c \leq \mathcal{K}_2^c.$$

A recursive argument yields the desired result:

$$\begin{aligned} & \sum_{i,j \in \mathbb{Z}} \left( \Delta y \left| \rho_{i+1,j}^{c,n} - \rho_{i,j}^{c,n} \right| + \Delta x \left| \rho_{i,j+1}^{c,n} - \rho_{i,j}^{c,n} \right| \right) \\ & \leq e^{2n \Delta t \mathcal{K}_1^c} \sum_{i,j \in \mathbb{Z}} \left( \Delta x \left| \rho_{i,j+1}^{c,0} - \rho_{i,j}^{c,0} \right| + \Delta y \left| \rho_{i+1,j}^{c,0} - \rho_{i,j}^{c,0} \right| \right) + \frac{2\mathcal{K}_2^c}{\mathcal{K}_1^c} \left( e^{2n \Delta t \mathcal{K}_1^c} - 1 \right). \end{aligned}$$

□

**Corollary 3.7. (BV estimate in space and time)** *Let  $\rho_o \in (\mathbf{L}^\infty \cap \mathbf{BV})(\Omega; \mathbb{R}_+^N)$ . Let  $(\mathbf{v})$ ,  $(\mathbf{H})$ ,  $(\boldsymbol{\eta})$ , (3.13) hold. Then, for all  $c \in \{1, \dots, N\}$  and  $t > 0$ ,  $\rho_\Delta^c$  constructed through Algorithm 3.1 satisfies the following estimate: for all  $n = 1, \dots, N_T$ ,*

$$\sum_{m=0}^{n-1} \sum_{i,j \in \mathbb{Z}} \Delta t \left( \Delta y \left| \rho_{i+1,j}^{c,m} - \rho_{i,j}^{c,m} \right| + \Delta x \left| \rho_{i,j+1}^{c,m} - \rho_{i,j}^{c,m} \right| \right) + \sum_{m=0}^{n-1} \sum_{i,j \in \mathbb{Z}} \Delta x \Delta y \left| \rho_{i,j}^{c,m+1} - \rho_{i,j}^{c,m} \right| \leq \mathcal{C}_{xt}^c(t^n) \quad (3.35)$$

where

$$\mathcal{C}_{xt}^c(t) = t (\mathcal{C}_x^c(t) + 2\mathcal{C}_t^c(t)), \quad (3.36)$$

with  $\mathcal{C}_x^c$  as in (3.15) and  $\mathcal{C}_t^c$  as in (3.38).

**Proof.** By Proposition 3.5 we have

$$\sum_{m=0}^{n-1} \sum_{i,j \in \mathbb{Z}} \Delta t \left( \Delta y \left| \rho_{i+1,j}^{c,m} - \rho_{i,j}^{c,m} \right| + \Delta x \left| \rho_{i,j+1}^{c,m} - \rho_{i,j}^{c,m} \right| \right) \leq n \Delta t \mathcal{C}_x^c(n \Delta t). \quad (3.37)$$

Since

$$\left| \rho_{i,j}^{c,m+1} - \rho_{i,j}^{c,m} \right| \leq \left| \rho_{i,j}^{c,m+1} - \rho_{i,j}^{c,m+1/2} \right| + \left| \rho_{i,j}^{c,m+1/2} - \rho_{i,j}^{c,m} \right|,$$

we focus first on

$$\sum_{i,j \in \mathbb{Z}} \Delta x \Delta y \left| \rho_{i,j}^{c,m+1/2} - \rho_{i,j}^{c,m} \right|.$$

By the scheme (3.5), we have, using the notation (3.9),

$$\begin{aligned} \rho_{i,j}^{c,m+1/2} - \rho_{i,j}^{c,m} &= -\lambda_x \left[ V_1(x_{i+1/2,j}, \rho_{i,j}^{c,m}, \rho_{i+1,j}^{c,m}) + F^c(\rho_{i,j}^{c,m}, \rho_{i+1,j}^{c,m}, J_1^{c,m}(x_{i+1/2,j})) \right. \\ &\quad \left. - V_1(x_{i-1/2,j}, \rho_{i-1,j}^{c,m}, \rho_{i,j}^{c,m}) - F^c(\rho_{i-1,j}^{c,m}, \rho_{i,j}^{c,m}, J_1^{c,m}(x_{i-1/2,j})) \right] \\ &= -\lambda_x \left[ v_{i+1/2,j} \rho_{i,j}^{c,m} + \min \left\{ 0, v_{i+1/2,j} \right\} (\rho_{i+1,j}^{c,m} - \rho_{i,j}^{c,m}) \right. \\ &\quad \left. - v_{i-1/2,j} \rho_{i-1,j}^{c,m} - \min \left\{ 0, v_{i-1/2,j} \right\} (\rho_{i,j}^{c,m} - \rho_{i-1,j}^{c,m}) \right. \\ &\quad \left. + J_1^{c,m}(x_{i+1/2,j}) \rho_{i,j}^{c,m} + \min \left\{ 0, J_1^{c,m}(x_{i+1/2,j}) \right\} (\rho_{i+1,j}^{c,m} - \rho_{i,j}^{c,m}) \right. \\ &\quad \left. - J_1^{c,m}(x_{i-1/2,j}) \rho_{i-1,j}^{c,m} - \min \left\{ 0, J_1^{c,m}(x_{i-1/2,j}) \right\} (\rho_{i,j}^{c,m} - \rho_{i-1,j}^{c,m}) \right. \\ &\quad \left. \pm v_{i-1/2,j} \rho_{i,j}^{c,m} \pm J_1^{c,m}(x_{i-1/2,j}) \rho_{i,j}^{c,m} \right] \\ &= -\lambda_x \left[ \Delta x \partial_x v_1^{stat}(\xi_i, y_j) \rho_{i,j}^{c,m} \right. \\ &\quad \left. + \left( v_{i-1/2,j} - \min \left\{ 0, v_{i-1/2,j} \right\} \right) (\rho_{i,j}^{c,m} - \rho_{i-1,j}^{c,m}) \right. \\ &\quad \left. + \min \left\{ 0, v_{i+1/2,j} \right\} (\rho_{i+1,j}^{c,m} - \rho_{i,j}^{c,m}) \right. \\ &\quad \left. + \left( J_1^{c,m}(x_{i+1/2,j}) - J_1^{c,m}(x_{i-1/2,j}) \right) \rho_{i,j}^{c,m} \right. \\ &\quad \left. + \left( J_1^{c,m}(x_{i-1/2,j}) - \min \left\{ 0, J_1^{c,m}(x_{i-1/2,j}) \right\} \right) (\rho_{i,j}^{c,m} - \rho_{i-1,j}^{c,m}) \right] \end{aligned}$$

$$\begin{aligned}
& + \min \left\{ 0, J_1^{c,m}(x_{i+1/2,j}) \right\} (\rho_{i+1,j}^{c,m} - \rho_{i,j}^{c,m}) \Big] \\
& \leq \Delta t \left( \left\| \partial_x v_1^{stat} \right\|_{\mathbf{L}^\infty} + \varepsilon_c \left( 2 \left\| \nabla^2 \eta_c \right\|_{\mathbf{L}^\infty} + L_H \left\| \nabla \tilde{\eta}_c \right\|_{\mathbf{L}^\infty} \right) \left\| r_{\Omega,o} \right\|_{\mathbf{L}^1} \right) \rho_{i,j}^{c,m} \\
& \quad + \lambda_x \left( \left\| v_1^{stat} \right\|_{\mathbf{L}^\infty} + \varepsilon_c \right) \left( \left| \rho_{i,j}^{c,m} - \rho_{i-1,j}^{c,m} \right| + \left| \rho_{i+1,j}^{c,m} - \rho_{i,j}^{c,m} \right| \right),
\end{aligned}$$

where  $\xi_i \in ]x_{i-1/2}, x_{i+1/2}[$  and we used (A.1) and (A.2). Therefore,

$$\begin{aligned}
& \sum_{i,j \in \mathbb{Z}} \Delta x \Delta y \left| \rho_{i,j}^{c,m+1/2} - \rho_{i,j}^{c,m} \right| \\
& \leq \Delta t \left( \left\| \partial_x v_1^{stat} \right\|_{\mathbf{L}^\infty} + \varepsilon_c \left( 2 \left\| \nabla^2 \eta_c \right\|_{\mathbf{L}^\infty} + L_H \left\| \nabla \tilde{\eta}_c \right\|_{\mathbf{L}^\infty} \right) \left\| r_{\Omega,o} \right\|_{\mathbf{L}^1} \right) \left\| \rho_o^c \right\|_{\mathbf{L}^1} \\
& \quad + 2 \Delta t \left( \left\| v_1^{stat} \right\|_{\mathbf{L}^\infty} + \varepsilon_c \right) \sum_{i,j \in \mathbb{Z}} \Delta y \left| \rho_{i+1,j}^{c,m} - \rho_{i,j}^{c,m} \right| \\
& \leq \Delta t \left( \left\| \partial_x v_1^{stat} \right\|_{\mathbf{L}^\infty} + \varepsilon_c \left( 2 \left\| \nabla^2 \eta_c \right\|_{\mathbf{L}^\infty} + L_H \left\| \nabla \tilde{\eta}_c \right\|_{\mathbf{L}^\infty} \right) \left\| r_{\Omega,o} \right\|_{\mathbf{L}^1} \right) \left\| \rho_o^c \right\|_{\mathbf{L}^1} \\
& \quad + 2 \Delta t \left( \left\| v_1^{stat} \right\|_{\mathbf{L}^\infty} + \varepsilon_c \right) \mathcal{C}_x^c(m \Delta t) \\
& \leq \Delta t \mathcal{C}_t^c(m \Delta t),
\end{aligned}$$

where we set

$$\begin{aligned}
\mathcal{C}_t^c(s) &= 2 \left( \left\| \mathbf{v}^{stat} \right\|_{\mathbf{L}^\infty} + \varepsilon_c \right) \mathcal{C}_x^c(s) \\
& \quad + \left( \left\| \nabla \mathbf{v}^{stat} \right\|_{\mathbf{L}^\infty} + \varepsilon_c \left( 2 \left\| \nabla^2 \eta_c \right\|_{\mathbf{L}^\infty} + L_H \left\| \nabla \tilde{\eta}_c \right\|_{\mathbf{L}^\infty} \right) \left\| r_{\Omega,o} \right\|_{\mathbf{L}^1} \right) \left\| \rho_o^c \right\|_{\mathbf{L}^1}.
\end{aligned} \tag{3.38}$$

Analogously, we get

$$\begin{aligned}
& \sum_{i,j \in \mathbb{Z}} \Delta x \Delta y \left| \rho_{i,j}^{c,m+1} - \rho_{i,j}^{c,m+1/2} \right| \\
& \leq \Delta t \left( \left\| \partial_y v_2^{stat} \right\|_{\mathbf{L}^\infty} + \varepsilon_c \left( 2 \left\| \nabla^2 \eta_c \right\|_{\mathbf{L}^\infty} + L_H \left\| \nabla \tilde{\eta}_c \right\|_{\mathbf{L}^\infty} \right) \left\| r_{\Omega,o} \right\|_{\mathbf{L}^1} \right) \left\| \rho_o^c \right\|_{\mathbf{L}^1} \\
& \quad + 2 \Delta t \left( \left\| v_2^{stat} \right\|_{\mathbf{L}^\infty} + \varepsilon_c \right) \sum_{i,j \in \mathbb{Z}} \Delta x \left| \rho_{i,j+1}^{c,m+1/2} - \rho_{i,j}^{c,m+1/2} \right| \\
& \leq \Delta t \left( \left\| \partial_y v_2^{stat} \right\|_{\mathbf{L}^\infty} + \varepsilon_c \left( 2 \left\| \nabla^2 \eta_c \right\|_{\mathbf{L}^\infty} + L_H \left\| \nabla \tilde{\eta}_c \right\|_{\mathbf{L}^\infty} \right) \left\| r_{\Omega,o} \right\|_{\mathbf{L}^1} \right) \left\| \rho_o^c \right\|_{\mathbf{L}^1} \\
& \quad + 2 \Delta t \left( \left\| v_2^{stat} \right\|_{\mathbf{L}^\infty} + \varepsilon_c \right) \mathcal{C}_x^c(m \Delta t) \\
& \leq \Delta t \mathcal{C}_t^c(m \Delta t).
\end{aligned}$$

Hence

$$\sum_{m=0}^{n-1} \sum_{i,j \in \mathbb{Z}} \Delta x \Delta y \left| \rho_{i,j}^{c,m+1} - \rho_{i,j}^{c,m} \right| \leq 2n \Delta t \mathcal{C}_t^c(n \Delta t), \quad (3.39)$$

which, together with (3.37), completes the proof.  $\square$

### 3.5 Discrete entropy inequalities

Following [1], see also [6, 7], introduce the following notation: for  $i, j \in \mathbb{Z}$ ,  $n = 0, \dots, N_T - 1$  and  $\kappa \in \mathbb{R}$ ,

$$\begin{aligned} \Phi_{i+1/2,j}^{c,n}(u, v) &= V_1(x_{i+1/2,j}, u \vee \kappa, v \vee \kappa) + F^c(u \vee \kappa, v \vee \kappa, J_1^{c,n}(x_{i+1/2,j})) \\ &\quad - V_1(x_{i+1/2,j}, u \wedge \kappa, v \wedge \kappa) - F^c(u \wedge \kappa, v \wedge \kappa, J_1^{c,n}(x_{i+1/2,j})), \\ \Gamma_{i,j+1/2}^{c,n}(u, v) &= V_2(x_{i,j+1/2}, u \vee \kappa, v \vee \kappa) + F^c(u \vee \kappa, v \vee \kappa, J_2^{c,n}(x_{i,j+1/2})) \\ &\quad - V_2(x_{i,j+1/2}, u \wedge \kappa, v \wedge \kappa) - F^c(u \wedge \kappa, v \wedge \kappa, J_2^{c,n}(x_{i,j+1/2})), \end{aligned}$$

with  $V_1$ ,  $V_2$  and  $F$  defined as in (3.2), (3.4) and (3.3) respectively. We can then apply the following Lemma to conclude the convergence towards a weak entropy solution

**Lemma 3.8. (Discrete entropy condition)** Fix  $\rho_o \in (\mathbf{L}^\infty \cap \mathbf{BV})(\Omega; \mathbb{R}^+)$ . Let  $(\mathbf{v})$ ,  $(\mathbf{H})$ ,  $(\boldsymbol{\eta})$ , (3.13) hold. Then, the solution  $\rho_\Delta^c$  constructed through Algorithm 3.1 satisfies the following discrete entropy inequality: for  $i, j \in \mathbb{Z}$ , for  $n = 0, \dots, N_T - 1$  and  $\kappa \in \mathbb{R}$ ,

$$\begin{aligned} &\left| \rho_{i,j}^{c,n+1} - \kappa \right| - \left| \rho_{i,j}^{c,n} - \kappa \right| + \lambda_x \left( \Phi_{i+1/2,j}^{c,n}(\rho_{i,j}^{c,n}, \rho_{i+1,j}^{c,n}) - \Phi_{i-1/2,j}^{c,n}(\rho_{i-1,j}^{c,n}, \rho_{i,j}^{c,n}) \right) \\ &\quad + \lambda_x \operatorname{sgn}(\rho_{i,j}^{c,n+1/2} - \kappa) \left( v_1^{\text{stat}}(x_{i+1/2,j}) - v_1^{\text{stat}}(x_{i-1/2,j}) \right) \kappa \\ &\quad + \lambda_x \operatorname{sgn}(\rho_{i,j}^{c,n+1/2} - \kappa) \left( J_1^{c,n}(x_{i+1/2,j}) - J_1^{c,n}(x_{i-1/2,j}) \right) \kappa \\ &+ \lambda_y \left( \Gamma_{i,j+1/2}^{c,n}(\rho_{i,j}^{c,n+1/2}, \rho_{i,j+1}^{c,n+1/2}) - \Gamma_{i,j-1/2}^{c,n}(\rho_{i,j-1}^{c,n+1/2}, \rho_{i,j}^{c,n+1/2}) \right) \\ &\quad + \lambda_y \operatorname{sgn}(\rho_{i,j}^{c,n+1} - \kappa) \left( v_2^{\text{stat}}(x_{i,j+1/2}) - v_2^{\text{stat}}(x_{i,j-1/2}) \right) \kappa \\ &\quad + \lambda_y \operatorname{sgn}(\rho_{i,j}^{c,n+1} - \kappa) \left( J_2^{c,n}(x_{i,j+1/2}) - J_2^{c,n}(x_{i,j-1/2}) \right) \kappa \leq 0. \end{aligned}$$

The proof is omitted, being entirely analogous to that of [2, Proposition 2.8], see also [1, Lemma 2.8].

### 3.6 Lipschitz continuous dependence on initial data

**Proposition 3.9.** Fix  $T > 0$ . Let  $(\mathbf{v})$ ,  $(\mathbf{H})$  and  $(\boldsymbol{\eta})$  hold. Let  $\rho_o, \sigma_o \in (\mathbf{L}^\infty \cap \mathbf{BV})(\Omega; \mathbb{R}_+^N)$ . Call  $\rho^c$  and  $\sigma^c$  the corresponding entropy weak solutions to (1.1) for  $c \in \{1, \dots, N\}$ . Then the following estimate holds:

$$\left\| \rho^c(t) - \sigma^c(t) \right\|_{\mathbf{L}^1(\Omega; \mathbb{R})} \leq \mathcal{K}^c(t) \left\| \rho_o^c - \sigma_o^c \right\|_{\mathbf{L}^1(\Omega; \mathbb{R})}$$

where  $\mathcal{K}^c(t)$  as defined in [9, Theorem 2] is given by

$$\mathcal{K}^c(t) = e^{\int_0^t \mathcal{K}_c(s) ds}.$$

where  $\mathcal{K}^c$  depends on  $\varepsilon, \|\nabla \eta_c\|_{\mathbf{L}^\infty}, \|\nabla^2 \eta_c\|_{\mathbf{L}^\infty}, L_H, \|\nabla \tilde{\eta}\|_{\mathbf{L}^\infty}$  and  $\|\nabla^2 \tilde{\eta}\|_{\mathbf{L}^\infty}$ .

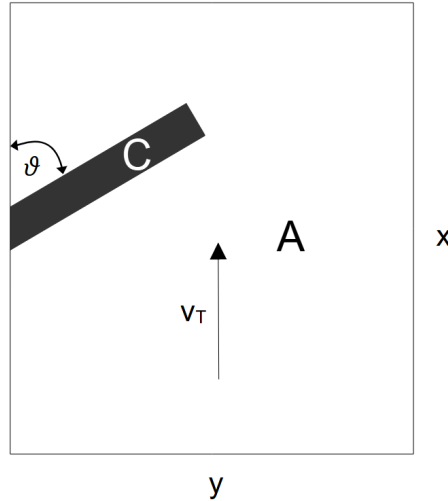
**Proof.** For this proof we rely on a result in [9], where the stability with respect to the initial data was shown under less restrictive assumptions. To show the Lipschitz estimate with respect to the initial data it is only necessary to verify the assumptions  $(\nu)$  and  $(\mathcal{J})$  of [9, Theorem 2.4]. In our setting it follows by assumptions  $(\mathbf{v}), (\mathbf{H})$  and  $(\boldsymbol{\eta})$  that  $\mathbf{v}^{stat} + \mathbf{v}_c^{dyn} \in \mathcal{C}^2(\mathbb{R}^2, \mathbb{R}^2)$  and  $\|(\mathbf{v}^{stat} + \mathbf{v}_c^{dyn})(t, \cdot, \cdot)\|_{\mathcal{C}^2}$  is uniformly bounded for all  $t \geq 0$  and for all  $c = 1, \dots, N$ . Since  $\eta_c \in (\mathbf{C}^3 \cap \mathbf{W}^{3,\infty})(\mathbb{R}^2; \mathbb{R}^+)$  for all  $c \in \{1, \dots, N\}$  the nonlocal operator  $\mathbf{I}_c : \mathbf{L}^\infty(\mathbb{R}^2; \mathbb{R}^{N+1}) \rightarrow \mathbf{C}^2(\mathbb{R}^2; \mathbb{R}^m)$ . Together with Lemma A.1 the assumptions of [9, Theorem 2.4] are met and the Lipschitz continuity with respect to the initial data follows.  $\square$

## 4 Numerical results

We follow the test setting given in [14] for two classes of objects with different parameters  $\sigma_c$ ,  $c \in \{1, 2\}$ , using the Roe scheme detailed in Algorithm 3.1

### 4.1 Test setting

We simulate a setting where particles in the shape of metal cylinders are transported on a conveyor belt moving with speed  $\|\mathbf{v}^{stat}\| = v_T = 0.1$  m/s and are redirected by a diverter which is positioned at an angle  $\vartheta = 55$  degrees. The static velocity field is given by  $\mathbf{v}^{stat} = (0.1 \ 0)^T$ . Figure 1 provides an illustration of the conveyor belt and the obstacle. Setting the mass of the diverter (Region C) high enough ( $> \rho_{max}$ ) and the velocity field to 0 inside region C prohibits parts from passing through the diverter, see [4] and [8].



**Figure 1** – Schematic view of the static field of the conveyor belt.

### 4.2 Discretisation and solution properties

For the numerical model, we introduce a uniform cartesian grid with  $\Delta x = \Delta y$  on the selected area of the conveyor belt. Initial conditions for the density at time  $t = 0$  are given by experimental data, where a total of  $N = 192$  particles were placed on the conveyor belt in the area  $\Omega_0 = [0, 0.4] \times [0, 0.6]$ . For given initial positions of the cargo  $(\mathbf{x}_i^0)_{i \leq N_n} \subset \Omega_0$  the initial

density is calculated by

$$\rho_0(\mathbf{x}) = \frac{\gamma}{2\pi\rho_{max}} \sum_{i=1}^N e^{-\frac{1}{2}\gamma\|\mathbf{x}-\mathbf{x}_i^0\|_2^2}, \quad \gamma = 0.02, \quad \rho_{max} = 2004.$$

following [10]. We then divide the initial mass vertically at  $x = 1/3$  into two separate species  $c \in \{1, 2\}$ , where we assign each population a different  $\sigma_c$ . The PCE coefficients  $\alpha_c$  are set to  $\alpha_1 = 2$  and  $\alpha_2 = 1$ . The mollifier  $\eta_c$ , which is used in the operator  $\mathbf{I}_c[\rho]$  given in (1.2), is chosen as follows

$$\eta_c(x) = \frac{\sigma_c}{2\pi} e^{-\frac{1}{2}\sigma_c\|x\|_2^2}, \quad c = 1, 2,$$

with  $\sigma_1 = 10000$  and  $\sigma_2 = 3\sigma_1 = 30000$ . The mollifier  $\tilde{\eta}$ , which is used in the Heaviside function  $H$ , see (1.2), is defined in the same way:

$$\tilde{\eta}(x) = \frac{\sigma_3}{2\pi} e^{-\frac{1}{2}\sigma_3\|x\|_2^2},$$

where we take  $\sigma_3 = 3\sigma_2$ . In the original model formulation [10], the Heaviside function was introduced to avoid densities larger than  $\rho_{max}$ . In this work we use the approximation  $H_t$  (atan), obtained using the inverse tangent.

$$H_t(u) = \frac{\arctan(50(u-1))}{\pi} + 0.5. \quad (4.1)$$

Using the inverse tangent approximation corresponds to activating the collision operator  $\mathbf{I}_c[\rho]$  very close to  $\rho_{max}$ , see [14]. Due to the smoothing of  $H$  and the introduction of the mollifier  $\tilde{\eta}$ , the maximal density constraint is not satisfied. Thus, there is no maximum principle.

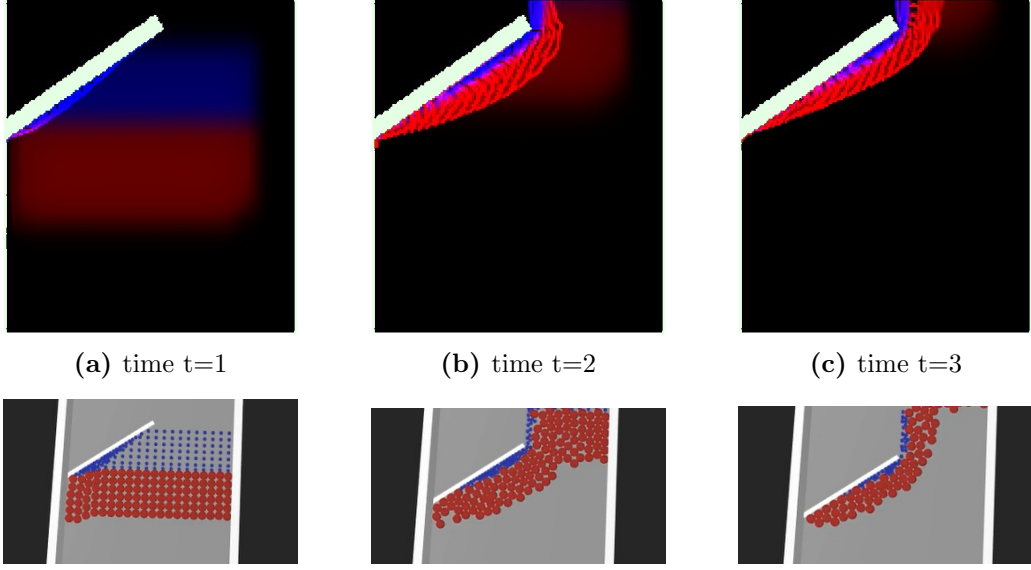
We consider two configurations. First we simulate the case where the upstream particle class is  $c = 1$  and the downstream particles are  $c = 2$ . This corresponds to placing the larger objects behind the smaller ones, so that the smaller particles arrive first at the obstacle. As it can be seen in the top row of Figure 2, the smaller particles are pushed by the incoming larger ones to the edge of the obstacle. We can also observe that the particles do not mix in the outflow, with the smaller particles (blue) staying on the left side. We observe the same effect in the microscopic simulation in Figure 2, bottom.

In the second case we swap the classes positions. The macroscopic simulation shows that the incoming smaller particles penetrate the mass of the downstream larger particles until they reach the diverter, see Figure 3. The different particle classes mix in the outflow with the smaller particles (blue) overtaking some of the larger ones. This effect can also be seen in the outflow diagram depicted in Figure 4b. The same effect can be observed in the microscopic simulation, see Figure 3, bottom.

We also analyze the amount of mass that pass the obstacle, i.e. the outflow at the end of the conveyor belt for each particle class. The normalised time-dependent mass function describing the outflow to the solution of the conservation law is given by

$$U_\rho^c(t) = \frac{1}{\int_\Omega \rho^c(\mathbf{x}, 0) \, d\mathbf{x}} \int_\Omega \rho^c(\mathbf{x}, t) \, d\mathbf{x}. \quad (4.2)$$

where  $\Omega$  is whole domain of the conveyor belt. The outflow curves for both test cases are shown in Figure 4a and Figure 4b, respectively.



**Figure 2** – The smaller particles (blue) start first and are pushed to the edge of the obstacle by the coming larger particles (red).

## 5 Conclusion

In this paper, we have proved the well-posedness of a non-local model describing heterogeneous flow on conveyor belts using the Roe scheme. Based on arguments borrowed from [14], the results are generalized to the system case. The key challenge lies in the treatment of the discontinuity occurring in the flux function. This is solved by introducing an additional convolution in  $H$ . The numerical simulations show that the model captures effects for the mixing of different particle sizes, which can be observed in the physical simulations.

## 6 Acknowledgements

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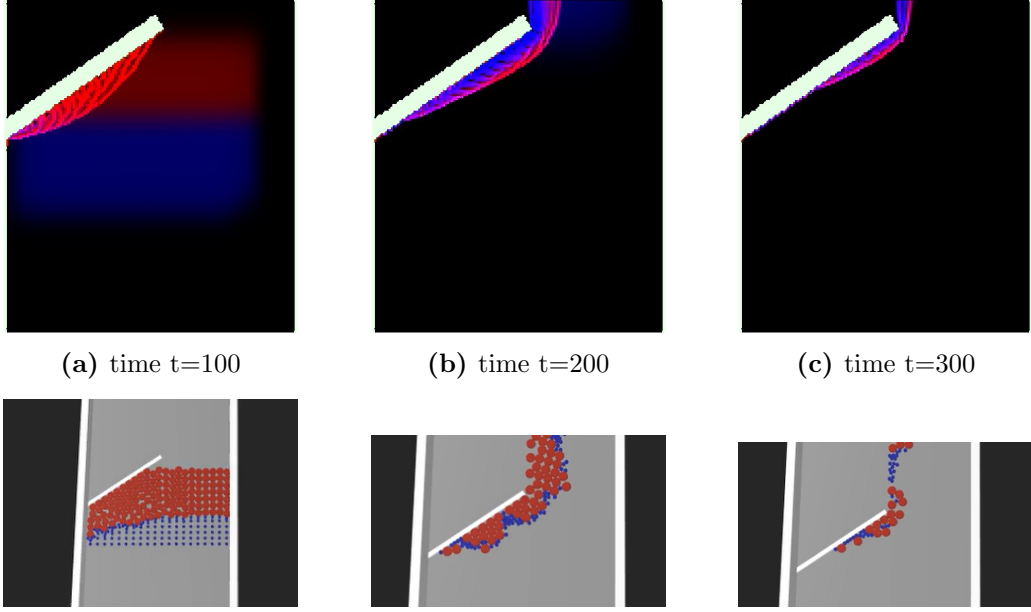
## Appendix A Technical Lemma

**Lemma A.1.** *Let  $\eta_c \in (\mathbf{C}^3 \cap \mathbf{W}^{3,\infty})(\mathbb{R}^2; \mathbb{R})$  for all  $c \in \{1, \dots, N\}$ . Then, for  $n = 0, \dots, N_T$ , for  $i, j \in \mathbb{Z}$ , the following estimates hold:*

$$\|J_\ell^{c,n}\|_{\mathbf{L}^\infty} \leq \varepsilon_c \quad \text{for } \ell = 1, 2, \quad (\text{A.1})$$

$$\left| J_1^{c,n}(x_{i+1/2,j}) - J_1^{c,n}(x_{i-1/2,j}) \right| \leq \left( 2 \left\| \nabla^2 \eta_c \right\|_{\mathbf{L}^\infty} + L_H \left\| \nabla \tilde{\eta}_c \right\|_{\mathbf{L}^\infty} \right) \varepsilon_c \Delta x \left\| r_\Omega^n \right\|_{\mathbf{L}^1}, \quad (\text{A.2})$$

$$\left| J_1^{c,n}(x_{i+1/2,j}) - J_1^{c,n}(x_{i+1/2,j+1}) \right| \leq \left( 2 \left\| \nabla^2 \eta_c \right\|_{\mathbf{L}^\infty} + L_H \left\| \nabla \tilde{\eta}_c \right\|_{\mathbf{L}^\infty} \right) \varepsilon_c \Delta y \left\| r_\Omega^n \right\|_{\mathbf{L}^1}, \quad (\text{A.3})$$



**Figure 3** – The larger particles (red) start first with the following smaller particles (blue) creeping through the mass of larger particles.

$$\begin{aligned} \left| J_2^{c,n}(x_{i,j+1/2}) - J_2^{c,n}(x_{i,j-1/2}) \right| &\leq \left( 2 \left\| \nabla^2 \eta_c \right\|_{\mathbf{L}^\infty} + L_H \left\| \nabla \tilde{\eta}_c \right\|_{\mathbf{L}^\infty} \right) \varepsilon_c \Delta y \left\| r_\Omega^n \right\|_{\mathbf{L}^1}, \\ \left| J_2^{c,n}(x_{i+1,j+1/2}) - J_2^{c,n}(x_{i,j+1/2}) \right| &\leq \left( 2 \left\| \nabla^2 \eta_c \right\|_{\mathbf{L}^\infty} + L_H \left\| \nabla \tilde{\eta}_c \right\|_{\mathbf{L}^\infty} \right) \varepsilon_c \Delta x \left\| r_\Omega^n \right\|_{\mathbf{L}^1}, \end{aligned}$$

and

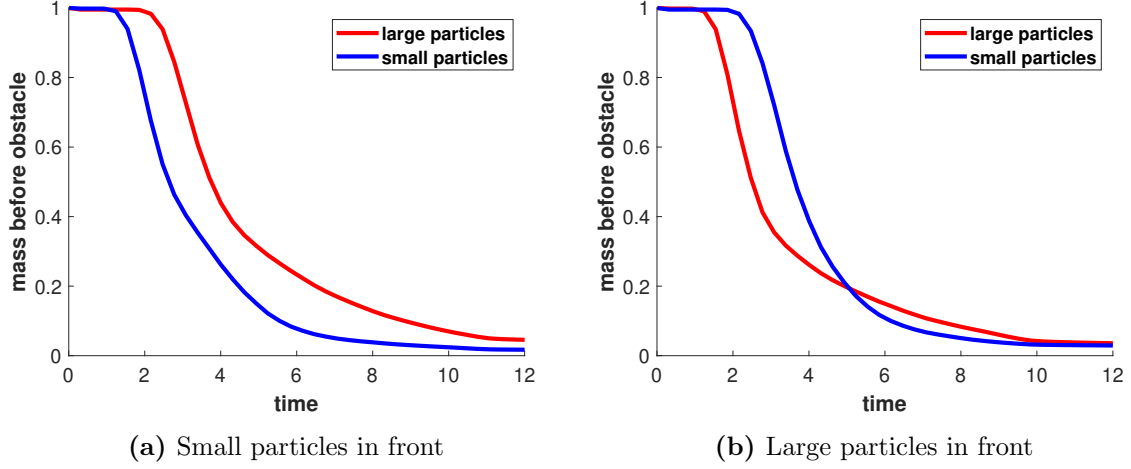
$$\left| J_1^{c,n}(x_{i+3/2,j}) - 2 J_1^{c,n}(x_{i+1/2,j}) - J_1^{c,n}(x_{i-1/2,j}) \right| \leq 2 \varepsilon_c (\Delta x)^2 \left( c_1^c \left\| r_\Omega^n \right\|_{\mathbf{L}^1} + c_2^c \left\| r_\Omega^n \right\|_{\mathbf{L}^1}^2 \right), \quad (\text{A.4})$$

$$\begin{aligned} \left| J_2^{c,n}(x_{i,j+3/2}) - 2 J_2^{c,n}(x_{i,j+1/2}) - J_2^{c,n}(x_{i,j-1/2}) \right| &\leq 2 \varepsilon_c (\Delta y)^2 \left( c_1^c \left\| r_\Omega^n \right\|_{\mathbf{L}^1} + c_2^c \left\| r_\Omega^n \right\|_{\mathbf{L}^1}^2 \right), \\ \left| J_1^{c,n}(x_{i+1/2,j}) - J_1^{c,n}(x_{i+1/2,j+1}) - J_1^{c,n}(x_{i-1/2,j}) - J_1^{c,n}(x_{i-1/2,j+1}) \right| &\leq 2 \varepsilon_c \Delta x \Delta y C_c, \quad (\text{A.5}) \\ \left| J_2^{c,n}(x_{i,j+1/2}) - J_2^{c,n}(x_{i+1,j+1/2}) - J_2^{c,n}(x_{i,j-1/2}) - J_2^{c,n}(x_{i+1,j-1/2}) \right| &\leq 2 \varepsilon_c \Delta x \Delta y C_c, \end{aligned}$$

where we set  $r_\Omega^n = \sum_{k=1}^{N+1} \alpha_k \rho_\Omega^{k,n}$  and

$$\begin{aligned} C_c &= c_1^c \left\| r_\Omega^n \right\|_{\mathbf{L}^1} + c_2^c \left\| r_\Omega^n \right\|_{\mathbf{L}^1}^2, \quad (\text{A.6}) \\ c_1^c &= 2 \left\| \nabla^3 \eta_c \right\|_{\mathbf{L}^\infty}, \\ c_2^c &= 3 \left\| \nabla^2 \eta_c \right\|_{\mathbf{L}^\infty}^2 + 2 L_H \left\| \nabla \tilde{\eta}_c \right\|_{\mathbf{L}^\infty} \left\| \nabla^2 \eta_c \right\|_{\mathbf{L}^\infty} \end{aligned}$$

**Proof.** The proof of (A.1) is immediate.



**Figure 4** – Outflow diagram for both test cases. By penetrating the mass of large particles, most of the mass of smaller particles leaves the area around the obstacle faster than the larger particles even if the large particles start first.

Pass now to (A.2). For the sake of simplicity, introduce the following notation:

$$D_+^c = \sqrt{1 + \left\| (\nabla \eta_c * r_\Omega^n)(x_{i+1/2}, y_j) \right\|^2},$$

$$D_-^c = \sqrt{1 + \left\| (\nabla \eta_c * r_\Omega^n)(x_{i-1/2}, y_j) \right\|^2}.$$

Hence, setting  $r_{\Omega, h, \ell}^n = \sum_{k=1}^{N+1} \alpha_k \rho_{\Omega, h, \ell}^{k, n}$

$$\left| J_1^{c, n}(x_{i+1/2}, y_j) - J_1^{c, n}(x_{i-1/2}, y_j) \right| \tag{A.7}$$

$$= \varepsilon_c \left| H_{i+1/2} \frac{\Delta x \Delta y}{D_+^c} \sum_{h, \ell \in \mathbb{Z}} r_{\Omega, h, \ell}^n \partial_1 \eta_c(x_{i+1/2-k}, y_{j-\ell}) - H_{i-1/2} \frac{\Delta x \Delta y}{D_-^c} \sum_{h, \ell \in \mathbb{Z}} r_{\Omega, h, \ell}^n \partial_1 \eta_c(x_{i-1/2-h}, y_{j-\ell}) \right| \tag{A.8}$$

$$\leq \varepsilon_c \left| H_{i+1/2} \right| \frac{\Delta x \Delta y}{D_+^c} \sum_{h, \ell \in \mathbb{Z}} r_{\Omega, h, \ell}^n \left( \partial_1 \eta_c(x_{i+1/2-h}, y_{j-\ell}) - \partial_1 \eta_c(x_{i-1/2-h}, y_{j-\ell}) \right) \tag{A.9}$$

$$+ \varepsilon_c H_{i+1/2} \Delta x \Delta y \left| \frac{1}{D_+^c} - \frac{1}{D_-^c} \right| \sum_{h, \ell \in \mathbb{Z}} \left| r_{\Omega, h, \ell}^n \right| \left| \partial_1 \eta_c(x_{i-1/2-h}, y_{j-\ell}) \right| \tag{A.10}$$

$$+ \varepsilon_c \left( H_{i+1/2} - H_{i-1/2} \right) \Delta x \Delta y \left| \frac{1}{D_-^c} \right| \sum_{h, \ell \in \mathbb{Z}} \left| r_{\Omega, h, \ell}^n \right| \left| \partial_1 \eta_c(x_{i-1/2-h}, y_{j-\ell}) \right|. \tag{A.11}$$

where we define for  $\mu \in \{i - 1/2, i + 1/2\}$

$$H_\mu = H \left( \Delta x \Delta y \sum_{h, \ell \in \mathbb{Z}} r_{\Omega, h, \ell}^n \tilde{\eta}_c(x_{\mu-k}, y_{j-\ell}) - r_{max} \right)$$

Consider (A.9): since  $D_+^c \geq 1$ ,  $H_\mu \leq 1$  and

$$\left| \partial_1 \eta_c(x_{i+1/2-h}, y_{j-\ell}) - \partial_1 \eta_c(x_{i-1/2-h}, y_{j-\ell}) \right| \leq \int_{x_{i-1/2-h}}^{x_{i+1/2-h}} \left| \partial_{11}^2 \eta_c(x, y_{j-\ell}) \right| dx,$$

we obtain

$$[(A.9)] \leq \varepsilon_c \Delta x \left\| \partial_{11}^2 \eta_c \right\|_{\mathbf{L}^\infty} \left\| r_\Omega^n \right\|_{\mathbf{L}^1}. \quad (A.12)$$

On the other hand, to estimate (A.10), compute

$$\left| \frac{1}{D_+^c} - \frac{1}{D_-^c} \right| = \frac{|D_+^c - D_-^c|}{D_+^c D_-^c}.$$

Introduce  $a_c(x) = \nabla \eta_c * r_\Omega^n(x)$  and  $b(z) = (1 + \|z\|^2)^{1/2}$ , for  $z \in \mathbb{R}^2$ . In particular compute  $b'(z) = \frac{\|z\|}{(1 + \|z\|^2)^{1/2}}$  and observe that  $|b'(z)| \leq 1$ . Then

$$\begin{aligned} |D_+^c - D_-^c| &= \left| b(a_c(x_{i+1/2})) - b(a_c(x_{i-1/2})) \right| = \left| b'(a_c(\tilde{x}_i)) a'_c(\tilde{x}_i) (x_{i+1/2} - x_{i-1/2}) \right| \\ &= \left| \frac{a_c(\tilde{x}_i)}{(1 + a_c(\tilde{x}_i)^2)^{1/2}} (\partial_x \nabla \eta_c * r_\Omega^n)(\tilde{x}_i) \Delta x \right| \\ &\leq \Delta x \left\| r_\Omega^n \right\|_{\mathbf{L}^1} \left\| \nabla^2 \eta_c \right\|_{\mathbf{L}^\infty}. \end{aligned} \quad (A.13)$$

Therefore,

$$\begin{aligned} \varepsilon_c H_{i+1/2} \Delta x \Delta y \left| \frac{1}{D_+^c} - \frac{1}{D_-^c} \right| \sum_{h, \ell \in \mathbb{Z}} \left| r_{\Omega, h, \ell}^n \right| \left| \partial_1 \eta_c(x_{i-1/2-h}, y_{j-\ell}) \right| \\ \leq \varepsilon_c \Delta x \left\| \nabla^2 \eta_c \right\|_{\mathbf{L}^\infty} \left\| r_\Omega^n \right\|_{\mathbf{L}^1}. \end{aligned} \quad (A.14)$$

Considering (A.11), by assumption **(H)** and A.12 we get

$$\begin{aligned} H_{i+1/2} - H_{i-1/2} &\leq L_H \left( \Delta x \Delta y \sum_{h, \ell \in \mathbb{Z}} r_{\Omega, h, \ell}^n \left( \tilde{\eta}_c(x_{i+1/2-h}, y_{j-\ell}) - \tilde{\eta}_c(x_{i-1/2-h}, y_{j-\ell}) \right) \right) \\ &\leq L_H \Delta x \left\| \nabla \tilde{\eta}_c \right\|_{\mathbf{L}^\infty} \left\| r_\Omega^n \right\|_{\mathbf{L}^1} \end{aligned}$$

Since  $\frac{(\partial_1 \eta_c * r_\Omega^n)(x_{i-1/2, j})}{D_-^c} \leq 1$  we obtain therefore

$$\varepsilon_c \left( H_{i+1/2} - H_{i-1/2} \right) \Delta x \Delta y \left| \frac{1}{D_-^c} \right| \sum_{h, \ell \in \mathbb{Z}} \left| r_{\Omega, h, \ell}^n \right| \left| \partial_1 \eta_c(x_{i-1/2-h}, y_{j-\ell}) \right|$$

$$\leq \varepsilon_c L_H \Delta x \|\nabla \tilde{\eta}_c\|_{\mathbf{L}^\infty} \|r_\Omega^n\|_{\mathbf{L}^1} \quad (\text{A.15})$$

Inserting (A.12), (A.14) and (A.15) into the estimate of (A.7) yields the desired result.

Consider now (A.4). Introduce the following notation: for  $\mu \in \{-1; 1; 3\}$  set

$$D_\mu^c = \sqrt{1 + \left\| (\nabla \eta_c * r_\Omega^n)(x_{i+\mu/2}, y_j) \right\|^2}.$$

Thus

$$\begin{aligned} & J_1^{c,n}(x_{i+3/2,j}) - 2 J_1^{c,n}(x_{i+1/2,j}) + J_1^{c,n}(x_{i-1/2,j}) \\ = & -\varepsilon_c \left( \frac{H_{i+3/2}(\partial_1 \eta_c * r_\Omega^n)(x_{i+3/2,j})}{D_3^c} - 2 \frac{H_{i+1/2}(\partial_1 \eta_c * r_\Omega^n)(x_{i+1/2,j})}{D_1^c} \right. \\ & + \frac{H_{i-1/2}(\partial_1 \eta_c * r_\Omega^n)(x_{i-1/2,j})}{D_{-1}^c} \pm \frac{H_{i+3/2}(\partial_1 \eta_c * r_\Omega^n)(x_{i+3/2,j})}{D_1^c} \\ & \pm \frac{H_{i-1/2}(\partial_1 \eta_c * r_\Omega^n)(x_{i-1/2,j})}{D_1^c} \pm \frac{H_{i+1/2}(\partial_1 \eta_c * r_\Omega^n)(x_{i+3/2,j})}{D_1^c} \\ & \left. \pm \frac{H_{i+1/2}(\partial_1 \eta_c * r_\Omega^n)(x_{i-1/2,j})}{D_1^c} \right) \\ = & -\varepsilon_c \left( \left( \frac{1}{D_3^c} - \frac{1}{D_1^c} \right) H_{i+3/2}(\partial_1 \eta_c * r_\Omega^n)(x_{i+3/2,j}) \right. \\ & + \left( H_{i+3/2} - H_{i+1/2} \right) \frac{1}{D_1^c} (\partial_1 \eta_c * r_\Omega^n)(x_{i+3/2,j}) \\ & + \frac{1}{D_1^c} \left( H_{i+1/2}(\partial_1 \eta_c * r_\Omega^n)(x_{i+3/2,j}) - H_{i+1/2}(\partial_1 \eta_c * r_\Omega^n)(x_{i+1/2,j}) \right) \\ & + \frac{1}{D_1^c} \left( H_{i+1/2}(\partial_1 \eta_c * r_\Omega^n)(x_{i-1/2,j}) - H_{i+1/2}(\partial_1 \eta_c * r_\Omega^n)(x_{i+1/2,j}) \right) \\ & + \left( H_{i-1/2} - H_{i+1/2} \right) \frac{1}{D_1^c} (\partial_1 \eta_c * r_\Omega^n)(x_{i-1/2,j}) \\ & \left. + \left( \frac{1}{D_{-1}^c} - \frac{1}{D_1^c} \right) H_{i-1/2}(\partial_1 \eta_c * r_\Omega^n)(x_{i-1/2,j}) \right). \end{aligned}$$

Consider the terms separately, forgetting for a moment the  $\varepsilon_c$  in front of everything. Focus first on the terms with common denominator  $D_1^c$  and  $H_{i+1/2}$ :

$$\begin{aligned} & \frac{H_{i+1/2}}{D_1^c} \left( (\partial_1 \eta_c * r_\Omega^n)(x_{i+3/2,j}) - (\partial_1 \eta_c * r_\Omega^n)(x_{i+1/2,j}) \right) \\ & + (\partial_1 \eta_c * r_\Omega^n)(x_{i-1/2,j}) - (\partial_1 \eta_c * r_\Omega^n)(x_{i+1/2,j}) \\ = & H_{i+1/2} \frac{\Delta x \Delta y}{D_1^c} \sum_{h,\ell \in \mathbb{Z}} r_{\Omega,h,\ell}^n \left( \partial_1 \eta_c(x_{i+3/2-h}, y_{j-\ell}) - \partial_1 \eta_c(x_{i+1/2-h}, y_{j-\ell}) \right) \\ & + \partial_1 \eta_c(x_{i-1/2-h}, y_{j-\ell}) - \partial_1 \eta_c(x_{i+1/2-h}, y_{j-\ell}) \end{aligned}$$

$$\begin{aligned}
&= H_{i+1/2} \frac{\Delta x \Delta y}{D_1^c} \sum_{h, \ell \in \mathbb{Z}} r_{\Omega, h, \ell}^n \Delta x \left( \partial_{11}^2 \eta_c(\hat{x}_{i+1-h}, y_{j-\ell}) - \partial_{11}^2 \eta_c(\hat{x}_{i-h}, y_{j-\ell}) \right) \\
&= H_{i+1/2} \frac{\Delta x \Delta y}{D_1^c} \sum_{h, \ell \in \mathbb{Z}} r_{\Omega, h, \ell}^n \Delta x \int_{\hat{x}_{i-h}}^{\hat{x}_{i+1-h}} \partial_{111}^3 \eta_c(x, y_{j-\ell}) dx \\
&\leq 2(\Delta x)^2 \left\| \partial_{111}^3 \eta_c \right\|_{\mathbf{L}^\infty} \left\| r_\Omega^n \right\|_{\mathbf{L}^1}, \tag{A.16}
\end{aligned}$$

with  $\hat{x}_{i-h} \in ]x_{i-1/2-h}, x_{i+1/2-h}[$ . Next, we obtain, since  $\frac{1}{D_1^c} \leq 1$  and for  $r_\Omega^n = \sum_{k=1}^{N+1} \alpha_k \rho_\Omega^{k,n}$

$$\begin{aligned}
&\left( H_{i+3/2} - H_{i+1/2} \right) \frac{1}{D_1^c} (\partial_1 \eta_c * r_\Omega^n)(x_{i+3/2, j}) + \left( H_{i-1/2} - H_{i+1/2} \right) \frac{1}{D_1^c} (\partial_1 \eta_c * r_\Omega^n)(x_{i-1/2, j}) \\
&\leq L_H \left( \Delta x \Delta y \sum_{h, \ell} r_{\Omega, h, \ell}^n \left( \tilde{\eta}_c(x_{i+3/2-h, j-\ell}) - \tilde{\eta}_c(x_{i+1/2-h, j-\ell}) \right) \right) \frac{(\partial_1 \eta_c * r_\Omega^n)(x_{i+3/2, j})}{D_1^c} \\
&\quad + L_H \left( \Delta x \Delta y \sum_{h, \ell} r_{\Omega, h, \ell}^n \left( \tilde{\eta}_c(x_{i-1/2-h, j-\ell}) - \tilde{\eta}_c(x_{i+1/2-h, j-\ell}) \right) \right) \frac{(\partial_1 \eta_c * r_\Omega^n)(x_{i-1/2, j})}{D_1^c} \\
&\leq L_H \Delta x \|\nabla \tilde{\eta}_c\|_{\mathbf{L}^\infty} \left\| r_\Omega^n \right\|_{\mathbf{L}^1} \left( \frac{(\partial_1 \eta_c * r_\Omega^n)(x_{i+3/2, j})}{D_1^c} + \frac{(\partial_1 \eta_c * r_\Omega^n)(x_{i-1/2, j})}{D_1^c} \right) \\
&= L_H \Delta x \|\nabla \tilde{\eta}_c\|_{\mathbf{L}^\infty} \left\| r_\Omega^n \right\|_{\mathbf{L}^1} \frac{1}{D_1^c} \left( \Delta x \Delta y \sum_{h, \ell} r_{\Omega, h, \ell}^n \int_{x_{i-1/2}}^{x_{i+3/2}} \partial_{11}^2 \eta_c(x) dx \right) \\
&\leq L_H 2(\Delta x)^2 \|\nabla \tilde{\eta}_c\|_{\mathbf{L}^\infty} \left\| \nabla^2 \eta_c \right\|_{\mathbf{L}^\infty} \left\| r_\Omega^n \right\|_{\mathbf{L}^1}^2. \tag{A.17}
\end{aligned}$$

We are left with

$$H_{i+3/2} \left( \frac{1}{D_3^c} - \frac{1}{D_1^c} \right) (\partial_1 \eta_c * r_\Omega^n)(x_{i+3/2, j}) + H_{i-1/2} \left( \frac{1}{D_{-1}^c} - \frac{1}{D_1^c} \right) (\partial_1 \eta_c * r_\Omega^n)(x_{i-1/2, j}). \tag{A.18}$$

Add and subtract to (A.18)

$$H_{i+3/2} \left( \frac{1}{D_{-1}^c} - \frac{1}{D_1^c} \right) (\partial_1 \eta_c * r_\Omega^n)(x_{i+3/2, j}).$$

and

$$H_{i-1/2} \left( \frac{1}{D_{-1}^c} - \frac{1}{D_1^c} \right) (\partial_1 \eta_c * r_\Omega^n)(x_{i+3/2, j}).$$

Hence,

$$\left( \frac{1}{D_3^c} - 2 \frac{1}{D_1^c} + \frac{1}{D_{-1}^c} \right) H_{i+3/2} (\partial_1 \eta_c * r_\Omega^n)(x_{i+3/2, j}) \tag{A.19}$$

$$+ H_{i-1/2} \left( \frac{1}{D_{-1}^c} - \frac{1}{D_1^c} \right) \left( (\partial_1 \eta_c * r_\Omega^n)(x_{i-1/2, j}) - (\partial_1 \eta_c * r_\Omega^n)(x_{i+3/2, j}) \right). \tag{A.20}$$

$$+ \left( H_{i-1/2} - H_{i+3/2} \right) \left( \frac{1}{D_{-1}^c} - \frac{1}{D_1^c} \right) (\partial_1 \eta_c * r_\Omega^n)(x_{i+3/2,j}) \quad (\text{A.21})$$

Consider first (A.20): exploiting also (A.13), we obtain

$$\begin{aligned} [(\text{A.20})] &= H_{i-1/2} \frac{D_1^c - D_{-1}^c}{D_1^c D_{-1}^c} \Delta x \Delta y \\ &\quad \sum_{h,\ell \in \mathbb{Z}} r_{\Omega,h,\ell}^n \left( \partial_1 \eta_c(x_{i-1/2-h}, y_{j-\ell}) - \partial_1 \eta_c(x_{i+3/2-h}, y_{j-\ell}) \right) \\ &= H_{i-1/2} \frac{D_1^c - D_{-1}^c}{D_1^c D_{-1}^c} \Delta x \Delta y \sum_{h,\ell \in \mathbb{Z}} r_{\Omega,h,\ell}^n \int_{x_{i+3/2-h}}^{x_{i-1/2-h}} \partial_{11}^2 \eta_c(x, y_{j-\ell}) dx \\ &\leq 2 (\Delta x)^2 \left\| \nabla^2 \eta_c \right\|_{\mathbf{L}^\infty}^2 \left\| r_\Omega^n \right\|_{\mathbf{L}^1}^2. \end{aligned} \quad (\text{A.22})$$

As far as (A.19) is concerned, focus on the terms in the brackets:

$$\begin{aligned} \frac{1}{D_3^c} - 2 \frac{1}{D_1^c} + \frac{1}{D_{-1}^c} &= \frac{D_1^c D_{-1}^c - 2 D_3^c D_{-1}^c + D_3^c D_1^c}{D_3^c D_1^c D_{-1}^c} \\ &= \frac{D_{-1}^c (D_1^c - D_3^c) - D_3^c (D_{-1}^c - D_1^c) \pm D_3^c (D_1^c - D_3^c)}{D_3^c D_1^c D_{-1}^c} \\ &= \frac{(D_{-1}^c - D_3^c)(D_1^c - D_3^c)}{D_3^c D_1^c D_{-1}^c} - \frac{D_{-1}^c - 2 D_1^c + D_3^c}{D_1^c D_{-1}^c}. \end{aligned} \quad (\text{A.23})$$

Inserting the first addend of (A.23) back into (A.19) yields

$$\frac{(D_{-1}^c - D_3^c)(D_1^c - D_3^c)}{D_3^c D_1^c D_{-1}^c} H_{i+3/2} (\partial_1 \eta_c * r_\Omega^n)(x_{i+3/2,j}) \leq 2 (\Delta x)^2 \left\| \nabla^2 \eta_c \right\|_{\mathbf{L}^\infty}^2 \left\| r_\Omega^n \right\|_{\mathbf{L}^1}^2, \quad (\text{A.24})$$

where we exploit (A.13) twice and use the fact that  $\frac{(\partial_1 \eta_c * r_\Omega^n)(x_{i+3/2,j})}{D_3^c} \leq 1$ . Concerning the second addend of (A.23), focus on its numerator: with the notation introduced before (A.13),

$$\begin{aligned} D_{-1}^c - 2 D_1^c + D_3^c &= b \left( a_c(x_{i+3/2}) \right) - 2 b \left( a_c(x_{i+1/2}) \right) + b \left( a_c(x_{-1/2}) \right) \\ &= b' \left( a_c(\tilde{x}_{i+1}) \right) a'_c(\tilde{x}_{i+1})(x_{i+3/2} - x_{i+1/2}) - b' \left( a_c(\tilde{x}_i) \right) a'_c(\tilde{x}_i)(x_{i+1/2} - x_{i-1/2}) \\ &= \Delta x \left( b' \left( a_c(\tilde{x}_{i+1}) \right) (\partial_1 \nabla \eta_c * r_\Omega^n)(\tilde{x}_{i+1}) - b' \left( a_c(\tilde{x}_i) \right) (\partial_1 \nabla \eta_c * r_\Omega^n)(\tilde{x}_i) \right) \\ &\quad \pm \Delta x b' \left( a_c(\tilde{x}_i) \right) (\partial_1 \nabla \eta_c * r_\Omega^n)(\tilde{x}_{i+1}) \\ &= \Delta x \left[ b' \left( a_c(\tilde{x}_{i+1}) \right) - b' \left( a_c(\tilde{x}_i) \right) \right] (\partial_1 \nabla \eta_c * r_\Omega^n)(\tilde{x}_{i+1}) \\ &\quad + \Delta x b' \left( a_c(\tilde{x}_i) \right) \left[ (\partial_1 \nabla \eta_c * r_\Omega^n)(\tilde{x}_{i+1}) - (\partial_1 \nabla \eta_c * r_\Omega^n)(\tilde{x}_i) \right] \\ &= \Delta x b'' \left( a_c(\bar{x}_{i+1/2}) \right) a'_c(\bar{x}_{i+1/2}) (\tilde{x}_{i+1} - \tilde{x}_i) (\partial_1 \nabla \eta_c * r_\Omega^n)(\tilde{x}_{i+1}) \\ &\quad + \Delta x b' \left( a_c(\tilde{x}_i) \right) \Delta x \Delta y \\ &\quad \sum_{h,\ell \in \mathbb{Z}} r_{\Omega,h,\ell}^n (\partial_1 \nabla \eta_c(\tilde{x}_{i+1-h}, y_{j-\ell}) - \partial_1 \nabla \eta_c(\tilde{x}_{i-h}, y_{j-\ell})) \\ &= \Delta x b'' \left( a_c(\bar{x}_{i+1/2}) \right) a'_c(\bar{x}_{i+1/2}) (\tilde{x}_{i+1} - \tilde{x}_i) (\partial_1 \nabla \eta_c * r_\Omega^n)(\tilde{x}_{i+1}) \end{aligned}$$

$$+ \Delta x b'(a_c(\check{x}_i)) \Delta x \Delta y \sum_{h,\ell \in \mathbb{Z}} r_{\Omega,h,\ell}^n \int_{\check{x}_i-h}^{\check{x}_i+1-h} \partial_{11}^2 \nabla \eta_c(x, y_{j-\ell}) dx$$

where  $\check{x}_i \in ]x_{i-1/2}, x_{i+1/2}[$  and  $\bar{x}_{i+1/2} \in ]\check{x}_i, \check{x}_{i+1}[$ . Now insert this estimate back into (A.23) and (A.19): since  $|b''(z)| \leq 1$ ,

$$\begin{aligned} & \left| \frac{D_{-1}^c - 2D_1^c + D_3^c}{D_1^c D_{-1}^c} (\partial_1 \eta_c * r_{\Omega}^n)(x_{i+3/2,j}) \right| \\ & \leq 2(\Delta x)^2 \left[ \|\partial_1 \nabla \eta_c\|_{\mathbf{L}^\infty}^2 \|r_{\Omega}^n\|_{\mathbf{L}^1}^2 + \|\partial_{11}^2 \nabla \eta_c\|_{\mathbf{L}^\infty} \|r_{\Omega}^n\|_{\mathbf{L}^1} \right]. \end{aligned} \quad (\text{A.25})$$

Lastly, we consider (A.21). As before, by assumption ((**H**)) we get

$$\begin{aligned} H_{i+3/2} - H_{i-1/2} & \leq L_H \left( \tilde{\eta}_c(x_{i+3/2,j}) * r_{\Omega}^n - \tilde{\eta}_c(x_{i-1/2,j}) * r_{\Omega}^n \right) \\ & \leq L_H 2\Delta x \|\nabla \tilde{\eta}_c\|_{\mathbf{L}^\infty} \|r_{\Omega}^n\|_{\mathbf{L}^1} \end{aligned}$$

and with (A.14) follows

$$\begin{aligned} & (H_{i+3/2} - H_{i-1/2}) \left( \frac{1}{D_{-1}^c} - \frac{1}{D_1^c} \right) (\partial_1 \eta_c * r_{\Omega}^n)(x_{i+3/2,j}) \\ & \leq L_H 2(\Delta x)^2 \|\nabla \tilde{\eta}_c\|_{\mathbf{L}^\infty} \|\nabla^2 \eta_c\|_{\mathbf{L}^\infty} \|r_{\Omega}^n\|_{\mathbf{L}^1}^2. \end{aligned} \quad (\text{A.26})$$

Collecting together (A.16), (A.17), (A.22), (A.24), (A.25) and (A.26) yields

$$\begin{aligned} & \left| J_1^{c,n}(x_{i+3/2}, y_j) - 2J_1^{c,n}(x_{i+1/2}, y_j) + J_1^{c,n}(x_{i-1/2}, y_j) \right| \\ & \leq 2\varepsilon_c (\Delta x)^2 \left( 2 \|\nabla^3 \eta_c\|_{\mathbf{L}^\infty} \|r_{\Omega}^n\|_{\mathbf{L}^1} + 3 \|\nabla^2 \eta_c\|_{\mathbf{L}^\infty}^2 \|r_{\Omega}^n\|_{\mathbf{L}^1}^2 \right. \\ & \quad \left. + 2L_H \|\nabla \tilde{\eta}_c\|_{\mathbf{L}^\infty} \|\nabla^2 \eta_c\|_{\mathbf{L}^\infty} \|r_{\Omega}^n\|_{\mathbf{L}^1}^2 \right). \end{aligned}$$

The proofs of the other inequalities follow analogously. □

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